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ON AN INITIAL BOUNDARY–VALUE PROBLEM FOR A DEGENERATE EQUATION OF HIGH EVEN ORDER

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Abstract: In this paper, we formulate and study an initial boundary-value problem of the type of the third boundary condition for a degenerate partial differential equation of high even order in a rectangle. Using the Fouriers method, based on separation of variables, a spectral problem for an ordinary differential equation is obtained. Using the Green's function method, the latter problem is equivalently reduced to the Fredholm integral equation of the second kind with a symmetric kernel, which implies the existence of eigenvalues and a system of eigenfunctions of the spectral problem. Using the found integral equation and Mercer's theorem, the uniform convergence of certain bilinear series depending on the eigenfunctions is proved. The order of the Fourier coefficients has been established. The solution to the considered problem has been written as a sum of the Fourier series over the system of eigenfunctions of the spectral problem. The uniqueness of the solution to the problem was proved using the method of energy integrals. An estimate for solution of the problem was obtained, which implies its continuous dependence on the given functions.

Keywords: Degenerate equation, Initial boundary-value problem, Method of separation of variables, Spectral problem, Green's function method, Integral equation, Fourier series.

1. Introduction

Recently, researchers have been paying more and more attention to degenerate partial differential equations. This trend is primarily driven by the intrinsic requirements of the theory of partial differential equations. Additionally, a multitude of problems in gas dynamics, hydrodynamics [4, 5], the theory of infinitesimal bending of surfaces, and the momentless theory of shells with alternating curvature [17], as well as in the theory of oscillations [8, 9], mathematical biology [12], filtration theory, boundary layer theory, and technical mechanics, necessitate the investigation of degenerate partial differential equations.

Currently, intensive research is underway on initial boundary value problems in quadrangular domains for degenerate partial differential equations of high even order in spatial variables. For instance, in [3], initial boundary value problems in a rectangle were formulated and investigated for the following degenerate equation:

$$\frac{\partial^l u}{\partial t^l} = (-1)^k \frac{\partial^k}{\partial x^k}\left(x^\alpha \frac{\partial^k u}{\partial x^k}\right) + f(x,t), \quad l = \overline{1,2}, \quad \alpha \in (0, 2k). \tag{1.1}$$

Moreover, in [2] and [13], similar equations with generalizations were explored.

When considering initial boundary value problems for degenerate equations of type (1.1), the formulation of the problems is significantly influenced by the degree of degeneracy α [2, 3], and sometimes by the evenness and oddness of the number k. Additionally, as the order of the equation increases, the number of options for boundary conditions also increases. For instance, in [2, 3], when considering initial boundary value problems for equation (1.1) in the quadrilateral

$$\Omega = \{0 < x < 1,\ 0 < t < T\}$$

at $0 < \alpha < 1$, boundary conditions of the form

$$\left(\partial^j/\partial x^j\right) u\big|_{x=0} = 0, \quad j = \overline{0, k-1}; \quad \left(\partial^q/\partial x^q\right) u|_{x=1} = 0, \quad q = \overline{0, k-1} \tag{1.2}$$

were specified, at $\alpha \in (1, k)$, some boundary conditions at $x = 0$ are replaced by the boundedness condition, and at $\alpha \in (k, 2k)$ at $x = 0$ no boundary conditions were specified.

In [13], considering equation (1.1) for $\alpha \in (0, 1)$, boundary conditions of the form (1.2) were specified, but here $q = \overline{k, 2k-1}$.

In [6, 7], when considering a degenerate equation of a different type, boundary conditions (1.2) were adopted. In [15], for a specific degenerate equation, a problem with boundary conditions relating the values of the desired function and the derivatives with respect to x at $x = 0$ and $x = 1$ was formulated and studied. In [1] and [16], for equation (1.1) with $\alpha = 0$, $l = 2$, and for a degenerate fourth-order equation of type (1.1) respectively, conditions of the third type were specified for both $x = 0$ and $x = 1$. Moreover, in [14], a mixed problem was considered for a fourth-order degenerate equation with fractional case of l, namely for $1 < l < 2$, and the dependence of the degeneration degree of α to the formulation of the boundary conditions has been studied.

In this paper, an initial boundary value problem with conditions similar to the third boundary condition for a degenerate partial differential equation of high even order in a rectangle is formulated and investigated.

2. Formulation of the problem

In a rectangle

$$\Omega = \{(x, t) : 0 < x < 1;\ 0 < t < T\},$$

we consider the following degenerate equation of high even order

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^{2n}}{\partial x^{2n}} \left(x^\alpha \frac{\partial^{2n} u}{\partial x^{2n}} \right) = f(x, t), \tag{2.1}$$

where $u = u(x, t)$ is an unknown function, $f(x, t)$ is a given function, and α is a given real number, such that $0 < \alpha < 1$ and $n \in N$.

