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A REMARK AND AN IMPROVED VERSION ON RECENT RESULTS CONCERNING RATIONAL FUNCTIONS[1]

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Abstract: This paper extends as a lemma an auxiliary result obtained by Singh and Chanam. Using it, we prove a refinement of the Turán-type inequality for rational functions obtained recently by Akhter et al. Next, using examples, we discuss the result of Mir et al.

Keywords: Rational function, Polynomial, Inequalities in complex domain.

1. Introduction

Let $\mathbb{C}$ denote the set of complex numbers z, and let $\Re(z)$ be the real part of z. Let $\mathcal{P}_n$ be the set of all complex polynomials

$$g(z) := \sum_{k=0}^{n} d_k z^k$$

of degree at most n, and let $g'(z)$ be the derivative of $g(z)$. Let $S_l := \{z : |z| = l\}$, and let R_l^- and R_l^+ be the interior and exterior of S_l, respectively. For $\gamma_k \in \mathbb{C}$, let

$$w(z) := \prod_{k=1}^{n}(z-\gamma_k); \quad V(z) := \prod_{k=1}^{n}\left(\frac{1-\overline{\gamma_k}z}{z-\gamma_k}\right),$$

and let

$$\mathcal{R}_n = \mathcal{R}_n(\gamma_1, \gamma_2, \ldots, \gamma_n) := \left\{\frac{g(z)}{w(z)} : g \in \mathcal{P}_n\right\}$$

be the set of rational functions having a finite limit as $z \to \infty$ and poles $\gamma_1, \gamma_2, \ldots, \gamma_n$, such that $\gamma_k \in R_1^+$. The well-known result of Bernstein [4] states the following.

Theorem 1 [4]. *For any $z \in \mathbb{C}$, if $g \in \mathcal{P}_n$, then*

$$\max_{z\in S_1}|g'(z)| \le n \max_{z\in S_1}|g(z)|.$$

Confining himself to the set of polynomials whose zeros all lie in $S_1 \cup R_1^+$, Erdös conjectured, which was later confirmed by Lax [5], that

$$\max_{z\in S_1}|g'(z)| \le \frac{n}{2} \max_{z\in S_1}|g(z)|.$$

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If all zeros of $g(z)$ are in $S_1 \cup R_1^-$, Turán [9] proved that

$$\max_{z\in S_1}|g'(z)| \geq \frac{n}{2}\max_{z\in S_1}|g(z)|.$$

Li et al. [6] derived inequalities similar to Bernstein inequalities for rational functions $q \in \mathcal{R}_n$, considering prescribed poles $\gamma_1, \gamma_2, \ldots, \gamma_n$ and replacing z^n by the Blashke product $V(z)$. They established the following result featuring these poles.

Theorem 2 [6]. *If $q \in \mathcal{R}_n$ has all its zeros in $S_1 \cup R_1^+$, then, for $z \in S_1$,*

$$|q'(z)| \leq \frac{1}{2}|V'(z)||q(z)|.$$

Equality holds for $q(z) = a_0V(z) + b_0$ with $|a_0| = |b_0| = 1$.

Aziz and Shah [2] improved this inequality as follows.

Theorem 3 [2]. *Let $q \in \mathcal{R}_n$ and all its zeros lye in $S_1 \cup R_1^+$. If $e_1, e_2, \ldots, e_n$ are the zeros of $V(z) + \xi$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are the zeros of $V(z) - \xi$, $\xi \in S_1$, then, for $z \in S_1$,*

$$|q'(z)| \leq \frac{|V'(z)|}{2}\left\{\Big(\max_{1\leq k\leq n}|q(e_k)|\Big)^2 + \Big(\max_{1\leq k\leq n}|q(\epsilon_k)|\Big)^2\right\}^{1/2}. \tag{1.1}$$

Recently, Mir et al. [7] proved the following result, which gives a generalized and strengthened upper estimate than that in Theorem 3.

