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ON WIDTHS OF SOME CLASSES OF ANALYTIC FUNCTIONS IN A CIRCLE

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Abstract: We calculate exact values of some n-widths of the class $W_q^{(r)}(\Phi)$, $r \in \mathbb{Z}_+$, in the Banach spaces $\mathscr{L}_{q,\gamma}$ and $B_{q,\gamma}$, $1 \le q \le \infty$, with a weight γ. These classes consist of functions f analytic in the unit circle, their rth order derivatives $f^{(r)}$ belong to the Hardy space H_q, $1 \le q \le \infty$, and the averaged moduli of smoothness of boundary values of $f^{(r)}$ are bounded by a given majorant Φ at the system of points $\{\pi/(2k)\}_{k\in\mathbb{N}}$; more precisely,

$$\frac{k}{\pi-2}\int_0^{\pi/(2k)} \omega_2(f^{(r)}, 2t)_{H_{q,\rho}}\, dt \le \Phi\left(\frac{\pi}{2k}\right)$$

for all $k \in \mathbb{N}$, $k > r$.

Keywords: Modulus of smoothness, The best approximation, n-widths, The best linear method of approximation.

1. Introduction

There are many studies devoted to calculating exact values of various n-widths of classes of functions analytic in the unit circle both in the Hardy space H_q $(1 \le q \le \infty)$ and in the Bergman space B_q $(1 \le q \le \infty)$ (see, e.g., [1–36]). The present paper aims to obtain new results related to calculating exact values of various n-widths of some classes of functions analytic in the unit circle.

First, we introduce some notation and concepts. Define

$$U_\rho := \{z \in \mathbb{C} : |z| < \rho\}, \quad 0 < \rho \le 1,$$

Let $U := U_1$, let $\mathscr{A}(U_\rho)$ be the set of functions analytic in a circle U_ρ, and let H_q $(1 \le q \le \infty)$ be the Hardy space of functions $f \in \mathscr{A}(U)$ such that

$$\|f\|_{H_q} = \lim_{\rho\to 1-0} M_q(f,\rho),$$

where

$$M_q(f,\rho) := \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(\rho e^{it})|^q dt\right)^{1/q}, & 1 \le q < \infty, \\ \max\left\{|f(\rho e^{it})| : 0 < t \le 2\pi\right\}, & q = \infty; \end{cases}$$

the integral is understood in the Lebesgue sense.

It is known [26] that the norm $\|f\|_{H_q}$ is attained on angular boundary values $f(e^{it})$ of functions $f \in H_q$, which exist almost everywhere on $[0, 2\pi)$. We set

$$H_{q,\rho} := \left\{ f \in \mathscr{A}(U_\rho) : \|f(\cdot)\|_{H_{q,\rho}} := \|f(\rho\cdot)\|_{H_q} < \infty \right\}$$

and, for $r \in \mathbb{Z}_+$,

$$H_q^{(r)} := \left\{ f \in \mathscr{A}(U) : f^{(r)} \in H_q \right\} \quad (H_q^{(0)} \equiv H_q),$$

where

$$f^{(r)}(z) = \sum_{k=r}^{\infty} \alpha_{k,r} c_k(f) z^k,$$

$$\alpha_{k,r} = k(k-1)\cdots(k-r+1), \quad k \geq r, \quad k \in \mathbb{Z}_+, \quad \alpha_{k,0} \equiv 1,$$

and $c_k(f)$ are coefficients of the Taylor series

$$f(z) = \sum_{k=0}^{\infty} c_k(f) z^k.$$

Denote by

$$\mathscr{L}_q := \mathscr{L}_q(U) \quad (1 \leq q < \infty)$$

the Banach space of complex-valued functions f on U with finite norms

$$\|f\|_{\mathscr{L}_q} = \left(\frac{1}{2\pi} \iint_{(U)} |f(z)|^q dxdy \right)^{1/q} = \left(\frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \rho |f(\rho e^{it})|^q dt d\rho \right)^{1/q},$$

where the integral is understood in the Lebesgue sense.