We study the following initial boundary-value problem:

Problem *A*. Find a function $u(x, t)$ such that:

1) $u_t, \left(\partial^j/\partial x^j\right) u, \left(\partial^j/\partial x^j\right) \left[x^\alpha \left(\partial^{2n}/\partial x^{2n}\right) u\right] \in C(\bar{\Omega}), \quad j = \overline{0, 2n-1};$
$\left(\partial^{2n}/\partial x^{2n}\right) \left[x^\alpha \left(\partial^{2n}/\partial x^{2n}\right) u\right], \quad u_{tt} \in C(\Omega);$
2) it satisfies the equation (2.1) in the domain Ω;
3) it satisfies the following initial conditions

$$u(x, 0) = \varphi_1(x), \quad x \in [0, 1], \quad u_t(x, 0) = \varphi_2(x), \quad x \in [0, 1] \tag{2.2}$$

and boundary conditions

$$\left.\begin{array}{l} \dfrac{\partial^{2j}}{\partial x^{2j}} u(0, t) = \dfrac{\partial^{2j+1}}{\partial x^{2j+1}} u(0, t), \quad \dfrac{\partial^{2j}}{\partial x^{2j}} \left(x^\alpha \dfrac{\partial^{2n}}{\partial x^{2n}} u(x, t) \right)\Big|_{x=0} = \dfrac{\partial^{2j+1}}{\partial x^{2j+1}} \left(x^\alpha \dfrac{\partial^{2n}}{\partial x^{2n}} u(x, t) \right)\Big|_{x=0}; \\ \dfrac{\partial^{2j}}{\partial x^{2j}} u(1, t) = \dfrac{\partial^{2j+1}}{\partial x^{2j+1}} u(1, t), \quad \dfrac{\partial^{2j}}{\partial x^{2j}} \left(x^\alpha \dfrac{\partial^{2n}}{\partial x^{2n}} u(x, t) \right)\Big|_{x=1} = \dfrac{\partial^{2j+1}}{\partial x^{2j+1}} \left(x^\alpha \dfrac{\partial^{2n}}{\partial x^{2n}} u(x, t) \right)\Big|_{x=1}; \\ \qquad j = \overline{0, n-1}, \quad t \in [0, T], \end{array}\right\} \tag{2.3}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are given continuous functions.

3. Investigation of the spectral problem

By formally applying the Fourier method to the problem A, we get the following spectral problem:

$$M[v(x)] \equiv \left(x^{\alpha}v^{(2n)}(x)\right)^{(2n)} = \lambda v(x), \quad 0<x<1; \tag{3.1}$$

$$\left.\begin{array}{c} v^{(j)}(x), \left(x^{\alpha}v^{(2n)}(x)\right)^{(j)} \in C[0,1], \quad j=\overline{0,2n-1}; \\ v^{(2j)}(0)=v^{(2j+1)}(0), \quad \left[x^{\alpha}v^{(2n)}(x)\right]^{(2j)}\Big|_{x=0} = \left[x^{\alpha}v^{(2n)}(x)\right]^{(2j+1)}\Big|_{x=0}, \quad j=\overline{0,n-1}; \\ v^{(2j)}(1)=v^{(2j+1)}(1), \quad \left[x^{\alpha}v^{(2n)}(x)\right]^{(2j)}\Big|_{x=1} = \left[x^{\alpha}v^{(2n)}(x)\right]^{(2j+1)}\Big|_{x=1}, \quad j=\overline{0,n-1}. \end{array}\right\} \tag{3.2}$$

It is easy to verify that for any functions $v(x)$ and $w(x)$ satisfying the conditions (3.2), the equality

$$\int_0^1 w(x)M[v(x)]dx = \int_0^1 v(x)M[w(x)]dx$$

holds true. This implies that the problem with conditions $M[v(x)]=0$ and (3.2) is self-adjoint.

Let $v(x)$ be a function satisfying conditions $\{(3.1),(3.2)\}$. Then, multiplying the equation (3.1) with the function $v(x)$ and integrating the resulting equality over the interval $[0,1]$, and subsequently applying the integration by parts rule and considering equalities (3.2), we arrive at

$$\lambda\int_0^1 v^2(x)dx = \int_0^1 x^{\alpha}\left[v^{(2n)}(x)\right]^2 dx. \tag{3.3}$$

If $\lambda=0$, then from equality (3.3) it follows that

$$v^{(2n)}(x)=0, \quad 0<x<1.$$

Hence, due to the conditions

$$v^{(2j)}(0)=v^{(2j+1)}(0), \quad v^{(2j)}(1)=v^{(2j+1)}(1), \quad j=\overline{0,n-1},$$

we have $v(x)\equiv 0, \quad 0\le x\le 1$. If $\lambda<0$, then from (3.3) it immediately follows that $v(x)\equiv 0$, $0\le x\le 1$. Consequently, problem $\{(3.1),(3.2)\}$ can have nontrivial solutions only for $\lambda>0$.

Assuming $\lambda>0$, we prove the existence of eigenvalues of problem $\{(3.1),(3.2)\}$ using the Green's function method. The Green's function $G(x,s)$ of this problem has the following properties:

1) $\left(\partial^j/\partial x^j\right)G(x,s)$, $j=\overline{0,2n-1}$ and $\left(\partial^j/\partial x^j\right)\left[x^{\alpha}\left(\partial^{2n}/\partial x^{2n}\right)G(x,s)\right]$, $j=\overline{0,2n-2}$ are continuous for all $x,s\in[0,1]$;
2) in each of the intervals $[0,s)$ and $(s,1]$ there exists a continuous derivative $\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^{\alpha}\left(\partial^{2n}/\partial x^{2n}\right)G(x,s)\right]$, and at $x=s$ it has a jump:

$$\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^{\alpha}\left(\partial^{2n}/\partial x^{2n}\right)G(x,s)\right]_{x=s-0}^{x=s+0}=1; \tag{3.4}$$

