URAL MATHEMATICAL JOURNAL, Vol. 11, No. 1, 2025, pp. 104–113

DOI: 10.15826/umj.2025.1.007

A STUDY ON PERFECT ITALIAN DOMINATION OF GRAPHS AND THEIR COMPLEMENTS

Agnes Poovathingal[a,b,†], **Joseph Varghese Kureethara**[a,c,††]

[a]Christ University,
Bangalore-560029, Karnataka, India

[b]Christ College (Autonomous),
Christ Nagar, Irinjalakuda, Kerala 680125, India

[c]Kuriakose Elias College,
Kottayam, Mannanam, Kerala 686561, India

[†]agnes.poovathingal@res.christuniversity.in [††]frjoseph@christuniversity.in

Abstract: Perfect Italian Domination is a type of vertex domination which can also be viewed as a graph labelling problem. The vertices of a graph G are labelled by $0, 1$ or 2 in such a way that a vertex labelled 0 should have a neighbourhood with exactly two vertices in it labelled 1 each or with exactly one vertex labelled 2. The remaining vertices in the neighbourhood of the vertex labelled 0 should be all 0's. The minimum sum of all labels of the graph G satisfying these conditions is called its Perfect Italian domination number. We study the behaviour of graph complements and how the Perfect Italian Domination number varies between a graph and its complement. The *Nordhaus–Gaddum type* inequalities in the Perfect Italian Domination number are also discussed.

Keywords: Perfect Italian domination, Graph complement, Nordhaus–Gaddum type inequalities.

1. Introduction

Analysing how graph properties vary across each graph family is always fascinating. That is the manner in which a graph's structural characteristics, such as its number of vertices, edges, connectivity, symmetry, etc., affect graph parameters such as its chromatic number, clique number, domination number, etc. The variation of a graph parameter between a graph and its complement has also been researched since the seminal work of Nordhaus and Gaddum [7]. On n-vertex graphs, they determined an upper and lower bound for the sum (and product) of chromatic numbers of a graph and its complement. The problems that include determining the upper and lower bounds of the sum or product of certain graph properties are referred to as *Nordhaus–Gaddum type* studies.

Perfect Italian Domination is a domination concept defined by T.W. Haynes and M.A. Henning. It can be viewed as a vertex labelling problem, where vertices are labelled by $0, 1$ or by 2. A vertex in a Perfect Italian Dominated (PID) graph is labelled 0 if and only if it is adjacent to two vertices labelled 1 each or one vertex labelled 2, and the remaining vertices in its neighbourhood are labelled 0. The sum of the vertex labels on a graph G that satisfies the PID condition is determined and the term *PID number* of G denoted as $\gamma_I^p(G)$ refers to the smallest sum that may be computed for a graph G [5].

The graph $\overline{G}$ is called the complement of a graph G, when two vertices are neighbours in G if and only if they are not neighbours in $\overline{G}$. In this paper, we examine the variation in the Perfect Italian Domination (PID) number of a graph and its complement. We find some *Nordhaus–Gaddum type* inequalities of Perfect Italian Domination number and, also characterise some graph classes which attain the upper bound and lower bound. We have also considered a few graph classes whose PID numbers are found and are compared with the PID numbers of their complements.

2. PID on graph complements and Nordhaus–Gaddum inequalities

The Perfect Italian domination number of any graph G is at least two and is at most its order. Hence, for a graph G of order n,

$$4 \leq \gamma_I^p(G) + \gamma_I^p(\overline{G}) \leq 2n.$$

In this paper, we prove that these bounds are tight by constructing classes of graphs. The gap between the bounds is shortened when a few restrictions are made to the graphs considered. We consider a few cases where the upper bound is small. We arrive at a conclusion that if G is any graph such that $\gamma_I^p(G) = n$, then $\gamma_I^p(\overline{G}) \geq 5$ or equal to 2. If G is a connected graph, then $\gamma_I^p(\overline{G}) \geq 5$. We have also determined the PID number of certain graph cases and their complements. This helps in the study of determining the criteria that the graph must satisfy in order to maximise or reduce a graph PID value. This study can help us find extremal graphs which is an important area of study in graph theory. Some of these will also would lead to optimal solutions.

