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INTEGRAL ANALOGUE OF TURÁN-TYPE INEQUALITIES CONCERNING THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract: If $w(\zeta)$ is a polynomial of degree n with all its zeros in $|\zeta|\leq\Delta$, $\Delta\geq 1$ and any real $\gamma\geq 1$, Aziz proved the integral inequality [1]

$$\left\{\int_0^{2\pi}\left|1+\Delta^n e^{i\theta}\right|^{\gamma}d\theta\right\}^{1/\gamma}\max_{|\zeta|=1}|w'(\zeta)|\geq n\left\{\int_0^{2\pi}\left|w\left(e^{i\theta}\right)\right|^{\gamma}d\theta\right\}^{1/\gamma}.$$

In this article, we establish a refined extension of the above integral inequality by using the polar derivative instead of the ordinary derivative consisting of the leading coefficient and the constant term of the polynomial. Besides, our result also yields other intriguing inequalities as special cases.

Keywords: Polar derivative, Turán-type inequalities, Integral inequalities.

1. Introduction

In the late nineteenth century, renowned chemist Mendeleev became interested in the subject of the extremal properties of polynomials while searching for an upper bound of a quadratic polynomial. More specifically, he [14] established that, if $w(r)$ is a quadratic polynomial of real variable r with real coefficients, then for $-1\leq w(r)\leq 1$ and $-1\leq r\leq 1$,

$$\max_{-1\leq r\leq 1}|w'(r)|\leq 4.$$

While working on a problem in Approximation Theory, Bernstein needed an upper bound estimate of the maximum modulus $|w'(\zeta)|$ of a complex polynomial in terms of the maximum modulus of $|w(\zeta)|$, where $|\zeta|=1$, which is an analogue of above Mendeleev's problem in the complex domain. He [5] proved his famous inequality which states that, if $w(\zeta)$ is a n degree polynomial, then

$$\max_{|\zeta|=1}|w'(\zeta)|\leq n\max_{|\zeta|=1}|w(\zeta)|. \tag{1.1}$$

This inequality is sharp if and only if $w(\zeta)=\delta\zeta^n$, where

$$|\delta|=\max_{|\zeta|=1}|w(\zeta)|.$$

Inequality (1.1) is an immediate consequence of an inequality concerning trigonometric polynomials proved by him.

Paul Turán [21] was the first to estimate the maximum modulus for the derivative of a polynomial through a lower bound in terms of the maximum modulus of the polynomial. He established, in particular, that if $w(\zeta)$ is a n degree polynomial and all of its zeros lie in $|\zeta|\leq 1$, then

$$\max_{|\zeta|=1}w'(\zeta)\geq\frac{n}{2}\max_{|\zeta|=1}|w(\zeta)|. \tag{1.2}$$

Equality in (1.2) attains for $w(\zeta) = \delta\zeta^n + \beta$, where $|\delta| = |\beta|$. If $w(\zeta)$ is a n degree polynomial over the complex numbers $\mathbb{C}$, and for a real number $\gamma > 0$, the integral mean of $w(\zeta)$ is defined by

$$\|w\|_\gamma = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| w(e^{i\theta}) \right|^\gamma d\theta \right\}^{1/\gamma}.$$

Taking limit as $\gamma \to \infty$ and using the fact from the analysis [18, 20] that

$$\lim_{\gamma\to\infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| w(e^{i\theta}) \right|^\gamma d\theta \right\}^{1/\gamma} = \max_{|\zeta|=1} |w(\zeta)|,$$

we can legitimately denote

$$\|w\|_\infty = \max_{|\zeta|=1} |w(\zeta)|.$$

Aziz and Dawood [2] improved (1.2) into the form

$$\|w'\|_\infty \geq \frac{n}{2} \Big\{ \|w\|_\infty + \min_{|\zeta|=1} |w(\zeta)| \Big\}. \tag{1.3}$$

Throughout this paper, $\mathbb{P}_{n,s,\Delta}$ represents the class of all polynomials

$$w(\zeta) = \zeta^s \sum_{j=0}^{n=s} \alpha_j \zeta^j, \quad 0 \leq s \leq n,$$

with zero of multiplicity s at the origin having all its zeros in $|\zeta| \leq \Delta,\ \Delta \geq 1$ and $\mathbb{P}_{n,\Delta}$, the class of all polynomials

$$w(\zeta) = \sum_{j=0}^{n} \alpha_j \zeta^j$$

with all their zeros in $|\zeta| \leq \Delta,\ \Delta \geq 1$.

Applications and interest in inequality (1.2) have been substantial. Thus, it would be very interesting to determine its generalisation for polynomials whose zeros are all in $|\zeta| \leq \Delta,\ \Delta > 0$. For $0 < \Delta \leq 1$, Malik [13] proved

$$\|w'\|_\infty \geq \frac{n}{1+\Delta} \|w\|_\infty. \tag{1.4}$$

For $\Delta \geq 1$, Govil [9] found

$$\|w'\|_\infty \geq \frac{n}{1+\Delta^n} \|w\|_\infty. \tag{1.5}$$

Equality in (1.5) holds for $w(\zeta) = \zeta^n + \Delta^n, \Delta \geq 1$.

Govil [10] refined inequality (1.4) by proving that

$$\|w'\|_\infty \geq \frac{n}{1+\Delta} \Big(\|w\|_\infty + \frac{1}{\Delta^{n-1}} \min_{|\zeta|=\Delta} |w(\zeta)| \Big). \tag{1.6}$$

Equality in (1.6) holds for $w(\zeta) = (\zeta + \Delta)^n$.