Let $\gamma(|z|)$ be a nonnegative measurable function not equivalent to zero and summable on the set U. Denote by

$$\mathscr{L}_{q,\gamma} := \mathscr{L}(U, \gamma) \quad (1 \leq q < \infty)$$

the set of complex-valued functions f on U such that

$$\gamma^{1/q} f \in \mathscr{L}_q(U), \quad \|f\|_{\mathscr{L}_{q,\gamma}} := \|\gamma^{1/q} f\|_{\mathscr{L}_q}.$$

By $B_{q,\gamma}$ $(1 \leq q < \infty)$, we mean the Banach space of functions $f \in \mathscr{A}(U)$ such that $f \in \mathscr{L}_{q,\gamma}$. In this case,

$$\|f\|_{B_{q,\gamma}} = \left(\int_0^1 \rho \gamma(\rho) M_q^q(f, \rho) d\rho \right)^{1/q}.$$

In the particular case of $\gamma \equiv 1$, the space $B_q := B_{q,1}$ is the well-known Bergman space.

Assume that X is a Banach space, $\mathbb{B}$ is the unit ball in this space, $\mathfrak{M}$ is a convex centrally symmetric subset of X, $L_n \subset X$ is an n-dimensional linear subspace, $L^n \subset X$ is a subspace of codimension n, and $\Lambda : X \to L_n$ is a continuous linear operator from X into L_n. Define the best approximation to an element $f \in X$ by elements of the subspace $L_n \subset X$ as

$$E_n(f)_X := E(f, L_n)_X = \inf \left\{ \|f - \varphi\|_X : \varphi \in L_n \right\}.$$

The approximation to the fixed set $\mathfrak{M} \subset X$ by the fixed subspace $L_n \subset X$ is defined by

$$E(\mathfrak{M}, L_n)_X := \sup \left\{ E(f, L_n)_X : f \in \mathfrak{M} \right\}.$$

If the approximation is performed with a linear operator A then, we will study the sharp upper bound

$$\sup\{\|f-A(f)\|_X : f\in\mathfrak{M}\},$$

and the quantity

$$\mathscr{E}(\mathfrak{M},L_n)_X=\inf\{\sup\{\|f-A(f)\|_X : f\in\mathfrak{M}\} : AX\subset L_n\}, \tag{1.1}$$

which characterizes the best linear approximation of the set $\mathfrak{M}$ by elements of $L_n\subset X$. If there exists a linear operator A^*, $A^*X\subset L_n$ realizing the infimum in (1.1), i.e., an operator such that

$$\mathscr{E}(\mathfrak{M},L_n)_X=\sup\{\|f-A^*(f)\|_X : f\in\mathfrak{M}\},$$

then A^* is called the best linear method of approximation to $\mathfrak{M}$.

The quantities

$$\begin{gathered} b_n(\mathfrak{M},X):=\sup\big\{\sup\{\varepsilon>0;\ \varepsilon\mathbb{B}\cap L_{n+1}\subset\mathfrak{M}\} : L_{n+1}\subset X\big\},\\ d_n(\mathfrak{M},X):=\inf\{E(\mathfrak{M},L_n)_X : L_n\subset X\},\\ d^n(\mathfrak{M},X):=\inf\big\{\sup\{\|f\|_X : f\in\mathfrak{M}\cap L^n\} : L^n\subset X\big\},\\ \delta_n(\mathfrak{M},X):=\inf\{\mathscr{E}(\mathfrak{M},L_n)_X : L_n\subset X\}, \end{gathered} \tag{1.2}$$

are called *Bernstein*, *Kolmogorov*, *Gelfand*, and *linear n-widths*, respectively (see, for example, [8, Ch. II], [30, Ch. III]).

