ASYMPTOTIC EXPANSION OF A SOLUTION FOR ONE SINGULARLY PERTURBED OPTIMAL CONTROL PROBLEM IN $\mathbb{R}^n$ WITH A CONVEX INTEGRAL QUALITY INDEX

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Abstract: The paper deals with the problem of optimal control with a convex integral quality index for a linear steady-state control system in the class of piecewise continuous controls with a smooth control constraints. In a general case, for solving such a problem, the Pontryagin maximum principle is applied as the necessary and sufficient optimum condition. In this work, we deduce an equation to which an initial vector of the conjugate system satisfies. Then, this equation is extended to the optimal control problem with the convex integral quality index for a linear system with a fast and slow variables. It is shown that the solution of the corresponding equation as $\varepsilon \to 0$ tends to the solution of an equation corresponding to the limit problem. The results received are applied to study of the problem which describes the motion of a material point in $\mathbb{R}^n$ for a fixed period of time. The asymptotics of the initial vector of the conjugate system that defines the type of optimal control is built. It is shown that the asymptotics is a power series of expansion.

Keywords: Optimal control, Singularly perturbed problems, Asymptotic expansion, Small parameter.

Introduction

The paper is devoted to studying the asymptotics of the initial vector of a conjugated state and an optimal value of the quality index in the optimal control problem [1]–[3] for a linear system with a fast and slow variables (see review [4]), convex integral quality index [3, Chapter 3], and smooth geometrical constraints for control.

Singularly perturbed problems of optimal control have been considered in different settings in [5]–[7].

The method of boundary function that was developed in [4, 10] allows effectively constructing an asymptotics of solutions for problems with an open control area and smooth controlling actions.

The solving of problems with a closed and bounded control area meets certain difficulties. That is why the problems with fast and slow variables and closed constraints for control have been studied to a less extent. A significant contribution to solving these problems was made by Dontchev and Kokotovic.

Problems of fast operation and terminal control with constraints for control in the form of a polygon are dealt with in [5, 7]. The structure of such optimal control is a relay function with values in the apexes of the polygon. No optimal control with constraints in the form of a sphere, which is a continuous function with a finite and countable number of discontinuity points, has been considered so far.

The asymptotics of solutions of the perturbed control problem was formulated differently in papers [7, 9].

In the present work, the basic equation for searching for the asymptotics of the initial vector of the conjugated state of the problem under consideration and optimal control is obtained. General relationships are applied to the case of the optimal control with a point of a small mass in an n-dimensional space under the action of a bounded force.

1. General statement of problem and condition for optimality

Let us consider a problem that belongs to the class of piecewise continuous controls – optimal control problem for a linear stationary system with a convex integral quality index:

$$\begin{cases} \dot{z} = \mathcal{A}z + \mathcal{B}u, \quad z(0) = z^0, \quad \|u(t)\| \leqslant 1, \quad t \in [0;T], \\ J(u) = \varphi(z(T)) + \int\limits_0^T \|u(t)\|^2\, dt \to \min, \end{cases} \tag{1.1}$$

where $z \in \mathbb{R}^{\widetilde{n}}$, $u \in \mathbb{R}^r$, $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^r$, $\mathcal{A}$, $\mathcal{B}$ are constant matrices of the corresponding dimensional, and $\varphi(\cdot)$ is the convex function that is continuously differentiable in $\mathbb{R}^{\widetilde{n}}$.

Note that in the considered convex integral quality index J, where the first term can be interpreted as a fine for the control error at a finite time instant T, whereas the second, as an account of an energy spent for the realization of control.