For the polynomials which have all their zeros in $|\zeta| \leq \Delta$, $\Delta \leq 1$ with zero of multiplicity s at the origin, Aziz and Shah [4] obtained the following generalization of (1.4) that

$$\|w'\|_\infty \geq \frac{n + s\Delta}{1+\Delta} \|w\|_\infty.$$

The above inequality is sharp with the extremal polynomial being $w(\zeta) = \zeta^s (\zeta + \Delta)^{n-s}$, $0 \leq s \leq n$.

Using the same assumption, Govil [10] was able to improve (1.5) as

$$\|w'\|_\infty \geq \frac{n}{1+\Delta^n}\left\{\|w\|_\infty + \min_{|\zeta|=\Delta}|w(\zeta)|\right\}. \tag{1.7}$$

Inequality (1.7) attains equality for

$$w(\zeta) = \zeta^n + \Delta^n, \quad \Delta \geq 1.$$

Malik [12] extended inequality (1.2) for the first time in 1984 into its integral analogue by establishing that if $w(\zeta)$ is a n degree polynomial with all its zeros in $|\zeta| \leq 1$, then for $\gamma > 0$,

$$\|1+\zeta\|_\gamma \|w'\|_\infty \geq n\|w\|_\gamma.$$

The result is best possible for $w(\zeta) = (\zeta+1)^n$.

In 1988, Aziz [1] extended to integral form of (1.5) by establishing

Theorem 1. *If $w(\zeta) \in \mathbb{P}_{n,\Delta}$, then for $\gamma \geq 1$,*

$$\|1+\Delta^n\zeta\|_\gamma \|w'\|_\infty \geq n\|w\|_\gamma. \tag{1.8}$$

Equality in (1.8) holds for

$$w(\zeta) = \delta\zeta^n + \beta\Delta^n, \quad |\delta| = |\beta|.$$

For a n degree polynomial $w(\zeta)$ and any $\delta \in \mathbb{C}$, we define the polar derivative of the polynomial $w(\zeta)$ with regard to δ by

$$D_\delta w(\zeta) = nw(\zeta) + (\delta - \zeta)w'(\zeta).$$

Note that $D_\delta w(\zeta)$ has atmost $n-1$ degree, and it is a generalization of the ordinary derivative as

$$\lim_{\delta\to\infty} \frac{D_\delta w(\zeta)}{\delta} = w'(\zeta),$$

uniformly with respect to ζ for $|\zeta| \leq R, R > 0$.

Inequality (1.4) was first extended to the polar derivative by Aziz and Rather [3]. They obtained that if $w(\zeta)$ is a n degree polynomial with all its zeros in $|\zeta| \leq \Delta, \Delta \leq 1$, then for $\delta \in \mathbb{C}$, $|\delta| \geq \Delta$,

$$\|D_\delta w\|_\infty \geq n\left(\frac{|\delta|-\Delta}{1+\Delta}\right)\|w\|_\infty.$$

Besides, in the same article [3], they could extend (1.5) to polar derivative by proving that

$$\|D_\delta w\|_\infty \geq n\left(\frac{|\delta|-\Delta}{1+\Delta^n}\right)\|w\|_\infty, \tag{1.9}$$

where $\delta \in \mathbb{C}$ with $|\delta| \geq \Delta$.

Dewan et al. [7] obtained the polar derivative version of (1.7), which also sharpens (1.9) by proving that if $w(\zeta) \in \mathbb{P}_{n,\Delta}$, then for $\delta \in \mathbb{C}$ with $|\delta| \geq \Delta$,

$$\|D_\delta w\|_\infty \geq \frac{n}{1+\Delta^n}\Big\{(|\delta|-\Delta)\|w\|_\infty + \left(|\delta| + \frac{1}{\Delta^{n-1}}\right)\min_{|\zeta|=\Delta}|w(\zeta)|\Big\}. \tag{1.10}$$

The following generalization and improvement of (1.9) consisting of the polynomial's constant term and leading coefficient was recently established by Singh and Chanam [19].

Theorem 2. *If $w(\zeta) \in \mathbb{P}_{n,s,\Delta}$, then $\delta \in \mathbb{C}$ with $|\delta| \geq \Delta$,*

$$\|D_\delta w\|_\infty \geq \frac{|\delta| - \Delta}{1 + \Delta^n} \left\{ n + s + \frac{\sqrt{\Delta^{n-s}|\alpha_{n-s}|} - \sqrt{|\alpha_0|}}{\sqrt{\Delta^{n-s}|\alpha_{n-s}|}} \right\} \|w\|_\infty . \tag{1.11}$$

Milovanovic et al. [15] proved the following improvement and generalization of (1.9), (1.10) and (1.11).

Theorem 3. *If $w(\zeta) \in \mathbb{P}_{n,s,\Delta}$, then for $\delta \in \mathbb{C}$, $|\delta| \geq \Delta$,*

$$\begin{aligned} \|D_\delta w\|_\infty \geq \frac{n}{1 + \Delta^n} \Big\{ \big(|\delta| - \Delta\big) \|w\|_\infty + \Big(|\delta| + \frac{1}{\Delta^{n-1}}\Big) m \Big\} \\ + \frac{|\delta| - \Delta}{1 + \Delta^n} \left\{ s + \frac{\sqrt{\Delta^{n-s}|\alpha_{n-s}| - m} - \sqrt{|\alpha_0|}}{\sqrt{\Delta^{n-s}|\alpha_{n-s}| - m}} \right\} \Big(\|w\|_\infty - \frac{m}{\Delta^n} \Big), \end{aligned} \tag{1.12}$$

where $m = \min_{|\zeta|=\Delta} |w(\zeta)|$.

