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# TAUBERIAN THEOREM FOR GENERAL MATRIX SUMMABILITY METHOD

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**Abstract:** In this paper, we prove certain Littlewood–Tauberian theorems for general matrix summability method by imposing the Tauberian conditions such as slow oscillation of usual as well as matrix generated sequence, and the De la Vallée Poussin means of real sequences. Moreover, we demonstrate $(\bar{N}, p_n)$ and $(C, 1)$ — summability methods as the generalizations of our proposed general matrix method and establish an equivalence relation connecting them. Finally, we draw several remarks in view of the generalizations of some existing well-known results based on our results.

**Keywords:** Matrix summability, Weighted mean, Cesàro mean, Slow oscillation, Tauberian theorem.

## 1. Introduction and motivation

The study of Littlewood–Tauberian theorems has long been central to mathematical analysis, particularly in the theory of summability and asymptotic analysis. Tauberian theory was initially introduced by Tauber [27] and serves to establish essential connections between summability methods and classical convergence. Numerous researchers, including Littlewood [20], Hardy and Littlewood [11], Landau [19], Schmidt [25], and Hardy [10] contemplated Tauberian hypotheses for Abel and Cesàro summability means. Mostly, they concentrated on forcing the conditions on $(n\Delta u_n)$ to recuperate convergence of $(u_n)$ out of its Abel and Cesàro summability means. A few researchers, like Jakimovski [12] and Szász [26] are focused on imposing the conditions on the arithmetic means of $(n\Delta u_n)$, which is later signified by $(V_n^0 \Delta u_n)$.

The authors Jakimovski [12] and Szász [26] has obtained the convergence of sequences $(u_n)$ via a Littlewood–Tauberian theorem for Cesàro summability method under the conditions on the oscillatory behavior of the generated sequence $(V_n^0 \Delta u_n)$. Also, some authors namely, Çanak and Totur (see, [5, 7, 8]), Dik [9], Totur and Dik [28], Çanak (see, [2–4]), Jena *et al.* (see, [13–16]), Parida *et al.* (see, [22–24]) have proved Tauberian theorems for Abel and Cesàro summability methods by using Tauberian conditions, the oscillatory behaviour of the generator sequences $(V_n^0 \Delta u_n)$ and the De la Vallée Poussin means.

Moreover, it is well known that if the $(\bar{N}, p_n)$-summability exists for a sequence $(u_n)$, then the $(J, p)$-summability also exists. However, the $(J, p)$-summability does not imply $(\bar{N}, p_n)$-summability. During last decades of twentieth century, Tietz and Trautner [30] proved the converse part (that is, $(J, p) \to (\bar{N}, p_n)$) under certain Tauberian conditions. Subsequently, Kratz and Stadtmüller [17] proved $o-$ and $O-$types of Tauberian theorems for $(J, p)$-summability of sequence $(u_n)$ based on various conditions. Again also, Kratz and Stadtmüller [18] obtained a new result of Tauberian theorem for $(J, p)$-summability method under some specific conditions for the sequence $(p_n)$.

A few authors like, Hardy [10], Tietz [29], Tietz and Zeller [31], Ananda-Rau [1], Móricz and Rhoades [21] have obtained Tauberian theorems via $(\bar{N}, p_n)$-summability method based on different kinds of Tauberian conditions for the sequences $(u_n)$ (or $\omega_{n,p}^{(0)}(u)$). Recently, Çanak and Totur [6] also proved Tauberian theorems via $(\bar{N}, p_n)$-summability means under slow oscillation of the generated sequences $(V_n^0 \Delta u_n)$ and De la Vallée Poussin means. Furthermore, the authors Tietz [29], Tietz and Zeller [31] proved a Tauberian theorem for $(\bar{N}, p_n)$-summability method under the conditions of controlling the oscillatory behavior of sequence $(u_n)$. Again, Móricz and Rhoades [21] established necessary and sufficient conditions for $(\bar{N}, p_n)$-summablility of the sequence $(u_n)$ to be convergent, and they also proved the one-sided boundedness Tauberian theorem for $(\bar{N}, p_n)$-summability method of the sequence $(\omega_{n,p}^{(0)}(u))$ under certain specific Tauberian conditions. In fact, $(\omega_{n,p}^{(0)}(u)) = O(1)$ is a Tauberian condition for the $(\bar{N}, p_n)$-summability method which was earlier given by Hardy (see, [10, Theorem 67]).

