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CONTROL PROBLEM FOR A PARABOLIC SYSTEM WITH UNCERTAINTIES AND A NON-CONVEX GOAL[1]

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Abstract: We consider the control problem for a parabolic system that describes the heating of a given number of rods. Control is carried out through heat sources that are located at the ends of the rods (only at one end or at both). The density functions of the internal heat sources and exact values of the temperature at the right ends of some rods are unknown, and only the segments of their change are given. The goal of choosing control is to ensure that at a fixed time moment the weighted sum of the average temperatures of the rods belongs to a non-convex terminal set for any admissible unknown functions. After a change of variables, this problem reduces to a one-dimensional differential game. Necessary and sufficient conditions for the game termination are found.

Keywords: Control, Uncertainty, Parabolic system.

1. Introduction

Mathematical modelling of controlled processes of thermal conductivity, diffusion, filtration leads to problems of control of parabolic equations [2, 4, 8, 11, 13]. In applications, problems often arise about heating a rod at the ends of which there are controlled heat sources. In a formalized form, these problems are reduced to the study of the heat equation, the boundary conditions of which depend on the control functions (see, for example, [1, 10]).

Control processes for real dynamic systems often occur in conditions where some of the system parameters and boundary conditions are not precisely specified, and there is also influence from uncontrolled disturbances [3, 5, 18, 19].

To study such problems, the method of optimization of guaranteed result [9] can be applied. This method is based on the theory of differential games (see, for example, [12, 14]). Uncertainties and disturbances affecting the system are taken as the second player – the opponent. In [12, 14] control is constructed within the framework of the theory of positional differential games.

This article continues the research begun in [6, 15]. The work [15] considers the problem of heating a rod by controlling the rate of temperature change at its left end. The temperature at the right end of the rod is determined by an unknown function limited in value. The density function of the internal heat sources of the rod is not precisely known, and only the boundaries of its possible values are given. The goal of the control is to bring the average temperature of the rod at a fixed time moment to a given segment for any unknown temperature at the right end of the rod and for any function of the density of internal heat sources. The average temperature value is calculated as the integral of the product of temperature and a given function. In [6] the problem of controlling a

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parabolic system describing the heating of a given number of rods using point heat sources located at the ends of the rods is considered. The goal of choosing a control is to ensure that at a fixed time moment the modulus of the linear function, determined using the average temperatures of the rods, does not exceed a given value.

In this work, a modification of problems [6, 15] is solved. A finite set of desired temperature values is given. The goal of the control is to bring the weighted sum of the average temperatures of the rods into the ε–neighbourhood of one of the desired values. After changing variables, taking unknown functions as a control of the second player, the original problem is reduced to a single-type one-dimensional differential game. For the resulting differential game, a solvability set and corresponding player controls are constructed.

2. Problem statement

The heat equation

$$\frac{\partial T_i(x,t)}{\partial t} = \frac{\partial^2 T_i(x,t)}{\partial x^2} + f_i(x,t), \quad 0 \le t \le p, \quad 0 \le x \le 1, \quad i = \overline{1,n}, \tag{2.1}$$

describes the temperature distribution $T_i(x,t)$ in i-th $(i = \overline{1,n})$ homogeneous rod of unit length as a function of time t. At the initial time moment $t = 0$, the temperature distributions $T_i(x,0) = g_i(x)$, $i = \overline{1,n}$, are given, where $g_i(x)$ are continuous functions.

We assume that the controlled temperature $T_i(0,t)$ at the left end of i–th rod varies according to equation

$$\frac{dT_i(0,t)}{dt} = a_i^{(1)}(t) + a_i^{(2)}(t) G_i^{(1)} \overline{\xi}(t). \tag{2.2}$$

Here, $a_i^{(\zeta)}(t)$, $i = \overline{1,n}$, $\zeta = 1,2$, are continuous functions for $0 \le t \le p$, and $a_i^{(2)}(t) > 0$. The vector-function $\overline{\xi}(t) = (\xi_1(t), \xi_2(t), \dots, \xi_q(t))^* \in U$, where U is compact in $\mathbb{R}^q$, is a control. The symbol $*$ denotes the transposition operation. The choice of the corresponding one-dimensional controls $\xi_\iota(t)$ for the left end of each rod is given by the matrix $G^{(1)}$ of n by q dimension. $G_i^{(1)}$ denotes the i–th row of the corresponding matrix.

The temperature value $T_i(1,t)$ at the right end of the i-th rod is given as follows:

1. Determined by $\overline{\xi}$ control

$$\frac{dT_i(1,t)}{dt} = b_i^{(1)}(t) + b_i^{(2)}(t) G_i^{(2)} \overline{\xi}(t), \quad i = \overline{1,k}. \tag{2.3}$$

   Here, the functions $b_i^{(1)}(t)$ and $b_i^{(2)}(t)$, $i = \overline{1,k}$, are continuous for $0 \le t \le p$, and $b_i^{(2)}(t) > 0$. The choice of the corresponding one-dimensional controls $\xi_\iota(t)$ for the right end of the rods with indices $i = \overline{1,k}$ is given by the matrix $G^{(2)}$ of k by q dimension. $G_i^{(2)}$ denotes the i–th row of the corresponding matrix.
2. The temperature values $T_i(1,t)$, $i = \overline{k+1,l}$, which depend continuously on the time $t \in [0,p]$, are not exactly known, but the limits of their change are given

$$\beta_i^{(1)}(t) \le T_i(1,t) \le \beta_i^{(2)}(t), \quad 0 \le t \le p. \tag{2.4}$$

   Here $\beta_i^{(\zeta)}(t)$, $i = \overline{k+1,l}$, $\zeta = 1,2$, are continuous functions for $0 \le t \le p$.
3. $T_i(1,t)$, $i = \overline{l+1,n}$, are known continuous functions.