2. Main result

Below we derive the generalized integral extension of Theorem 2, which further improves Theorem 3 and also gives many other interesting results as special cases. In particular, we prove

Theorem 4. *If $w(\zeta) \in \mathbb{P}_{n,s,\Delta}$, then for $\delta \in \mathbb{C}$, $|\delta| \geq \Delta$ and $\lambda \in \mathbb{C}$, $|\lambda| < 1$ and $\gamma > 0$,*

$$\left\| D_\delta \left\{ w\big(e^{i\theta}\big) - \frac{m}{\Delta^n} \lambda e^{in\theta} \right\} \right\|_\gamma \geq \frac{|\delta| - \Delta}{2E_\gamma} A \left\| w\big(e^{i\theta}\big) - \frac{m}{\Delta^n} \lambda e^{in\theta} \right\|_\gamma , \tag{2.1}$$

where

$$m = \min_{|\zeta|=\Delta} |w(\zeta)|, \quad A = \left\{ n + s + \frac{\sqrt{\Delta^n |\alpha_{n-s}| - |\lambda| m} - \sqrt{\Delta^s |\alpha_0|}}{\sqrt{\Delta^n |\alpha_{n-s}| - |\lambda| m}} \right\}$$

and

$$E_\gamma = \frac{\left\{ \int_0^{2\pi} \left| 1 + \Delta^{n-s} e^{i\theta} \right|^\gamma d\theta \right\}^{1/\gamma}}{\left\{ \int_0^{2\pi} \left| 1 + e^{i\theta} \right|^\gamma d\theta \right\}^{1/\gamma}}.$$

Remark 1. Suppose $w(\zeta)$ has all its zeros in $|\zeta| \leq \Delta$, $\Delta \geq 1$. Now, for $|\zeta| = \Delta$

$$m = \min_{|\zeta|=\Delta} |w(\zeta)| \leq |w(\zeta)|. \tag{2.2}$$

As a consequence of Maximum Modulus Principle, we have

$$\max_{|\zeta|=\Delta} |w(\zeta)| \leq \Delta^n \max_{|\zeta|=1} |w(\zeta)|. \tag{2.3}$$

Using (2.3) to (2.2), we get

$$m \leq \Delta^n \max_{|\zeta|=1} |w(\zeta)|,$$

i.e.

$$\frac{m}{\Delta^n} \leq \max_{|\zeta|=1} |w(\zeta)|. \tag{2.4}$$

For arbitrary $\lambda \in \mathbb{C}$, $|\lambda| < 1$, we have

$$\frac{|\lambda| m}{\Delta^n} < \max_{|\zeta|=1} |w(\zeta)|. \tag{2.5}$$

Remark 2. Suppose $\gamma \to \infty$ in (2.1) and knowing the simple fact that

$$E_\gamma \to \frac{1+\Delta^n}{2} \quad \text{as} \quad \gamma \to \infty,$$

we get

$$\max_{|\zeta|=1} \left| D_\delta \left\{ w(\zeta) - \frac{m\lambda}{\Delta^n} \zeta^n \right\} \right| \geq \frac{|\delta| - \Delta}{1+\Delta^n} A \max_{|\zeta|=1} \left| w(\zeta) - \frac{m\lambda}{\Delta^n} \zeta^n \right|,$$

i.e.

$$\max_{|\zeta|=1} \left| D_\delta w(\zeta) - \frac{|\delta| m n \lambda}{\Delta^n} \zeta^{n-1} \right| \geq \frac{|\delta| - \Delta}{1+\Delta^n} A \max_{|\zeta|=1} \left| w(\zeta) - \frac{m\lambda}{\Delta^n} \zeta^n \right|. \tag{2.6}$$

Let ζ_0 on $|\zeta| = 1$ be such that

$$\max_{|\zeta|=1} \left| D_\delta w(\zeta) - \frac{|\delta| m n \lambda}{\Delta^n} \zeta^{n-1} \right| = \left| D_\delta w(\zeta_0) - \frac{|\delta| m n \lambda}{\Delta^n} \zeta_0^{n-1} \right|. \tag{2.7}$$

In the right side of (2.7), we can choose the argument of λ with

$$\left| D_\delta w(\zeta_0) - \frac{|\delta| m n \lambda}{\Delta^n} \zeta_0^{n-1} \right| = |D_\delta w(\zeta_0)| - \frac{n|\delta||\lambda|}{\Delta^n} m. \tag{2.8}$$

From (2.7) and (2.8), (2.6) becomes

$$|D_\delta w(\zeta_0)| - \frac{n|\delta||\lambda|}{\Delta^n} m \geq \frac{|\delta| - \Delta}{1+\Delta^n} A \max_{|\zeta|=1} \left| w(\zeta) - \frac{m\lambda}{\Delta^n} \zeta^n \right|. \tag{2.9}$$

Since

$$|D_\delta w(\zeta_0)| \leq \max_{|\zeta|=1} |D_\delta w(\zeta)|,$$

(2.9) gives

$$\max_{|\zeta|=1} |D_\delta w(\zeta)| - \frac{n|\delta||\lambda|}{\Delta^n} m \geq \frac{|\delta| - \Delta}{1+\Delta^n} A \max_{|\zeta|=1} \left| w(\zeta) - \frac{m\lambda}{\Delta^n} \zeta^n \right|. \tag{2.10}$$

Let ζ_1 on $|\zeta| = 1$ be such that $\max_{|\zeta|=1} |w(\zeta)| = |w(\zeta_1)|$. Then

$$\max_{|\zeta|=1} \left| w(\zeta) - \frac{m\lambda}{\Delta^n} \zeta^n \right| \geq \left| w(\zeta_1) - \frac{m\lambda}{\Delta^n} \zeta^n \right| \geq \left| |w(\zeta_1)| - \frac{m|\lambda|}{\Delta^n} \right|. \tag{2.11}$$