In view of the above mentioned literature, our motivation stems from the desire to generalize and unify these results by considering a broad class of matrix summability methods. Such generalizations are particularly valuable because they encapsulate various existing methods as special cases, offering a unified framework to analyze convergence and summability. A key focus of this work is on the interplay between matrix summability and Tauberian conditions, such as the slow oscillation of sequences and the behavior of the De la Vallée Poussin means. By introducing these conditions, we aim to bridge the gap between traditional Tauberian theory and modern summability methods, capturing the subtleties of sequences generated by matrices. Our investigation also demonstrates how well-known summability methods like $(\bar{N}, p_n)$ and $(C, 1)$ emerge as specific instances of the proposed general matrix method. This establishes an equivalence that enhances the applicability and depth of summability theory. Ultimately, our results not only generalize classical theorems but also provide new perspectives and tools for exploring broader classes of summability and their implications.

## 2. Preliminaries

Let us consider $u = (u_n)$ be a real sequence and $(A = a_{n,k})$ be an infinite lower triangular matrix with non-negative entries. The infinite matrix mean of $(u_n)$ is

$$\sigma_{n,k}^{(1)}(u) = \sum_{k=0}^{n} a_{n,k} u_n, \quad n, k = 0, 1, 2, \ldots. \tag{2.1}$$

It is noticed that, if $a_{n,k} = p_k / P_n$ in (2.1), then it reduces to the weighted $(\bar{N}, p_n)$ mean of $(u_n)$ and it is denoted by

$$\sigma_{p_n}^{(1)}(u) = \frac{1}{P_n} \sum_{k=0}^{n} p_k u_n, \quad n, k = 0, 1, 2, \ldots. \tag{2.2}$$

Subsequently, if $a_{n,k} = 1/(n+1)$ in (2.1), then it reduces to the Cesàro $(C,1)$ mean of $(u_n)$ and it is denoted by

$$\sigma_C^{(1)}(u) = \frac{1}{n+1}\sum_{k=0}^{n} u_n, \quad n,k = 0,1,2,\dots.$$

We shall also use the notation,

$$\Delta(u_n) = u_{n+1} - u_n.$$

A real sequence $(u_n)$ is summable to $s$ via infinite matrix mean $\sigma_{n,k}^{(1)}(u)$, if

$$\lim_{n\to\infty} \sigma_{n,k}^{(1)}(u) = s. \tag{2.3}$$

Note that, if the limit of sequence

$$\lim_{n\to\infty} (u_n) = s$$

exists, then (2.3) also exists. However, the converse is not true in general.

Next, in view of the proposed matrix summability method the converse part can be achieved under certain conditions, called Tauberian conditions, and the associated results are known as Tauberian theorems.

For each non-negative integer $m \geq 0$, we define

$$\sigma_{n,k}^{m}(u) = \begin{cases} \displaystyle\sum_{k=0}^{n} a_{n,k}\sigma_{n,k}^{(m-1)}u_n, & m \geq 1,\\ u_n, & m = 0.\end{cases}$$