If there exists a subspace $\bar{L}_{n+1}\subset X$, $\dim\bar{L}_{n+1}=n+1$, for which

$$b_n(\mathfrak{M},X):=\sup\{\varepsilon>0 : \varepsilon\mathbb{B}\cap L_{n+1}\subset\mathfrak{M}\},$$

then it is an extremal subspace for $b_n(\mathfrak{M},X)$. A subspace $L_n^*\subset X$, $\dim L_n^*=n$, on which the infimum in (1.2) is attained, i.e., $d_n(\mathfrak{M},X)=E(\mathfrak{M},L_n^*)$ is called an extremal subspace for the Kolmogorov n-width $d_n(\mathfrak{M},X)$. If there exist a subspace $L_*^n\subset X$ of codimention n such that

$$d^n(\mathfrak{M},X):=\sup\{\|f\|_X : f\in\mathfrak{M}\cap L_*^n\},$$

then L_*^n is said to be extremal for $d^n(\mathfrak{M},X)$. A subspace $\tilde{L}_n\subset X$, $\dim\tilde{L}_n=n$ such that

$$\delta_n(\mathfrak{M},X)=\mathscr{E}(\mathfrak{M},\tilde{L}_n),$$

if it exists, is called extremal for $\delta_n(\mathfrak{M},X)$. Finding extremal subspaces $\hat{L}_n\subset X$, $\dim\hat{L}_n=n$, such that

$$E(\mathfrak{M},\hat{L}_n)_X=\mathscr{E}(\mathfrak{M},\hat{L}_n)_X=d_n(\mathfrak{M},X)=\delta_n(\mathfrak{M},X)$$

is of special interest. The n-widths mentioned above satisfy the relations [8, 30]

$$b_n(\mathfrak{M},X)\le \begin{matrix} d_n(\mathfrak{M},X)\\ d^n(\mathfrak{M},X)\end{matrix} =\delta_n(\mathfrak{M},X). \tag{1.3}$$

2. Main theorem

Following [28, p. 652] and [14, p. 284], for an arbitrary function $f\in H_q$ $(1\le q\le\infty)$, we consider the modulus of smoothness

$$\omega_2(f,2x)_{H_q}:=\sup_{|t|\le x}\big\|f(e^{i(\cdot+t)})-2f(e^{i(\cdot)})+f(e^{i(\cdot-t)})\big\|_{L_q[0,2\pi]},$$

where the $L_q[0,2\pi]$-norm is defined by

$$\|f\|_{L_q[0,2\pi]} = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(e^{it})|^q dt\right)^{1/q}, & 1 \le q < \infty, \\ \operatorname*{ess\,sup}_{0\le t\le 2\pi} |f(e^{it})|, & q = \infty. \end{cases}$$

Let $\Phi(t)$, $t \ge 0$, be a continuous increasing function such that $\Phi(0) = 0$. Using Φ as a majorazing function, we consider the class of functions studied by Taikov [28]:

$$W_q^{(r)}(\Phi) := \left\{ f \in \mathscr{A}(U) : \ f^{(r)} \in H_q, \ \frac{k}{\pi - 2}\int_0^{\pi/(2k)} \omega_2(f^{(r)}, 2t)_{H_q} dt \le \Phi\left(\frac{\pi}{2k}\right), \ k \in \mathbb{N} \right\},$$

where $r \in \mathbb{Z}_+$ and $1 \le q \le \infty$.

In [28, Theorem 4], it is proved that, if the majorant $\Phi(t)$ for $0 < t \le \pi/2$ satisfies the inequality

$$\frac{\Phi(\lambda t)}{\Phi(t)} \ge \frac{\pi}{\pi - 2} \begin{cases} 1 - \dfrac{2}{\lambda\pi}\sin\dfrac{\lambda\pi}{2}, & \text{if} \quad 0 < \lambda \le 2, \\ 2\left(1 - \dfrac{1}{\lambda}\right), & \text{if} \quad \lambda \ge 2, \end{cases} \tag{2.1}$$

then the following equality holds for all $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$, and $1 \le q \le \infty$:

$$d_n\left(W_q^{(r)}(\Phi), H_q\right) = \frac{1}{\alpha_{n,r}}\,\Phi\left(\frac{\pi}{2(n-r)}\right).$$

It is also proved that the function $\Phi_*(t) = t^{2/(\pi-2)}$ satisfies constraint (2.1).