Theorem 4 [7]. *Let*

$$q(z) = \frac{g(z)}{w(z)} \in \mathcal{R}_n,$$

where

$$g(z) = z^s\sum_{k=0}^{m-s} d_kz^k$$

is an m-degree polynomial ($m \leq n$) having all its zeros in $S_l \cup R_l^+$, $l \geq 1$, except for a zero of multiplicity s at the origin. If $e_1, e_2, \ldots, e_n$ are the zeros of $V(z) + \xi$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are the zeros of $V(z) - \xi$, $\xi \in S_1$, then, for $z \in S_1$,

$$\begin{aligned}|q'(z)| \leq \frac{|V'(z)|}{2}\Bigg\{&\Big(\max_{1\leq k\leq n}|q(e_k)|\Big)^2 + \Big(\max_{1\leq k\leq n}|q(\epsilon_k)|\Big)^2\\ -4\Bigg(\frac{l}{1+l}\left(\frac{|d_0| - l^{m-s}|d_{m-s}|}{|d_0| + l^{m-s}|d_{m-s}|}\right) - \frac{sl}{1+l} - &\frac{2m - n(1+l)}{2(1+l)}\Bigg)\frac{|q(z)|^2}{|V'(z)|}\Bigg\}^{1/2}.\end{aligned} \tag{1.2}$$

Furthermore, Li et al. [6] obtained the following inequality for rational functions, which generalizes the polynomial inequality of Turán [9].

Theorem 5 [6]. *If $q \in \mathcal{R}_n$ has all its zeros in $S_1 \cup R_1^-$, then, for $z \in S_1$,*

$$|q'(z)| \geq \frac{1}{2}|V'(z)||q(z)|.$$

Recently, Akhter et al. [1] obtained the following result by introducing a complex parameter α which provides an improvement and a generalization of Theorem 5.

Theorem 6 [1]. *Assume that*

$$q(z) = \frac{g(z)}{w(z)} \in \mathcal{R}_n,$$

where

$$g(z) = z^s \sum_{k=0}^{m-s} d_k z^k$$

is an m-degree polynomial ($m \leq n$) *having all zeros in* $S_l \cup R_l^-$, $l \leq 1$, *and a zero of multiplicity s at the origin. Then, for every complex* δ, $|\delta| \leq 1$, *and* $z \in S_1$,

$$\left|zq'(z) + \frac{(m-s)\delta}{1+l}q(z)\right| \geq \frac{1}{2}\left\{|V'(z)| + \frac{1}{1+l}\Big(l(2s-n) + 2m - n + 2(m-s)\Re(\delta)\Big)\right\}|q(z)|.$$

In this paper, we first establish a refined inequality of Theorem 6 by including certain coefficients of the polynomial, and then discuss Theorem 4 due to Mir et al. [7] using counterexamples that they claim improve the bound given by Theorem 3. The paper is organized as follows. Section 2 presents the main result, some remarks, and corollaries. In addition, we discuss the result due to Mir et al. [7]. Section 3 presents some auxiliary results necessary to establish the main result. Section 4 provides a proof of the main result. Section 5 concerns the conclusion.

2. Main result and discussion

Here, we present the following result concerning rational functions, which generalizes and sharpens the polynomial inequality of Turán [9].

Theorem 7. *Let*

$$q(z) = \frac{g(z)}{w(z)} \in \mathcal{R}_n,$$

where

$$g(z) = z^s \sum_{k=0}^{m-s} d_k z^k$$

is an m-degree polynomial ($m \leq n$) *having all its zeros in* $S_l \cup R_l^-$, $l \leq 1$, *and a zero of multiplicity s at the origin. Then, for every complex* δ, $|\delta| \leq 1$, *and* $z \in S_1$,

$$\begin{aligned} \left|zq'(z) + \frac{(m-s)\delta}{1+l}q(z)\right| \geq \frac{1}{2}\Bigg\{|V'(z)| + \frac{1}{1+l}\Bigg(l(2s-n) + 2m - n \\ +2l\left(\frac{\sqrt{l^{m-s}|d_{m-s}|} - \sqrt{|d_0|}}{\sqrt{l^{m-s}|d_{m-s}|}}\right) + 2(m-s)\Re(\delta)\Bigg)\Bigg\}|q(z)|. \end{aligned} \tag{2.1}$$