The difference between $u_n$ and the infinite matrix summability mean $(\sigma_{n,k}^{(1)}(u))$ is known as matrix Kronecker identity (or matrix generator sequences $V_{n,k}^{(0)}(\Delta u)$), and is given by

$$u_n - \sigma_{n,k}^{(1)}(u) = V_{n,k}^{(0)}(\Delta u) = \sum_{k=1}^{n} \bar{a}_{n,k}\Delta u_k, \tag{2.4}$$

where

$$\bar{A} = \bar{a}_{n,k} = \sum_{r=k}^{n} a_{n,r}, \quad n,k = 0,1,2,\dots.$$

Similarly, we define $V_{n,k}^{(m)}(\Delta u)$ for each non-negative integer $m \geq 0$ as

$$V_{n,k}^{m}(\Delta u) = \begin{cases} \displaystyle\sum_{k=0}^{n} a_{n,k}V_{n,k}^{(m-1)}(\Delta u), & m \geq 1,\\ V_{n,k}^{(0)}(\Delta u), & m = 0.\end{cases}$$

Moreover, for the matrix De la Vallée Poussin means of $(u_n)$, we may define

$$\tau_{n,[\lambda n],k}(u_n) = \sum_{k=n+1}^{[\lambda n]} (a_{[\lambda n],k} - a_{n,k})u_k, \quad \lambda \in (1,\infty)$$

and

$$\tau_{n,[\lambda n],k}(u_n) = \sum_{k=[\lambda n]+1}^{n} (a_{n,k} - a_{[\lambda n],k})u_k, \quad \lambda \in (0,1).$$

A sequence $u = (u_n)$ is oscillating slowly [4], if

$$\lim_{\lambda\to 1^+} \limsup_n \max_{n\le k\le[\lambda n]} |u_k - u_n| = 0. \tag{2.5}$$

An equivalent reformulation of (2.5) can be given as follows:

$$\lim_{\lambda\to 1^-} \limsup_n \max_{[\lambda n]\le k\le n} |u_n - u_k| = 0.$$

## 3. Some auxiliary lemmas

Before we establish the Tauberian theorems via our purposed mean, first we need the following lemmas.

**Lemma 1** [9]. *The sequence $u = (u_n)$ is oscillating slowly if and only if $V^{(0)}_{n,k}(\Delta u)$ is bounded and oscillating slowly.*

**Lemma 2.** *Let $u = (u_n)$ be a sequence of real numbers*

(i) *for $\lambda > 1$,*

$$u_n-\sigma^{(1)}_{n,k}(u)=\left(\bar a_{[\lambda n],[\lambda n]}-\bar a_{n,[\lambda n]}\right)\left(\sigma^{(1)}_{[\lambda n],k}(u)-\sigma^{(1)}_{n,k}(u)\right)-\sum_{k=n+1}^{[\lambda n]}\left(a_{[\lambda n],k}-a_{n,k}\right)(u_k-u_n),$$

(ii) *for $0 < \lambda < 1$,*

$$u_n-\sigma^{(1)}_{n,k}(u)=\left(\bar a_{n,[\lambda n]}-\bar a_{[\lambda n],[\lambda n]}\right)\left(\sigma^{(1)}_{n,k}(u)-\sigma^{(1)}_{[\lambda n],k}(u)\right)-\sum_{k=[\lambda n]+1}^{n}\left(a_{n,k}-a_{[\lambda n],k}\right)(u_n-u_k).$$

P r o o f. For $\lambda > 1$, from the definition of de la Vallée Poussin means of $(u_n)$, we have

$$\begin{aligned}\tau_{n,[\lambda n],k}(u_n) &= \sum_{k=n+1}^{[\lambda n]}(a_{[\lambda n],k}-a_{n,k})u_k = \left(\bar a_{[\lambda n],[\lambda n]}-\bar a_{n,[\lambda n]}\right)\sigma^{(1)}_{[\lambda n],k}u_k-\left(\bar a_{[\lambda n],n}-\bar a_{n,n}\right)\sigma^{(1)}_{n,k}u_k\\ &= \sigma^{(1)}_{n,k}u_k+\left(\bar a_{[\lambda n],[\lambda n]}-\bar a_{n,[\lambda n]}\right)\left(\sigma^{(1)}_{[\lambda n],k}(u)-\sigma^{(1)}_{n,k}(u)\right).\end{aligned}$$

