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# TOPOLOGIES ON THE FUNCTION SPACE $Y^X$ WITH VALUES IN A TOPOLOGICAL GROUP

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**Abstract:** Let $Y^X$ denote the set of all functions from $X$ to $Y$. When $Y$ is a topological space, various topologies can be defined on $Y^X$. In this paper, we study these topologies within the framework of function spaces. To characterize different topologies and their properties, we employ generalized open sets in the topological space $Y$. This approach also applies to the set of all continuous functions from $X$ to $Y$, denoted by $C(X,Y)$, particularly when $Y$ is a topological group. In investigating various topologies on both $Y^X$ and $C(X,Y)$, the concept of limit points plays a crucial role. The notion of a topological ideal provides a useful tool for defining limit points in such spaces. Thus, we utilize topological ideals to study the properties and consequences for function spaces and topological groups.

**Keywords:** Topological group, Topological ideal, Function space $Y^X$.

## 1. Introduction

For any topological space $Z$ and topological group $H$ [6, 26], let $C(Z,H)$ denote the group of all continuous functions from $Z$ to $H$, equipped with the "pointwise group operations". That is, the product of $f\in C(Z,H)$ and $g\in C(Z,H)$ is the function $fg\in C(Z,H)$ defined by

$$fg(z)=f(z)g(z)$$

for all $z\in Z$, and the inverse of $f$ is the function $h\in C(Z,H)$ defined by

$$h(z)=(f(z))^{-1}$$

for all $z\in Z$. The space $C(Z,H)$ with the point-open topology was studied by Shakhmatov and Spěvák [25]. A set of the form

$$[z,V]^+=\big\{f\in C(Z,H)|\ f(z)\in V\big\},$$

where $z\in Z$ and $V$ is an open subset of $H$, is a subbase of the point-open topology on $C(Z,H)$. The space $C(Z,H)$ with the open-point topology has a subbase consisting of sets of the form

$$[U,r]^-=\big\{f\in C(Z,H)|\ f^{-1}(r)\cap U\neq\emptyset\big\},$$

where $r\in H$ and $U$ is an open subset of $Z$.

The space $C(Z,H)$ with the bi-point-open topology has a subbase consisting of sets of both kinds: $[z,V]^+$ and $[U,r]^-$, where $z\in Z$ and $V$ is an open subset of $H$, $U$ is an open subset of $Z$, and $r\in H$.

The following three propositions serve as necessary tools for the development of this paper.

**Proposition 1** [5]. *Let $\beta$ be a basis of a topological group $H$. The collection*

$$\big\{[z_1, B_1]^+ \cap \cdots \cap [z_n, B_n]^+ |\ n \in \mathbb{N},\ z_i \in Z,\ B_i \in \beta\big\}$$

*is a basis for the space $C(Z, H)$ equipped with the point-open topology.*

**Proposition 2** [26]. *Let $\beta$ be a basis of a topological space $X$. The collection*

$$\big\{[B_1, r_1]^- \cap \cdots \cap [B_n, r_n]^- |\ n \in \mathbb{N},\ r_i \in H,\ B_i \in \beta\big\}$$

*is a basis for the space $C(Z, H)$ equipped with the open-point topology.*

**Proposition 3** [26]. *Let $\beta_Z$ and $\beta_H$ be bases of a topological space $Z$ and a topological group $H$, respectively. The collection*

$$\big\{[z_1, B_1]^+ \cap \cdots \cap [z_n, B_n]^+ \cap [V_1, r_1]^- \cap \cdots \cap [V_m, r_m]^- | \\ z_i \in Z,\ r_j \in H,\ r_i \in H,\ B_i \in \beta_H,\ \text{and } V_j \in \beta_Z,\ 1 \le i \le n,\ 1 \le j \le m\big\}$$

*is a basis for the space $C(Z, H)$ equipped with the bi-point-open topology.*

General definition of the point-open topology on $Y^X$:

**Definition 1** [21]. *Given a point $x \in X$ and an open set $U$ in a topological space $Y$, define*

$$S(x, U) = \big\{f \in Y^X |\ f(x) \in U\big\}.$$

*The collection of all such sets $S(x, U)$ forms a subbasis for a topology on $Y^X$. This topology is called the* ***point-open topology on*** *$Y^X$.*

To obtain a topology on $Y^X$, it is not necessary that $Y$ be a topological space. That is, for any set $Y$, the following construction defines a topology on $Y^X$.

Let $x$ be a point of the set $X$ and $A$ be any subset of $Y$. Consider

$$S(x, A) = \big\{f \in Y^X |\ f(x) \in A\big\}.$$

The sets $S(x, A)$ form a subbasis for a topology on $Y^X$. Suppose $\mathfrak{F} \subseteq Y^X$.

The question is: Is $\mathfrak{F}$ open in the topology on $Y^X$ generated by the subbasis elements above?

Let $g \in \mathfrak{F}$. For any $x \in X$, we have $g(x) \in Y$. If $X$ is finite, then $g \in S(x, \{g(x)\}) \subseteq \mathfrak{F}$. Thus, the subbasis

$$\{S(x, A) |\ x \in X,\ A \in \wp(Y)\}$$

generates the discrete topology on $Y^X$ when $X$ is finite. If we take $A = Y$, then the subbasis

$$\big\{\emptyset\big\} \cup \big\{S(x, Y) |\ x \in X\big\}$$

generates the indiscrete topology on $Y^X$. If we restrict the subsets of $Y$ used in the subbasis, we obtain a weaker topology on $Y^X$. Therefore, we conclude that "$Y$ being a topological space" is not essential for defining a topology on $Y^X$. In particular, starting with the discrete topology on $Y$ yields the discrete topology on $Y^X$, while starting with the indiscrete topology on $Y$ yields the indiscrete topology on $Y^X$.

In this paper, we will discuss various topologies on $Y^X$. For this purpose, the following generalized open sets are important tools.

**Definition 2.** *A subset $A$ of a topological space $Y$ is said to be*

– *semi-open* [15] *if* $A \subseteq \mathrm{Co}(\mathrm{Io}(A))$;
– *preopen* [16] *if* $A \subseteq \mathrm{Io}(\mathrm{Co}(A))$;
– $\beta$*-open* [10] *or semi-preopen* [3] *if* $A \subseteq \mathrm{Co}(\mathrm{Io}(\mathrm{Co}(A)))$;
– $b$*-open* [4] *if* $A \subseteq \mathrm{Io}(\mathrm{Co}(A)) \cup \mathrm{Co}(\mathrm{Io}(A))$;
– $h$*-open* [1] *if, for every nonempty open set* $U \neq Y$, $A \subseteq \mathrm{Io}(A \cup U)$,

*where* Io *and* Co *denote the interior and closure operators, respectively.*

We denote the collection of all semi-open sets, preopen sets, $\beta$-open sets, and $b$-open sets in a topological space $Y$ by $SO(Y)$, $PO(Y)$, $\beta O(Y)$, and $BO(Y)$, respectively. These collections satisfy the following inclusion relations: the collection of open sets $\subseteq PO(Y) \subseteq BO(Y) \subseteq \beta O(Y)$ and the collection of open sets $\subseteq SO(Y) \subseteq BO(Y) \subseteq \beta O(Y)$.