Using (2.5) to (2.11), we get

$$\max_{|\zeta|=1} \left| w(\zeta) - \frac{m\lambda}{\Delta^n} \zeta^n \right| \geq \max_{|\zeta|=1} |w(\zeta)| - \frac{m|\lambda|}{\Delta^n}. \tag{2.12}$$

Using (2.12), (2.10) gives

$$\max_{|\zeta|=1} |D_\delta w(\zeta)| - \frac{n|\delta||\lambda|}{\Delta^n} m \geq \frac{|\delta| - \Delta}{1+\Delta^n} A \Big(\max_{|\zeta|=1} |w(\zeta)| - \frac{|\lambda|}{\Delta^n} m \Big). \tag{2.13}$$

When $|\lambda| \to l$ in (2.13), we have

$$\max_{|\zeta|=1} |D_\delta w(\zeta)| - \frac{n|\delta| l}{\Delta^n} m \geq \frac{|\delta| - \Delta}{1+\Delta^n} A \Big(\max_{|\zeta|=1} |w(\zeta)| - \frac{l}{\Delta^n} m \Big),$$

which becomes the following result on simply taking limit $l \to 1$.

Corollary 1. *If $w(\zeta) \in \mathbb{P}_{n,s,\Delta}$, then for $\delta \in \mathbb{C}$, $|\delta| \geq \Delta$,*

$$\begin{aligned}\|D_\delta w\|_\infty \geq \frac{n}{1+\Delta^n}\Big\{(|\delta|-\Delta)\|w\|_\infty + \Big(|\delta| + \frac{1}{\Delta^{n-1}}\Big)m\Big\} \\ + \frac{|\delta|-\Delta}{1+\Delta^n}\left(s + \frac{\sqrt{\Delta^n|\alpha_{n-s}|-m} - \sqrt{\Delta^s|\alpha_0|}}{\sqrt{\Delta^n|\alpha_{n-s}|-m}}\right)\left(\|w\|_\infty - \frac{m}{\Delta^n}\right),\end{aligned} \tag{2.14}$$

where $m = \min_{|\zeta|=\Delta}|w(\zeta)|$.

Remark 3. Using the three facts (2.4), (4.1) and (4.3) in (2.14), it is obvious that Corollary 1 improves (1.10).

Remark 4. Also, the function

$$f(x) = \frac{\sqrt{\Delta^{n-s}|\alpha_{n-s}|-x} - \sqrt{|\alpha_0|}}{\sqrt{\Delta^{n-s}|\alpha_{n-s}|-x}}$$

is non-increasing for x. Therefore, for $\Delta \geq 1$

$$f\left(\frac{m}{\Delta^s}\right) \geq f(m),$$

that is,

$$\frac{\sqrt{\Delta^n|\alpha_{n-s}|-m} - \sqrt{\Delta^s|\alpha_0|}}{\sqrt{\Delta^n|\alpha_{n-s}|-m}} \geq \frac{\sqrt{\Delta^{n-s}|\alpha_{n-s}|-m} - \sqrt{|\alpha_0|}}{\sqrt{\Delta^{n-s}|\alpha_{n-s}|-m}}.$$

This shows that Corollary 1 is an improvement of (1.12).

Remark 5. If we divide both sides of (2.14) by $|\delta|$ and let $|\delta| \to \infty$, the next result which improves (1.7), is obtained.

Corollary 2. *If $w(\zeta) \in \mathbb{P}_{n,s,\Delta}$, then*

$$\|w'\|_\infty \geq \frac{n}{1+\Delta^n}(\|w\|_\infty + m) + \frac{1}{1+\Delta^n}\left(s + \frac{\sqrt{\Delta^n|\alpha_{n-s}|-m} - \sqrt{\Delta^s|\alpha_0|}}{\sqrt{\Delta^n|\alpha_{n-s}|-m}}\right)\left(\|w\|_\infty - \frac{m}{\Delta^n}\right), \tag{2.15}$$

where $m = \min_{|\zeta|=\Delta}|w(\zeta)|$.

Remark 6. If we divide both sides of (2.1) of Theorem 4 by $|\delta|$ and let $|\delta| \to \infty$, the following generalized integral extension of Corollary 2 is obtained.

Corollary 3. *If $w(\zeta) \in \mathbb{P}_{n,s,\Delta}$, then for each $\lambda \in \mathbb{C}$, $|\lambda| < 1$ and $\gamma > 0$,*

$$\left\|w'(e^{i\theta}) - \frac{mn}{\Delta^n}\lambda e^{i(n-1)\theta}\right\|_\gamma \geq \frac{A}{2E_\gamma}\left\|w(e^{i\theta}) - \frac{m}{\Delta^n}\lambda e^{in\theta}\right\|_\gamma,$$

where m, A and E_r are defined in Theorem 4.

Remark 7. When $\lambda = 0$ in (2.1) of Theorem 4, the below integral extension of Theorem 2 yields an improved and generalised integral analogue for polar derivative of Theorem 1.

Corollary 4. *If $w(\zeta) \in \mathbb{P}_{n,s,\Delta}$, then for $\delta \in \mathbb{C}$, $|\delta| \geq \Delta$ and $\gamma > 0$,*

$$\|D_\delta w\|_\gamma \geq \frac{|\delta| - \Delta}{2E_\gamma}\left(n + s + \frac{\sqrt{\Delta^{n-s}|\alpha_{n-s}|} - \sqrt{|\alpha_0|}}{\sqrt{\Delta^{n-s}|\alpha_{n-s}|}}\right)\|w\|_\gamma\,, \tag{2.16}$$

where E_γ is defined in Theorem 4.