The difference $(\tau_{n,[\lambda n],k}(u_n) - \sigma^1_{n,k}(u_n))$ can be written as

$$\tau_{n,[\lambda n],k}(u_n)-\sigma^1_{n,k}(u_n)=\left(\bar a_{[\lambda n],[\lambda n]}-\bar a_{n,[\lambda n]}\right)\left(\sigma^{(1)}_{[\lambda n],k}(u)-\sigma^{(1)}_{n,k}(u)\right). \tag{3.1}$$

Subtracting $(\sigma^{(1)}_{n,k}(u))$ from both sides of the identity

$$u_n=\tau_{n,[\lambda n],k}(u_n)-\sum_{k=n+1}^{[\lambda n]}\left(a_{[\lambda n],k}-a_{n,k}\right)(u_k-u_n),$$

we have

$$u_n-\sigma^{(1)}_{n,k}(u)=\left(\tau_{n,[\lambda n],k}(u_n)-\sigma^{(1)}_{n,k}(u)\right)-\sum_{k=n+1}^{[\lambda n]}\left(a_{[\lambda n],k}-a_{n,k}\right)(u_k-u_n). \tag{3.2}$$

Considering equations (3.1) and (3.2), we have

$$u_n-\sigma^{(1)}_{n,k}(u)=\left(\bar a_{[\lambda n],[\lambda n]}-\bar a_{n,[\lambda n]}\right)\left(\sigma^{(1)}_{[\lambda n],k}(u)-\sigma^{(1)}_{n,k}(u)\right)-\sum_{k=n+1}^{[\lambda n]}\left(a_{[\lambda n],k}-a_{n,k}\right)(u_k-u_n).$$

Next, for $0 < \lambda < 1$, the remaining part, that is, Lemma 2 (ii) can be proved in the similar lines of the proof of Lemma 2 (i). Thus, we skip the details. $\square$

## 4. Main results

In this section, we establish four theorems along with their associated corollaries. The first theorem proves a Tauberian theorem under the infinite matrix mean of order 1, specifically under the $(A, 1)$-summability mean, based on the slow oscillation of the sequence $u = (u_n)$. The second theorem extends this result to the infinite matrix mean of order $m$ that is, under $(A, m)$- summability mean, also relying on the slow oscillation of the sequence $u = (u_n)$. The third theorem establishes and proves a Tauberian theorem under the infinite matrix mean of order 1 but focuses on the slow oscillation of a generalized sequence $V = V^{(0)}_{n,k}(\Delta u)$. Similarly, the fourth theorem generalizes this result to the infinite matrix mean of order $m$ again utilizing the slow oscillation of the generalized sequence $V = V^{(0)}_{n,k}(\Delta u)$. Subsequently, we state and prove three corollaries that demonstrate how the results recover or extend earlier established results in the literature. This structured progression highlights the depth and generality of the Tauberian theorems under various summability means.

**Theorem 1.** *If $u = (u_n)$ is matrix summable to $s$, and so also oscillating slowly, then $u_n \to s$ as $n \to \infty$.*

P r o o f. Suppose $u = (u_n)$ is matrix summable to $s$, this implies that $(\sigma^{(1)}_{n,k}(u))$ is matrix summable to $s$. From equation (2.4), we have

$$V^{(0)}_{n,k}(\Delta u) = \sum_{k=1}^{n} \bar{a}_{n,k} \Delta u_k,$$

which is also matrix summable to zero.