The following is one way to obtain weaker and stronger topologies on $Y^X$; it serves as an introductory result of the paper.

**Lemma 1.** *Suppose $\sigma$ and $\sigma'$ are two topologies on the set $Y$ such that $\sigma \subsetneq \sigma'$. Then, the point-open topology induced by $\sigma'$ is finer than the point-open topology induced by $\sigma$.*

P r o o f. Let $\beta_\tau$ and $\beta_{\tau'}$ be bases for the point-open topologies $\tau$ and $\tau'$ induced by $\sigma$ and $\sigma'$, respectively, on $Y^X$. Let

$$B = S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n)$$

be a member of $\beta_\tau$, and suppose $f \in B$. Then $f \in S(x_i, U_i)$ for all $i = 1, 2, \ldots, n$. This implies that $f \in S(x_i, U_i')$, where $U_i = U_i'$ for all $i = 1, 2, \ldots, n$. So,

$$f \in S(x_1, U_1') \cap S(x_2, U_2') \cap \cdots \cap S(x_n, U_n') = B' \in \beta_{\tau'}$$

as $U_1', U_2', \ldots, U_n'$ are open subsets of $(Y, \sigma')$. Thus, for every $f \in B$, there exists $B' \in \beta_{\tau'}$ such that $B' \subseteq B$. This completes the proof. □

Note that if $\sigma'$ is strictly finer than $\sigma$, then the point-open topology induced by $\sigma'$ is strictly finer than the point-open topology induced by $\sigma$.

Our aim is to discuss different point-open topologies for various operators in topological spaces. Thus, for various operators, we consider a topological ideal [2, 14].

An ideal $\mathbb{I}$ on a topological space $(Y, \sigma)$ is a collection of subsets of $Y$ satisfying:

(i) If $A \subseteq B \in \mathbb{I}$, then $A \in \mathbb{I}$;
(ii) If $A, B \in \mathbb{I}$, then $A \cup B \in \mathbb{I}$.

This concept of an ideal on a topological space was first introduced by Kuratowski [14] in 1933. The study of the local function (or the generalization of limit points) is an important aspect of the theory of topological ideals. It is defined as follows:

$$A^* = \{y \in Y |\ U_y \cap A \notin \mathbb{I},\ U_y \in \sigma(y)\},$$

where $\sigma(y)$ is the collection of all open sets of $(Y, \sigma)$ containing $y$. The set-valued set function [20] associated with the operator $()^*$ is the operator $\psi$ [18, 22], which is defined by the relation $\psi(A) = Y \setminus (Y \setminus A)^*$.

Throughout this paper, $(Y, \sigma, \mathbb{I})$ denotes an ideal topological space. Furthermore, an ideal $\mathbb{I}$ on the topological space $(Y, \sigma)$ is called a codense ideal [9] (or, equivalently, the ideal topological space $(Y, \sigma, \mathbb{I})$ is called an H-S space [8]) if $\mathbb{I} \cap \sigma = \{\emptyset\}$.

## 2. Topologies on $Y^X$

In this section, we consider $X$ as a set and $Y$ as a topological space (or simply, a space).

**Lemma 2.** *Given a point $x \in X$ and a subset $A$ of the topological space $Y$, define*

$$S(x, \mathrm{Io}(A)) = \{f \in Y^X \mid f(x) \in \mathrm{Io}(A)\}.$$

*The sets $S(x, \mathrm{Io}(A))$ form a subbasis for a topology on $Y^X$.*

P r o o f. Let $f \in Y^X$. Then

$$f \in S(x, Y) = S(x, \mathrm{Io}(Y)) \subseteq \bigcup_i S(x_i, \mathrm{Io}(A_i)),$$

where $x_i \in X$ and $A_i$ are subsets of $Y$. So,

$$f \in \bigcup_i S(x_i, \mathrm{Io}(A_i)).$$

Thus,

$$Y^X \subseteq \bigcup_i S(x_i, \mathrm{Io}(A_i)).$$

Hence, the sets $S(x_i, \mathrm{Io}(A_i))$ form a subbasis for a topology on $Y^X$. □

The topology generated by the above subbasis is called the ***point-interior topology*** on $Y^X$.

As is well known, the operator Co is the set-valued set function [20] associated with Io. Thus, if we define the sets $S(x, \mathrm{Io}(A))$ by

$$\{f \in Y^X \mid f(x) \in X \setminus \mathrm{Co}(X \setminus A)\}$$

or

$$\{f \in Y^X \mid f(x) \notin \mathrm{Co}(X \setminus A)\},$$

then we obtain the same topology.

Now we state that the operator Co independently generates a topology on $Y^X$ as follows.

**Lemma 3.** *Given a point $x \in X$ and a subset $A$ of the topological space $Y$, define*

$$S(x, \mathrm{Co}(A)) = \{f \in Y^X \mid f(x) \in \mathrm{Co}(A)\}.$$

*The sets $S(x, \mathrm{Co}(A))$ form a subbasis for a topology on $Y^X$.*

The topology generated by the above subbasis is called the ***point-closure topology*** on $Y^X$.

As $\mathrm{Io} \sim^Y \mathrm{Co}$ [20], one can rewrite the above Lemma using the Io operator. The point-open topology and the point-interior topology on $Y^X$ coincide. However, the point-interior topology and the point-closure topology are not comparable.

*Example 1.* Let $X = \{a, b\}$ and $(Y, \sigma)$ be a topological space, where $Y = \{1, 2, 3\}$ and $\sigma = \{\emptyset, Y, \{2\}, \{3\}, \{2, 3\}\}$. All possible functions from $X$ to $Y$ are defined by

$$f_1(a) = 1, \quad f_1(b) = 2; \quad f_2(a) = 1, \quad f_2(b) = 3; \quad f_3(a) = 2, \quad f_3(b) = 3;$$
$$f_4(a) = 2, \quad f_4(b) = 1; \quad f_5(a) = 3, \quad f_5(b) = 1; \quad f_6(a) = 3, \quad f_6(b) = 2;$$
$$f_7(a) = 1, \quad f_7(b) = 1; \quad f_8(a) = 2, \quad f_8(b) = 2; \quad f_9(a) = 3, \quad f_9(b) = 3.$$

Then, a basis of the point-interior topology $\tau$ on $Y^X$ is

$$\beta_\tau = \big\{\emptyset, Y^X, \{f_3\}, \{f_6\}, \{f_8\}, \{f_9\}, \{f_3, f_8\}, \{f_6, f_9\}, \{f_6, f_8\}, \{f_3, f_9\}, \\ \{f_3, f_4, f_8\}, \{f_5, f_6, f_9\}, \{f_1, f_6, f_8\}, \{f_2, f_3, f_9\}, \{f_3, f_6, f_8, f_9\}, \\ \{f_3, f_4, f_5, f_6, f_8, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}\big\}.$$

A basis of the point-closure topology $\tau'$ on $Y^X$ is

$$\beta_{\tau'} = \big\{\emptyset, Y^X, \{f_7\}, \{f_1, f_7\}, \{f_2, f_7\}, \{f_4, f_7\}, \{f_5, f_7\}, \\ \{f_1, f_2, f_7\}, \{f_4, f_5, f_7\}, \{f_1, f_4, f_7, f_8\}, \{f_2, f_3, f_4, f_7\}, \{f_1, f_5, f_6, f_7\}, \{f_2, f_5, f_7, f_9\}, \\ \{f_1, f_2, f_3, f_4, f_7, f_8\}, \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_1, f_4, f_5, f_6, f_7, f_8\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}\big\}.$$

Here, $f_6 \in \{f_6\} \in \beta_\tau$ but there does not exist any $B' \in \beta_{\tau'}$ such that $f_6 \in B' \subseteq \{f_6\}$. Thus, $\tau'$ is not finer than $\tau$.