It is also of interest to calculate the exact values of the above n-widths for the classes $W_q^{(r)}(\Phi)$ in the spaces $\mathscr{L}_{q,\gamma}$ and $B_{q,\gamma}$, $r \in \mathbb{Z}_+$, $1 \le q < \infty$.

For this purpose, we specify the extremal subspaces L_n^*, L_*^n, and $\bar{L}_{n+1}$ and the best linear approximation method Λ_{n-1}^* already mentioned in the first section.

We set

$$\begin{aligned} L_n^* &:= \operatorname{span}\left\{ \{z^k\}_{k=0}^{r-1}, \left[\left\{1 + \frac{\rho^{2(n-k)}\alpha_{k,r}}{\alpha_{2n-k,r}}\left[\beta_{k,r}\left(1 - \left(\frac{k-r}{2n-k-r}\right)^2\right) - 1\right]\right\} z^k\right]_{k=r}^{n-1} \right\}, \\ \Lambda_{n-1}^* &:= \sum_{k=0}^{r-1} c_k(f) z^k + \sum_{k=r}^{n-1} \left\{1 + \frac{\rho^{2(n-k)}\alpha_{k,r}}{\alpha_{2n-k,r}}\left[\beta_{k,r}\left(1 - \left(\frac{k-r}{2n-k-r}\right)^2\right) - 1\right]\right\} c_k(f) z^k, \end{aligned} \tag{2.2}$$

where

$$\beta_{k,r} := \frac{2(n-r)}{\pi - 2}\int_0^{\pi/2(n-r)} (1 - \sin(n-r)x)\cos(k-r)x\,dx, \quad k \ge n > r, \quad k, n \in \mathbb{N}, \quad r \in \mathbb{Z}_+.$$

Theorem 1. *Let $r \in \mathbb{Z}_+$, $1 \le q < \infty$, and let the majorant Φ satisfies condition (2.1). Then, the following equalities hold for all $n \in \mathbb{N}$, $n > r$:*

$$\begin{aligned} b_n\left(W_q^{(r)}(\Phi), B_{q,\gamma}\right) &= b_n\left(W_q^{(r)}(\Phi), \mathscr{L}_{q,\gamma}\right) = d^n\left(W_q^{(r)}(\Phi), B_{q,\gamma}\right) = d^n\left(W_q^{(r)}(\Phi), \mathscr{L}_{q,\gamma}\right), \\ d_n\left(W_q^{(r)}(\Phi), B_{q,\gamma}\right) &= d_n\left(W_q^{(r)}(\Phi), \mathscr{L}_{q,\gamma}\right) = E\left(W_q^{(r)}(\Phi); L_n^*\right)_{\mathscr{L}_{q,\gamma}} = \mathscr{E}\left(W_q^{(r)}(\Phi); L_n^*\right)_{\mathscr{L}_{q,\gamma}} \\ &= \sup\left\{ \|f - \Lambda_{n-1}^*(f)\|_{\mathscr{L}_{q,\gamma}} : f \in W_q^{(r)}(\Phi) \right\} = \frac{1}{\alpha_{n,r}}\,\Phi\left(\frac{\pi}{2(n-r)}\right)\left(\int_0^1 \rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q}. \end{aligned} \tag{2.3}$$

Moreover,

(1) *the subspace* L_n^* *is extremal in the case of* n*-widths* $d_n\big(W_q^{(r)}(\Phi),\mathscr{L}_{q,\gamma}\big)$ *and* $\delta_n\big(W_q^{(r)}(\Phi),\mathscr{L}_{q,\gamma}\big)$;

(2) *the continuous linear operator* Λ_{n-1}^* *is the best linear approximation method for* $W_q^{(r)}(\Phi)$ *in* $\mathscr{L}_{q,\gamma}$;

(3) *the subspace* L_*^n *is extremal for the* n*-width* $d^n\left(W_q^{(r)}(\Phi),B_{q,\gamma}\right)$;

(4) *the subspace* $\bar{L}_{n+1}$ *is extremal for the* n*-width* $b_n\left(W_q^{(r)}(\Phi),B_{q,\gamma}\right)$.