We examine graphs that correspond to a specific PID number and analyze the PID number of its complement. We will start by considering graphs G with $\gamma_I^p(G) = 2, 3, 4$ and later $\gamma_I^p(G) \geq 5$.

The only possible graphs of order $n = 2$ are $2K_1$ and K_2. We know that PID number of each of them is 2 and they are complement to each other. When $n \geq 3$, $\gamma_I^p(G) = 2$ if and only if there is a universal vertex or if there exist two non adjacent vertices adjacent to all the remaining vertices of G. A universal vertex of G forms an isolated vertex in $\overline{G}$. Similarly, the non adjacent vertices adjacent to all the remaining vertices in G form a K_2 component. Hence when $n \geq 3$ if $\gamma_I^p(G) = 2$, then $\gamma_I^p(\overline{G})$ is always greater than or equal to 3.

Let G be any graph of order n and $\gamma_I^p(G) = 2$. Then $\overline{G}$ is a disconnected graph with

$$2 \leq \gamma_I^p(G) \leq n.$$

The following realization problem shows that for any integer $2 \leq a \leq n$, we can find a graph such that its PID number is 2 whereas the PID number of its complement is a.

Theorem 1. *For any $a \in \mathbb{N} - \{1\}$, there exists a graph G such that $\gamma_I^p(G) = 2$ and $\gamma_I^p(\overline{G}) = a$.*

Proof. Let G be a graph obtained from the join of a path complement graph- $\overline{P}_{2a-3}$ and K_1, $(\overline{P}_{2a-3} + K_1)$, where (see [8])

$$\gamma_I^p(\overline{P}_{2a-3} + K_1) = 2.$$

Then $\overline{G}$ will be $P_{2a-3} \cup K_1$. For any path P_n, (see [6])

$$\gamma_I^p(P_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Hence,

$$\gamma_I^p(\overline{G}) = \gamma_I^p(P_{2a-1} \cup K_1) = \left\lceil \frac{2a-3+1}{2} + 1 \right\rceil = a.$$

□

Proposition 1. *Let G be a graph such that $\gamma_I^p(G) = 3$. Then $\gamma_I^p(\overline{G}) \leq 6$.*

P r o o f. A graph G with $\gamma_I^p(G) > 2$ has $\gamma_I^p(G) = 3$ if and only if $\overline{G}$ has a perfect dominating set of size 3 [6]. This implies that $\gamma_I^p(\overline{G}) \leq 6$. □

From the above results it is clear that $\gamma_I^p(G) = 3$ and $\gamma_I^p(\overline{G}) = 2$ if and only if G is a disconnected graph.

Corollary 1. *Let G be a connected graph such that $\gamma_I^p(G) = 3$. Then $3 \leq \gamma_I^p(\overline{G}) \leq 6$.*

Proposition 2. *Let G be a graph such that $\gamma_I^p(G) = 4$. Then $\gamma_I^p(\overline{G}) \leq 4$.*

P r o o f. If G is a graph such that $\gamma_I^p(G) = 4$, then either of the following is true.

1) There exists a vertex set S in G consisting of four vertices $\{u_i\}$ for $i = 1, 2, 3, 4$ such that the remaining vertices in G are adjacent to exactly any two vertices of the set S.
2) There exists a set S in G consisting of two vertices, u_1, u_2 such that the remaining vertices in G are adjacent to exactly any one vertex of the set S.
3) There exists a set S in G consisting of three vertices, u_1, u_2, u_3 such that any other vertex, v belonging to G satisfies one of the following:
   (a) $N(v) \cap S = \{u_1\}$
   (b) $N(v) \cap S = \{u_2, u_3\}$.

If G satisfies 1), then the vertices belonging to $N(u_i) \cap N(u_j)$ in G will not be adjacent to u_i, u_j in $\overline{G}$, but will be adjacent to u_k where $k \neq i, j$. Hence labelling all the u_i's by 1 and the remaining vertices by 0 satisfies the PID condition. Thus, $\gamma_I^p(\overline{G}) \leq 4$.

If the graph G satisfies 2), then the vertices adjacent to $u_1 \in G$ are not adjacent to $u_1 \in \overline{G}$ but will be adjacent to u_2. Similar is the case of neighbours of u_2. Hence labelling u_1, u_2 by 2 and the remaining vertices by 0 satisfies the PID condition, i.e., $\gamma_I^p(\overline{G}) \leq 4$.