As $V^{(0)}_{n,k}(\Delta u)$ is oscillating slowly, so from Lemma 1 and Lemma 2 (i), we get

$$\begin{aligned} V^{(0)}_{n,k}(\Delta u) - \sigma^{(1)}_{n,k}(V^{(0)}_{n,k}(\Delta u)) &= \left(\bar{a}_{[\lambda n],[\lambda n]} - \bar{a}_{n,[\lambda n]}\right)\left(\sigma^{(1)}_{[\lambda n],k}(V^{(0)}_{n,k}(\Delta u)) - \sigma^{(1)}_{n,k}(V^{(0)}_{n,k}(\Delta u))\right) \\ &\quad - \sum_{k=n+1}^{[\lambda n]} \left(a_{[\lambda n],k} - a_{n,k}\right)(V^{(0)}_{n,k}(\Delta u) - V^{(0)}_{n,n}(\Delta u)). \end{aligned}$$

Now,

$$\begin{aligned} \left|V^{(0)}_{n,k}(\Delta u) - \sigma^{(1)}_{n,k}(V^{(0)}_{n,k}(\Delta u))\right| &\leq \left|\bar{a}_{[\lambda n],[\lambda n]} - \bar{a}_{n,[\lambda n]}\right|\left|\sigma^{(1)}_{[\lambda n],k}(V^{(0)}_{n,k}(\Delta u)) - \sigma^{(1)}_{n,k}(V^{(0)}_{n,k}(\Delta u))\right| \\ &\quad + \Big|\sum_{k=n+1}^{[\lambda n]} (a_{[\lambda n],k} - a_{n,k})\big(V^{(0)}_{n,k}(\Delta u) - V^{(0)}_{n,n}(\Delta u)\big)\Big|. \end{aligned} \tag{4.1}$$

Next, taking $\limsup$ to both sides of equation (4.1) as $n \to \infty$, and $\sigma^{(1)}_{n,k}(V^{(0)}_{n,k}(\Delta u))$ being convergent, so in view of equation (2.5), we obtain

$$\begin{aligned} &\limsup_{n} \left|V^{(0)}_{n,k}(\Delta u) - \sigma^{(1)}_{n,k}(V^{(0)}_{n,k}(\Delta u))\right| \\ &\leq \limsup_{n} \max_{n+1 \leq k \leq [\lambda n]} \Big|\sum_{k=n+1}^{[\lambda n]} (a_{[\lambda n],k} - a_{n,k})\big(V^{(0)}_{n,k}(\Delta u) - V^{(0)}_{n,n}(\Delta u)\big)\Big|. \end{aligned} \tag{4.2}$$

Letting $\lambda \to 1^+$ in (4.2), we have

$$\limsup_{n} \left|V^{(0)}_{n,k}(\Delta u) - \sigma^{(1)}_{n,k}(V^{(0)}_{n,k}(\Delta u))\right| \leq 0.$$

This implies that

$$V_{n,k}^{(0)}(\Delta u) = o(1) \quad (n \to \infty).$$

Moreover, since $(u_n)$ is matrix summable to $s$ and $V_{n,k}^{(0)}(\Delta u) = o(1)$, consequently,

$$\lim_{n\to\infty} u_n = s.$$

This completes the proof. □

**Remark 1.** *For $\lambda \to 1^+$, Theorem 1 can also be proved by using (2.4) and Lemma 2.*

**Theorem 2.** *If $u = (u_n)$ is $(A, m)$ summable to $s$ and so also $(u_n)$ is oscillating slowly, then $\lim_{n\to\infty} u_n = s$.*

P r o o f. Let $u = (u_n)$ be oscillating slowly, then $\sigma_{n,k}^m(u)$ is oscillating slowly (by Lemma 1). Since $u = (u_n)$ is $(A, m)$ summable to $s$,

$$\lim_{n\to\infty} \sigma_{n,k}^m(u) = s. \tag{4.3}$$

We next write,

$$\sigma_{n,k}^m(u) = \sigma_{n,k}^1(u)(\sigma_{n,k}^{m-1}(u)). \tag{4.4}$$

Clearly from equations (4.3) and (4.4), $u = (u_n)$ is $(A, m-1)$ summable to $s$. Again, $(\sigma_{n,k}^{m-1}(u))$ is oscillating slowly (by Lemma 1). Thus, we have

$$\lim_{n\to\infty} \sigma_{n,k}^{m-1}(u) = s,$$

and continuing in this way, we get $\lim_{n\to\infty}(u_n) = s$. □

**Corollary 1.** *If $u = (u_n)$ is $(\bar{N}, p_n)$ summable to $s$ and so also $(u_n)$oscillating slowly, then $u_n \to s$ as $n \to \infty$.*