Similarly, $f_7 \in \{f_7\} \in \beta_{\tau'}$ but there does not exist any $B \in \beta_\tau$ such that $f_7 \in B \subseteq \{f_7\}$. Thus, $\tau$ is not finer than $\tau'$.

Hence, the point-interior topology and the point-closure topology on $Y^X$ are not comparable.

**Lemma 4.** *Given a point $x \in X$ and a subset $A$ of the topological space $Y$, define*

$$S(x, \mathrm{Io}(\mathrm{Co}(A))) = \big\{f \in Y^X \mid f(x) \in \mathrm{Io}(\mathrm{Co}(A))\big\}.$$

*The sets $S(x, \mathrm{Io}(\mathrm{Co}(A)))$ form a subbasis for a topology on $Y^X$.*

P r o o f. Let $f \in Y^X$. Then

$$f \in S(x, Y) = S(x, \mathrm{Io}(\mathrm{Co}(Y))) \subseteq \bigcup_i S(x_i, \mathrm{Io}(\mathrm{Co}(A_i))),$$

where $x_i \in X$ and $A_i$ are subsets of $Y$. Therefore,

$$f \in \bigcup_i S(x_i, \mathrm{Io}(\mathrm{Co}(A_i))).$$

Thus,

$$Y^X \subseteq \bigcup_i S(x_i, \mathrm{Io}(\mathrm{Co}(A_i))).$$

Hence, the sets $S(x_i, \mathrm{Io}(\mathrm{Co}(A_i)))$ form a subbasis for a topology on $Y^X$. □

The topology generated by the above subbasis is called the ***point-interior-closure topology*** on $Y^X$. Since $\mathrm{Io}\,\mathrm{Co} \sim^Y \mathrm{Co}\,\mathrm{Io}$ [20], we may rewrite the subbasis of the point-interior-closure topology on $Y^X$ using the $\mathrm{Co}\,\mathrm{Io}$ operator.

**Proposition 4.** *Suppose $Y$ is a topological space. Then, the point-open topology on $Y^X$ is finer than the point-interior-closure topology on $Y^X$.*

P r o o f. Let $\beta_\tau$ and $\beta_{\tau'}$ be bases for the point-interior-closure topology and the point-open topology on $Y^X$, respectively. Let

$$B = S(x_1, \mathrm{Io}(\mathrm{Co}(A_1))) \cap S(x_2, \mathrm{Io}(\mathrm{Co}(A_2))) \cap \cdots \cap S(x_n, \mathrm{Io}(\mathrm{Co}(A_n)))$$

be a member of $\beta_\tau$, and let $f \in B$. Then

$$f \in S(x_i, \mathrm{Io}(\mathrm{Co}(A_i))) \quad \forall i = 1, 2, \ldots, n.$$

This implies that $f \in S(x_i, U_i)$, where

$$U_i = \mathrm{Io}(\mathrm{Co}(A_i)) \quad \forall i = 1, 2, \ldots, n.$$

Therefore,

$$f \in S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n) = B' \in \beta_{\tau'},$$

as $U_1, U_2, \ldots, U_n$ are open subsets of $Y$. Thus, for every $f \in B$, there exists $B' \in \beta_{\tau'}$ such that $B' \subseteq B$. □

For the converse of this proposition, we have the following.

Let

$$B'_1 = S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n)$$

be a member of $\beta_{\tau'}$, and let $g \in B'_1$. Then

$$g \in S(x_i, U_i) \Rightarrow g \in S(x_i, \mathrm{Io}(\mathrm{Co}(U_i))) \quad (\text{as} \quad U_i \subseteq \mathrm{Co}(U_i) \Rightarrow U_i \subseteq \mathrm{Io}(\mathrm{Co}(U_i))), \quad \forall i = 1, 2, \ldots, n.$$

So,

$$g \in S(x_1, \mathrm{Io}(\mathrm{Co}(U_1))) \cap S(x_2, \mathrm{Io}(\mathrm{Co}(U_2))) \cap \cdots \cap S(x_n, \mathrm{Io}(\mathrm{Co}(U_n))) = B_1 \in \beta_\tau.$$

Thus, for each $B'_1 \in \beta_{\tau'}$, there exists $B_1 \in \beta_\tau$. However, $B_1 \subseteq B'_1$ does not hold in general. To justify this statement, we give the following example.

*Example 2.* Let $(Y, \sigma)$ be a topological space, where $Y = \{a, b, c\}$ and $\sigma = \{\emptyset, Y, \{c\}\}$. Then

$$\{\mathrm{Io}(\mathrm{Co}(A)) \mid A \subseteq Y\} = \{\emptyset, Y\}.$$

Thus,

$$\{\mathrm{Io}(\mathrm{Co}(A)) \mid A \subseteq Y\}$$

is not equal to $\sigma$.

**Lemma 5.** *Given a point $x \in X$ and a subset $A$ of the topological space $Y$, define*

$$S(x, \mathrm{Co}(\mathrm{Io}(A))) = \{f \in Y^X \mid f(x) \in \mathrm{Co}(\mathrm{Io}(A))\}.$$

*The sets $S(x, \mathrm{Co}(\mathrm{Io}(A)))$ form a subbasis for a topology on $Y^X$.*

P r o o f. Let $f \in Y^X$. Then

$$f \in S(x, Y) = S(x, \mathrm{Co}(\mathrm{Io}(Y))) \subseteq \bigcup_i S(x_i, \mathrm{Co}(\mathrm{Io}(A_i))),$$

where $x_i \in X$ and $A_i$ are subsets of $Y$. So,

$$f \in \bigcup_i S(x_i, \mathrm{Co}(\mathrm{Io}(A_i))).$$

Thus,

$$Y^X \subseteq \bigcup_i S(x_i, \mathrm{Co}(\mathrm{Io}(A_i))).$$

Hence, the sets $S(x_i, \mathrm{Co}(\mathrm{Io}(A_i)))$ form a subbasis for a topology on $Y^X$. □

The topology generated by the above subbasis is called the ***point-closure-interior topology*** on $Y^X$.

The following example shows that the point-interior-closure topology and the point-closure-interior topology on $Y^X$ are not comparable.