In addition, we know estimates of the continuous functions $f_i(x,t)$, which are the densities of internal heat sources of the rods:

$$f_i^{(1)}(x,t) \le f_i(x,t) \le f_i^{(2)}(x,t), \quad 0 \le t \le p, \quad 0 \le x \le 1, \quad i = \overline{1,n}. \tag{2.5}$$

Here functions $f_i^{(\zeta)}(x,t)$, $i = \overline{1,n}$, $\zeta = 1,2$, are continuous.

Assumption 1. *Each function $f_i : [0,1] \times [0,p] \to \mathbb{R}$, $i = \overline{1,n}$, is such that for any numbers $0 \le \tau < \nu$ and for any continuous functions $\varrho_i^{(\zeta)} : [\tau,\nu] \to \mathbb{R}$, $\zeta = 1,2$, $\mu_i : [0,1] \to \mathbb{R}$ such that the matching condition $\varrho_i^{(1)}(\tau) = \mu_i(0)$, $\varrho_i^{(2)}(\tau) = \mu_i(1)$ is satisfied, the first boundary value problem*

$$\frac{\partial Q_i(x,t)}{\partial t} = \frac{\partial^2 Q_i(x,t)}{\partial x^2} + f_i(x,t), \tag{2.6}$$

$$Q_i(0,t) = \varrho_i^{(1)}(t), \quad Q_i(1,t) = \varrho_i^{(2)}(t), \quad \tau \le t \le \nu; \tag{2.7}$$

$$Q_i(x,\tau) = \mu_i(x), \quad 0 \le x \le 1 \tag{2.8}$$

has a unique solution $Q_i(x,t)$ continuous for $0 \le x \le 1$, $\tau \le t \le \nu$.

Let numbers α_s, $s = \overline{1,r}$, and $\varepsilon \ge 0$ be such that $\alpha_{s+1} - \alpha_s = \Delta > 0$, $s = \overline{1,r-1}$ and $\Delta > 2\varepsilon$, and vector $\overline{\lambda} = (\lambda_1, \dots, \lambda_n)^* \in \mathbb{R}^n$ such that $\lambda_i > 0$, $i = \overline{1,n}$, be given. The goal of choosing control $\overline{\xi}(t)$ in (2.2), (2.3) is to implement the inclusion

$$\sum_{i=1}^{n} \lambda_i \int_0^1 T_i(x,p)\sigma_i(x)dx \in Z(\varepsilon) = \bigcup_{s=\overline{1,r}} [\alpha_s - \varepsilon, \alpha_s + \varepsilon] \tag{2.9}$$

for any continuous functions $T_i(1,t)$ (2.4), $i = \overline{k+1,l}$, and for any continuous functions $f_i(x,t)$ (2.5), $i = \overline{1,n}$, satisfying Assumption 1.

Here continuous functions $\sigma_i : [0,1] \to \mathbb{R}$, $i = \overline{1,n}$ are given and satisfy the conditions

$$\sigma_i(0) = \sigma_i(1) = 0. \tag{2.10}$$

3. Problem formalization

Let us describe an admissible rule for choosing control $\overline{\xi}(t)$. It means that for each time moment $0 \le \nu < p$ and for each admissible temperature distribution

$$\overline{T}(x,\nu) = (T_1(x,\nu), T_2(x,\nu), \dots, T_n(x,\nu))$$

at this time moment, a measurable vector-function $\overline{\xi}(t)$ such that $\overline{\xi} : [\nu,p] \to U$ is choosing. We will denote such a rule as

$$\overline{\xi}(t) = N(t, \overline{T}(\cdot,\nu)), \quad t \in [\nu,p]. \tag{3.1}$$

Fix a partition $\omega : 0 = t_0 < t_1 < \ldots < t_j < t_{j+1} < \ldots < t_{m+1} = p$ of the segment $[0,p]$ with diameter

$$d(\omega) = \max_{0 \le j \le m} (t_{j+1} - t_j).$$

Let the temperature distribution $\overline{T}^{(\omega)}(x,t_j)$, $0 \le x \le 1$ be realized at time moment t_j, $j = \overline{0,m}$. Denote $\overline{\xi}^{(j)}(t) = N(t, \overline{T}^{(\omega)}(\cdot,t_j))$, $t \in [t_j,p]$. Let continuous functions (2.4) $T_i(1,t) = \varrho_i^{(2)}(t)$ for $t_j \le t \le t_{j+1}$, $i = \overline{k+1,l}$, for which $\varrho_i^{(2)}(t_j) = T_i^{(\omega)}(1,t_j)$, and continuous functions $f_i(x,t)$ (2.5), $i = \overline{1,n}$, for $t_j \le t \le t_{j+1}$, $0 \le x \le 1$, be realized.