3) in the intervals $(0,s)$ and $(s,1)$ with respect to the argument x there exists a continuous derivative $MG(x,s)$ and the equality $MG(x,s)=0$ holds;
4) for $s\in(0,1)$ with respect to x it satisfies the conditions

$$\left.\begin{array}{c} \dfrac{\partial^{2j}G(0,s)}{\partial x^{2j}} = \dfrac{\partial^{2j+1}G(0,s)}{\partial x^{2j+1}}, \\ \dfrac{\partial^{2j}}{\partial x^{2j}}\left(x^{\alpha}\dfrac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right)\Big|_{x=0} = \dfrac{\partial^{2j+1}}{\partial x^{2j+1}}\left(x^{\alpha}\dfrac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right)\Big|_{x=0}, \quad j=\overline{0,n-1}; \end{array}\right\}$$

$$\left.\begin{aligned} \frac{\partial^{2j} G(1,s)}{\partial x^{2j}} &= \frac{\partial^{2j+1} G(1,s)}{\partial x^{2j+1}}, \\ \frac{\partial^{2j}}{\partial x^{2j}}\Big(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\Big)\Big|_{x=1} &= \frac{\partial^{2j+1}}{\partial x^{2j+1}}\Big(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\Big)\Big|_{x=1}, \quad j=\overline{0,n-1}. \end{aligned}\right\}$$

As proven above, problem $\{(3.1),(3.2)\}$ for $\lambda=0$ has only a trivial solution. Then, according to [11, p. 39], there exists a unique Green's function $G(x,s)$ for this problem. Let us now prove that the Green's function $G(x,s)$, satisfying the above conditions 1–4, is symmetric with respect to its arguments.

Let

$$v(x),\ h(x) \in C^{2n-1}[0,1]; \quad x^{\alpha}v^{(2n)}(x),\ x^{\alpha}h^{(2n)}(x) \in C^{2n-1}[0,1] \cap C^{2n}(0,1).$$

Let us introduce the following notation:

$$M[v(x)] \equiv \big(x^{\alpha}v^{(2n)}(x)\big)^{(2n)} = f(x), \quad M[h(x)] \equiv \big(x^{\alpha}h^{(2n)}(x)\big)^{(2n)} = g(x).$$

Then the following equality holds true

$$\begin{gathered} h(x)M[v(x)] - v(x)M[h(x)] = h(x)\big(x^{\alpha}v^{(2n)}(x)\big)^{(2n)} - v(x)\big(x^{\alpha}h^{(2n)}(x)\big)^{(2n)} \\ = \sum_{j=0}^{2n-1}\frac{d}{dx}\Big\{(-1)^j\Big[h^{(j)}(x)\big(x^{\alpha}v^{(2n)}(x)\big)^{(2n-1-j)} - v^{(j)}(x)\big(x^{\alpha}h^{(2n)}(x)\big)^{(2n-1-j)}\Big]\Big\} \\ = f(x)h(x) - g(x)v(x), \quad 0<x<1. \end{gathered} \tag{3.5}$$

If we assume $v(x)=G(x,s)$ and $h(x)=G(x,\xi)$, then at all the points of the interval $(0,1)$, except points $x\neq\xi,\ x\neq s$, the equalities $M[v(x)]=0$ and $M[h(x)]=0$ hold. Then equality (3.5) takes the form

$$\begin{gathered} \sum_{j=0}^{2n-1}\frac{d}{dx}\Big\{(-1)^j\Big[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,s)\Big) \\ -\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big]\Big\} = 0, \quad x\in(0,1)/\{s,\xi\} \end{gathered} \tag{3.6}$$

Without loss of generality, we assume that $s<\xi$. Then the segment $[0,1]$ is divided into three segments: $[0,s],\ [s,\xi],\ [\xi,1]$. Integrating the equality (3.6) over these segments, we obtain

$$\begin{aligned} &\sum_{j=0}^{2n-1}\Big\{(-1)^j\Big[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,s)\Big) \\ &\qquad -\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big]\Big\}\Big|_{x=0}^{x=s-0} \\ &+\sum_{j=0}^{2n-1}\Big\{(-1)^j\Big[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,s)\Big) \\ &\qquad -\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big]\Big\}\Big|_{x=s+0}^{x=\xi-0} \\ &+\sum_{j=0}^{2n-1}\Big\{(-1)^j\Big[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,s)\Big) \\ &\qquad -\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big]\Big\}\Big|_{x=\xi+0}^{x=1} = 0. \end{aligned}$$

If we consider the properties 1 and 4 of the Green's function $G(x,s)$, then the last equality takes the form:

$$-\Big[G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\Big)\Big]\Big|_{x=s-0}^{x=s+0}+\Big[G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big]_{x=s-0}^{x=s+0}$$
$$-\Big[G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\Big)\Big]_{x=\xi-0}^{x=\xi+0}+\Big[G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big]\Big|_{x=\xi-0}^{x=\xi+0}=0.$$

According to the property 2 of the function $G(x,\eta)$, the derivative of $\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^\alpha\left(\partial^{2n}/\partial x^{2n}\right)G(x,\eta)\right]$ is continuous at $x\neq\eta$. Therefor we have the equality

$$\Big[G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\Big)\Big|_{x=s-0}-G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\Big)\Big|_{x=s+0}\Big]$$
$$+\Big[G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big|_{x=\xi+0}-G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big|_{x=\xi-0}\Big]=0.$$

Hence, by virtue of equality (3.4), the equality

$$-G(s,\xi)+G(\xi,s)=0,$$

follows, which we need to prove.