Condition 1. Let us assume that a pair $(\mathcal{A},\mathcal{B})$ is quite controllable,

$$rank\big(\mathcal{B}, \mathcal{A}\mathcal{B}, \dots, \mathcal{A}^{\widetilde{n}-1}\mathcal{B}\big) = \widetilde{n}.$$

Under the conditions stated, the Pontryagin maximum principle in the problem (1.1) is the necessary and sufficient criterion of optimality. In this case, the problem has the unique solution [3, p. 3.5, Theorem 14]: if z, η is the unique solution to (1.1) and

$$\dot{\eta} = -\mathcal{A}^*\eta, \quad \eta(T) = -\nabla\varphi(z(T)), \tag{1.2}$$

then the optimal control u^o is determined from the maximum principle

$$-\|u^o(t)\|^2 + \langle \mathcal{B}^*\eta(t), u^o(t)\rangle = \max_{\|u\|\leqslant 1} \big(-\|u\|^2 + \langle \mathcal{B}^*\eta(t), u\rangle\big). \tag{1.3}$$

Here $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^r$.

Calculating maximum in (1.3), we find

$$u^o(t) = \frac{\mathcal{B}^*\eta(t)}{S(\|\mathcal{B}^*\eta(t)\|)}, \qquad \text{where} \quad S(\xi) := \begin{cases} 2, & 0 \leqslant \xi \leqslant 2, \\ \xi, & \xi > 2. \end{cases} \tag{1.4}$$

Note that the determination of function $S(\cdot)$ leads to the validity of inequality

$$\forall\, w_1, w_2 \in \mathbb{R}^r \qquad \left\| \frac{w_1}{S(\|w_1\|)} - \frac{w_2}{S(\|w_2\|)} \right\| \leqslant \|w_1 - w_2\|. \tag{1.5}$$

Let $\lambda := \eta(T)$. Then

$$\eta(t) = e^{-\mathcal{A}^*(t-T)}\lambda, \quad z(t) = e^{\mathcal{A}t}z^0 + \int\limits_0^t e^{\mathcal{A}(t-s)}\mathcal{B}u^o(s)ds.$$

At a finite time instant $t = T$ we have

$$z(T) = e^{\mathcal{A}T}z^0 + \int\limits_0^T \frac{e^{\mathcal{A}(T-s)}\mathcal{B}\mathcal{B}^* e^{\mathcal{A}^*(T-s)}\lambda}{S(\|\mathcal{B}^* e^{\mathcal{A}^*(T-s)}\lambda\|)} ds.$$

Replacing the variable $\tau := T - s$, we obtain

$$z(T) = e^{\mathcal{A}T} z^0 + \int\limits_0^T \frac{e^{\mathcal{A}\tau}\mathcal{B}\mathcal{B}^* e^{\mathcal{A}^*\tau}\lambda}{S(\|\mathcal{B}^* e^{\mathcal{A}^*\tau}\lambda\|)} d\tau.$$

Thus, the following is valid:

Statement 1. Let condition 1 be valid, $z(t)$, $u(t)$ be a solution of the system from Problem (1.1), and $\eta(t)$ be a solution of the system (1.2). Then $z(t)$, $\eta(t)$, $u(t)$ is the solution of the maximum principle problem (1.1), (1.2), (1.3) if and only if when $\eta(T) = \lambda$, $u(t)$ is determined by the formula (1.4), and a vector λ is the unique solution of equation

$$-\lambda = \nabla\varphi\left(e^{\mathcal{A}T} z^0 + \int\limits_0^T e^{\mathcal{A}\tau}\mathcal{B} \frac{\mathcal{B}^* e^{\mathcal{A}^*\tau}\lambda}{S(\|\mathcal{B}^* e^{\mathcal{A}^*\tau}\lambda\|)} d\tau \right). \tag{1.6}$$

Besides $u(t)$ is the unique optimal control in the problem (1.1).

The vector λ that satisfies the equation (1.6) will be called as *a vector determining the optimal control* in the problem (1.1).