*Example 3.* In Example 1, a basis of the point-interior-closure topology $\tau$ on $Y^X$ is

$$\beta_\tau = \{\emptyset, Y^X, \{f_3\}, \{f_6\}, \{f_8\}, \{f_9\}, \{f_3, f_4, f_8\}, \{f_5, f_6, f_9\}, \{f_1, f_6, f_8\}, \{f_2, f_3, f_9\}\}.$$

A basis of the point-closure-interior topology $\tau'$ on $Y^X$ is

$$\beta_{\tau'} = \{\emptyset, Y^X, \{f_7\}, \{f_1, f_7\}, \{f_2, f_7\}, \{f_4, f_7\}, \{f_5, f_7\},$$
$$\{f_1, f_2, f_7\}, \{f_4, f_5, f_7\}, \{f_1, f_4, f_7, f_8\}, \{f_2, f_3, f_4, f_7\}, \{f_1, f_5, f_6, f_7\}, \{f_2, f_5, f_7, f_9\},$$
$$\{f_1, f_2, f_3, f_4, f_7, f_8\}, \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_1, f_4, f_5, f_6, f_7, f_8\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}\}.$$

Here, $f_3 \in \{f_3\} \in \beta_\tau$, but there does not exist any $B' \in \beta_{\tau'}$ such that $f_3 \in B' \subseteq \{f_3\}$. Thus, $\tau'$ is not finer than $\tau$.

Similarly, $f_7 \in \{f_7\} \in \beta_{\tau'}$, but there does not exist any $B_1 \in \beta_\tau$ such that $f_7 \in B_1 \subseteq \{f_7\}$. Thus, $\tau$ is not finer than $\tau'$.

Hence, the point-interior-closure topology and the point-closure-interior topology of $Y^X$ are not comparable.

**Lemma 6.** *Let $Y$ be a topological space. Given a point $x \in X$ and a subset $A \in SO(Y)$ (resp. $PO(Y)$, $\beta O(Y)$, $BO(Y)$), define*

$$S(x, A) = \{f \in Y^X \mid f(x) \in A\}.$$

*The sets $S(x, A)$ form a subbasis for a topology on $Y^X$.*

The topology generated by the above subbasis is called the ***point-semi-open*** (resp. ***point-preopen, point-$\beta$-open, point-$b$-open***) topology on $Y^X$.

**Theorem 1.** *Suppose $Y$ is a topological space. Then, the point-preopen topology on $Y^X$ is finer than the point-open topology on $Y^X$.*

P r o o f. Let $\beta_\tau$ and $\beta_{\tau'}$ be bases for the point-open topology and the point-preopen topology on $Y^X$, respectively. Let

$$B = S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n)$$

be a member of $\beta_\tau$, and let $f \in B$. Then $f \in S(x_i, U_i)$ for all $i = 1, 2, \ldots, n$. This implies that $f \in S(x_i, U_i)$, where $U_i \in PO(Y)$ for all $i = 1, 2, \ldots, n$ (since $U_i$ are open in $Y$). So,

$$f \in S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n) = B' \in \beta_{\tau'}$$

as $U_1, U_2, \ldots, U_n \in PO(Y)$ are open subsets of $Y$. Thus, for every $f \in B$, there exists $B' \in \beta_{\tau'}$ such that $B' \subseteq B$. Hence, the proof is complete. □

For the converse of Theorem 1, we always obtain a set $B_1 \in \beta_\tau$ for any $B_1' \in \beta_{\tau'}$, but it is not necessarily the case that $B_1 \subseteq B_1'$. To illustrate this, we present the following example.

*Example 4.* Let $(Y, \sigma)$ be a topological space, where $Y = \{a, b, c\}$ and $\sigma = \{\emptyset, Y, \{c\}\}$. Then

$$\{A \subseteq Y | \ A \in PO(Y)\} = \big\{\emptyset, Y, \{c\}, \{a, c\}, \{b, c\}\big\}.$$

Thus,

$$\{A \subseteq Y | \ A \in PO(Y)\} \neq \sigma.$$

However, the two topologies will be equal when $PO(Y) = \sigma$.

**Theorem 2.** *Suppose $Y$ is a topological space. Then, the point-semi-open (resp. point-$\beta$-open, point-b-open) topology on $Y^X$ is finer than the point-open topology on $Y^X$.*

The proof of this theorem follows from the fact that open sets in $Y$ are contained in $SO(Y)$ (resp. $\beta O(Y)$, $BO(Y)$). The reader should not conclude that, for any collection $\mathcal{A}$ containing the collection of open sets of $Y$, the point-open topology with respect to $\mathcal{A}$ is necessarily finer than the point-open topology on $Y^X$. However, the result of Theorem 2 holds because every open set is a preopen (resp. semi-open, $b$-open, $\beta$-open) set.

Therefore, a common generalization is discussed in the following lemma.

**Lemma 7.** *Suppose a collection $\mathcal{G} \subseteq \wp(Y)$ (the power set of $Y$) satisfies the following conditions:*

1) $\emptyset$, $Y \in \mathcal{G}$;
2) $\mathcal{G}$ *is closed under arbitrary unions.*

*Let $h : \mathcal{G} \to \mathcal{G}$ and $k : \mathcal{G} \to \mathcal{G}$ be two set-valued set functions* [20] *such that $h(A) = Y \setminus k(Y \setminus A)$ for all $A \in \wp(Y)$ and $h(\emptyset) = \emptyset$, $h(Y) = Y$.*

*Given a point $x \in X$ and a subset $A \subseteq h \circ k(A)$, define*

$$S(x, A) = \{f \in Y^X \mid \ f(x) \in A\}.$$

*The sets $S(x, A)$ form a subbasis for a topology on $Y^X$.*

P r o o f.

$$h \circ k(Y) = h(Y \setminus h(Y \setminus Y)) = h(Y \setminus h(\emptyset)) = h(Y) \quad (\text{as} \quad h(\emptyset) = \emptyset) = Y.$$

Thus, $Y \subseteq h \circ k(Y)$.

Let $f \in Y^X$. Then

$$f \in S(x, Y) \subseteq \bigcup_i S(x_i, (A_i)),$$

where $x_i \in X$ and $A_i \subseteq h \circ k(A_i)$. So,

$$f \in \bigcup_i S(x_i, (A_i)).$$

Thus,

$$Y^X \subseteq \bigcup_i S(x_i, (A_i)).$$

Hence, the sets $S(x_i, (A_i))$ form a subbasis for a topology on $Y^X$. □

The topology generated by the above subbasis is called the ***point-associated topology*** on $Y^X$. The following is an example of this topology.

*Example 5.* By taking $h$ and $k$ to be the Io and Co operators, respectively, we see that Lemma 7 coincides with Lemma 4.