In the special case when $n=1$, the Green's function $G\left(x,s\right)$ takes the following form:

$$G\left(x,s\right)=\begin{cases}\dfrac{sx^{3-\alpha}}{(2-\alpha)_2}+\dfrac{sx^{2-\alpha}}{(1-\alpha)_2}+\Big(\dfrac{s^{3-\alpha}}{(2-\alpha)_2}+\dfrac{s}{3-\alpha}+\dfrac{1}{3-\alpha}\Big)\left(x+1\right), & 0\le x\le s,\\[2mm] \dfrac{xs^{3-\alpha}}{(2-\alpha)_2}+\dfrac{xs^{2-\alpha}}{(1-\alpha)_2}+\Big(\dfrac{x^{3-\alpha}}{(2-\alpha)_2}+\dfrac{x}{3-\alpha}+\dfrac{1}{3-\alpha}\Big)\left(s+1\right), & s\le x\le 1.\end{cases}$$

Now, applying the method used in [11], it is easy to verify that problem $\{(3.1),(3.2)\}$ is equivalent to study of the following integral equation

$$v(x)=\lambda\int_0^1 G(x,s)v(s)ds. \tag{3.7}$$

Since the kernel is continuous, symmetric and positive, the integral equation (3.7), and therefore, the problem $\{(3.1),(3.2)\}$ both have a countable set of eigenvalues

$$0<\lambda_1\le\lambda_2\le\lambda_3\le\ldots\le\lambda_k<\ldots,\quad \lambda_k\to+\infty,$$

and the corresponding system of eigenfunctions $v_1(x)$, $v_2(x)$, $v_3(x)$, $\ldots$, $v_k(x)\ldots$ forms an orthonormal system in the space $L_2(0,1)$ [10].

In addition, it is not difficult to verify that the system of functions $x^{\alpha/2}v_k^{(2n)}(x)/\sqrt{\lambda_k}$, $k=1,2,\ldots$ also forms an orthonormal system in $L_2(0,1)$.

Lemma 1. *Let the function $g(x)$ satisfy the conditions (3.2) and $Mg(x)\in C(0,1)\cap L_2(0,1)$. Then, $g(x)$ can be expanded on the segment $[0,1]$ into the absolutely and uniformly convergent series in the system of eigenfunctions of the problem $\{(3.1),(3.2)\}$.*

P r o o f. Using the integration by parts rule, the properties of the Green's function $G(x,s)$, and the conditions imposed on the function $g(x)$, it is straightforward to verify the equality:

$$\int_0^1 G(x,s)Mg(s)ds=\int_0^1 G(x,s)\left[s^\alpha g^{(2n)}(s)\right]^{(2n)}ds=g(x).$$

Since $Mg(x) \in L_2(0,1)$, it follows from the last equality that $g(x)$ is a function representable through the kernel $G(x,s)$. Additionally, the function $G(x,s)$, i.e. the kernel of equation (3.7), is continuous in $\bar{\Omega}$. Then, based on Theorem 2 in [10, p. 153], the statement of Lemma 1 holds true. □

Lemma 2. *The following series converge uniformly on segment* $[0,1]$:

$$\sum_{k=1}^{+\infty}\left[v_k^{(j)}(x)\right]^2/\lambda_k, \quad \sum_{k=1}^{+\infty}\left(\left[x^\alpha v_k^{(2n)}(x)\right]^{(j)}\right)^2/\lambda_k^2, \quad j=\overline{0,2n-1} \tag{3.8}$$

P r o o f. Considering the equality (3.1) and the properties of the function $G(x,s)$, from (3.7) at $v(x) \equiv v_k(x)$, we obtain

$$v_k^{(j)}(x) = \lambda_k \int_0^1 \frac{\partial^j}{\partial x^j} G(x,s) v_k(s) ds = \int_0^1 \left[s^\alpha v_k^{(2n)}(s)\right]^{(2n)} \frac{\partial^j}{\partial x^j} G(x,s) ds, \quad j=\overline{0,2n-1}.$$

Hence, applying the rule of integration by parts $2n$ times, and then considering the conditions (3.2), we have

$$v_k^{(j)}(x) = \int_0^1 s^\alpha v_k^{(2n)}(s) \frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s) ds, \quad j=\overline{0,2n-1},$$

which, due to $\lambda_k > 0$, implies the equality

$$\frac{v_k^{(j)}(x)}{\sqrt{\lambda_k}} = \int_0^1 \left(s^{\alpha/2} \frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s)\right)\left(\frac{s^{\alpha/2} v_k^{(2n)}(s)}{\sqrt{\lambda_k}}\right) ds, \quad j=\overline{0,2n-1}. \tag{3.9}$$

From (3.9) it follows that $v_k^{(j)}(x)/\sqrt{\lambda_k}$ is the Fourier coefficient of the function by the orthonormal system

$$\left\{s^{\alpha/2} v_k^{(2n)}(s)/\sqrt{\lambda_k}\right\}_{k=1}^{+\infty}.$$

Therefore, according to Bessel's inequality [10], we obtain

$$\sum_{k=1}^{+\infty}\left[v_k^{(j)}(x)\right]^2/\lambda_k \le \int_0^1 s^\alpha \left[\frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s)\right]^2 ds, \quad j=\overline{0,2n-1}. \tag{3.10}$$

The integral on the right-hand side (3.10) can be rewritten as

$$\int_0^1 s^\alpha \left[\frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s)\right]^2 ds = \int_0^1 s^{-\alpha}\left[\frac{\partial^j}{\partial x^j}\left(s^\alpha \frac{\partial^{2n}}{\partial s^{2n}} G(x,s)\right)\right]^2 ds, \quad j=\overline{0,2n-1}.$$

Since

$$s^\alpha \frac{\partial^{2n} G(x,s)}{\partial s^{2n}}, \ \frac{\partial^j G(x,s)}{\partial x^j} \in C(\bar{\Omega}), \quad j=\overline{0,2n-1},$$

the function in the square bracket is continuous on $\bar{\Omega}$. Then, due to $0<\alpha<1$, the integral on the right-hand side, and therefore the integral in (3.10), is uniformly bounded at $j=\overline{0,2n-1}$, which implies that the first series in (3.8) converges uniformly.