Remark 8. In case $r \to \infty$ in (2.16), Corollary 4, in particular, becomes Theorem 2 and dividing both sides by $|\delta|$ and making $|\delta| \to \infty$, we have an improved form of (1.5).

Corollary 5. *If $w(\zeta) \in \mathbb{P}_{n,s,\Delta}$, then*

$$\|w'\|_\infty \geq \frac{1}{1 + \Delta^n}\left(n + s + \frac{\sqrt{\Delta^{n-s}|\alpha_{n-s}|} - \sqrt{|\alpha_0|}}{\sqrt{\Delta^{n-s}|\alpha_{n-s}|}}\right)\|w\|_\infty. \tag{2.17}$$

Remark 9. If degree n of polynomial $w(\zeta)$ is greater than or equal to 1, the leading coefficient α_n is different from zero, and using the fact (4.1), it follows obviously that inequality (2.17) always provides better bounds than that of (1.5). When $\Delta = 1$, (2.15) and (2.17) sharpen (1.3) and (1.2) respectively.

3. Example with numerical illustration

Example. Consider $w(\zeta) = \zeta(\zeta + 1)$ with all zeros $0, -1$. Now, all the zeros lie in the closed disk $|\zeta| \leq 1$. On the unit circle $|\zeta| = 1$,

$$\left|w(e^{i\theta})\right| = \sqrt{2 + 2\cos\theta}.$$

Since the non-negative function

$$f(\theta) = 2 + 2\cos\theta, \quad 0 \leq \theta < 2\pi,$$

attains its maximum at $\theta = 0$,

$$\max_{|\zeta|=1}\left|w(\zeta)\right| = 2.$$

For each fixed $\Delta = \Delta_0$,

$$|w(\Delta_0 e^{i\theta})| = \Delta_0\sqrt{\Delta_0^2 + 2\Delta_0\cos\theta + 1}.$$

Since the function

$$g(\theta) = \Delta_0^2 + 2\Delta_0\cos\theta + 1, \quad 0 \leq \theta < 2\pi,$$

attains its minimum at $\theta = \pi$,

$$m = \min_{|\zeta|=\Delta_0}|w(\zeta)| = \Delta_0(\Delta_0 - 1).$$

If we take $\Delta_0 = 1.95$ and $|\delta| = 10$, then by using Theorem 2, we have

$$\|D_{10}w\|_\infty \geq \frac{10 - 1.95}{1 + 1.95^2}\left\{2 + 1 + \frac{\sqrt{1.95 \times 1} - \sqrt{1}}{\sqrt{1.95 \times 1}}\right\} \times 2 \approx 11.009,$$

while by Theorem 3,

$$\|D_{10}w\|_\infty \geq \frac{2}{1+1.95^2}\Big\{(10-1.95)2+\Big(10+\frac{1}{1.95}\Big)1.95(1.95-1)\Big\}$$
$$+\frac{10-1.95}{1+1.95^2}\Big\{1+\frac{\sqrt{1.95\times 1-1.95(1.95-1)}-\sqrt{1}}{\sqrt{1.95\times 1-1.95(1.95-1)}}\Big\}\Big(2-\frac{1.95(1.95-1)}{1.95^2}\Big)\approx 11.7657$$

Meanwhile, if we use Corollary 1, we get

$$\|D_{10}w\|_\infty \geq \frac{2}{1+1.95^2}\Big\{(10-1.95)2+\Big(10+\frac{1}{1.95}\Big)1.95(1.95-1)\Big\}$$
$$+\frac{10-1.95}{1+1.95^2}\Big\{1+\frac{\sqrt{1.95^2\times 1-1.95(1.95-1)}-\sqrt{1.95\times 1}}{\sqrt{1.95^2\times 1-1.95(1.95-1)}}\Big\}\Big\{2-\frac{1.95(1.95-1)}{1.95^2}\Big\}\approx 17.351,$$

which is larger than the bounds obtained by using Theorems 2 and 3. In other words, the bound of Corollary 1 improves over those of Theorems 2 and 3 respectively due to Singh and Chanam [19] and Milovanovic et al. [15] by about 57.61% and 47.47%. From this, it is easy to see that by making appropriate choices of the polynomial $w(\zeta)$, and the parameters Δ and δ, this improvement can be scaled up.

4. Lemmas

We need the following auxiliary results to prove the theorem and its corollaries. For a n degree polynomial $w(\zeta)$, we will use

$$q(\zeta)=\zeta^n\overline{w\left(1/\bar{\zeta}\right)}.$$

Lemma 1 [13]. *If $w(\zeta)$ is a n degree polynomial with all its zeros in $|\zeta|\leq\Delta$, $\Delta\leq 1$, then for $|\zeta|=1$,*

$$|q'(\zeta)|\leq\Delta|w'(\zeta)|.$$

Lemma 2. *If $w(\zeta)$ is a n degree polynomial, then for $R\geq 1$ and $\gamma>0$,*

$$\left\{\int_0^{2\pi}\left|w\left(Re^{i\theta}\right)\right|^\gamma d\theta\right\}^{1/\gamma}\leq R^n\left\{\int_0^{2\pi}\left|w\left(e^{i\theta}\right)\right|^\gamma d\theta\right\}^{1/\gamma}.$$