We denote by $T_i^{(\omega)}(x,t)$ for $0 \le x \le 1$, $t_j \le t \le t_{j+1}$ the solution $Q_i(x,t)$ of the problem (2.6)–(2.8) for $\tau = t_j$, $\nu = t_{j+1}$ and for the following initial and boundary conditions:

$$\beta_i(x) = T_i^{(\omega)}(x,t_j), \quad x \in [0,1]; \tag{3.2}$$

$$Q_i(0,t) = T_i^{(\omega)}(0,t_j) + \int_{t_j}^{t} (a_i^{(1)}(r) + a_i^{(2)}(r) G_i^{(1)} \overline{\xi}^{(j)}(r))dr, \quad t \in [t_j, t_{j+1}]; \tag{3.3}$$

$$Q_i(1,t) = T_i^{(\omega)}(1,t_j) + \int_{t_j}^{t} (b_i^{(1)}(r) + b_i^{(2)}(r) G_i^{(2)} \overline{\xi}^{(j)}(r))dr, \quad i = \overline{1,k}, \tag{3.4}$$

$$Q_i(1,t) = T_i(1,t), \quad i = \overline{k+1,n}, \quad t \in [t_j, t_{j+1}]. \tag{3.5}$$

Definition 1. *We say that control of the form (3.1) guarantees the fulfilment of the stated goal (2.9), if for any number $\gamma \in (\varepsilon, \Delta/2)$ there exists a number $\delta > 0$ such that for any partition ω with diameter $d(\omega) < \delta$, for any continuous functions $f_i(x,t)$ (2.5), $i = \overline{1,n}$, that satisfy Assumption 1, and for any continuous functions $T_i(1,t)$ (2.4), $i = \overline{k+1,l}$, the inclusion*

$$\sum_{i=1}^{n} \lambda_i \int_0^1 T_i^{(\omega)}(x,p)\sigma_i(x)dx \in Z(\gamma) = \bigcup_{s=\overline{1,r}} [\alpha_s - \gamma, \alpha_s + \gamma] \tag{3.6}$$

holds.

Note that when inequality $\gamma < \Delta/2$ is satisfied, segments $[\alpha_s - \gamma, \alpha_s + \gamma]$, $s = \overline{1,r}$, do not intersect.

4. Reduction to a one-dimensional problem

Let us denote by $\psi_i(x,\tau)$ for $0 \le x \le 1$, $0 \le \tau \le p$ solutions of the following first boundary value problems

$$\frac{\partial \psi_i(x,\tau)}{\partial \tau} = \frac{\partial^2 \psi_i(x,\tau)}{\partial x^2}, \quad \psi_i(x,0) = \sigma_i(x), \quad \psi_i(0,\tau) = \psi_i(1,\tau) = 0, \quad i = \overline{1,n}. \tag{4.1}$$

Equality (2.6) implies that the matching conditions at the ends of the segment in problems (2.9) are satisfied.

Using the conditions (2.4), it can be shown that [15]

$$\begin{gathered}\left\{ \int_0^1 f_i(x,t)\psi_i(x,p-t)dx : f_i^{(1)}(x,t) \le f_i(x,t) \le f_i^{(2)}(x,t) \right\} = \\ = \{c_i^{(1)}(t) + c_i^{(2)}(t) s_i(t) : |s_i(t)| \le 1\}, \quad i = \overline{1,n},\end{gathered} \tag{4.2}$$

where

$$c_i^{(1)}(t) = \frac{1}{2}\int_0^1 (f_i^{(1)}(x,t) + f_i^{(2)}(x,t))\psi_i(x,p-t)dx,$$
$$c_i^{(2)}(t) = \frac{1}{2}\int_0^1 (f_i^{(2)}(x,t) - f_i^{(1)}(x,t))|\psi_i(x,p-t)|dx.$$

Note that the functions $c_i^{(1)}(t)$ and $c_i^{(2)}(t)$, $i = \overline{1,n}$, are continuous for $0 \le t \le p$ and $c_i^{(2)}(t) \ge 0$.

The inequalities (2.4) imply

$$\{T_i(1,t)\} = \left\{ \frac{\beta_i^{(1)}(t) + \beta_i^{(2)}(t)}{2} + \frac{\beta_i^{(2)}(t) - \beta_i^{(1)}(t)}{2} \widehat{\eta}_i(t) : \ |\widehat{\eta}_i(t)| \le 1 \right\} \tag{4.3}$$

for $i = \overline{k+1, l}$.

Introduce new variables

$$\begin{gathered} y_i(t) = \int_0^1 T_i(x,t)\psi_i(x,p-t)dx + T_i(0,t)\int_t^p \frac{\partial\psi_i(0,p-r)}{\partial x}dr + \\ + \int_t^p \left(a_i^{(1)}(\tau)\int_\tau^p \frac{\partial\psi_i(0,p-r)}{\partial x}dr + c_i^{(1)}(\tau)\right)d\tau - \theta_i(t), \quad i = \overline{1,n}, \end{gathered} \tag{4.4}$$

where

$$\theta_i(t) = T_i(1,t)\int_t^p \frac{\partial\psi_i(1,p-r)}{\partial x}dr + \int_t^p b_i^{(1)}(\tau)\int_\tau^p \frac{\partial\psi_i(1,p-r)}{\partial x}drd\tau, \quad i = \overline{1,k},$$

$$\theta_i(t) = \int_t^p \left(\frac{\beta_i^{(1)}(\tau)+\beta_i^{(2)}(\tau)}{2}\frac{\partial\psi_i(1,p-\tau)}{\partial x}\right)d\tau, \quad i = \overline{k+1,l},$$

$$\theta_i(t) = \int_t^p \left(T_i(1,\tau)\frac{\partial\psi_i(1,p-\tau)}{\partial x}\right)d\tau, \quad i = \overline{l+1,n}.$$