Statement 2. Let $u^o(t)$ be the optimal control in (1.1). Then $u^o(t)$ is continuous on $[0;T]$ and infinitely differentiable at points $\widetilde{t}$ such that $\|\mathcal{B}^* e^{\mathcal{A}^*(T-\widetilde{t})}\lambda\| \neq 2$. Here λ is a vector determining the optimal control in problem (1.1).

P r o o f. The validity of statement follows from (1.4) and analytical form of the matrix exponent $e^{\mathcal{A}^* t}$. □

2. Optimal control problem with fast and slow variables

Consider a particular case of problem (1.1), when the system under control contains fast and slow variables and the terminal part of the quality index depends only on slow variables:

$$\begin{cases} \dot{x}_\varepsilon = A_{11}x_\varepsilon + A_{12}y_\varepsilon + B_1 u, & t \in [0,T], \quad \|u\| \leqslant 1, \\ \varepsilon\dot{y}_\varepsilon = A_{21}x_\varepsilon + A_{22}y_\varepsilon + B_2 u, & x_\varepsilon(0) = x^0, \quad y_\varepsilon(0) = y^0, \\ J(u) := \sigma(x_\varepsilon(T)) + \int\limits_0^T \|u(t)\|^2\, dt \to \min, & \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^r$; A_{ij}, B_i $(i,j = 1,2)$ are the constant matrices of the corresponding dimensions, and $\sigma(\cdot)$ is the convex function that is continuously differentiable in $\mathbb{R}^n$.

Condition 2. All eigenvalues of matrix A_{22} have negative real parts.

For each fixed $\varepsilon > 0$ the problem (2.1) coincides with the problem (1.1):

$$z_\varepsilon(t) = \begin{pmatrix} x_\varepsilon(t) \\ y_\varepsilon(t) \end{pmatrix}, \quad z_\varepsilon^0 = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}, \quad \mathcal{A}_\varepsilon = \begin{pmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{pmatrix}, \quad \mathcal{B}_\varepsilon = \begin{pmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{pmatrix},$$

$$\widetilde{n} = n + m, \quad \varphi(z_\varepsilon) = \sigma(x_\varepsilon).$$

As a limit problem for (2.1), the following problem is introduced

$$\begin{cases} \dot{x}_0 = A_0 x_0 + B_0 u, & t \in [0,T], \qquad \|u\| \leqslant 1, \\ A_0 := A_{11} - A_{12}A_{22}^{-1}A_{21}, & B_0 := B_1 - A_{12}A_{22}^{-1}B_2, \quad x_0(0) = x^0, \\ J(u) := \sigma(x_0(T)) + \int\limits_0^T \|u(t)\|^2\, dt \to \min\,. \end{cases} \tag{2.2}$$

Condition 3. Pairs (A_0, B_0) and (A_{22}, B_2) are quite controllable.

If the Conditions 2-3 are satisfied, then there exists $\varepsilon_0 > 0$ such that the pair $(\mathcal{A}_\varepsilon, B_\varepsilon)$ is quite controllable at any $\varepsilon : 0 < \varepsilon \leqslant \varepsilon_0$ [5, Theorem 1].

Note that since $\nabla\varphi(z_\varepsilon) = \begin{pmatrix} \nabla\sigma(x_\varepsilon) \\ 0 \end{pmatrix}$, then the vector λ_ε, which determines the optimal control in the problem (2.1), has the form $\lambda_\varepsilon = \begin{pmatrix} l_\varepsilon \\ 0 \end{pmatrix}$, $l_\varepsilon \in \mathbb{R}^n$.

The vector l_ε also will be called *as determining the optimal control in problem* (2.1).