Remark 1. Since the zeros of the polynomial

$$h(z) = \frac{g(z)}{z^s} = \sum_{k=0}^{m-s} d_k z^k$$

are in $S_l \cup R_l^-$, $l \leq 1$, we have

$$\left|\frac{d_0}{d_{m-s}}\right| \leq l^{m-s},$$

which is equivalent to

$$\sqrt{l^{m-s}|d_{m-s}|} \geq \sqrt{|d_0|}. \tag{2.2}$$

On the right-hand side of inequality (2.1) of Theorem 7, there is an extra term contributed by the quantity

$$2l\left(\frac{\sqrt{l^{m-s}|d_{m-s}|}-\sqrt{|d_0|}}{\sqrt{l^{m-s}|d_{m-s}|}}\right),$$

which in view of (2.2) is nonnegative, and hence Theorem 7 refines Theorem 6.

Taking $\delta = 0$ and $m = n$ in Theorem 7, we obtain the following interesting result, which gives a generalization and an improvement of Theorem 5 due to Li et al. [6], and an improvement of the result established by Akhter et al. [1, Corollary 2.2].

Corollary 1. *Let*

$$q(z) = \frac{g(z)}{w(z)} \in \mathcal{R}_n,$$

where

$$g(z) = z^s \sum_{k=0}^{n-s} d_k z^k$$

is an n-degree polynomial having all its zeros in $S_l \cup R_l^-$, $l \leq 1$, and a zero of multiplicity s at the origin. Then, for $z \in S_1$,

$$|q'(z)| \geq \frac{1}{2}\left\{|V'(z)| + \frac{1}{1+l}\left(2ls + n(1-l) + 2l\left(\frac{\sqrt{l^{n-s}|d_{n-s}|}-\sqrt{|d_0|}}{\sqrt{l^{n-s}|d_{n-s}|}}\right)\right)\right\}|q(z)|.$$

Moreover, taking $l = 1$ in Theorem 7, we obtain a result that improves the known result [1, Corollary 2.4] obtained by Akhter et al.

Corollary 2. *Let*

$$q(z) = \frac{g(z)}{w(z)} \in \mathcal{R}_n,$$

where

$$g(z) = z^s \sum_{k=0}^{m-s} d_k z^k$$

is an m-degree polynomial ($m \leq n$) having all its zeros in $S_1 \cup R_1^-$ and a zero of multiplicity s at the origin. Then, for every complex δ, $|\delta| \leq 1$, and $z \in S_1$,

$$\left|zq'(z) + \frac{(m-s)\delta}{2}q(z)\right| \geq \frac{1}{2}\left\{|V'(z)| + (s+m-n) + \left(\frac{\sqrt{|d_{m-s}|}-\sqrt{|d_0|}}{\sqrt{|d_{m-s}|}}\right) + (m-s)\Re(\delta)\right\}|q(z)|.$$

Next, the claim that the bound in inequality (1.2) of Theorem 4 proved by Mir et al. [7] sharpens the bound in inequality (1.1) of Theorem 3 due to Aziz and Shah [2] follows in the case when the quantity

$$\left(\frac{l}{1+l}\left(\frac{|d_0| - l^{m-s}|d_{m-s}|}{|d_0| + l^{m-s}|d_{m-s}|}\right) - \frac{sl}{1+l} - \frac{2m-n(1+l)}{2(1+l)}\right) = A$$

on the right-hand side of inequality (1.2) of Theorem 4 is nonnegative. But this is not always the case, as the following counterexamples illustrate.

Example 1. Let $q \in \mathcal{R}_6$, where $g(z) = z^3(z^3 - z^2 + z - 1)$ has the zeros $\{1, i, -i\}$ on $|z| = 1$ and the remaining zeros at the origin. It can be easily seen that this polynomial gives $A = -1.5$ in Theorem 4.

Example 2. Let $q \in \mathcal{R}_5$, where $g(z) = z^3(z^2 - 4)$ has the zeros $\{-2, 2\}$ on $|z| = 2$ and the remaining zeros at the origin. For this polynomial, we have $A = -1.166\bar{6}$.

3. Lemmas

We must incorporate the following lemmas into our proof to demonstrate the theorem. Aziz and Zargar [3] established the first.