To prove the theorem, we need the following lemma.

Lemma 1. *The following inequality holds for an arbitrary function* $f\in H_q^{(r)}$ $(r\in\mathbb{Z}_+,$ $1\le q<\infty)$:

$$E_{n-1}(f)_{\mathscr{L}_{q,\gamma}}\le\frac{1}{\alpha_{n,r}}\left(\int_0^1\rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q}E_{n-r-1}(f^{(r)})_{H_q}. \tag{2.4}$$

Inequality (2.4) *turns into an equality at the function* $f_0(z)=z^n$, $n>r$.

P r o o f. Relation (2.3) from [14] with $s=0$ implies that, for an arbitrary function $f\in H_q^{(r)}$ $(r\in\mathbb{Z}_+,\ 1\le q<\infty)$, there exists a polynomial $p_{n-1}\in\mathscr{P}_{n-1}$ satisfying the following inequality for $n\in\mathbb{N}$, $n>r$, and $0<\rho\le1$:

$$\left\|f(\rho e^{it})-p_{n-1}(\rho e^{it})\right\|_{H_q}\le\frac{\rho^n}{\alpha_{n,r}}E_{n-r-1}(f^{(r)})_{H_q}. \tag{2.5}$$

We raise both sides of (2.5) to the power q $(1\le q<\infty)$, multiply both sides by $\rho\gamma(\rho)$, integrate with respect to ρ over $[0,1]$, and raise the obtained result to the power $1/q$ $(1\le q<\infty)$. Finally, we have

$$\|f-p_{n-1}\|_{\mathscr{L}_{q,\gamma}}\le\frac{1}{\alpha_{n,r}}\left(\int_0^1\rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q}E_{n-r-1}(f^{(r)})_{H_q}.$$

This implies inequality (2.4). The equality in (2.4) for the function $f_0(z)=z^n$ is verified by direct calculation. The proof of lemma is complete. □

P r o o f of Theorem 1. Taikov proved [28, p. 288] the following inequality for an arbitrary function $f\in H_q$ $(1\le q\le\infty)$:

$$E_{n-1}(f)_{H_q}\le\frac{n}{\pi-2}\int_0^{\pi/(2n)}\omega_2(f,2t)_{H_q}dt; \tag{2.6}$$

and the equality in (2.6) for the function $f_0(z)=z^n$, $n\in\mathbb{N}$.

Replacing in (2.6) the number n with $n-r$ and the function f with $f^{(r)}\in H_q$, we obtain the following inequality for any function $f\in H_q^{(r)}$:

$$E_{n-r-1}(f^{(r)})_{H_q}\le\frac{n-r}{\pi-2}\int_0^{\pi/2(n-r)}\omega_2(f^{(r)},2t)_{H_q}dt. \tag{2.7}$$

In view of (2.7), we can write inequality (2.4) in the form

$$E_{n-1}(f)_{\mathscr{L}_{q,\gamma}}\le\frac{1}{\alpha_{n,r}}\left(\int_0^1\rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q}\frac{n-r}{\pi-2}\int_0^{\pi/2(n-r)}\omega_2(f^{(r)},2t)_{H_q}dt. \tag{2.8}$$

From (2.8), assuming that $f\in W_q^{(r)}(\Phi)$, we obtain

$$E_{n-1}(f)_{\mathscr{L}_{q,\gamma}}\le\frac{1}{\alpha_{n,r}}\left(\int_0^1\rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q}\Phi\left(\frac{\pi}{2(n-r)}\right).$$