Let

$$e^{\mathcal{A}_\varepsilon t} := \begin{pmatrix} \mathcal{W}_\varepsilon^{11}(t) & \mathcal{W}_\varepsilon^{12}(t) \\ \mathcal{W}_\varepsilon^{21}(t) & \mathcal{W}_\varepsilon^{22}(t) \end{pmatrix}, \tag{2.3}$$

then, by virtue of (2.3) the equation (1.6) transforms into

$$-l_\varepsilon = \nabla\sigma\Bigg(\mathcal{W}_\varepsilon^{11}(T)x^0 + \mathcal{W}_\varepsilon^{12}(T)y^0 + \int\limits_0^T \big(\mathcal{W}_\varepsilon^{11}(t)B_1 + \varepsilon^{-1}\mathcal{W}_\varepsilon^{12}(t)B_2\big) \frac{\big(B_1^*(\mathcal{W}_\varepsilon^{11}(t))^* + \varepsilon^{-1}B_2^*(\mathcal{W}_\varepsilon^{12}(t))^*\big)l_\varepsilon}{S\big(\big\|\big(B_1^*(\mathcal{W}_\varepsilon^{11}(t))^* + \varepsilon^{-1}B_2^*(\mathcal{W}_\varepsilon^{12}(t))^*\big)l_\varepsilon\big\|\big)} dt \Bigg). \tag{2.4}$$

Note that the optimal control $u_\varepsilon^o(t)$ in the problem (2.1) is expressed through the vector l_ε as follows:

$$u_\varepsilon^o(T-t) = \frac{\big(B_1^*(\mathcal{W}_\varepsilon^{11}(t))^* + \varepsilon^{-1}B_2^*(\mathcal{W}_\varepsilon^{12}(t))^*\big)l_\varepsilon}{S\big(\big\|\big(B_1^*(\mathcal{W}_\varepsilon^{11}(t))^* + \varepsilon^{-1}B_2^*(\mathcal{W}_\varepsilon^{12}(t))^*\big)l_\varepsilon\big\|\big)}. \tag{2.5}$$

Theorem 1. Let the Conditions 2 and 3 be valid. Then $l_\varepsilon \to l_0$ as $\varepsilon \to +0$, where l_ε is the unique solution of the equation (2.4), and l_0 is the unique solution of the equation

$$-l_0 = \nabla\sigma\left(e^{A_0 T}x^0 + \int\limits_0^T e^{A_0 t}B_0 \frac{B_0^* e^{A_0^* t} l_0}{S\big(\|B_0^* e^{A_0^* t} l_0\|\big)} dt \right). \tag{2.6}$$

P r o o f. It is known that the attainability set for the controllable system under control from (2.1) is uniformly bounded by the time instant T at $\varepsilon \in (0;\varepsilon_0]$ (see, for example, [6, theorem 3.1]). Hence, by virtue of (2.4) vectors $\{l_\varepsilon\}$ are also bounded at $\varepsilon \in (0;\varepsilon_0]$. Therefore, to prove the theorem, it is sufficient to show that all partial limits $\{l_\varepsilon\}$ as $\varepsilon \to +0$ are equal to l_0.

As follows from the A. B. Vasil'eva's results (see, for example [10, Chapter 3]) there is $\gamma > 0$ such that

$$\begin{aligned} &\mathcal{W}_\varepsilon^{11}(t) = e^{A_0 t} + O(\varepsilon), \qquad \mathcal{W}_\varepsilon^{12}(t) = -\varepsilon e^{A_0 t}A_{12}A_{22}^{-1} + O(\varepsilon e^{-\gamma t/\varepsilon}) + O(\varepsilon^2), \\ &\mathcal{W}_\varepsilon^{21}(t) = -A_{22}^{-1}A_{21}e^{A_0 t} + O(e^{-\gamma t/\varepsilon}) + O(\varepsilon), \qquad \mathcal{W}_\varepsilon^{22}(t) = O(e^{-\gamma t/\varepsilon}). \end{aligned} \tag{2.7}$$

Moreover, asymptotic estimates are uniform in $t \in [0;T]$.