We fix a partition ω of the segment $[0,p]$ and a control (3.1). Let us substitute the realized functions $T_i^{(\omega)}(x,t)$, $i = \overline{1,n}$, into formula (4.4). Further, taking into account formulas (2.1), (3.2)–(3.5) and (4.1)–(4.3), we obtain

$$\begin{gathered} \dot{y}_i^{(\omega)}(t) = \left(a_i^{(2)}(t)\int_t^p \frac{\partial\psi_i(0,p-r)}{\partial x}dr\right)G_i^{(1)}\overline{\xi}^{(j)}(t) - \\ - \left(b_i^{(2)}(t)\int_t^p \frac{\partial\psi_i(1,p-r)}{\partial x}dr\right)G_i^{(2)}\overline{\xi}^{(j)}(t) + c_i^{(2)}(t)s_i(t), \quad i = \overline{1,k}, \end{gathered} \tag{4.5}$$

$$\begin{gathered} \dot{y}_i^{(\omega)}(t) = \left(a_i^{(2)}(t)\int_t^p \frac{\partial\psi_i(0,p-r)}{\partial x}dr\right)G_i^{(1)}\overline{\xi}^{(j)}(t) - \\ - \left(\frac{\beta_i^{(2)}(t)-\beta_i^{(1)}(t)}{2}\frac{\partial\psi_i(1,p-t)}{\partial x}\right)\widehat{\eta}_i(t) + c_i^{(2)}(t)s_i(t), \quad i = \overline{k+1,l}, \end{gathered} \tag{4.6}$$

$$\dot{y}_i^{(\omega)}(t) = \left(a_i^{(2)}(t)\int_t^p \frac{\partial\psi_i(0,p-r)}{\partial x}dr\right)G_i^{(1)}\overline{\xi}^{(j)}(t) + c_i^{(2)}(t)s_i(t), \quad i = \overline{l+1,n}. \tag{4.7}$$

Next, we rewrite (4.5)–(4.7) in the matrix form

$$\dot{\overline{y}}^{(\omega)}(t) = -A(t)\overline{\xi}^{(j)}(t) + B(t)\overline{\eta}(t), \quad \overline{\xi}^{(j)}(t) \in U, \quad \overline{\eta}(t) \in \Pi(n). \tag{4.8}$$

Here

$$\overline{y}^{(\omega)}(t) = \left(y_1^{(\omega)}(t), y_2^{(\omega)}(t), \ldots, y_n^{(\omega)}(t)\right)^*;$$

$$\Pi(n) = \{\overline{s} = (s_1, s_2, \ldots, s_n)^* \in \mathbb{R}^n : |s_i| \le 1, i = \overline{1,n}\};$$

$$A_i(t) = -\left(a_i^{(2)}(t)\int_t^p \frac{\partial\psi_i(0,p-r)}{\partial x}dr\right)G_i^{(1)} + \left(b_i^{(2)}(t)\int_t^p \frac{\partial\psi_i(1,p-r)}{\partial x}dr\right)G_i^{(2)}$$

for $i = \overline{1,k}$,

$$A_i(t) = -\left(a_i^{(2)}(t)\int_t^p \frac{\partial\psi_i(0,p-r)}{\partial x}dr\right)G_i^{(1)}, \quad \text{for} \quad i = \overline{k+1,n};$$

$$\begin{gathered} B(t) = \mathrm{diag}\Bigg\{c_1^{(2)}(t), \ldots, c_k^{(2)}(t), c_{k+1}^{(2)}(t) + \frac{\beta_{k+1}^{(2)}(t)-\beta_{k+1}^{(1)}(t)}{2}\left|\frac{\partial\psi_{k+1}(1,p-t)}{\partial x}\right|, \\ \ldots, c_l^{(2)}(t) + \frac{\beta_l^{(2)}(t)-\beta_l^{(1)}(t)}{2}\left|\frac{\partial\psi_l(1,p-t)}{\partial x}\right|, c_{l+1}^{(2)}(t), \ldots, c_n^{(2)}(t)\Bigg\}. \end{gathered}$$

Denote by $\langle\cdot,\cdot\rangle$ the operation of the scalar product of two vectors. Define

$$a_-(t) = \min_{\overline{\xi}\in U}\langle\overline{\lambda}, A(t)\overline{\xi}\rangle, \quad a_+(t) = \max_{\overline{\xi}\in U}\langle\overline{\lambda}, A(t)\overline{\xi}\rangle, \quad b(t) = \max_{\overline{\eta}\in\Pi(n)}\langle\overline{\lambda}, B(t)\overline{\eta}\rangle.$$

Note that these functions are continuous.