**Lemma 8.** *Let $Y$ be a topological space. Given a point $x \in X$ and a subset $A \in \mathcal{D}(Y)$ (the set of all dense sets in $Y$), define*

$$S(x, A) = \{f \in Y^X | \ f(x) \in A\}.$$

*The sets $S(x, A)$ form a subbasis for a topology on $Y^X$.*

P r o o f. Let $f \in Y^X$. Then

$$f \in S(x, Y) \subseteq \bigcup_i S(x_i, (A_i)),$$

where $x_i \in X$ and $A_i \in \mathcal{D}(Y)$ (as $Y \in \mathcal{D}(Y)$). So, $f \in \bigcup\limits_i S(x_i, (A_i))$. Thus,

$$Y^X \subseteq \bigcup_i S(x_i, (A_i)).$$

Hence, the sets $S(x_i, (A_i))$ form a subbasis for a topology on $Y^X$. □

The topology generated by the above subbasis is called the **point-dense topology** on $Y^X$.

*Example 6.*

1. In Example 1, a basis of the point-open topology $\tau$ on $Y^X$ is

$$\beta_\tau = \big\{\emptyset, Y^X, \{f_3\}, \{f_6\}, \{f_8\}, \{f_9\}, \{f_3, f_8\}, \{f_3, f_9\}, \{f_6, f_8\}, \{f_6, f_9\}, \{f_3, f_4, f_8\}, \{f_5, f_6, f_9\}, \{f_1, f_6, f_8\}, \{f_2, f_3, f_9\}, \{f_3, f_6, f_8, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}, \{f_3, f_4, f_5, f_6, f_8, f_9\}\big\}.$$

A basis of the point-dense topology $\tau'$ on $Y^X$ is

$$\beta_{\tau'} = \big\{Y^X, \{f_3, f_6, f_8, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}, \{f_3, f_4, f_5, f_6, f_8, f_9\}\big\}.$$

In this case, the point-open topology is strictly finer than the point-dense topology.

2. In Example 1 with $\sigma = \{\emptyset, Y, \{3\}\}$, a basis of the point-open topology $\tau$ on $Y^X$ is

$$\beta_\tau = \big\{\emptyset, Y^X, \{f_9\}, \{f_5, f_6, f_9\}, \{f_2, f_3, f_9\}\big\}.$$

A basis of the point-dense topology $\tau'$ on $Y^X$ is

$$\beta_{\tau'} = \big\{Y^X, \{f_9\}, \{f_2, f_9\}, \{f_3, f_9\}, \{f_5, f_9\}, \{f_6, f_9\}, \{f_2, f_3, f_9\}, \{f_5, f_6, f_9\}, \{f_1, f_2, f_6, f_9\}, \{f_2, f_5, f_7, f_9\}, \{f_3, f_4, f_5, f_9\}, \{f_3, f_6, f_8, f_9\}, \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}, \{f_3, f_4, f_5, f_6, f_8, f_9\}\big\}.$$

In this case, the point-dense topology is strictly finer than the point-open topology.

Hence, we conclude that the point-open topology and the point-dense topology on $Y^X$ are not comparable.

## 3. Topologies on $Y^X$ due to ideal

It is known from [7, 11, 17, 18] that $\psi$ is not an interior operator. The following lemma shows that a noninterior operator may also serve as an essential tool in obtaining a topology on $Y^X$.

**Lemma 9.** *Let $\mathbb{I}$ be an ideal on the topological space $Y$. Given a point $x \in X$ and a subset $A$ of the topological space $Y$, define*

$$S_{\mathbb{I}}(x, \psi(A)) = \{f \in Y^X \mid f(x) \in \psi(A)\}.$$

*The sets $S_{\mathbb{I}}(x, \psi(A))$ form a subbasis for a topology on $Y^X$.*

P r o o f. Let $f \in Y^X$. Then

$$f \in S_{\mathbb{I}}(x, Y) = S_{\mathbb{I}}(x, \psi(Y)) \subseteq \bigcup_i S_{\mathbb{I}}(x_i, \psi(A_i)),$$

where $x_i \in X$ and $A_i$ are subsets of $Y$. So,

$$f \in \bigcup_i S_{\mathbb{I}}(x_i, \psi(A_i)).$$

Thus,

$$Y^X \subseteq \bigcup_i S_{\mathbb{I}}(x_i, \psi(A_i)).$$

Hence, the sets $S_{\mathbb{I}}(x_i, \psi(A_i))$ form a subbasis for a topology on $Y^X$. □

The topology generated by the above subbasis is called the ***point-$\psi$ topology*** on $Y^X$.

Since $\psi \sim^Y *$ [20], the subbasis for the point-$\psi$ topology on $Y^X$ can be equivalently rewritten in terms of the $*$-operator.

Comparison of the point-$\psi$ topology with other topologies on $Y^X$ are as follows.

**Proposition 5.** *Suppose $\mathbb{I}$ is an ideal on the topological space $Y$. Then, the point-open topology on $Y^X$ is finer than the point-$\psi$ topology on $Y^X$.*

P r o o f. Let $\beta_\tau$ and $\beta_{\tau'}$ be bases for the point-$\psi$ topology and the point-open topology on $Y^X$, respectively. Let

$$B = S_{\mathbb{I}}(x_1, \psi(A_1)) \cap S_{\mathbb{I}}(x_2, \psi(A_2)) \cap \cdots \cap S_{\mathbb{I}}(x_n, \psi(A_n))$$

be a member of $\beta_\tau$, and let $f \in B$. Then $f \in S_{\mathbb{I}}(x_i, \psi(A_i))$ for all $i = 1, 2, \ldots, n$. This implies that $f \in S(x_i, U_i)$, where $U_i = \psi(A_i)$ (since for each $i$, $\psi(A_i)$ is open by [11, 18]), for all $i = 1, 2, \ldots, n$. Hence,

$$f \in S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n) = B' \in \beta_{\tau'}$$

since $U_1,\ U_2, \ldots,\ U_n$ are open subsets of $Y$. Thus, for each $f \in B$, there exists $B' \in \beta_{\tau'}$ such that $B' \subseteq B$. □

For the converse relation of this proposition, we give the following example.

*Example 7.* Consider Example 1 with $\sigma = \{\emptyset, Y, \{3\}, \{1,3\}, \{2,3\}\}$ and $\mathbb{I} = \{\emptyset, \{1\}\}$. Then, a basis of the point-$\psi$ topology $\tau$ on $Y^X$ is

$$\beta_\tau = \{\emptyset, Y^X, \{f_2, f_5, f_7, f_9\}, \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}\}.$$

A basis of the point-open topology $\tau'$ on $Y^X$ is

$$\beta_{\tau'} = \big\{\emptyset, Y^X, \{f_9\}, \{f_2, f_9\}, \{f_3, f_9\}, \{f_5, f_9\}, \{f_6, f_9\},$$
$$\{f_2, f_3, f_9\}, \{f_5, f_6, f_9\}, \{f_1, f_2, f_6, f_9\}, \{f_2, f_5, f_7, f_9\}, \{f_3, f_4, f_5, f_9\}, \{f_3, f_6, f_8, f_9\},$$
$$\{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}, \{f_3, f_4, f_5, f_6, f_8, f_9\}\big\}.$$

Here, $f_9 \in \{f_9\} \in \beta_{\tau'}$, but there does not exist any $B_1 \in \beta_\tau$ such that $f_9 \in B_1 \subseteq \{f_9\}$. Thus, $\tau$ is not finer than $\tau'$.