Hence, by virtue of (2.2) which determines the matrices A_0 and B_0 and by formulas (2.7) the expression standing $\nabla\sigma$ for the formula (2.4) has the form

$$e^{A_0T}x^0 + O(\varepsilon) + \int_0^T \big(e^{A_0t}B_0 + O(e^{-\gamma t/\varepsilon}) + O(\varepsilon)\big)\frac{\big(B_0^*e^{A_0^*t} + O(e^{-\gamma t/\varepsilon}) + O(\varepsilon)\big)l_\varepsilon}{S\big(\big\|\big(B_0^*e^{A_0^*t} + O(e^{-\gamma t/\varepsilon}) + O(\varepsilon)\big)l_\varepsilon\big\|\big)}dt. \tag{2.8}$$

Let us divide the integral from (2.8) into two terms $\int_0^T = \int_0^{\sqrt{\varepsilon}} + \int_{\sqrt{\varepsilon}}^T$. Then, taking into account that the expression under integral is uniformly constrained and that $O(e^{-\gamma/\sqrt{\varepsilon}}) = O(\varepsilon^\alpha)$ as $\varepsilon \to 0$ for any $\alpha > 0$, we obtain from (2.4) and (2.8)

$$-l_\varepsilon = \nabla\sigma\left(e^{A_0T}x^0 + O(\varepsilon) + O(\sqrt{\varepsilon}) + \int_{\sqrt{\varepsilon}}^T e^{A_0t}B_0\frac{\big(B_0^*e^{A_0^*t} + O(\varepsilon)\big)l_\varepsilon}{S\big(\big\|\big(B_0^*e^{A_0^*t} + O(\varepsilon)\big)l_\varepsilon\big\|\big)}dt\right). \tag{2.9}$$

Let $\bar{l}$ be a partial limit of the vectors $\{l_\varepsilon\}$ as $\varepsilon \to +0$, i.e. $l_{\varepsilon_k} \to \bar{l}$ for a certain $\{\varepsilon_k\}$ so that $\varepsilon_k \to +0$. Going to the limit as $k \to \infty$ in (2.9) we obtain that $\bar{l}$ is the solution of (2.6). Because of the uniqueness of such a solution we have $\bar{l} = l_0$. □

The main problem for (2.1) is to find the complete asymptotic expansion in powers of small parameter ε of the optimal control, optimal values of the quality index, and the optimal process. Formulas (2.5) and (1.5) show that if one manages to gain the complete asymptotic expansion of vector l_ε, which determines the optimal control in problem (2.1), this vector can be used for the asymptotic expansions of the above values as well.

3. Construction of complete asymptotic expansion of vector l_ε for an optimal control problem with fast and slow variables

Consider a partial case of problem (2.1):

$$\begin{cases} \dot{x}_\varepsilon = y_\varepsilon, \quad t \in [0,T], \quad \|u\| \leqslant 1, \\ \varepsilon\dot{y}_\varepsilon = -y_\varepsilon + u, \quad x_\varepsilon(0) = x^0, \quad y_\varepsilon(0) = y^0, \\ J(u) := \frac{1}{2}\|x_\varepsilon(T)\|^2 + \int_0^T \|u(t)\|^2\,dt \to \min, \end{cases} \tag{3.1}$$

where x_ε, y_ε, $u \in \mathbb{R}^n$.

Problem (3.1) simulates a motion of a material point of small mass $\varepsilon > 0$ with the coefficient of the medium resistance equals to 1 in the space $\mathbb{R}^n$ under action of the constrained control force $u(t)$.

Here $A_{11} = 0$, $A_{12} = I$, $A_{21} = 0$, $A_{22} = -I$, $B_1 = 0$, $B_2 = I$, and 0 and I are the zero and the identity matrices of dimensional $n \times n$, respectively. For the limit problem we have $A_0 = 0$, $B_0 = I$ and thus, Conditions 2 and 3 are valid.