Hence, by relations (1.3), we write upper estimates for the Bernstein and Kolmogorov n-widths:

$$\begin{aligned} b_n\left(W_q^{(r)}(\Phi),\mathscr{L}_{q,\gamma}\right) &\le d_n\left(W_q^{(r)}(\Phi),\mathscr{L}_{q,\gamma}\right) \le E_{n-1}\left(W_q^{(r)}(\Phi)\right)_{\mathscr{L}_{q,\gamma}} \\ &\le \frac{1}{\alpha_{n,r}}\left(\int_0^1 \rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q}\Phi\left(\frac{\pi}{2(n-r)}\right). \end{aligned} \tag{2.9}$$

To obtain a similar upper estimate for the linear n-width, we will use a result of Vakarchuk [36, p. 324]. He proved the following inequality for an arbitrary function $f \in W_q^{(r)}(\Phi)$ ($r \in \mathbb{Z}_+$, $1 \le q \le \infty$) for all $n \in \mathbb{N}$ and $0 < \rho \le 1$:

$$\left\| f(\rho e^{i(\cdot)}) - \Lambda^*_{n-1}(f,\rho e^{i(\cdot)})\right\|_{H_q} \le \frac{\rho^n}{\alpha_{n,r}}\Phi\left(\frac{\pi}{2(n-r)}\right),$$

Hence, we obtain an upper estimate for the linear n-widths:

$$\begin{aligned} &\delta_n\left(W_q^{(r)}(\Phi),\mathscr{L}_{q,\gamma}\right) \le \mathscr{E}_{n-1}\left(W_q^{(r)}(\Phi)\right)_{\mathscr{L}_{q,\gamma}} \\ &= \sup\left\{\|f-\Lambda^*_{n-1}(f)\|_{\mathscr{L}_{q,\gamma}} : f \in W_q^{(r)}(\Phi)\right\} \le \frac{1}{\alpha_{n,r}}\left(\int_0^1 \rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q}\Phi\left(\frac{\pi}{2(n-r)}\right). \end{aligned} \tag{2.10}$$

Relations (2.9) and (2.10) imply the following upper estimates for the n-widths $b_n(\cdot)$, $d_n(\cdot)$, and $\delta_n(\cdot)$:

$$\begin{aligned} \lambda_n\left(W_q^{(r)}(\Phi),\mathscr{L}_{q,\gamma}\right) &\le E_{n-1}\left(W_q^{(r)}(\Phi);L_n^*\right)_{\mathscr{L}_{q,\gamma}} \le \mathscr{E}_{n-1}\left(W_q^{(r)}(\Phi);L_n^*\right)_{\mathscr{L}_{q,\gamma}} \\ &\le \frac{1}{\alpha_{n,r}}\left(\int_0^1 \rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q}\Phi\left(\frac{\pi}{2(n-r)}\right), \end{aligned} \tag{2.11}$$

where $\lambda(\cdot)$ is any of the n-widths $b_n(\cdot)$, $d_n(\cdot)$, or $\delta_n(\cdot)$.

It is known [8, Ch. II, Sect. 3] that, if X and Y are linear normed spaces and X is the subspace of Y ($X \subset Y$), then $d^n(\mathfrak{N}, X) = d^n(\mathfrak{N}, Y)$, where $\mathfrak{N} \subset X$. Consequently, we can write

$$d^n\left(W_q^{(r)}(\Phi),\mathscr{L}_{q,\gamma}\right) = d^n\left(W_q^{(r)}(\Phi),B_{q,\gamma}\right).$$

By definition of the Bernstein n-width, we write

$$b_n\left(W_q^{(r)}(\Phi),\mathscr{L}_{q,\gamma}\right) \ge b_n\left(W_q^{(r)}(\Phi),B_{q,\gamma}\right).$$

In view of relation (1.3), to complete the proof of Theorem 1, it remains to obtain the inequality

$$b_n\left(W_q^{(r)}(\Phi),B_{q,\gamma}\right) \ge \frac{1}{\alpha_{n,r}}\left(\int_0^1 \rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q}\Phi\left(\frac{\pi}{2(n-r)}\right).$$

To this end, let us introduce the $(n+1)$-dimensional ball of polynomials

$$\mathbb{B}_{n+1} := \left\{p_n \in \mathscr{P}_n : \|p_n\|_{B_{q,\gamma}} \le \frac{1}{\alpha_{n,r}}\left(\int_0^1 \rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q}\Phi\left(\frac{\pi}{2(n-r)}\right)\right\}$$

and prove the possibility of the embedding $\mathbb{B}_{n+1} \subset W_q^{(r)}(\Phi)$.