However, the set $\{\psi(A) : A \subseteq Y\}$ does not form a topology on $Y$.

*Example 8.* Let $(Y, \sigma, \mathbb{I})$ be an ideal topological space, where $Y = \{a, b, c\}$, $\sigma = \big\{\emptyset, Y, \{c\}, \{a, c\}, \{b, c\}\big\}$, and $\mathbb{I} = \{\emptyset, \{a\}\}$. Then $\{\psi(A) |\ A \subseteq Y\} = \{\emptyset, Y, \{a, c\}\}$. In this example, it is clear that

$$\{\psi(A) |\ A \subseteq Y\} \neq \sigma$$

on $Y$.

As a consequences of the above results and Theorem 46.7 of [21], we have the following.

**Theorem 3.** *Suppose $\mathbb{I}$ is an ideal on the metric space $(Y, d)$ and $Y$ is a topological space. For the function space $Y^X$, the following inclusions of topologies hold*:

$$(\textit{uniform}) \supset (\textit{compact convergence}) \supset (\textit{point-open}) = (\textit{point-interior}) \supseteq (\textit{point-}\psi).$$

**Proposition 6.** *Suppose $\mathbb{I}$ is a codense ideal on the topological space $Y$. Given a point $x \in X$ and a subset $A$ of the topological space $Y$, define*

$$S_{\mathbb{I}}(x, A^*) = \{f \in Y^X \mid f(x) \in A^*\}.$$

*The sets $S_{\mathbb{I}}(x, A^*)$ form a subbasis for a topology on $Y^X$.*

P r o o f. Let $f \in Y^X$. Then

$$f \in S_{\mathbb{I}}(x, Y) = S_{\mathbb{I}}(x, Y^*) \quad (\text{since } \mathbb{I} \text{ is a codense ideal}) \subseteq \bigcup_i S_{\mathbb{I}}(x_i, A_i^*),$$

where $x_i \in X$ and $A_i$ are subsets of $Y$. Thus,

$$f \in \bigcup_i S_{\mathbb{I}}(x_i, A_i^*).$$

Thus,

$$Y^X \subseteq \bigcup_i S_{\mathbb{I}}(x_i, A_i^*).$$

Hence, the sets $S_{\mathbb{I}}(x_i, A_i^*)$ form a subbasis for a topology on $Y^X$. □

The topology generated by the above subbasis is called the ***point-$*$ topology*** on $Y^X$.

*Example 9.* Consider Example 1 with $\sigma = \big\{\emptyset, Y, \{3\}, \{1, 3\}, \{2, 3\}\big\}$ and $\mathbb{I} = \{\emptyset, \{1\}\}$. Then, a basis of the point-$\psi$ topology $\tau$ on $Y^X$ is

$$\beta_\tau = \big\{\emptyset, Y^X, \{f_2, f_5, f_7, f_9\}, \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}\big\}.$$

A basis of the point-$*$ topology $\tau'$ on $Y^X$ is

$$\beta_{\tau'} = \{\emptyset, Y^X, \{f_8\}, \{f_3, f_4, f_8\}, \{f_1, f_6, f_8\}\}.$$

Here, $f_2 \in \{f_2, f_5, f_7, f_9\} \in \beta_\tau$, but there does not exist any $B' \in \beta_{\tau'}$ such that $f_2 \in B' \subseteq B$. Thus, $\tau'$ is not finer than $\tau$.

Similarly, $f_8 \in \{f_8\} \in \beta_{\tau'}$, but there does not exist any $B_1 \in \beta_\tau$ such that $f_8 \in B_1 \subseteq \{f_8\}$. Thus, $\tau$ is not finer than $\tau'$.

Hence, the point-$\psi$ topology and point-$*$ topology of $Y^X$ are not comparable.

To discuss further topologies on $Y^X$, we make use of the notion of $\psi$-sets in an ideal topological space. This concept was introduced by Modak and Bandyopadhyay in [7], whose definition is as follows.

Let $\mathbb{I}$ be an ideal on a topological space $Y$. A subset $A$ of $Y$ is called a $\psi$-set if $A \subset \mathrm{Io}(\mathrm{Co}(\psi(A)))$. The collection of all $\psi$-sets in the ideal topological space $Y$ is denoted by $\psi^Y(Y)$.

**Theorem 4.** *Let $\mathbb{I}$ be an ideal on the topological space $Y$. Given a point $x \in X$ and $A \in \psi^Y(Y)$, define*

$$S_{\mathbb{I}}(x, A) = \{f \in Y^X \mid f(x) \in A\}.$$

*The sets $S_{\mathbb{I}}(x, A)$ form a subbasis for a topology on $Y^X$.*

Before proceeding to the proof of this theorem, we make a few remarks on $\psi$-sets. The collection $\psi^Y(Y)$ forms a topology on $Y$ whenever the ideal $\mathbb{I}$ is a codense ideal or a $\sigma$-boundary ideal [23] on $Y$. Modak and Bandyopadhyay studied this topology in [7] and showed that this topology coincides with the $\alpha$-topology [24] of the $*$-topology [12] generated by $\sigma$. Thus, we say that the topology obtained in Theorem 4 is the point-open topology for $\psi^Y(Y)$ (forms a topology on $Y$). If we denote the $\sigma^*$-topology generated by $\sigma$ by $*$-topology, the topology constructed in Theorem 4 is the point-open topology of $(\sigma^*)^\alpha$. We also note that codenseness is not essential for the proof of Theorem 4. However, if we consider the point-open topology of $Y^X$ arising from $(\sigma^*)^\alpha$, then codenseness is required. We omit the proof of this theorem, leaving it as an exercise for the reader.

For our next discussion, we will refer to the topology obtained in Theorem 4 as the ***point-*Co$_\psi$ *topology*** on $Y^X$.

The following gives a comparison of the point-Co$_\psi$ topology on $Y^X$.

**Corollary 1.** *Suppose $\mathbb{I}$ is an ideal on the topological space $Y$. Then, the point-*Co$_\psi$ *topology on $Y^X$ is finer than the point-open topology on $Y^X$.*

P r o o f The proof of this corollary is only meaningful when $\mathbb{I}$ is not a codense ideal on $Y$; otherwise, the result follows immediately from Lemma 1. □

**Theorem 5.** *Suppose $\mathbb{I}$ is an ideal on the topological space $Y$. Then, the point-*Co$_\psi$ *topology on $Y^X$ is finer than the point-$\psi$ topology on $Y^X$.*

P r o o f. Let $\beta_\tau$ and $\beta_{\tau'}$ be bases for the point-$\psi$ topology and the point-Co$_\psi$ topology on $Y^X$, respectively. Let

$$B = S_{\mathbb{I}}(x_1, \psi(A_1)) \cap S_{\mathbb{I}}(x_2, \psi(A_2)) \cap \cdots \cap S_{\mathbb{I}}(x_n, \psi(A_n))$$

be a member of $\beta_\tau$, and let $f \in B$. Then

$$f \in S_{\mathbb{I}}(x_i, \psi(A_i)), \quad \forall i = 1, 2, \ldots, n.$$

This implies that $f \in S_{\mathbb{I}}(x_i, U_i)$, where $U_i = \psi(A_i)$ for all $i = 1, 2, \ldots, n$. Therefore,

$$f \in S_{\mathbb{I}}(x_1, U_1) \cap S_{\mathbb{I}}(x_2, U_2) \cap \cdots \cap S_{\mathbb{I}}(x_n, U_n) = B' \in \beta_{\tau'}$$

(as $U_1, U_2, \ldots, U_n$ are open subsets of $Y$ and $U_i \in \psi^Y(Y)$). Thus, for every $f \in B$, there exists $B' \in \beta_{\tau'}$ such that $B' \subseteq B$. This completes the proof. □

The converse of this theorem does not necessarily hold in general.