Calculating $e^{\mathcal{A}_\varepsilon t}$ and $\nabla(\frac{1}{2}\|x_\varepsilon(T)\|^2)$, we obtain

$$\mathcal{W}_\varepsilon^{11}(t) = I, \quad \mathcal{W}_\varepsilon^{12}(t) = \varepsilon(1-e^{-t/\varepsilon})I, \quad \mathcal{W}_\varepsilon^{21}(t) = 0, \quad \mathcal{W}_\varepsilon^{22}(t) = e^{-t/\varepsilon}I, \quad \nabla\Big(\frac{1}{2}\|x_\varepsilon(T)\|^2\Big) = x_\varepsilon(T).$$

Therefore, equations (2.4) and (2.6) for l_ε and l_0 take the form

$$-l_\varepsilon = x^0 + \varepsilon(1 - e^{-T/\varepsilon})y^0 + \int_0^T \frac{(1 - e^{-t/\varepsilon})^2 l_\varepsilon}{S\big(\big\|(1 - e^{-t/\varepsilon})l_\varepsilon\big\|\big)}dt, \quad -l_0 = x^0 + T\frac{l_0}{S(\|l_0\|)}. \tag{3.2}$$

If the vector-function $f_\varepsilon(t)$ is such that $f_\varepsilon(t) = O(\varepsilon^\alpha)$ as $\varepsilon \to 0$ for any $\alpha > 0$ uniformly with respect to $t \in [0;T]$ then instead of $f_\varepsilon(t)$ we will write $\mathbb{O}$. In particular, $e^{-\gamma T/\varepsilon} = \mathbb{O}$.

From (3.2) we obtain

$$\begin{aligned} &1.\ \|x^0\| < T+2 \Longrightarrow l_0 = -\frac{2}{2+T}\,x^0 \qquad \text{and} \qquad \|l_0\| < 2, \\ &2.\ \|x^0\| > T+2 \Longrightarrow l_0 = -\frac{\|x^0\| - T}{\|x^0\|}\,x^0 \quad \text{and} \qquad \|l_0\| > 2. \end{aligned} \tag{3.3}$$

1. Consider first the case: $\|x^0\| < T+2$.

By virtue of (3.3) and Theorem 1 the inequality $\|l_\varepsilon\| < 2$ is valid for any sufficiently small ε. Taking into account that $(1 - e^{-t/\varepsilon}) \leqslant 1$ at any $t \geqslant 0$ and $\varepsilon > 0$, from (3.2) we obtain for l_ε the equation

$$-l_\varepsilon = x^0 + \varepsilon y^0 + \mathbb{O} + \frac{1}{2}\int_0^T (1 - e^{-t/\varepsilon})^2\, dt\, l_\varepsilon. \tag{3.4}$$

Calculating the integral $\int_0^T (1 - e^{-t/\varepsilon})^2\, dt \ = T - 3/(2\varepsilon) + \mathbb{O}$, from (3.4) we find

$$l_\varepsilon = -\frac{4(x^0 + \varepsilon y^0 + \mathbb{O})}{4 + 2T - 3\varepsilon}.$$

It follows from this representation that l_ε is expanded in the asymptotic series in powers of ε.

Statement 3. Let $\|x^0\| < T+2$. Then the vector l_ε which determines the optimal control in problem (3.1), is expanded as $\varepsilon \to 0$ in the power asymptotic series

$$l_\varepsilon \overset{as}{=} l_0 + \sum_{k=1}^{\infty} \varepsilon^k l_k, \text{ where, in particular, } l_1 = -\frac{3l_0 + 4y^0}{4 + 2T}.$$