The convergence of the remaining series can be proved similarly.

Lemma 2 has been proved. □

Lemma 3. *Let the conditions*

$$g^{(j)}(x) \in C[0,1], \quad j=\overline{0,2n-1}, \quad x^{\alpha/2}g^{(2n)}(x) \in C(0,1)\cap L_2(0,1);$$
$$g^{(2j)}(0)=g^{(2j+1)}(0), \quad g^{(2j)}(1)=g^{(2j+1)}, \quad j=\overline{0,n-1}$$

be fulfilled, then the inequality

$$\sum_{k=1}^{+\infty}\lambda_k g_k^2 \le \int_0^1 x^\alpha\left[g^{(2n)}(x)\right]^2 dx \tag{3.11}$$

holds true. Specifically, the series on the left-hand side converges, where

$$g_k=\int_0^1 g(x)v_k(x)dx, \quad k\in N.$$

P r o o f. By utilizing equation (3.1), we can write

$$\lambda_k^{1/2}g_k=\lambda_k^{1/2}\int_0^1 g(x)v_k(x)dx=\lambda_k^{-1/2}\int_0^1 g(x)\left[x^\alpha v_k^{(2n)}(x)\right]^{(2n)}dx.$$

Hence, by applying the integration by parts rule $2n$ times and considering the properties of the functions $g(x)$ and $v_k(x)$, we derive

$$\lambda_k^{1/2}g_k=\int_0^1\left\{x^{\alpha/2}g^{(2n)}(x)\right\}\left\{\lambda_k^{-1/2}x^{\alpha/2}v_k^{(2n)}(x)\right\}dx.$$

This implies that $\lambda_k^{1/2}g_k$ is the Fourier coefficient of the function $x^{\alpha/2}g^{(2n)}(x)$ by the orthonormal system $\left\{x^{\alpha/2}v^{(2n)}(x)/\sqrt{\lambda_k}\right\}_{k=1}^{+\infty}$. Therefore, according to Bessel's inequality [10], inequality (3.11) holds true. Lemma 3 has been proved. □

Lemma 4. *Let the function $g(x)$ satisfy the conditions (3.2) and let*

$$Mg(x)\in C(0,1)\cap L_2(0,1),$$

then the following inequality holds true

$$\sum_{k=1}^{+\infty}\lambda_k^2 g_k^2 \le \int_0^1 [Mg(x)]^2 dx. \tag{3.12}$$

Specifically, the series on the left side converges, where

$$g_k=\int_0^1 g(x)v_k(x)dx, \quad k\in N.$$

P r o o f. By virtue of the formula for g_k and equation (3.1), the equality

$$\lambda_k g_k=\lambda_k\int_0^1 g(x)v_k(x)dx=\int_0^1 g(x)\left[x^\alpha v_k^{(2n)}(x)\right]^{(2n)}dx$$

is valid.

Applying the rule of integration by parts $4n$ times to the integral on the right side and considering the properties of the functions $g(x)$ and $v_k(x)$, we get

$$\lambda_k g_k = \int_0^1 \left[x^\alpha g^{(2n)}(x)\right]^{(2n)} v_k(x)dx = \int_0^1 \left[Mg(x)\right] v_k(x)dx.$$

This implies that the value $\lambda_k g_k$ is the Fourier coefficient of the function $Mg(x)$ in the orthonormal system of functions $\{v_k(x)\}_{k=1}^{+\infty}$. Then, according to Bessel's inequality [10], inequality (3.12) holds true. Lemma 4 has been proved. □

Similarly to Lemma 3, one can prove the following

Lemma 5. *If the function $g(x)$ satisfies the conditions (3.2) and*

$$[Mg(x)]^{(j)} \in C[0,1], \quad j=\overline{0,2n-1}; \quad x^{\alpha/2}[Mg(x)]^{(2n)} \in C(0,1)\cap L_2(0,1);$$
$$[Mg(x)]^{(2j)}\big|_{x=0} = [Mg(x)]^{(2j+1)}\big|_{x=0}, \quad [Mg(x)]^{(2j)}\big|_{x=1} = [Mg(x)]^{(2j+1)}\big|_{x=1}, \quad j=\overline{0,n-1},$$

then the inequality

$$\sum_{k=1}^{+\infty} \lambda_k^3 g_k^2 \le \int_0^1 x^\alpha \left\{[Mg(x)]^{(2n)}\right\}^2 dx$$

holds true, particularly, the series on the left side converges, where

$$g_k = \int_0^1 g(x)v_k(x)dx, \quad k\in N.$$

4. Existence, uniqueness and stability of a solution to Problem A

We will seek a solution to problem A in the form

$$u(x,t) = \sum_{k=1}^{+\infty} u_k(t)v_k(x), \tag{4.1}$$

where $v_k(x)$, $k\in N$ are the eigenfunctions of the problem $\{(3.1),(3.2)\}$, and $u_k(t)$, $k\in N$ are the unknown functions to be determined.