If we replace the Co operator with $()^*$ operator, we obtain another topology on $Y^X$. To this end, we introduce Modak's $\dot{\psi}^*$-set [17]. Its formal definition is as follows.

Let $\mathbb{I}$ be an ideal on a space $Y$. A subset $A$ of $Y$ is called a $\dot{\psi}^*$-set if $A \subseteq \mathrm{Io}((\psi(A))^*)$. The collection of all $\dot{\psi}^*$-sets in an ideal topological space $Y$ is denoted by $\dot{\psi}^*(Y)$.

**Theorem 6.** *Let $\mathbb{I}$ be a codense ideal on the topological space $Y$. Given a point $x \in X$ and a subset $A \in \dot{\psi}^*(Y)$, define*

$$S_{\mathbb{I}}(x, A) = \{f \in Y^X \mid f(x) \in A\}.$$

*The sets $S(x, A)$ form a subbasis for a topology on $Y^X$.*

P r o o f. Since $Y$ is open, $Y \subseteq \psi(Y)$. Then $Y = Y^*$ (as $\mathbb{I}$ is codense)$\subseteq (\psi(Y))^*$. This implies $Y = \mathrm{Io}(Y) \subseteq \mathrm{Io}((\psi(Y))^*)$, and hence, $Y \in \dot{\psi}^*(Y)$.

Let $f \in Y^X$. Then

$$f \in S_{\mathbb{I}}(x, Y) \subseteq \bigcup_i S_{\mathbb{I}}(x_i, (A_i)),$$

where $x_i \in X$ and $A_i \in \dot{\psi}^*(Y)$. Therefore,

$$f \in \bigcup_i S(x_i, (A_i)).$$

Hence,

$$Y^X \subseteq \bigcup_i S_{\mathbb{I}}(x_i, (A_i)).$$

Thus, the sets $S_{\mathbb{I}}(x_i, (A_i))$ form a subbasis for a topology on $Y^X$. □

The topology generated by the above subbasis is called the ***point-$\dot{\psi}^*$ topology*** on $Y^X$.

Moreover, if $\mathbb{I}$ is a codense ideal on $Y$, then the collections $\psi^Y(Y)$ and $\dot{\psi}^*(Y)$ both represent the $\alpha$-sets of the $*$-topology of $\sigma$ (see [7]). Thus, the point-open topologies induced by $\psi^Y(Y)$ and $\dot{\psi}^*(Y)$ coincide.

**Definition 3** [19]. *Let $(Y, \sigma, \mathbb{I})$ be an ideal topological space, and $A \subseteq Y$. Then $A$ is called $h^\psi$-open if, for every nonempty open set $U \neq Y$, it holds $A \subseteq \psi(A \cup U)$.*

**Theorem 7.** *Let $(Y, \sigma, \mathbb{I})$ be an ideal topological space. Given a point $x \in X$ and a $h^\psi$-open set $A$ of the topological space $Y$, define*

$$S_{\mathbb{I}}(x, A) = \{f \in Y^X \mid f(x) \in A\}.$$

*The sets $S_{\mathbb{I}}(x, A)$ form a subbasis for a topology on $Y^X$.*

P r o o f . This follows from the fact that the collection of $h^\psi$-open sets forms a topology on $Y$. □

The topology generated by the above subbasis is called the ***point-$h^\psi$-open topology*** on $Y^X$.

**Theorem 8.** *Suppose $\mathbb{I}$ is an ideal on the topological space $Y$. Then, the point-$h^\psi$-open topology on $Y^X$ is finer than the point-open topology on $Y^X$.*

P r o o f . This follows directly from the fact that the topology generated by the $h^\psi$-open sets is finer than the topology $\sigma$ on $Y$. □

**Theorem 9.** *Let $Y$ be a topological space. Given a point $x \in X$ and an $h$-open set $A$ of the space $Y$, define*

$$S(x, A) = \{f \in Y^X \mid f(x) \in A\}.$$

*The sets $S(x, A)$ form a subbasis for a topology on $Y^X$.*

The topology generated by the above subbasis is called the ***point-$h$-open topology*** on $Y^X$.

**Theorem 10.** *The point-$h$-open topology on $Y^X$ is finer than the point-open topology on $Y^X$.*

From the above theorems, we conclude the following common phenomenon.

**Corollary 2.** *Let $\mathbb{I}$ be an ideal on a topological space $Y$. Then, the point-open topology on $Y^X$ is contained in the point-$h$-open topology on $Y^X$, which in turn is contained in the point-$h^\psi$-topology on $Y^X$.*

## 4. Topologies on $Y^X$ induced by continuous functions

In this section, we discuss the interrelation among the open-point topology, the point-open topology, and the bi-point topology [13, 26].

**Theorem 11.** *Let $C_{op}(Z, H)$ be the group of all continuous open functions from $Z$ to $H$. Then, the open-point topology on $C_{op}(Z, H)$ is finer than the point-open topology on $C_{op}(Z, H)$.*

P r o o f. Let $\beta_\tau$ and $\beta_{\tau'}$ be bases for the open-point and point-open topologies on $C_{op}(Z, H)$, respectively. Let $B' = [z_1, V_1]^+ \cap \cdots \cap [z_n, V_n]^+$, where $n \in \mathbb{N}$, $z_i \in Z$, and each $V_i$ is an open subset of $H$, be a member of $\beta_{\tau'}$, and let $f \in B'$. Then $f \in [z_i, V_i]^+$ for all $i = 1, 2, \ldots, n$, and $f : Z \to H$ is continuous. Hence, $z_i \in B_i$, where $B_i = f^{-1}(V_i)$ for all $i = 1, 2, \ldots, n$ (as $f^{-1}(V_i)$ are open in $Z$). Let $r_i \in V_i$ be such that $f(z_i) = r_i$. Then $z_i \in f^{-1}(r_i)$. Therefore,

$$z_i \in f^{-1}(r_i) \cap B_i \quad \forall i = 1, 2, \ldots, n.$$

Thus,

$$f \in [B_1, r_1]^- \cap \cdots \cap [B_n, r_n]^- = B \in \beta_\tau.$$

Therefore, for every $f \in B'$, there exists $B \in \beta_\tau$.