P r o o f. If we substitute

$$a_{n,k} = \frac{p_k}{P_n},$$

then the matrix transform reduces to the weighted transform $(\bar{N}, p_n)$, where $(p_k)$ is the positive real sequence and $P_n = \sum_{k=0}^n p_k$. Next, the proof is similar to that of the proof of Theorem 1, thus we skip the details. □

**Corollary 2.** *If $u = (u_n)$ is $(C, 1)$-summable to $s$ and so also $(u_n)$ is oscillating slowly, then $u_n \to s$ as $n \to \infty$.*

P r o o f. If we substitute

$$a_{n,k} = \frac{1}{n+1},$$

then the matrix transform reduces to the Cesàro transform $(C, 1)$. Next, the proof is similar to that of the proof of Theorem 1, thus we skip the details. □

**Corollary 3.** *The following embedding equivalence relations are valid for a sequence* $(u_n)$*:*

(i) *Matrix summability* $(A, 1) \Longrightarrow (\bar{N}, p_n)$*-summability;*
(ii) $(\bar{N}, p_n)$*-summability* $\Longrightarrow (C, 1)$*-summability;*
(iii) *Matrix summability* $(A, 1) \Longrightarrow (C, 1)$*-summability.*

P r o o f. (i) Suppose $u = (u_n)$ is $(A, 1)$ summable to $s$. This implies that

$$\sigma^{(1)}_{n,k}(u) = \sum_{k=0}^{n} a_{n,k} u_k$$

is convergent to $s$ under matrix summability. If we substitute $a_{n,k} = p_k/P_n$, then $\sigma^{(1)}_{n,k}(u)$ mean reduces to $\sigma^{(1)}_{p_n}(u)$ mean, which is the the weighted mean (2.2). Thus, $(\sigma^{(1)}_{p_n}(u))$ is convergent to $s$.

(ii) If $u = (u_n)$ is $(\bar{N}, p_n)$ summable to $s$. This implies that

$$\sigma^{(1)}_{n,p_n}(u) = \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k$$

is convergent to $s$ under $(\bar{N}, p_n)$-summability mean. If we substitute $p_n = 1$, then $\sigma^{(1)}_{p_n}(u)$ mean reduces to $\sigma^{(1)}_{C}(u)$ mean, which is the Cesàro mean. Thus, $(\sigma^{(1)}_{C}(u))$ is convergent to $s$.

(iii) If $u = (u_n)$ is matrix summable to $s$. This implies that

$$\sigma^{(1)}_{n,k}(u) = \sum_{k=0}^{n} a_{n,k} u_n$$

is convergent to $s$ under matrix summability. If we substitute $a_{n,k} = 1/(n+1)$, then $(\sigma^{(1)}_{n,k}(u))$ mean reduces to $(\sigma^{(1)}_{C}(u))$ mean, which is the Cesàro mean. Thus, $(\sigma^{(1)}_{C}(u))$ is convergent to $s$. $\square$

**Theorem 3.** *If* $u = (u_n)$ *is matrix summable to* $s$ *and* $V = V^{(0)}_{n,k}(\Delta u)$ *is oscillating slowly, then* $u_n \to s$ *as* $n \to \infty$*.*

P r o o f. As $u = (u_n)$ is matrix summable to $s$, so $\sigma^{(1)}_{n,k}(u)$ is also matrix summable to $s$. Therefore by equation (2.4), $V^{(0)}_{n,k}(\Delta u)$ is matrix summable to zero. Also, considering the identity under (2.4) for $V^{(0)}_{n,k}(\Delta u)$, we fairly say that $V(V^{(0)}_{n,k}(\Delta u))$ is matrix summable to zero. Consequently, $V(V^{(0)}_{n,k}(\Delta u))$ is oscillating slowly by Lemma 1.