It is difficult to trace the origin of Lemma 2. However, it could be followed from a famous result of Hardy [11], by which for any function $f(\zeta)$ analytic in $|\zeta|<t_0$, and for each $\gamma>0$,

$$\left\{\int_0^{2\pi}\left|f\left(xe^{i\theta}\right)\right|^\gamma d\theta\right\}^{1/\gamma}$$

is non-decreasing for $x\in(0,t_0)$. If $w(\zeta)$ is a n degree polynomial, then

$$f(\zeta)=\zeta^n\overline{w\left(1/\bar{\zeta}\right)}$$

is a polynomial of degree at most n and is an entire function, and by Hardy's result for $\gamma>0$,

$$\left\{\int_0^{2\pi}\left|f\left(xe^{i\theta}\right)\right|^\gamma d\theta\right\}^{1/\gamma}\leq\left\{\int_0^{2\pi}\left|f\left(e^{i\theta}\right)\right|^\gamma d\theta\right\}^{1/\gamma},$$

for $x=1/R<1$, and hence Lemma 2.

Lemma 3 [19]. *If $w(\zeta) \in P_{n,s,1}$, then for $|\zeta| = 1$,*

$$\left|w'(\zeta)\right| \geq \frac{1}{2}\left\{n + s + \frac{\sqrt{|\alpha_{n-s}|} - \sqrt{|\alpha_0|}}{\sqrt{|\alpha_{n-s}|}}\right\}|w(\zeta)|.$$

Lemma 4 [6, 16]. *If $w(\zeta)$ is a n degree polynomial and $w(\zeta) \neq 0$ in $|\zeta| < 1$, then for $R \geq 1$ and $\gamma > 0$,*

$$\left\{\int_0^{2\pi} \left|w\left(Re^{i\theta}\right)\right|^{\gamma} d\theta\right\}^{1/\gamma} \leq B_{\gamma}\left\{\int_0^{2\pi} \left|w\left(e^{i\theta}\right)\right|^{\gamma} d\theta\right\}^{1/\gamma},$$

where

$$B_{\gamma} = \frac{\left\{\int_0^{2\pi} |1 + R^n e^{i\theta}|^{\gamma} d\theta\right\}^{1/\gamma}}{\left\{\int_0^{2\pi} |1 + e^{i\theta}|^{\gamma} d\theta\right\}^{1/\gamma}}.$$

This is due to Boas and Rahman [6] for $\gamma \geq 1$. Later, Rahman and Schmeisser [16] verified validity for $0 < \gamma < 1$.

Lemma 5 [8]. *If $w(\zeta)$ is a n degree polynomial and $w(\zeta) \neq 0$ in $|\zeta| < \Delta$, $\Delta > 0$, then for $|\zeta| < \Delta$*

$$|w(\zeta)| > m,$$

where $m = \min_{|\zeta|=\Delta} |w(\zeta)|$.

Lemma 6. *If*

$$w(\zeta) = \zeta^s\left(\sum_{j=0}^{n-s} \alpha_j \zeta^j\right), \quad 0 \leq s \leq n,$$

is a polynomial with all its zeros in $|\zeta| \leq \Delta$, $\Delta > 0$, then for $\lambda \in \mathbb{C}$, $|\lambda| < 1$

$$\sqrt{\Delta^{n-s}|\alpha_{n-s}| - |\lambda|\frac{m}{\Delta^s}} - \sqrt{|\alpha_0|} \geq 0, \tag{4.1}$$

where $m = \min_{|\zeta|=\Delta} |w(\zeta)|$.

P r o o f. By hypothesis,

$$w(\zeta) = \zeta^s h(\zeta) = \zeta^s\left(\sum_{j=0}^{n-s} \alpha_j \zeta^j\right), \quad 0 \leq s \leq n,$$

has all its zeros in $|\zeta| \leq \Delta$, $\Delta > 0$. Then, the polynomial $W(\zeta) = e^{-i \arg \alpha_{n-s}} h(\zeta)$ has the same zeros as $h(\zeta)$.
Now,

$$\begin{aligned} W(\zeta) &= e^{-i \arg \alpha_{n-s}}\left\{\alpha_0 + \alpha_1\zeta + \cdots + \alpha_{n-s-1}\zeta^{n-s-1} + |\alpha_{n-s}| e^{i \arg \alpha_{n-s}} \zeta^{n-s}\right\} \\ &= e^{-i \arg \alpha_{n-s}}\left\{\alpha_0 + \alpha_1\zeta + \cdots + \alpha_{n-s-1}\zeta^{n-s-1}\right\} + |\alpha_{n-s}|\zeta^{n-s}. \end{aligned}$$

Now, on $|\zeta| = \Delta$ for $\lambda \in \mathbb{C}$, $|\lambda| < 1$ and $m = \min_{|\zeta|=\Delta} w(\zeta) \neq 0$, we have

$$\left|\frac{m\lambda}{\Delta^n}\zeta^{n-s}\right| < \frac{m}{\Delta^s} = \min_{|\zeta|=\Delta} |h(\zeta)| = \min_{|\zeta|=\Delta} |W(\zeta)| \leq |W(\zeta)|.$$

Then by Rouche's theorem,

$$R(\zeta) = W(\zeta) - \frac{m|\lambda|}{\Delta^n}\zeta^{n-s}$$

has all its zeros in $|\zeta| < \Delta$. By Vieta's formula applied to $R(\zeta)$, we get

$$\frac{|\alpha_0|}{\big||\alpha_{n-s}| - m|\lambda|/\Delta^n\big|} < \Delta^{n-s},$$

that is,

$$\left(\frac{|\alpha_0|}{\big||\alpha_{n-s}| - m|\lambda|/\Delta^n\big|}\right)^{1/2} < \Delta^{(n-s)/2}. \tag{4.2}$$