Then the connectedness of the compact sets U, $\Pi(n)$ and the symmetry of $\Pi(n)$ imply

$$\langle\overline{\lambda}, A(t)\overline{\xi}\rangle = \frac{1}{2}(a_+(t) + a_-(t)) + a(t)u, \quad |u| \le 1, \quad a(t) = \frac{1}{2}(a_+(t) - a_-(t)) \ge 0; \tag{4.9}$$

$$\langle\overline{\lambda}, B(t)\overline{\eta}\rangle = b(t)v, \quad |v| \le 1. \tag{4.10}$$

We introduce a new one-dimensional variable

$$z = \langle\overline{\lambda}, \overline{y}\rangle. \tag{4.11}$$

Taking into account (4.11), we obtain a polygonal line $z^{(\omega)}(t)$, which satisfies the equality

$$z^{(\omega)}(p) = \sum_{i=1}^{n}\lambda_i\int_0^1 T_i^{(\omega)}(x,p)\sigma_i(x)dx.$$

It follows that inclusion (2.9) takes the form

$$z^{(\omega)}(p) \in Z(\gamma). \tag{4.12}$$

Differentiate z, taking into account formulas (4.8)–(4.10). Taking the uncertain function v as a control of the second player, we obtain the following one-dimensional differential game

$$\dot{z}^{(\omega)}(t) = -a(t)u + b(t)v, \quad |u| \le 1, \quad |v| \le 1, \quad z(p) \in Z(\varepsilon). \tag{4.13}$$

5. Termination conditions

Define function

$$g(t) = \int_t^p (a(r) - b(r))dr$$

for $t \le p$ and denote

$$q_1(\varepsilon) = \inf\big\{t < p : \varepsilon + g(\tau) < \Delta - \varepsilon - g(\tau) \text{ for all } t < \tau \le p\big\},$$
$$q_2(\varepsilon) = \inf\big\{t < p : 0 \le \varepsilon + g(\tau) \text{ for all } t < \tau \le p\big\},$$
$$q_3(\varepsilon) = \inf\big\{t < p : \alpha_1 - \varepsilon - g(\tau) \le \alpha_r + \varepsilon + g(\tau) \text{ for all } t < \tau \le p\big\}.$$

Let us define the set $W(t,\varepsilon)$ for $t \le p$ as follows:

$$W(t,\varepsilon) = \begin{cases} \bigcup\limits_{s=\overline{1,r}}[\alpha_s - \varepsilon - g(t), \alpha_s + \varepsilon + g(t)] & \text{for} \quad \max(q_1(\varepsilon), q_2(\varepsilon)) \le t \le p, \\ [\alpha_1 - \varepsilon - g(t), \alpha_r + \varepsilon + g(t)] & \text{for} \quad q_3(\varepsilon) \le t < q_1(\varepsilon), \quad q_2(\varepsilon) < q_1(\varepsilon), \\ \varnothing & \text{for} \quad \max(t, q_1(\varepsilon)) < q_2(\varepsilon) \text{ or } t < q_3(\varepsilon). \end{cases} \tag{5.1}$$

Here $\varnothing$ denotes the empty set.

Theorem 1. *Let the initial temperature distributions $T_i(x,0) = g_i(x)$ be such that the inclusion*

$$z(0) \in W(0,\varepsilon) \tag{5.2}$$

holds. Then there exists a control $\overline{\xi}$ that guarantees the fulfillment of the stated goal (2.9) for any unknown functions (2.4), (2.5).

P r o o f. *Case 1.* Let $\max(q_1(\varepsilon), q_2(\varepsilon)) \le 0 \le p$. Then, according to (5.1), inclusion (5.2) implies conditions

$$-g(\tau) \le \varepsilon \quad \text{for all} \quad 0 < \tau \le p, \quad z(0) \in [\alpha_s - \varepsilon - g(0), \alpha_s + \varepsilon + g(0)] \tag{5.3}$$

for some $s \in \overline{1, r}$.

Let's make a change of variable $z_* = z - \alpha_s$ and rewrite (5.3) as follows

$$F(z_*(0)) \le \varepsilon, \tag{5.4}$$

where

$$F(z) = \max\left(|z| - g(0), -\min_{0 \le \tau \le p} g(\tau)\right).$$

Define $\overline{\xi}_0(t) = N(t, \overline{T}(\cdot, \tau))$, $t \in [\tau, p]$ as the solution of problem

$$\langle \overline{\lambda}, A(t)\overline{\xi}(t)\rangle \text{sign } z_*(t) \to \max_{\overline{\xi}(t) \in U}.$$

Here and henceforth sign $0 = 1$.

Next, taking into account (4.9), we substitute the control $\xi_0(t)$ into (4.13) with $z = z_*$. We get that

$$\dot{z}_*^{(\omega)}(t) = -a(t)\text{sign } z_*(t_j) + b(t)v(t), \quad |v(t)| \le 1. \tag{5.5}$$

Here $v(t)$ satisfies the conditions: $|v(t)| \le 1$ if $b(t) = 0$, and

$$v(t) = \frac{\langle \overline{\lambda}, B(t)\overline{\eta}(t)\rangle}{b(t)} \quad \text{for} \quad b(t) > 0.$$

Each measurable function $v : [0, p] \to [-1, 1]$ with $z_*^{(\omega)}(0) = z_*(0)$ defines a polygonal line $z_*^{(\omega)}(t)$ satisfying equation (5.5). The family of these polygonal lines defined on the interval $[0, p]$ is uniformly bounded and equicontinuous [16, p. 46]. According to Arzel's theorem [7, p. 104] from any sequence of these polygonal lines we can select a subsequence uniformly converging on the segment $[0, p]$. The limit function $z_*(t)$ satisfies [16, Theorem 8.1] the inequality

$$|z_*(p)| \le F(z_*(0)). \tag{5.6}$$

Fix a number $\gamma \in (\varepsilon, \Delta/2)$. Let us show that there exists a number $\delta > 0$ such that inclusion (4.12) holds for any polygonal line $z^{(\omega)}(t)$ with partition diameter $d(\omega) < \delta$.