Next by Lemma 2 (i),

$$\begin{gathered} V(V^{(0)}_{n,k}(\Delta u)) - \sigma^{(1)}_{n,k}(V(V^{(0)}_{n,k}(\Delta u))) = \\ \left(\bar{a}_{[\lambda n],[\lambda n]} - \bar{a}_{n,[\lambda n]}\right) \left(\sigma^{(1)}_{[\lambda n],k}(V(V^{(0)}_{n,k}(\Delta u))) - \sigma^{(1)}_{n,k}(V(V^{(0)}_{n,k}(\Delta u)))\right) \\ - \sum_{k=n+1}^{[\lambda n]} (a_{[\lambda n],k} - a_{n,k})\big(V(V^{(0)}_{n,k}(\Delta u)) - V(V^{(0)}_{n,n}(\Delta u))\big). \end{gathered} \tag{4.5}$$

Also by (4.5),

$$\begin{aligned} &\left|V(V^{(0)}_{n,k}(\Delta u)) - \sigma^{(1)}_{n,k}(V(V^{(0)}_{n,k}(\Delta u)))\right| \\ \leq &\left|\bar{a}_{[\lambda n],[\lambda n]} - \bar{a}_{n,[\lambda n]}\right|\left|\sigma^{(1)}_{[\lambda n],k}(V(V^{(0)}_{n,k}(\Delta u))) - \sigma^{(1)}_{n,k}(V(V^{(0)}_{n,k}(\Delta u)))\right| \\ &+\left|\sum_{k=n+1}^{[\lambda n]} (a_{[\lambda n],k} - a_{n,k})\big(V(V^{(0)}_{n,k}(\Delta u)) - V(V^{(0)}_{n,n}(\Delta u))\big)\right|. \end{aligned} \tag{4.6}$$

Now taking lim sup to both sides of equation (4.7) as $n \to \infty$, and $\sigma^{(1)}_{n,k}(V(V^{(0)}_{n,k}(\Delta u)))$ being convergent, so in view of equation (2.5), we have

$$\begin{aligned} &\limsup_n \left|V(V^{(0)}_{n,k}(\Delta u)) - \sigma^{(1)}_{n,k}(V(V^{(0)}_{n,k}(\Delta u)))\right| \\ \leq &\limsup_n \max_{n+1\leq k\leq[\lambda n]} \left|\sum_{k=n+1}^{[\lambda n]} \left(a_{[\lambda n],k} - a_{n,k}\right) \left(V(V^{(0)}_{n,k}(\Delta u) - V(V^{(0)}_{n,n}(\Delta u)))\right)\right|. \end{aligned} \tag{4.7}$$

Letting $\lambda \to 1^+$, we have

$$\limsup_n \left|V(V^{(0)}_{n,k}(\Delta u)) - \sigma^{(1)}_{n,k}(V(V^{(0)}_{n,k}(\Delta u)))\right| \leq 0.$$

Thus, $V(V^{(0)}_{n,k}(\Delta u)) = o(1)$ as $n \to \infty$. Since $(u_n)$ is matrix summable to $s$ and $V(V^{(0)}_{n,k}(\Delta u)) = o(1)$ as $n \to \infty$, $\lim_{n\to\infty} u_n = s$. This completes the proof. □

**Theorem 4.** *If $(u_n)$ is $(A, m)$ summable to $s$ and $V^{(0)}_{n,k}(\Delta u)$ is oscillating slowly, then $\lim_{n\to\infty} u_n = s$.*