Lemma 1 [3]. *If*

$$V(z) = \prod_{k=1}^{n} \left(\frac{1 - \overline{\gamma_k} z}{z - \gamma_k} \right),$$

then, for $z \in S_1$,

$$\Re\left(\frac{zw'(z)}{w(z)}\right) = \frac{n - |V'(z)|}{2}.$$

Lemma 2. *If* $0 \le a \le 1$, $0 \le b \le 1$, *and* $0 \le l \le 1$, *then*

$$\frac{2}{1+a} \ge 1 + l\sqrt{b} - l\sqrt{ab}.$$

P r o o f. For $a = 1$, the inequality follows trivially. So, take $a < 1$, then

$$\frac{1+\sqrt{a}}{1+a} > 1 \ge l\sqrt{b};$$

that is,

$$\frac{1-a}{1+a} > l\sqrt{b}\frac{1-a}{1+\sqrt{a}} = l\sqrt{b} - l\sqrt{ab}.$$

Hence,

$$\frac{2}{1+a} > 1 + l\sqrt{b} - l\sqrt{ab}.$$

□

The following lemma we prove is a generalization of a finding by Singh and Chanam [8].

Lemma 3. *If* $g \in \mathcal{P}_n$ $(n \ge 1)$ *has all its zeros in* $S_l \cup R_l^-$, $l \le 1$, *then, for* $z \in S_1$ *such that* $g(z) \ne 0$,

$$\Re\left(z\frac{g'(z)}{g(z)}\right) \ge \frac{1}{1+l}\left\{n + l\left(\frac{\sqrt{l^n|d_n|} - \sqrt{|d_0|}}{\sqrt{l^n|d_n|}}\right)\right\}. \tag{3.1}$$

Remark 2. As the abstract mentioned, for $l = 1$, this lemma reduces to Lemma 2 of Singh and Chanam [8].

P r o o f. For simplicity, suppose that $d_n = 1$. We apply mathematical induction on the degree of $g(z)$.

If $n = 1$, then $g(z) = z - z_0$, $z_0 \in S_l \cup R_l^-$, and, for $z \in S_1$ and $z \neq z_0$, we have

$$\Re\left(z\frac{g'(z)}{g(z)}\right) = \Re\left(\frac{z}{z - z_0}\right) \geq \frac{1}{1+|z_0|}.$$

By basic computation, we can show that, for $z_0 \in S_l \cup R_l^-$,

$$\frac{1}{1+|z_0|} \geq \frac{1}{1+l}\left\{1 + l\left(\frac{\sqrt{l} - \sqrt{|z_0|}}{\sqrt{l}}\right)\right\}.$$

So,

$$\Re\left(z\frac{g'(z)}{g(z)}\right) \geq \frac{1}{1+l}\left\{1 + l\left(\frac{\sqrt{l} - \sqrt{|z_0|}}{\sqrt{l}}\right)\right\},$$

which is inequality (3.1) for $n = 1$.

Suppose that (3.1) holds for all polynomials of degree $\leq M$.

Let $g(z) = (z - w)G(z)$, $w \in S_l \cup R_l^-$, where

$$G(z) = \sum_{k=0}^{M} d_k z^k$$

is a polynomial of degree M having all its zeros in $S_l \cup R_l^-$, then

$$\Re\left(z\frac{g'(z)}{g(z)}\right) = \Re\left(\frac{z}{z-w}\right) + \Re\left(z\frac{G'(z)}{G(z)}\right) \geq \frac{1}{1+|w|} + \frac{1}{1+l}\left\{M + l\left(\frac{\sqrt{l^M} - \sqrt{|d_0|}}{\sqrt{l^M}}\right)\right\}$$

for all $z \in S_1$ such that $g(z) \neq 0$.