Since $W(\zeta)$ is a polynomial of degree $n-s$ with all its zeros in $|\zeta| \le \Delta$, then

$$Q(\zeta) = \zeta^{n-s}\overline{W\left(1/\overline{\zeta}\right)}$$

is a polynomial having atmost $n-s$ degree having no zero in $|\zeta| < 1/\Delta$. Using Lemma 5 to $Q(\zeta)$, we obtain

$$|\alpha_{n-s}| = |Q(0)| > \min_{|\zeta|=1/\Delta} |Q(\zeta)| = \frac{1}{\Delta^{n-s}} \min_{|\zeta|=\Delta} |W(\zeta)| = \frac{m}{\Delta^n},$$

i.e.

$$|\alpha_{n-s}| > \frac{m}{\Delta^n}. \tag{4.3}$$

Using (4.3) to (4.2), we have

$$\sqrt{\Delta^{n-s}|\alpha_{n-s}| - |\lambda|\frac{m}{\Delta^s}} - \sqrt{|\alpha_0|} > 0.$$

For $m = \min_{|\zeta|=\Delta} |w(\zeta)| = 0$, the result becomes trivial, simply by the similar reasoning of inequality (4.2) to

$$h(\zeta) = \sum_{j=0}^{n-s} \alpha_j \zeta^j,$$

i.e.

$$\sqrt{\Delta^{n-s}|\alpha_{n-s}|} - \sqrt{|\alpha_0|} \ge 0.$$

□

5. Proof of Theorem 4

By assumption, $w(\zeta)$ has all its zeros in $|\zeta| \le \Delta$, $\Delta \ge 1$. For $m = \min_{|\zeta|=\Delta} |w(\zeta)| \ne 0$, consider

$$R(\zeta) = w(\zeta) - \frac{m}{\Delta^n}\lambda\zeta^n,$$

where $\lambda \in \mathbb{C}$, $|\lambda| < 1$. Now, on $|\zeta| = \Delta$

$$\left|\frac{m}{\Delta^n}\lambda\zeta^n\right| < \frac{m}{\Delta^n}\Delta^n \le |w(\zeta)|.$$

Consequently, from Rouche's theorem, $R(\zeta)$ has all its zeros in $|\zeta| < \Delta$. When $m = 0$, $R(\zeta) = w(\zeta)$. Therefore, $R(\zeta)$ has all its zeros in $|\zeta| \le \Delta$ in any case. Then, all the zeros of $W(\zeta) = R(\Delta\zeta)$ are in $|\zeta| \le 1$. It is a simple fact that for $|\zeta| = 1$

$$|Q'(\zeta)| = \left|nW(\zeta) - \zeta W'(\zeta)\right|, \tag{5.1}$$

where

$$Q(\zeta) = \zeta^n \overline{W\left(1/\bar{\zeta}\right)}.$$

Using Lemma 1 to $W(\zeta)$, we have for $|\zeta| = 1$

$$\left|Q'(\zeta)\right| \leq \left|W'(\zeta)\right|. \tag{5.2}$$

Using (5.1) and (5.2), we have for $|\delta/\Delta| \geq 1$ and $|\zeta| = 1$

$$\begin{aligned}\left|D_{\delta/\Delta}W(\zeta)\right| &= \left|nW(\zeta) + \Big(\frac{\delta}{\Delta} - \zeta\Big)W'(\zeta)\right| \geq \left|\frac{\delta}{\Delta}\right|\left|W'(\zeta)\right| - \left|nW\left(\zeta\right) - \zeta W'\left(\zeta\right)\right| \\ &= \left|\frac{\delta}{\Delta}\right|\left|W'(\zeta)\right| - \left|Q'\left(\zeta\right)\right| \geq \Big(\left|\frac{\delta}{\Delta}\right| - 1\Big)\left|W'(\zeta)\right|.\end{aligned} \tag{5.3}$$

Applying Lemma 3 to $W(\zeta)$, we have for $|\zeta| = 1$

$$\left|W'(\zeta)\right| \geq \frac{1}{2}\Big\{n + s + \frac{\sqrt{\Delta^{n-s}|\alpha_{n-s} - (m/\Delta^n)\lambda|} - \sqrt{|\alpha_0|}}{\sqrt{\Delta^{n-s}|\alpha_{n-s} - (m/\Delta^n)\lambda|}}\Big\}|W(\zeta)|.$$

Since $f(x) = (x - |a|)/x$ is non-decreasing and in view of (4.3), we get

$$\left|W'(\zeta)\right| \geq \frac{1}{2}\Big\{n + s + \frac{\sqrt{\Delta^n|\alpha_{n-s}| - |\lambda|m} - \sqrt{\Delta^s|\alpha_0|}}{\sqrt{\Delta^n|\alpha_{n-s}| - |\lambda|m}}\Big\}|W(\zeta)|. \tag{5.4}$$

Combining (5.4) and (5.3), we get

$$\left|D_{\delta/\Delta}W\left(\zeta\right)\right| \geq \frac{|\delta| - \Delta}{2\Delta}\Big\{n + s + \frac{\sqrt{\Delta^n|\alpha_{n-s}| - |\lambda|m} - \sqrt{\Delta^s|\alpha_0|}}{\sqrt{\Delta^n|\alpha_{n-s}| - |\lambda|m}}\Big\}|W(\zeta)|.$$