P r o o f. As $V^{(0)}_{n,k}(\Delta u)$ is oscillating slowly, setting $u = (u_n)$ in place of $V^{(0)}_n(\Delta u)$, $\sigma^{(m)}_{n,k}(V^{(0)}_{n,k}(\Delta u))$ is oscillating slowly (by Lemma 1). Again, as $V^{(0)}_{n,k}(\Delta u)$ is $(A, m)$ summable to $s$, so by Theorem 3, we have

$$\lim_{n\to\infty} \sigma^{(m)}_{n,k}(V^{(11)}_{n,k}(\Delta u)) = s. \tag{4.8}$$

By definition,

$$\sigma^{(m)}_{n,k}(V^{(0)}_{n,k}(\Delta u)) = \sigma^{(0)}_{n,k}(V^{(0)}_{n,k}(\Delta u))(\sigma^{(m-1)}_{n,k}(V^{(0)}_{n,k}(\Delta u)). \tag{4.9}$$

From (4.8) and (4.9), we have $V^{(0)}_{n,k}(\Delta u)$ is $(A, m-1)$ summable to $s$.

Again by Lemma 1, since $(\sigma^{(m-1)}_{n,k}(V^{(0)}_{n,k}(\Delta u)))$ is oscillating slowly, so we have by Theorem 3

$$\lim_{n\to\infty} \sigma^{(m-1)}_{n,k}(V^{(0)}_{n,k}(\Delta u)) = s.$$

Continuing in this way, we get $\lim_{n\to\infty}(V^{(0)}_{n,k}(\Delta u)) = s$. □

## 5. Concluding remarks and observations

In this concluding part of our investigation, we draw several observations and further remarks concerning various results which we have established in this article.

**Remark 2.** *If $\bar{a}_{n,k} = p_k/P_n$, then matrix generator sequence $(V_{n,k}^{(0)}(\Delta u_n))$ reduces to the weighted generator sequence $(V_{n,p_n}^{(0)}(\Delta u_n))$, that is,*

$$V_{n,k}^{(0)}(\Delta u_n) = \sum_{k=1}^{n} \bar{a}_{n,k}\Delta u_k, \quad n,k = 0,1,2,...$$

*reduces to*

$$V_{n,p_n}^{(0)}(\Delta u_n) = \frac{1}{P_n}\sum_{r=k}^{n} p_r \Delta u_r.$$

**Remark 3.** *If $p_r = 1$ and $\sum_{r=k}^{n} P_r = n+1$, then the weighted generator sequence $V_{n,p_n}^{(0)}(\Delta u_n)$ reduces to the Cesàro generator sequence $V_C^{(0)}(\Delta u_n)$, that is,*

$$V_{n,p_n}^{(0)}(\Delta u_n) = \frac{1}{P_n}\sum_{r=k}^{n} P_k \Delta u_k$$

*reduces to*

$$V_C^{(0)}(\Delta u_n) = \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_n.$$

**Remark 4.** *If $\bar{a}_{n,k} = k/n+1$, then the matrix generator sequence $(V_{n,k}^{(0)}(\Delta u_n))$ reduces to the Cesàro generator sequence $V_{n,c}^{(0)}(\Delta u_n)$, that is,*

$$V_{n,k}^{(0)}(\Delta u_n) = \sum_{r=k}^{n} \bar{a}_{n,r}\Delta u_r, \quad n,r = 0,1,2,...$$

*reduces to*

$$V_C^{(0)}(\Delta u_n) = \frac{1}{n+1}\sum_{k=1}^{n} k\Delta u_n.$$

**Remark 5.** *If $u = (u_n)$ is $(\bar{N}, p_n)$ summable to $s$ and $V = V_{n,p_n}^{(0)}(\Delta u)$ is oscillating slowly, then $u_n \to s$ as $n \to \infty$.*

**Remark 6.** *If $u = (u_n)$ is $(C,1)$ summable to $s$ and $V = V_C^{(0)}(\Delta u)$ is oscillating slowly, then $u_n \to s$ as $n \to \infty$.*

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