2. Now consider the case: $\|x^0\| > T+2$.

By virtue of (3.3) and Theorem 1, the inequality $\|l_\varepsilon\| < 2$ is valid for all sufficiently small ε. Since for a fixed ε the function $(1 - e^{-t/\varepsilon})\|l_\varepsilon\|$ increases monotonically from 0 at $t = 0$ into $(1 - e^{-t/\varepsilon})\|l_\varepsilon\|$ at $t = T$ (which for sufficiently small ε gives the inequality $(1 - e^{-t/\varepsilon})\|l_\varepsilon\| > 2$), there is the unique $t_{1,\varepsilon} \in (0;T)$ such that $(1 - e^{-t_{1,\varepsilon}/\varepsilon})\|l_\varepsilon\| = 2$, or

$$(1 - e^{-t_{1,\varepsilon}/\varepsilon})\|l_\varepsilon\| = 2, \qquad t_{1,\varepsilon} = -\varepsilon \ln\Big(1 - \frac{2}{\|l_\varepsilon\|}\Big). \tag{3.5}$$

Therefore, the equation (3.2) takes the form

$$-l_\varepsilon = x^0 + \varepsilon(1 - e^{-T/\varepsilon})y^0 + \frac{1}{2}\int_0^{t_{1,\varepsilon}} \Big(1 - e^{-t/\varepsilon}\Big)^2 dt\, l_\varepsilon + \int_{t_{1,\varepsilon}}^T \Big(1 - e^{-t/\varepsilon}\Big)\, dt \frac{l_\varepsilon}{\|l_\varepsilon\|}. \tag{3.6}$$

Calculating the integrals in (3.6) and transposing $(-l_\varepsilon)$ into the right part, we obtain

$$\begin{aligned} 0 = F(\varepsilon, l_\varepsilon) :=\, & l_\varepsilon + x^0 + \varepsilon(1 - e^{-T/\varepsilon})y^0 - \varepsilon\Big(\frac{1}{\|l_\varepsilon\|} + \frac{1}{\|l_\varepsilon\|^2} + \frac{1}{2}\ln\Big(1 - \frac{2}{\|l_\varepsilon\|}\Big)\Big) l_\varepsilon \\ & + \Big(T + \varepsilon \ln\Big(1 - \frac{2}{\|l_\varepsilon\|}\Big) + \varepsilon e^{-T/\varepsilon} - \varepsilon + \varepsilon\frac{2}{\|l_\varepsilon\|}\Big)\frac{l_\varepsilon}{\|l_\varepsilon\|}. \end{aligned} \tag{3.7}$$

Theorem 2. Let $\|x^0\| > T+2$. Then the vector l_ε which determines the optimal control in problem (3.1) is expanded into a power asymptotic series (for $\varepsilon \to 0$)

$$l_\varepsilon \overset{as}{=} l_0 + \sum_{k=1}^{\infty} \varepsilon^k l_k.$$

P r o o f. Consider the equation $0 = F(\varepsilon, l)$, where $F(\cdot,\cdot)$ is defined in (3.7). Additionally predetermine $e^{-T/\varepsilon}$ at the point $\varepsilon = 0$ as zero. Then we obtain that $0 = F(0, l_0)$ and $F(\cdot,\cdot)$ is infinitely differentiable in ε and l in a certain neighborhood of the point $(0; l_0)$. Since

$$\mathcal{F}\rho := \frac{\partial F(\varepsilon, l)}{\partial l}\Big|_{\varepsilon=0, l=l_0} \rho = \rho + \frac{\|l_0\|^2\rho - \langle l_0, \rho\rangle l_0}{\|l_0\|^3} T,$$

then operator $\mathcal{F}$ is continuously reversible and

$$\mathcal{F}^{-1} g = \left(g + \frac{T\langle l_0, g\rangle l_0}{\|l_0\|^3} \right) \frac{\|l_0\|}{T + \|l_0\|}. \tag{3.8}$$

In this way, the theorem of implicitly specified function is applicable, which means that l_ε (as a function of ε) is infinitely differentiable in ε for all small ε and, therefore, l_ε is expanded into the asymptotic series. The coefficients of this series can be found via the standard procedure: substituting the series into the equation (3.7), expanding values dependent on ε into asymptotic series in power of ε, and equaling terms of the same order of smallness with respect to ε, we obtain an equation of the $\mathcal{F} l_k = g_k$ with the known right parts. Then, by the formula (3.8) we find l_k.