We also introduce the notation

$$(1-\cos nx)_* := \left\{1-\cos nx, \text{ if } 0 < nx \le \pi;\ 2, \text{ if } nx > \pi\right\}.$$

The following inequality was proved in [27] for an arbitrary polynomial $p_n \in \mathscr{P}_n$:

$$\|p_n^{(r)}\|_{H_q} \le \alpha_{n,r}\|p_n\|_{H_q}, \quad n > r, \quad n \in \mathbb{N}, \quad r \in \mathbb{Z}_+.$$

We also need the inequality

$$\rho^{nq}\|p_n\|_{H_q}^q \le M_q^q(p_n,\rho) \quad (n \in \mathbb{N}, \quad 1 \le q \le \infty, \quad 0 < \rho \le 1), \tag{2.12}$$

which follows from the inequality

$$\int_{|z|=1} |p_n(z)|^q |dz| \le \rho^{-(nq+1)} \int_{|z|=\rho} |p_n(z)|^q |dz|$$

established by Hille, Szegő, and Tamarkin (see, for example, [25]). Multiplying both sides of (2.12) by $\rho\gamma(\rho)$ and integrating with respect to ρ over $[0,1]$, we obtain

$$\left(\int_0^1 \rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q} \|p_n\|_{H_q} \le \|p_n\|_{B_{q,\gamma}}$$

and hence

$$\|p_n\|_{H_q} \le \left(\int_0^1 \rho^{nq+1}\gamma(\rho)d\rho\right)^{-1/q} \|p_n\|_{B_{q,\gamma}}. \tag{2.13}$$

To prove that the ball $\mathbb{B}_{n+1}$ belongs to the class $W_q^{(r)}(\Phi)$, we will use the inequality

$$\omega_2(p_n^{(r)}, 2t)_{H_q} \le 2\alpha_{n,r}(1 - \cos(n-r)t)_* \|p_n\|_{H_q} \tag{2.14}$$

obtained from one of Taikov's result [28].

Consider two cases: $2k \ge n-r$ and $2k < n-r$.

Let $2k \ge n-r$. By (2.13) and (2.14), for an arbitrary polynomial $p_n \in \mathbb{B}_{n+1}$, we have

$$\begin{aligned} &\frac{k}{\pi-2}\int_0^{\pi/(2k)} \omega_2(p_n^{(r)}, 2t)_{H_q} dt \le 2\alpha_{n,r}\left(\int_0^1 \rho^{nq+1}\gamma(\rho)d\rho\right)^{-1/q} \|p_n\|_{B_{q,\gamma}} \\ &\times \frac{k}{\pi-2}\int_0^{\pi/(2k)} (1-\cos(n-r)t)dt \le \frac{\pi}{\pi-2}\left(1 - \frac{2k}{\pi(n-r)}\sin\frac{\pi(n-r)}{2k}\right)\Phi\left(\frac{\pi}{2(n-r)}\right). \end{aligned} \tag{2.15}$$

Using (2.15) and the first inequality from (2.1) with

$$t = \frac{\pi}{2(n-r)}, \quad \lambda = \frac{n-r}{k}, \quad \lambda t = \frac{\pi}{2k}, \tag{2.16}$$

we obtain

$$\frac{k}{\pi-2}\int_0^{\pi/(2k)} \omega_2(p_n^{(r)}, 2t)_{H_q} dt \le \Phi\left(\frac{\pi}{2k}\right). \tag{2.17}$$