Replacing $W(\zeta)$ by $R(\Delta\zeta)$, this inequality gives

$$\left|nR(\Delta\zeta) + \Big(\frac{\delta}{\Delta} - \zeta\Big)\Delta R'(\Delta\zeta)\right| \geq \frac{|\delta| - \Delta}{2\Delta}A|R(\Delta\zeta)|, \tag{5.5}$$

where

$$A = \{n + s + \frac{\sqrt{\Delta^n|\alpha_{n-s}| - |\lambda|m} - \sqrt{\Delta^s|\alpha_0|}}{\sqrt{\Delta^n|\alpha_{n-s}| - |\lambda|m}}\}.$$

Inequality (5.5) becomes

$$\left|nR(\Delta\zeta) + \left(\delta - \Delta\zeta\right)R'(\Delta\zeta)\right| \geq \frac{|\delta| - \Delta}{2\Delta}A|R(\Delta\zeta)|,$$

therefore for any $\gamma > 0$, we have

$$\left|D_\delta R\left(\Delta e^{i\theta}\right)\right|^\gamma \geq \Big(\frac{|\delta| - \Delta}{2\Delta}A\Big)^\gamma\left|R\left(\Delta e^{i\theta}\right)\right|^\gamma, \quad 0 \leq \theta < 2\pi.$$

Equivalently,

$$\Big\{\int_0^{2\pi}\left|D_\delta R\left(\Delta e^{i\theta}\right)\right|^\gamma d\theta\Big\}^{1/\gamma} \geq \frac{|\delta| - \Delta}{2\Delta}A\Big\{\int_0^{2\pi}\left|R\left(\Delta e^{i\theta}\right)\right|^\gamma d\theta\Big\}^{1/\gamma}. \tag{5.6}$$

We have,

$$W(\zeta) = R(\Delta\zeta) = \alpha_0\Delta^s\zeta^s + \alpha_1\Delta^{s+1}\zeta^{s+1} + ... + (\alpha_{n-s}\Delta^n - m\lambda)\zeta^n,$$

and

$$Q(\zeta) = \zeta^n \overline{W\left(1/\bar{\zeta}\right)}. \tag{5.7}$$

Applying Lemma 4 to $Q(\zeta)$, we get

$$\Big\{\int_0^{2\pi} \big|Q\big(\Delta e^{i\theta}\big)\big|^\gamma d\theta\Big\}^{1/\gamma} \le E_\gamma \Big\{\int_0^{2\pi} \big|Q\big(e^{i\theta}\big)\big|^\gamma d\theta\Big\}^{1/\gamma}. \tag{5.8}$$

Now, it follows readily that $\big|Q(\Delta e^{i\theta})\big| = \Delta^n \big|R\left(e^{i\theta}\right)\big|$ and $\big|Q\left(e^{i\theta}\right)\big| = \big|R\left(\Delta e^{i\theta}\right)\big|$. Using the two relations, (5.8) gives

$$\Delta^n \Big\{\int_0^{2\pi} \big|R\big(e^{i\theta}\big)\big|^\gamma d\theta\Big\}^{1/\gamma} \le E_\gamma \Big\{\int_0^{2\pi} \big|R\big(\Delta e^{i\theta}\big)\big|^\gamma d\theta\Big\}^{1/\gamma}. \tag{5.9}$$

Since $D_\delta R(\zeta)$ is a polynomial of degree at most $(n-1)$, by Lemma 2 to $D_\delta R(\zeta)$, $R = \Delta \ge 1$, we have

$$\frac{1}{\Delta^{n-1}} \Big\{\int_0^{2\pi} \big|D_\delta R\big(\Delta e^{i\theta}\big)\big|^\gamma d\theta\Big\}^{1/\gamma} \le \left\{\int_0^{2\pi} \big|D_\delta R\big(e^{i\theta}\big)\big|^\gamma d\theta\right\}^{1/\gamma}. \tag{5.10}$$

Using (5.10) to (5.6), we get

$$\Delta^{n-1} \left\{\int_0^{2\pi} \big|D_\delta R\big(e^{i\theta}\big)\big|^\gamma d\theta\right\}^{1/\gamma} \ge \frac{|\delta| - \Delta}{2\Delta} A \left\{\int_0^{2\pi} \big|R(\Delta e^{i\theta})\big|^\gamma d\theta\right\}^{1/\gamma}. \tag{5.11}$$

Combining (5.9) and (5.11), we have

$$\left\{\int_0^{2\pi} \big|D_\delta R\big(e^{i\theta}\big)\big|^\gamma d\theta\right\}^{1/\gamma} \ge \frac{|\delta| - \Delta}{2E_\gamma} A \left\{\int_0^{2\pi} \big|R\big(e^{i\theta}\big)\big|^\gamma d\theta\right\}^{1/\gamma},$$

which is equivalent to

$$\left\{\int_0^{2\pi} \left|D_\delta \left\{w\big(e^{i\theta}\big) - \frac{m}{\Delta^n}\lambda e^{in\theta}\right\}\right|^\gamma d\theta\right\}^{1/\gamma} \ge \frac{|\delta| - \Delta}{2E_\gamma} A \left\{\int_0^{2\pi} \left|w\big(e^{i\theta}\big) - \frac{m}{\Delta^n}\lambda e^{in\theta}\right|^\gamma d\theta\right\}^{1/\gamma}.$$

This proves Theorem 4. □

6. Conclusion

For the set of n degree polynomials with all their zeros in $|\zeta| \le \Delta$, $\Delta \ge 1$, there has been no integral analogue of Turán-type inequalities for about 19 years until 2017 that Rather and Bhat [17] had extended inequality (1.9) to integral mean setting. In this paper, we provide an integral mean version of Theorem 2 by using some techniques different from those followed by Rather and Bhat [17]. Our result also implicates various existing known results in the literature.

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