In particular, for l_1 we obtain the equation

$$\mathcal{F} l_1 = g_1 := -x^0 - y^0 + \left(\frac{1}{\|l_0\|} + \frac{1}{\|l_0\|^2} + \frac{1}{2} \ln\left(1 - \frac{2}{\|l_0\|}\right) \right) l_0 - \left(\ln\left(1 - \frac{2}{\|l_0\|}\right) - 1 + \frac{2}{\|l_0\|} \right) \frac{l_0}{\|l_0\|}.$$

Hence, by virtue of (3.8) we obtain

$$l_1 = \left(g_1 + \frac{T\langle l_0, g_1\rangle l_0}{\|l_0\|^3} \right) \frac{\|l_0\|}{T + \|l_0\|}.$$

□

4. Remarks

1. Both in the first and the second cases under consideration, from (3.2), (3.5) and asymptotic expansion of l_ε, the asymptotic expansions of both the quality index and optimal control as well as optimal state of the system are conventionally obtained. With this, the asymptotic expansions of the optimal control and optimal state of the system will be exponentially decreasing boundary layers in the neighborhood of point $t = 0$. Moreover, if $t \geqslant \varepsilon^\beta$ and $\beta \in (0; 1)$, the optimal control $u^o(t)$ is a constant plus the asymptotic zero.

2. It follows from the formula (3.7) that l_ε lies in the subspace Π created by vectors x^0 and y^0. Therefore, for all $t \in [0; T]$ and $u^o_\varepsilon(t)$, $x_\varepsilon(t)$ and $y_\varepsilon(t)$ lie in the same subspace Π. In this way, the problem (3.1) is equivalent to the corresponding two-dimensional problem.

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REFERENCES

1. **Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mishchenko E.F.** The Mathematical Theory of Optimal Processes. NY–London–Sydney: J. Wiley and Sons inc., 1962. 360 pp.
2. **Krasovskii N.N.** Theory of Control of Movement. Linear Systems. M.: Nauka, 1968. 476 pp. [in Russian]
3. **Lee E.B., Markus L.** Foundations of Optimal Control Theory. New York–London–Sydney: John Wiley and Sons, Inc., 1967. 576 pp.
4. **Vasil'eva A.B., Dmitriev M.G.** Mathematical analysis. The results of science and technology. M.:VINITI, 1982. Vol. 20. P. 3–77. [in Russian]
5. **Kokotovic P.V., Haddad A.H.** Controllability and time-optimal control of systems with slow and fast models // IEEE Trans. Automat. Control. 1975. Vol. 20, no. 1. P. 111–113. DOI: 10.1109/TAC.1975.1100852
6. **Dontchev A.L.** Perturbations, approximations and sensitivity analisis of optimal control systems. Berlin–Heidelberg–New York–Tokio: Springer-Verlag, 1983. 156 pp. DOI: 10.1007/BFb0043612
7. **Kalinin A.I., Semenov K.V.** Asymptotic Optimization Method for Linear Singularly Perturbed Systems with Multidimensional Control // Computational Mathematics and Mathematical Physics. 2004. Vol. 44, no. 3. P. 407–418.
8. **Danilin A.R., Parysheva Y.V.** Asymptotics of the optimal cost functional in a linear optimal control problem // Doklady Mathematics. 2009. Vol. 80, no. 1. P. 478–481.
9. **Danilin A.R., Kovrizhnykh O.O.** Time-optimal control of a small mass point without environmental resistance // Doklady Mathematics. 2013. Vol. 88, no. 1. P. 465–467.
10. **Vasil'eva A.B., Butuzov V.F.** Asymptotic expansions of a solutions of singularly perturbed equations. M.: Nauka, 1973. [in Russian]