It remains to show that $B \subseteq B'$. Let $f \in B$. Then

$$f^{-1}(r_i) \cap B_i \neq \emptyset.$$

Let

$$z_i \in f^{-1}(r_i) \cap B_i.$$

Then

$$f(z_i) \in f[f^{-1}(r_i) \cap B_i] \subseteq f(f^{-1}(r_i)) \cap f(B_i) \subseteq f(B_i) = V_i.$$

Hence, $f(z_i) \in V_i$, which implies $f \in B'$. This show that $B \subseteq B'$. This completes the proof. □

Openness of a function is a necessary condition for Theorem 11. To illustrate this, we give the following example.

*Example 10.* Let $(Z, \tau)$ and $(Y, \sigma)$ be two topological spaces, where $Z = \{a, b\}$, $\tau = \{\emptyset, Z, \{a\}\}$, $Y = \{1, 2, 3\}$, and $\sigma = \big\{\emptyset, Y, \{2\}, \{3\}, \{2, 3\}\big\}$. All possible functions from $Z$ to $Y$ are given by

$$\begin{array}{llllll} f_1(a) = 1, & f_1(b) = 2; & f_2(a) = 1, & f_2(b) = 3; & f_3(a) = 2, & f_3(b) = 3; \\ f_4(a) = 2, & f_4(b) = 1; & f_5(a) = 3, & f_5(b) = 1; & f_6(a) = 3, & f_6(b) = 2; \\ f_7(a) = 1, & f_7(b) = 1; & f_8(a) = 2, & f_8(b) = 2; & f_9(a) = 3, & f_9(b) = 3. \end{array}$$

Now,

$$C(Z, Y) = \{f_4, f_5, f_7, f_8, f_9\}.$$

Here, $f_7$ is not an open map since $f_7(\{a\}) = \{1\} \notin \sigma$. We have

$$[a, \{2\}]^+ = \{f_4, f_8\}, \quad [a, \{3\}]^+ = \{f_5, f_9\}, \quad [a, \{2, 3\}]^+ = \{f_4, f_5, f_8, f_9\},$$
$$[b, \{2\}]^+ = \{f_8\}, \quad [b, \{3\}]^+ = \{f_9\}, \quad [b, \{2, 3\}]^+ = \{f_8, f_9\},$$
$$[a, Y]^+ = [b, Y]^+ = \{f_4, f_5, f_7, f_8, f_9\}.$$

Then, a basis for the point-open topology on $C(Z, Y)$ is

$$\beta' = \big\{\emptyset, \{f_4, f_8\}, \{f_5, f_9\}, \{f_8\}, \{f_9\}, \{f_8, f_9\}, \{f_4, f_5, f_8, f_9\}, \{f_4, f_5, f_7, f_8, f_9\}\big\}.$$

Also,

$$[\{a\}, 1]^- = \{f_7\}, \quad [\{a\}, 2]^- = \{f_4, f_8\}, \quad [\{a\}, 3]^- = \{f_5, f_9\},$$
$$[Z, 1]^- = [Z, 2]^- = [Z, 3]^- = \{f_4, f_5, f_7, f_8, f_9\}.$$

Then, a basis for the open-point topology on $C(Z, Y)$ is

$$\beta = \big\{\emptyset, \{f_7\}, \{f_4, f_8\}, \{f_5, f_9\}, \{f_4, f_5, f_7, f_8, f_9\}\big\}.$$

In this example, we see that the open-point topology on $C(Z, Y)$ is not finer than the point-open topology on $C(Z, Y)$.

**Theorem 12.** *Let $C_{op}(Z, H)$ be the group of all continuous open functions from $Z$ to $H$. Then, the bi-point-open topology on $C_{op}(Z, H)$ is finer than the point-open topology on $C_{op}(Z, H)$.*

P r o o f. Let $\beta_\tau$, $\beta_{\tau'}$, and $\beta_{\tau''}$ be bases for the open-point topology on $C_{op}(Z, H)$, the point-open topology on $C_{op}(Z, H)$, and the bi-point-open topology on $C_{op}(Z, H)$, respectively. Let

$$B' = [z_1, V_1]^+ \cap \cdots \cap [z_n, V_n]^+,$$

where $n \in \mathbb{N}$, $z_i \in Z$, and each $V_i$ is an open subset of $H$, be a member of $\beta_{\tau'}$, and let $f \in B'$. Then, from Theorem 11, there exists

$$B = [B_1, r_1]^- \cap \cdots \cap [B_n, r_n]^- \in \beta_\tau,$$

where $n \in \mathbb{N}$, $r_i \in H$, and $B_i$ are open subsets of $Z$ such that $f \in B$. Thus,

$$f \in [z_1, V_1]^+ \cap \cdots \cap [z_n, V_n]^+ \cap [B_1, r_1]^- \cap \cdots \cap [B_n, r_n]^- = B'' \in \beta_{\tau''}.$$

Clearly, $B'' \subseteq B'$. This completes the proof. □

**Theorem 13.** *Let $C_{op}(Z,H)$ be the group of all continuous open functions from $Z$ to $H$. Then, the bi-point-open topology on $C_{op}(Z,H)$ is finer than the open-point topology on $C_{op}(Z,H)$.*

P r o o f. Let $\beta_\tau$, $\beta_{\tau'}$ and $\beta_{\tau''}$ be bases for the open-point topology on $C_{op}(Z,H)$, the point-open topology on $C_{op}(Z,H)$, and the bi-point-open topology on $C_{op}(Z,H)$, respectively. Let

$$B = [B_1, r_1]^- \cap \cdots \cap [B_n, r_n]^- \in \beta_\tau,$$

where $n \in \mathbb{N}$, each $r_i \in H$, and each $B_i$ is an open subset of $Z$, be a member of $\beta_\tau$ and $f \in B$. Then

$$f^{-1}(r_i) \cap B_i \neq \emptyset.$$

Let $z_i \in f^{-1}(r_i) \cap B_i$. Then

$$f(z_i) \in f[f^{-1}(r_i) \cap B_i] \subseteq f f^{-1}(r_i) \cap f(B_i) \subseteq f(B_i) = V_i,$$

where each $V_i$ is open in $H$. Therefore, $f(z_i) \in V_i$. This implies that

$$f \in B' = [z_1, V_1]^+ \cap \cdots \cap [z_n, V_n]^+ \in \beta_{\tau'}.$$

Thus,

$$f \in [z_1, V_1]^+ \cap \cdots \cap [z_n, V_n]^+ \cap [B_1, r_1]^- \cap \cdots \cap [B_n, r_n]^- = B'' \in \beta_{\tau''}.$$

Clearly, $B'' \subseteq B'$. This completes the proof. □

## 5. Conclusion

In this paper, the role of generated open sets in defining topologies on $Y^X$ has been discussed. The interrelations among these topologies were also explored. We have shown that the concept of a topological ideal provides a useful framework for studying such topologies on $Y^X$. Furthermore, for a topological group $H$ and a space $Z$, the relationship between the point-open topology and the bi-point-open topology on $C(Z,H)$ was also examined.

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