Substituting (4.1) into equation (2.1) and the initial conditions (2.2), with respect to $u_k(t)$, $k\in N$, we obtain the following problem

$$u_k''(t) + \lambda_k u_k(t) = f_k(t), \quad t\in(0,T), \quad k\in N,$$
$$u_k(0)=\varphi_{1k}, \quad u_k'(0)=\varphi_{2k},$$

where

$$\varphi_{jk} = \int_0^1 \varphi_j(x)v_k(x)dx, \quad j=\overline{1,2}; \quad f_k(t) = \int_0^1 f(x,t)v_k(x)dx, \quad k\in N.$$

It is known that the solution to the last problem exists, is unique and is determined by the following formula:

$$u_k(t) = \varphi_{1k}\cos\left(t\sqrt{\lambda_k}\right) + \varphi_{2k}\lambda_k^{-1/2}\sin\left(t\sqrt{\lambda_k}\right) + \lambda_k^{-1/2}\int_0^t f_k(\tau)\sin\left[(t-\tau)\sqrt{\lambda_k}\right]d\tau, \quad 0\le t\le T. \tag{4.2}$$

From here, the following estimate

$$|u_k(t)| \le |\varphi_{1k}| + \frac{1}{\sqrt{\lambda_k}}|\varphi_{2k}| + \frac{1}{\sqrt{\lambda_k}}\sqrt{\int_0^T f_k^2(\tau)d\tau}, \quad 0\le t\le T \tag{4.3}$$

easily follows.

Theorem 1. *Let the function $\varphi_1(x)$ satisfy the conditions of Lemma 5, the function $\varphi_2(x)$ satisfy the conditions of Lemma 4, and the function $f(x,t)$ satisfy the conditions of Lemma 4 with respect to the argument x uniformly in t. Then series (4.1), the coefficients of which are defined by the equalities (4.2), determines the solution to problem A.*

P r o o f. To do this, it is necessary to prove the uniform convergence in $\bar{\Omega}$ of series (4.1) and the following series, formally obtained from (4.1):

$$u_t(x,t) = \sum_{k=1}^{+\infty} u_k'(t) v_k(x),$$

$$\frac{\partial^j u(x,t)}{\partial x^j} = \sum_{k=1}^{+\infty} u_k(t) v_k^{(j)}(x), \quad j = \overline{1, 2n-1},$$

$$\frac{\partial^j}{\partial x^j}\Big(x^\alpha \frac{\partial^{2n} u(x,t)}{\partial x^{2n}}\Big) = \sum_{k=1}^{+\infty} u_k(t) \big(x^\alpha v_k^{(2n)}(x)\big)^{(j)}, \quad j = \overline{0, 2n-1}$$

and uniform convergence in any compact set of $\Omega_0 \subset \Omega$ the series

$$\frac{\partial^{2n}}{\partial x^{2n}}\Big(x^\alpha \frac{\partial^{2n} u(x,t)}{\partial x^{2n}}\Big) = \sum_{k=1}^{+\infty} u_k(t) \big(x^\alpha v_k^{(2n)}(x)\big)^{(2n)}, \tag{4.4}$$

$$u_{tt}(x,t) = \sum_{k=1}^{+\infty} u_k''(t) v_k(x). \tag{4.5}$$

Let us consider series (4.1). By virtue of (4.3) from (4.1), for any $(x,t) \in \bar{\Omega}$ we have

$$|u(x,t)| \le \sum_{k=1}^{+\infty} |u_k(t)|\,|v_k(x)| \le \sum_{k=1}^{+\infty} \frac{|v_k(x)|}{\sqrt{\lambda_k}} \left(\sqrt{\lambda_k}\,|\varphi_{1k}| + |\varphi_{2k}| + \sqrt{\int_0^T f_k^2(\tau) d\tau} \right).$$

From here, applying the Cauchy–Schwarz inequality, we obtain

$$|u(x,t)| \le \sqrt{\sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k}} \left(\sqrt{\sum_{k=1}^{+\infty} \lambda_k \varphi_{1k}^2} + \sqrt{\sum_{k=1}^{+\infty} \varphi_{2k}^2} + \sqrt{\int_0^T \sum_{k=1}^{+\infty} [f_k(\tau)]^2 \, d\tau} \right). \tag{4.6}$$

The series on the right-hand sides of this inequality, due to the conditions of Theorem 1, according to Lemmas 2 and 3, converges uniformly. Therefore, the series on the left side, i.e. series (4.1), converges uniformly in $\bar{\Omega}$.

Now, we consider the series (4.4). By virtue of equation (3.1), in any compact set Ω_0 the series in (4.4) may be written in the form

$$\sum_{k=1}^{+\infty} \lambda_k u_k(t) v_k(x). \tag{4.7}$$

To prove the uniform convergence of series (4.7), according to (4.3), it is enough to prove the absolute and uniform convergence of the series

$$\sum_{k=1}^{+\infty} \lambda_k \varphi_{1k} v_k(x), \quad \sum_{k=1}^{+\infty} \sqrt{\lambda_k} \varphi_{2k} v_k(x), \quad \sum_{k=1}^{+\infty} \sqrt{\lambda_k} \sqrt{\int_0^T [f_k(\tau)]^2 \, d\tau v_k(x)}. \tag{4.8}$$