Indeed, let us assume the opposite. Then there exists a sequence of polygonal lines $z^{(\omega_k)}(t)$ with diameters $d(\omega_k) \to 0$ such that $z^{(\omega_k)}(p) \notin Z(\gamma)$ or what is the same

$$|z^{(\omega_k)}(p) - \alpha_s| > \gamma$$

for all $s \in \overline{1, r}$. We can assume that the functions $z^{(\omega_k)}(t)$ converge on the segment $[0, p]$ uniformly to the function $z(t)$ (otherwise we move on to a subsequence). Then

$$|z(p) - \alpha_s| \ge \gamma$$

for all $s \in \overline{1, r}$. This inequality contradicts inequalities (5.4) and (5.6).

Case 2. Let $q_3(\varepsilon) \le 0 < q_1(\varepsilon)$, $q_2(\varepsilon) < q_1(\varepsilon)$. Then, according to (5.1), inclusion (5.2) implies conditions

$$\alpha_1 - \varepsilon - g(\tau) \le \alpha_r + \varepsilon + g(\tau) \text{ for all } 0 < \tau \le p, \quad z(0) \in \left[\alpha_1 - \varepsilon - g(0), \alpha_r + \varepsilon + g(0)\right]. \tag{5.7}$$

Define $\overline{\xi}^0(t) = N(t, \overline{T}(\cdot,\tau))$, $t \in [\tau, p]$ as the solution of problem

$$\langle \overline{\lambda}, A(t)\overline{\xi}(t)\rangle \mathrm{sign}(z(t) - 0.5(\alpha_1 + \alpha_r)) \to \max_{\overline{\xi}(t)\in U}.$$

Taking into account (4.9), we substitute the control $\xi^0(t)$ into (4.13)

Next, reasoning by analogy with case 1 of the proof and relying on the results of work [17], it can be shown that when conditions (5.7) are satisfied, the limit function $z(t)$ satisfies the inclusion

$$z(q_1(\varepsilon)) \in \big[\alpha_1 - \varepsilon - g(q_1(\varepsilon)), \alpha_r + \varepsilon + g(q_1(\varepsilon))\big].$$

According to the definition of $q_1(\varepsilon)$, equality

$$\big[\alpha_1 - \varepsilon - g(q_1(\varepsilon)), \alpha_r + \varepsilon + g(q_1(\varepsilon))\big] = \bigcup_{s=\overline{1,r}} \big[\alpha_s - \varepsilon - g(q_1(\varepsilon)), \alpha_s + \varepsilon + g(q_1(\varepsilon))\big]$$

holds, and, therefore,

$$z(q_1(\varepsilon)) \in \big[\alpha_s - \varepsilon - g(q_1(\varepsilon)), \alpha_s + \varepsilon + g(q_1(\varepsilon))\big] \tag{5.8}$$

holds for some $s \in \overline{1,r}$.

Since $q_2(\varepsilon) < q_1(\varepsilon)$, then inequality $-g(\tau) \le \varepsilon$ holds for all $q_1(\varepsilon) < \tau \le p$. From here and from (5.8) we fall into the condition of case 1.

□

Now we consider the case when function

$$\overline{\eta}_*(t) = \mathrm{sign}(z^{(\omega)}(t_j) - 0.5(\alpha_1 + \alpha_r))(1, 1, \ldots, 1)^*$$

is realized in (4.8) for $t_j < t < t_{j+1}$.

Taking (4.10) into account, let us substitute this function $\overline{\eta}_*(t)$ into (4.13). We get that

$$\dot{z}^{(\omega)}(t) = -a(t)u_j(t) + b(t)\,\mathrm{sign}\,(z^{(\omega)}(t_j) - 0.5(\alpha_1 + \alpha_r)), \tag{5.9}$$

where

$$a(t)u_j(t) = \langle \overline{\lambda}, A(t)\overline{\xi}^{(j)}(t)\rangle.$$

Choosing arbitrary measurable functions $\xi^{(j)}(t) \in U$ and solving equation (5.9) with $z^{(\omega)}(0) = z(0)$, we obtain a family of polygonal lines $z^{(\omega)}(t)$.

Theorem 2. *Let at least one of the following inequalities be satisfied:*

$$z(0) < \alpha_1 - \gamma - g(0), \quad \alpha_r + \gamma + g(0) < z(0), \quad 0 < q_3(\gamma). \tag{5.10}$$

Then there exists a number $\delta > 0$ such that $z^{(\omega)}(p) \notin Z(\gamma)$ for any polygonal line $z^{(\omega)}(t)$ (5.9) with partition diameter $d(\omega) < \delta$.

P r o o f. Let's assume the opposite. Let's take a sequence of numbers $\delta_k \to 0$. Then there exists a sequence of polygonal lines $z^{(\omega_k)}(t)$ with diameter $d(\omega_k) < \delta_k$ and $z^{(\omega_k)}(t) \in Z(\gamma)$. The family of polygonal lines (5.9) with $z^{(\omega)}(0) = z(0)$ satisfies the conditions of Arzela's theorem. Passing, if necessary, to a subsequence, we can assume that the sequence of polygonal lines $z^{(\omega_k)}(t)$ converges to $z(t)$ uniformly.