It is required to show that, for $z \in S_1$,

$$\Re\left(z\frac{g'(z)}{g(z)}\right) \geq \frac{1}{1+l}\left\{M + 1 + l\left(\frac{\sqrt{l^{M+1}} - \sqrt{|w||d_0|}}{\sqrt{l^{M+1}}}\right)\right\}. \tag{3.2}$$

Clearly, inequality (3.2) holds if

$$\frac{1}{1+|w|} + \frac{1}{1+l}\left\{M + l\left(\frac{\sqrt{l^M} - \sqrt{|d_0|}}{\sqrt{l^M}}\right)\right\} \geq \frac{1}{1+l}\left\{M + 1 + l\left(\frac{\sqrt{l^{M+1}} - \sqrt{|w||d_0|}}{\sqrt{l^{M+1}}}\right)\right\},$$

which is equivalent to

$$\frac{1+l}{1+|w|} \geq 1 + l\sqrt{\frac{|d_0|}{l^M}} - l\sqrt{\frac{|w||d_0|}{l^{M+1}}}. \tag{3.3}$$

As the zeros of $g(z)$ are in $S_l \cup R_l^-$ and

$$0 \leq l \leq 1, \quad 0 \leq \frac{|d_0|}{l^M} \leq 1, \quad 0 \leq \frac{|w|}{l} \leq 1,$$

by Lemma 2,

$$\frac{2l}{l+|w|} \geq 1 + l\sqrt{\frac{|d_0|}{l^M}} - l\sqrt{\frac{|w||d_0|}{l^{M+1}}}. \tag{3.4}$$

Also,

$$\frac{1+l}{1+|w|} \geq \frac{2l}{l+|w|}. \tag{3.5}$$

From (3.4) and (3.5), inequality (3.3) follows, and this proves Lemma 3. □

4. Proof of the main result

Proof of Theorem 7. Since

$$q(z)=\frac{z^s h(z)}{w(z)}\in\mathcal{R}_n,$$

where

$$h(z)=\sum_{k=0}^{m-s} d_k z^k,$$

for every complex δ, $|\delta|\le 1$, we have

$$\frac{zq'(z)}{q(z)}+\frac{(m-s)\delta}{1+l}=s+\frac{zh'(z)}{h(z)}-\frac{zw'(z)}{w(z)}+\frac{(m-s)\delta}{1+l}.$$

Equivalently,

$$\Re\left(\frac{zq'(z)}{q(z)}+\frac{(m-s)\delta}{1+l}\right)=s+\Re\left(\frac{zh'(z)}{h(z)}\right)-\Re\left(\frac{zw'(z)}{w(z)}\right)+\frac{(m-s)\Re(\delta)}{1+l}.$$

Specially for $z\in S_1$, using Lemmas 3 and 1, we have

$$\Re\left(\frac{zq'(z)}{q(z)}+\frac{(m-s)\delta}{1+l}\right)\ge s+\frac{1}{1+l}\left\{m-s+l\left(\frac{\sqrt{l^{m-s}|d_{m-s}|}-\sqrt{|d_0|}}{\sqrt{l^{m-s}|d_{m-s}|}}\right)\right\}$$
$$-\left(\frac{n-|V'(z)|}{2}\right)+\frac{(m-s)\Re(\delta)}{1+l}$$
$$=\frac{1}{2}\left\{|V'(z)|+\frac{1}{1+l}\left(l(2s-n)+2m-n+2l\left(\frac{\sqrt{l^{m-s}|d_{m-s}|}-\sqrt{|d_0|}}{\sqrt{l^{m-s}|d_{m-s}|}}\right)+2(m-s)\Re(\delta)\right)\right\},$$

from which it is obvious that

$$\left|zq'(z)+\frac{(m-s)\delta}{1+l}q(z)\right|$$
$$\ge\frac{1}{2}\left\{|V'(z)|+\frac{1}{1+l}\left(l(2s-n)+2m-n+2l\left(\frac{\sqrt{l^{m-s}|d_{m-s}|}-\sqrt{|d_0|}}{\sqrt{l^{m-s}|d_{m-s}|}}\right)+2(m-s)\Re(\delta)\right)\right\}|q(z)|.$$

This proves Theorem 7. □

5. Conclusion

This paper investigates the bounds of the derivative of a class of rational functions on the unit disk while considering the contribution of certain coefficients of the underlying polynomial. We also discuss the result by Mir et al., recently published in the Ural Mathematical Journal, using some counterexamples.

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