Let $2k < n-r$. By (2.14) and (2.13), for an arbitrary polynomial $p_n \in \mathbb{B}_{n+1}$, we have

$$\begin{aligned} &\frac{k}{\pi-2}\int_0^{\pi/(2k)} \omega_2(p_n^{(r)}, 2t)_{H_q} dt \\ &\le \Phi\left(\frac{\pi}{2(n-r)}\right)\frac{k}{\pi-2}\left(\int_0^{\pi/(n-r)} 2(1-\cos(n-r)t)dt + \int_{\pi/(n-r)}^{\pi/(2k)} 4dt\right) \\ &= \frac{2\pi}{\pi-2}\left(1 - \frac{k}{n-r}\right)\Phi\left(\frac{\pi}{2(n-r)}\right). \end{aligned} \tag{2.18}$$

Using (2.16) and the second inequality from (2.1) with (2.18), we obtain equality (2.17). The inclusion $\mathbb{B}_{n+1} \subset W_q^{(r)}(\Phi)$ is proved. Then, by definition of the Bernstein n-width, we obtain

$$b_n\left(W_q^{(r)}(\Phi), B_{q,\gamma}\right) \geq b_n(\mathbb{B}_{n+1}, B_{q,\gamma}) \geq \frac{1}{\alpha_{n,r}}\,\Phi\left(\frac{\pi}{2(n-r)}\right)\left(\int_0^1 \rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q}. \tag{2.19}$$

Comparing relations (2.11) and (2.19), we obtain the required equality (2.3).

It follows from the proof of Theorem 1 that the subspace L_n^* is extremal for the class $W_q^{(r)}(\Phi)$ in the space $\mathscr{L}_{q,\gamma}$ in the case of exact values of the Kolmogorov n-width $d_n(\cdot)$ and the linear n-width $\delta_n(\cdot)$. The subspace $\bar{L}_{n+1}$ is extremal for the Bernstein n-width $b_n(\cdot)$. The linear continuous operator Λ_{n-1}^* defined by equality (2.2) is the best linear approximation method for the class $W_q^{(r)}(\Phi)$ in $\mathscr{L}_{q,\gamma}$. By definition of the Gelfand n-width, the last inequality in (2.10) particularly implies the following inequality for an arbitrary function $f \in W_q^{(r)}(\Phi)$ in the case $c_k(f) = 0$, $k = \overline{0, n-1}$:

$$\begin{aligned} d^n\left(W_q^{(r)}(\Phi), B_{q,\gamma}\right) &\leq \sup\left\{\|f\|_{B_{q,\gamma}} : f \in W_q^{(r)}(\Phi) \cup L_*^n\right\} \\ &\leq \frac{1}{\alpha_{n,r}}\,\Phi\left(\frac{\pi}{2(n-r)}\right)\left(\int_0^1 \rho^{nq+1}\gamma(\rho)d\rho\right)^{1/q}. \end{aligned} \tag{2.20}$$

Comparing inequalities (2.19) and (2.20) and taking into account relation (1.3), we see that the subspace L_*^n of codimension n is extremal for the Gelfand n-widths $d^n(\cdot)$. Theorem 1 is proved. □

3. Conclusion

In the Banach spaces $\mathscr{L}_{q,\gamma}$ and $B_{q,\gamma}$, $1 \leq q \leq \infty$, with a weight γ, exact values of some n-widths of the classes $W_q^{(r)}(\Phi)$, $r \in \mathbb{Z}_+$, have been calculated. It was proved that the subspace L_n^* is extremal for the Kolmogorov and linear n-widths in the class $W_q^{(r)}(\Phi)$, the continuous linear operator Λ_{n-1}^* is the best linear approximation method for $W_q^{(r)}(\Phi)$ in $\mathscr{L}_{q,\gamma}$, and the subspace L_*^n is extremal for the n-width $d^n(W_q^{(r)}(\Phi), B_{q,\gamma})$. The subspace $\bar{L}_{n+1}$ is extremal for the n-width $b_n(W_q^{(r)}(\Phi), B_{q,\gamma})$.

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