In Ω_0, we apply the Cauchy-Schwarz inequality to each of these series:

$$\Big|\sum_{k=1}^{+\infty}\lambda_k\varphi_{1k}v_k(x)\Big| \le \sum_{k=1}^{+\infty}\Big|\sqrt{\lambda_k^3}\varphi_{1k}\frac{v_k(x)}{\sqrt{\lambda_k}}\Big| \le \Big[\sum_{k=1}^{+\infty}\lambda_k^3\varphi_{1k}^2\sum_{k=1}^{\infty}\frac{v_k^2(x)}{\lambda_k}\Big]^{1/2},$$

$$\Big|\sum_{k=1}^{+\infty}\sqrt{\lambda_k}\varphi_{2k}v_k(x)\Big| \le \sum_{k=1}^{+\infty}\Big|\lambda_k\varphi_{2k}\frac{v_k(x)}{\sqrt{\lambda_k}}\Big| \le \Big[\sum_{k=1}^{+\infty}\lambda_k^2\varphi_{2k}^2\cdot\sum_{k=1}^{\infty}\frac{v_k^2(x)}{\lambda_k}\Big]^{1/2},$$

$$\Big|\sum_{k=1}^{+\infty}\sqrt{\lambda_k}\sqrt{\int_0^T[f_k(\tau)]^2d\tau}\cdot v_k(x)\Big| \le \sum_{k=1}^{+\infty}\Big|\sqrt{\lambda_k^2\int_0^T[f_k(\tau)]^2d\tau}\cdot\frac{v_k(x)}{\sqrt{\lambda_k}}\Big|$$

$$\le \Big[\int_0^T\sum_{k=1}^{+\infty}\lambda_k^2[f_k(\tau)]^2d\tau\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\Big]^{1/2}.$$

The series on the right-hand sides of these inequalities, due to the conditions of Theorem 1, according to Lemmas 2, 4 and 5, converges uniformly. Then the series located on the left sides, i.e. series (4.8) converges absolutely and uniformly in Ω_0. Therefore, the series (4.7), and therefore the series in (4.4), converges uniformly in the compact set Ω_0. The uniform convergence in Ω_0 of series (4.5) follows from the convergence of series (4.4) and the validity of equation (2.1).

The uniform convergence of the remaining series is similarly proved. Theorem 1 has been proved. □

Theorem 2. *A problem a cannot have more than one solution.*

P r o o f. Let us assume that there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ of problem A. We denote their difference by $u(x,t)$. Then the function $u(x,t)$ satisfies the equation (2.1) for $f(x,t)\equiv 0$, and conditions (2.2) and (2.3) for $\varphi_1(x)\equiv\varphi_2(x)\equiv 0$.

Let $\forall\, T_0\in(0,T]$,

$$\Omega_0=\{(x,t):0<x<1,\ 0<t<T_0\}.$$

It is obvious that $\bar\Omega_0\subset\bar\Omega$. Let us introduce the following function:

$$\omega(x,t)=-\int_t^{T_0}u(x,\xi)d\xi,\quad (x,t)\in\bar\Omega_0.$$

This function has the following properties:

1) $\omega_t,\,\omega_{tt},\,\dfrac{\partial^j\omega}{\partial x^j},\,\dfrac{\partial^j}{\partial x^j}\left(x^\alpha\dfrac{\partial^{2n}\omega}{\partial x^{2n}}\right)\in C\left(\bar\Omega_0\right),\quad j=\overline{0,2n-1};$

2) it satisfies the conditions (2.3) at $t\in[0,T_0]$.

Let us consider the equation (2.1) for $f(x,t)\equiv 0$ and multiply it by the function $\omega(x,t)$, and then integrate the resulting equality over the domain Ω_0 :

$$\int_{\Omega_0}\omega(x,t)\Big\{u_{tt}(x,t)+\frac{\partial^{2n}}{\partial x^{2n}}\Big[x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\Big]\Big\}dtdx=0.$$

We rewrite this equality as

$$\int_0^{T_0}dt\int_0^1\omega(x,t)\frac{\partial^{2n}}{\partial x^{2n}}\Big[x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\Big]dt+\int_0^1dx\int_0^{T_0}\omega(x,t)u_{tt}(x,t)dt=0.$$

Now, applying the rule of integration by parts, we obtain

$$\int_0^{T_0}\left[\omega(x,t)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)-\frac{\partial\omega(x,t)}{\partial x}\frac{\partial^{2n-2}}{\partial x^{2n-2}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)+\ldots\right.$$
$$\left.+\ldots-\frac{\partial^{2n-1}\omega(x,t)}{\partial x^{2n-1}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)\right]_{x=0}^{x=1}dt+\int_0^{T_0}dt\int_0^1 x^\alpha\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}dx+$$
$$+\int_0^1\left[\omega(x,t)\frac{\partial u(x,t)}{\partial t}\bigg|_{t=0}^{t=T_0}-\int_0^{T_0}\frac{\partial\omega(x,t)}{\partial t}\frac{\partial u(x,t)}{\partial t}\right]dx=0,$$

from which, due to the properties of functions $\omega(x,t)$ and $u(x,t)$, the equality

$$\int_0^{T_0}dt\int_0^1 x^\alpha\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}dx-\int_0^1 dx\int_0^{T_0}\frac{\partial\omega(x,t)}{\partial t}\frac{\partial u(x,t)}{\partial t}dt=0$$

follows.