Let's make a change of variables

$$\tilde{z} = z - 0.5(\alpha_1 + \alpha_r)$$

and rewrite conditions (5.10) in the following form:

$$|\tilde{z}(0)| > \gamma + g(0) + 0.5(\alpha_r - \alpha_1) \quad \text{or} \quad \gamma + 0.5(\alpha_r - \alpha_1) + \min_{0\le\tau\le p} g(\tau) < 0.$$

Hence,

$$0 \le \gamma < F(\tilde{z}(0)) - 0.5(\alpha_r - \alpha_1). \tag{5.11}$$

On the other hand, the limit function satisfies the inequality $|\tilde{z}(p)| \ge F(\tilde{z}(0))$ [16, Theorem 8.2]. From this and from (5.11) we obtain that

$$|\tilde{z}^{(\omega_k)}(p)| > \gamma + 0.5(\alpha_r - \alpha_1)$$

for any sufficiently large number k. After a reverse change of variables, we obtain one of the inequalities

$$z^{(\omega_k)}(p) < \alpha_1 - \gamma \quad \text{or} \quad z^{(\omega_k)}(p) > \alpha_r + \gamma.$$

Thus, we get a contradiction.

□

Next, consider the case when the function

$$\overline{\eta}^*(t) = -\text{sign}\left(z^{(\omega)}(t_j) - 0.5(\alpha_s + \alpha_{s+1})\right)(1, 1, \ldots, 1)^*$$

is realized in (4.8) for $t_j < t < t_{j+1}$. Here, number $s \in \overline{1, r-1}$ can be calculated as the solution of the minimization problem

$$\min_{s\in\overline{1,r-1}} |z(0) - 0.5(\alpha_s + \alpha_{s+1})|.$$

Taking (4.10) into account, let us substitute this function $\overline{\eta}^*(t)$ into (4.13). We obtain

$$\dot{z}^{(\omega)}(t) = -a(t)u_j(t) - b(t)\,\text{sign}\,(z^{(\omega)}(t_j) - 0.5(\alpha_s + \alpha_{s+1})). \tag{5.12}$$

Further, we define a family of polygonal lines $z^{(\omega)}(t)$ for equation (5.12) by analogy with (5.9).

Theorem 3. *Let the following inequalities be satisfied:*

$$\alpha_s + \gamma + g(0) < z(0) < \alpha_{s+1} - \gamma - g(0), \quad q_1(\gamma) < 0. \tag{5.13}$$

Then there exists a number $\delta > 0$ such that $z^{(\omega)}(p) \notin Z(\gamma)$ for any polygonal line $z^{(\omega)}(t)$ (5.12) with partition diameter $d(\omega) < \delta$.

P r o o f. Let's assume the opposite. By analogy with the proof of Theorem 2, we construct a sequence of polygonal lines $z^{(\omega_k)}(t)$ that converges to $z(t)$ uniformly.

Let's introduce the variable

$$\widehat{z} = z - 0.5(\alpha_s + \alpha_{s+1})$$

and write inequalities (5.13) as follows:

$$|\widehat{z}(0)| < 0.5\Delta - \gamma - g(0), \quad 0 < 0.5\Delta - \gamma - \max_{0\le\tau\le p} g(\tau).$$

From here we obtain

$$G(\widehat{z}(0)) = \max\left\{|\widehat{z}(0)| - \int_0^p (b(r) - a(r))dr, -\min_{0\le\tau\le p}\int_\tau^p (b(r) - a(r))dr\right\} < 0.5\Delta - \gamma \qquad (5.14)$$

On the other hand, applying [16, Theorem 8.1] from the point of view of the second player (in variables $\widehat{z}$ the roles of the players change, and the second player becomes the pursuer), we obtain that the limit function satisfies the inequality $|\widehat{z}(p)| \le G(\widehat{z}(0))$. From this and from (5.14) we obtain that

$$|\widehat{z}^{(\omega_k)}(p)| < 0.5\Delta - \gamma$$

for all sufficiently large numbers k. After a reverse change of variables, we obtain the inequalities

$$\alpha_s + \gamma < z^{(\omega_k)}(p) < \alpha_{s+1} - \gamma.$$

Thus, we get a contradiction. □

Remark 1. Let $q_3(\gamma) \le 0 < q_1(\gamma) < q_2(\gamma)$. Let's substitute an arbitrary function $v(t, z)$ ($|v(t, z)| \le 1$) into (4.13) and, by analogy with the proof of Theorem 2, define $z(t)$ as the uniform limit of a sequence of polygonal lines. Then there exists a time moment $t_* \in (q_1(\gamma), q_2(\gamma))$ such that $\gamma + g(\tau) < 0$ for all $\tau \in [t_*, q_2(\gamma))$. Then one of the following conditions is satisfied:

$$\alpha_s + \gamma + g(t_*) < z(t_*) < \alpha_{s+1} - \gamma - g(t_*) \quad \text{for some} \quad s \in \overline{1, r-1};$$
$$z(t_*) < \alpha_1 - \gamma - g(t_*); \quad \alpha_r + \gamma + g(t_*) < z(t_*).$$

From here we find ourselves in the conditions of Theorem 2 or 3 with the initial time moment t_*.

Corollary 1. *Theorems 1, 2, 3 and Remark 1 imply that the set $W(0, \varepsilon)$ determines the necessary and sufficient termination conditions in the differential game (4.13).*

6. Conclusion

This paper considers the problem of controlling a parabolic system that describes the heating of a given number of rods, with a non-convex one-dimensional terminal set, which is defined as the union of a finite number of disjoint segments of equal length. Necessary and sufficient conditions have been found under which there exists a control (3.1) that guarantees the achievement of the stated goal (2.9) for all continuous functions (2.4) and for all density functions of internal heat sources (2.5) that satisfy the Assumption 1.

In the future, it is planned to consider a version of this problem with an arbitrary n-dimensional non-convex terminal set. This will require the development of approximate algorithms for solving differential games: constructing a solvability set and restoring the corresponding control $\overline{\xi}$.

REFERENCES

1. Barseghyan V., Solodusha S. The problem of boundary control of the thermal process in a rod. *Mathematics,* 2023. Vol. 11, No. 13. Art. no. 2881. DOI: 10.3390/math11132881
2. Casas E., Yong J. Optimal control of a parabolic equation with memory. *ESAIM Control Optim. Calc. Var.*, 2023. Vol. 29. Art. no. 23. DOI: 10.1051/cocv/2023013
3. Dai J., Ren B. UDE-based robust boundary control of heat equation with unknown input disturbance. *IFAC-PapersOnLine,* 2017. Vol. 50, No. 1. P. 11403–11408. DOI: 10.1016/j.ifacol.2017.08.1801

4. Egorov A. I. *Optimal'noe upravlenie teplovymi i diffuzionnymi protsessami* [Optimal Control of Thermal and Diffusion Processes]. Moscow: Nauka, 1978. 464 p. (in Russian)
5. Feng H., Xu Ch.-Z., Yao P.-F. Observers and disturbance rejection control for a heat equation. *IEEE Trans. Automat. Control*, 2020. Vol. 65, No. 11. P. 4957–4964. DOI: 10.1109/TAC.2020.3022849
6. Izmest'ev I. V., Ukhobotov V. I. On one problem of controlling the heating of a rod system under uncertainty. *Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki*, 2022. Vol. 32, Iss. 4. P. 546–556. DOI: 10.35634/vm220404 (in Russian)
7. Kolmogorov A. N., Fomin S. V. *Elementy teorii funktsii i funktsional'nogo analiza* [Elements of the Theory of Functions and Functional Analysis]. Moscow: Nauka, 1972. 496 p. (in Russian)
8. Korotkii A. I., Osipov Iu. S. Approximation in problems of position control of parabolic systems. *J. Appl. Math. Mech.*, 1978. Vol. 42, No. 4. P. 631–637. DOI: 10.1016/0021-8928(78)90004-7
9. Krasovskii N. N. *Upravlenie dinamicheskoi sistemoi* [Control of a Dynamical System]. Moscow: Nauka, 1985. 520 p. (in Russian)
10. Lohéac J. Nonnegative boundary control of 1D linear heat equations. *Vietnam J. Math.*, 2021. Vol. 49, No. 3. P. 845–870. DOI: 10.1007/s10013-021-00497-5
11. Maksimov V. I. On the reconstruction of an input disturbance in a reaction-diffusion system. *Comput. Math. Math. Phys.*, 2023. Vol. 63, No. 6. P. 990–1000. DOI: 10.1134/S0965542523060143
12. Okhezin S. P. Differential encounter-evasion game for a parabolic system under integral constraints on the player's controls. *J. Appl. Math. Mech.*, 1977. Vol. 41, No. 2. P. 194–201. DOI: 10.1016/0021-8928(77)90002-8
13. Osipov Iu. S. Position control in parabolic systems. *J. Appl. Math. Mech.*, 1977. Vol. 41, No. 2. P. 187–193. DOI: 10.1016/0021-8928(77)90001-6
14. Osipov Yu. S., Okhezin S. P. On the theory of differential games in parabolic systems. *Dokl. Akad. Nauk SSSR*, 1976. Vol. 226, No. 6. P. 1267–1270.
15. Ukhobotov V. I., Izmest'ev I. V. A control problem for a rod heating process with unknown temperature at the right end and unknown density of the heat source. *Trudy Inst. Mat. Mekh. UrO RAN,* 2019. Vol. 25, No. 1. P. 297–305. DOI: 10.21538/0134-4889-2019-25-1-297-305 (in Russian)
16. Ukhobotov V. I. *Metod odnomernogo proektirovaniya v lineinykh differentsial'nykh igrakh s integral'nymi ogranicheniyami* [Method of One-Dimensional Projecting in Linear Differential Games with Integral Constraints]. Chelyabinsk: Chelyabinsk State University, 2005. 124 p. (in Russian)
17. Ukhobotov V. I. One type differential games with convex goal. *Trudy Inst. Mat. Mekh. UrO RAN,* 2010. Vol. 16, No. 5. P. 196–204. (in Russian)
18. Wang S., Qi J., Diagne M. Adaptive boundary control of reaction-diffusion PDEs with unknown input delay. *Automatica,* 2021. Vol. 134. Art. no. 109909. DOI: 10.1016/j.automatica.2021.109909
19. Zheng G., Li J. Stabilization for the multi-dimensional heat equation with disturbance on the controller. *Automatica,* 2017. Vol. 82. P. 319–323. DOI: 10.1016/j.automatica.2017.04.011