Hence, taking into account equalities

$$u=\frac{\partial\omega}{\partial t},\quad \frac{\partial^{2n}u}{\partial x^{2n}}=\frac{\partial^{2n+1}\omega}{\partial x^{2n}\partial t},$$

we have

$$\int_0^1 x^\alpha dx\int_0^{T_0}\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n+1}\omega(x,t)}{\partial x^{2n}\partial t}dt-\int_0^1 dx\int_0^{T_0}u(x,t)\frac{\partial u(x,t)}{\partial t}dt=0.$$

Further, taking into account the equalities

$$u(x,t)\frac{\partial u(x,t)}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}[u(x,t)]^2,\quad \frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n+1}\omega(x,t)}{\partial x^{2n}\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\right]^2,$$

and applying the rule of integration by parts to integrals over t, taking into account $\omega\,(x,T_0)=0$, $u(x,0)=0$, we obtain

$$\int_0^1 u^2\,(x,T_0)\,dx+\int_0^1 x^\alpha\left[\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\right]^2_{t=0}dx=0.$$

It follows that $u\,(x,T_0)\equiv 0,\ x\in[0,1]$. Since we considered $\forall\,T_0\in[0,T]$, then $u(x,t)\equiv 0$, $(x,t)\in\bar{\Omega}$. Then $u_1(x,t)\equiv u_2(x,t),\ (x,t)\in\bar{\Omega}$. Theorem 2 is proven. □

Theorem 3. *Let functions $\varphi_1(x)$, $\varphi_2(x)$ and $f(x,t)$ satisfy the conditions of Theorem 1. Then for the solution of Problem A the following estimates*

$$\|u(x,t)\|^2_{L_2(0,1)}\le K_0\big[\|\varphi_1(x)\|^2_{L_2(0,1)}+\|\varphi_2(x)\|^2_{L_2(0,1)}+\|f(x,t)\|^2_{L_2(\Omega)}\big],\tag{4.9}$$
$$B\|u(x,t)\|_{C(\Omega)}\le K_1\big[\|\varphi_1^{(2n)}(x)\|_{L_{2,r}(0,1)}+\|\varphi_2(x)\|_{L_2(0,1)}+\|f(x,t)\|_{L_2(\Omega)}\big],\tag{4.10}$$

are valid, where

$$\|\varphi_1(x)\|_{L_{2,r}(0,1)}=\left[\int_0^1 x^\alpha\,[\varphi_1(x)]^2\,dx\right]^{1/2}$$

and $r=r(x)=x^\alpha$, and K_0 and K_1 are some real positive numbers.

P r o o f. Here, taking into account the orthonormality of the system $\{v_k(x)\}_{k=1}^{+\infty}$ and inequality (4.3) followed from (4.1), we obtain

$$\|u(x,t)\|^2_{L_2(0,1)} = \sum_{k=1}^{+\infty} u_k^2(t) \le \sum_{k=1}^{+\infty} \left[|\varphi_{1k}| + \frac{1}{\sqrt{\lambda_k}} |\varphi_{2k}| + \frac{1}{\sqrt{\lambda_k}} \|f_k(t)\|_{L_2(0,T)} \right]^2$$

$$\le 3 \sum_{k=1}^{+\infty} \left[\varphi_{1k}^2 + \frac{1}{\lambda_k} \varphi_{2k}^2 + \frac{1}{\lambda_k} \|f_k(t)\|^2_{L_2(0,T)} \right] \le 3 \sum_{k=1}^{+\infty} \left[\varphi_{1k}^2 + \frac{1}{\lambda_1} \varphi_{2k}^2 + \frac{1}{\lambda_1} \|f_k(t)\|^2_{L_2(0,T)} \right].$$

Hence, considering Bessel's inequality, we get

$$\|u(x,t)\|^2_{L_2(0,1)} \le K_0 \Big(\|\varphi_1(x)\|^2_{L_2(0,1)} + \|\varphi_2(x)\|^2_{L_2(0,1)} + \sum_{k=1}^{+\infty} \|f_k(t)\|^2_{L_2(0,T)} \Big), \tag{4.11}$$

where $K_0 = 3C,\ C = \max(1, 1/\lambda_1)$.

Taking into account the following easily verifiable equality

$$\|f(x,t)\|^2_{L_2(\Omega)} = \sum_{n=1}^{+\infty} \|f_k(t)\|^2_{L_2(0,T)},$$

from (4.11), we obtain inequality (4.9).

Further, according to the statements of Lemmas 2 and 3, from (4.6) it follows

$$\|u(x,t)\|_{C(\bar{)}} = \sup_{\Omega} |u(x,t)| \le K_1 \left\{ \sqrt{\int_0^1 x^\alpha [\varphi_1^{(2n)}(x)]^2 dx} + \sqrt{\sum_{k=1}^{+\infty} \varphi_{2k}^2} + \sqrt{\int_0^{T+\infty} \sum_{k=1}^{+\infty} [f_k(\tau)]^2 \, d\tau} \right\},$$

where

$$K_1 = \sup_{[0,1]} \sqrt{\sum_{k=1}^{+\infty} v_k^2(x)/\lambda_k}\,.$$

From here, due to the introduced notation, inequality (4.10) follows. Theorem 3 has been proved. □

5. Conclusion

In a quadrilateral, an initial boundary-value problem has been considered for a high-order partial differential equation that degenerates at the boundary of the domain. The uniqueness of the solution to the problem was proved by the method of energy integrals. The solution to the problem was found in the form of a Fourier series. The sufficient conditions for the given functions have been identified that ensure the existence of a solution to the problem. The estimates for the solution of the problem in spaces $L_2[0,1]$ and $C[0,1]$ have been obtained.

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