If G satisfies 3), then the vertices belonging to $N(u_1)$ in G are not adjacent to u_1 but are adjacent to u_2, u_3 in $\overline{G}$. Similarly the vertices belonging to $N(u_2) \cup N(u_3)$ are not adjacent to u_2, u_3 but are adjacent to u_1. Hence labelling u_1 by 2 and u_2, u_3 by 1 gives a PID labelling, i.e., $\gamma_I^p(\overline{G}) \leq 4$. □

Corollary 2. *Let G be a connected graph such that $\gamma_I^p(G) = 4$. Then $\gamma_I^p(\overline{G}) = 3$ or 4.*

If G is a connected graph with a PID number greater than or equal to 7, then from the above results, PID number of $\overline{G}$ cannot be $2, 3$ or 4. This implies that PID number of $\overline{G}$ is greater than or equal to 5 but less than or equal to the order of G.

The following realisation problem shows that the upper bound is tight.

Theorem 2. *For any $k \geq 5$, there exists a graph G of order n such that $\gamma_I^p(G) = k$ and $\gamma_I^p(\overline{G}) = n$.*

P r o o f. Let G be a graph constructed by the following steps:

Take k copies of P_4 where k is any integer greater than or equal to 5. Label each path as $Q_1, Q_2, ..., Q_k$. Let us consider a K_k whose vertices are $u_1, u_2, ..., u_k$. Then make each vertex of the path Q_i adjacent to u_i, u_{i+1} where $i = 1, 2, ..., (k-1)$. The vertices of Q_k are adjacent to u_1 and u_k. An illustration of the construction when $k = 5$ is given in Figure 1. This is a connected graph of order $5k$.

Since each vertex of the path P_i is adjacent to exactly two vertices among the $u_i's$, labelling all the $u_i's$ 1 and the vertices belonging to the paths 0 gives a PID labelling where

$$\gamma_I^p(G) \leq k \longrightarrow (a).$$

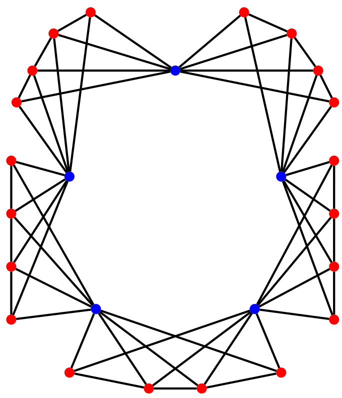

Figure 1. An illustration of construction of Graph G, where $k = 5$.

Obviously, degree of u_i is 8 which coincides with $\Delta(G)$. But from [3], we have

$$\gamma_I^p(G) \geq \gamma_I(G) \geq \frac{2(5k)}{\Delta(G)+2}, \quad \text{i. e.,} \quad \gamma_I^p(G) \geq k \longrightarrow (b).$$

From (a) and (b), $\gamma_I^p(G) = k$.

Since $\{u_1, u_2 ... u_k\}$ is a set of independent vertices in G, they induce a clique K_k in $\overline{G}$. As P_4 is a self-complementary graph, each Q_i remains the same in $\overline{G}$. Each vertex u_i is adjacent to the vertices of all the paths except P_{i-1}, P_i $j \neq i-1, i$ and $i, j = 2, 3, \ldots k$. The vertex u_1 is adjacent to the vertices of all the paths except P_k and P_1. Each vertex of the path P_i will be adjacent to all the vertices of the paths P_j where $j \neq i$ and $i, j = 1, 2, 3 \ldots k$.

Since G and $\overline{G}$ are connected graphs, $\gamma_I^p(\overline{G}) > 2$. Let us consider the following cases of possible labellings for $\overline{G}$:

1. Let a vertex v_i belonging to a path Q_s be labelled 0. Then, at most two vertices in its neighbourhood, say x, y, are non-zero labelled and the remaining vertices in its neighbourhood are zero labelled. Since each vertex in a path is of degree at least $5k-5$, there exist two vertices among the u_i's and at most two vertices in the path Q_s that are non-adjacent to the vertex v_i. If any one among this, say z is non zero labelled, then there exists at least one vertex on a path Q_i labelled 0 adjacent to x, y and z. This violates the perfect Italian domination condition. This implies that no vertex among the non adjacent vertices of v_i can be non-zero labelled. Hence, all remaining vertices in the graph are labelled 0. This contradicts $\gamma_I^p(\overline{G}) > 2$. Hence, no vertex on the path Q_i can be labelled 0 and its non adjacent vertices can be non-zero labelled. The remaining vertices in the graph are labelled 0. Since each vertex in a path is of degree of at least $5k-5$, there exist two vertices among the $u_i's$ and at most 2 vertices in the path Q_s that are non adjacent to the vertex v_i. If any one among this is non zero labelled, then there exists at least one vertex labelled 0 among the paths P_j where $j \neq k$ adjacent to all the vertices not labelled zero. This is a contradiction to the PID condition. Hence no vertex on an induced path P_i of the G can be labelled 0.

2. Each vertex u_i is adjacent to all the vertices of $k-2$ induced paths. From the above case we know that no vertex on an induced path of the graph G is labelled 0. Since $k \geq 5$, this implies that no vertex u_i can be labelled 0.

This shows that no vertex in $\overline{G}$ can be labelled 0. i.e., $\gamma_I^p(\overline{G}) = 5k$, the order of graph G. □

The following is a summary of the results mentioned above.

Remark 1. Let G be a connected graph of order n,

1. If $\gamma_I^p(G) = 3$, then $\gamma_I^p(\overline{G}) \in \{3, 4, 5, 6\}$.
2. If $\gamma_I^p(G) = 4$, then $\gamma_I^p(\overline{G}) \in \{3, 4\}$.
3. If $\gamma_I^p(G) \in \{5, 6\}$, then $\gamma_I^p(\overline{G}) \in \mathbb{N} - \{1, 2, 4\}$.
4. If $\gamma_I^p(G) \geq 7$, then $5 \leq \gamma_I^p(\overline{G}) \leq n$.

Based on the results above, we can deduce the following *Nordhaus–Gaddum type inequalities.*

Remark 2. Let G be a connected graph of order $n \geq 3$ and $\gamma_I^p(G) = 3$. Then,

$$6 \leq \gamma_I^p(G) + \gamma_I^p(\overline{G}) \leq 9, \quad 9 \leq \gamma_I^p(G) \cdot \gamma_I^p(\overline{G}) \leq 18.$$

Remark 3. Let G be a connected graph of order $n \geq 3$ and $\gamma_I^p(G) = 4$. Then,

$$7 \leq \gamma_I^p(G) + \gamma_I^p(\overline{G}) \leq 8, \quad 12 \leq \gamma_I^p(G) \cdot \gamma_I^p(\overline{G}) \leq 16.$$

Remark 4. Let G be a connected graph of order $n \geq 3$ and $7 \leq \gamma_I^p(G) \leq n$. Then,

$$12 \leq \gamma_I^p(G) + \gamma_I^p(\overline{G}) \leq 2n, \quad 35 \leq \gamma_I^p(G) \cdot \gamma_I^p(\overline{G}) \leq n^2.$$

Remark 5. Let G and $\overline{G}$ be connected graphs of order n. Then

$$6 \leq \gamma_I^p(G) + \gamma_I^p(\overline{G}) \leq 2n, \quad 6 \leq \gamma_I^p(G) \cdot \gamma_I^p(\overline{G}) \leq n^2.$$

3. PID of some graph classes and their complements

A vertex in a graph G is said to be dominated if it is either belonging to or is adjacent to a vertex belonging to the Dominating set S of G. A Perfect Dominating set, S_p of a graph G is a set of vertices such that any vertex of G not belonging to this set is dominated by exactly one vertex from S_p. The least number of vertices that can exist in such a set S_p is called Perfect Domination number $\gamma_p(G)$. [4].

Theorem 3 [2]. *For a path P_n, the perfect domination number,*

$$\gamma_p(P_n) = \begin{cases} \frac{n}{3}, & n \equiv 0 \pmod 3, \\ \frac{n+1}{3}, & n \equiv 2 \pmod 3, \\ \frac{n+2}{3}, & n \equiv 1 \pmod 3. \end{cases}$$

Theorem 4 [1]. *For a cycle C_n, the perfect domination number,*

$$\gamma_p(C_n) = \begin{cases} \frac{n}{3}, & n \equiv 0 \pmod 3, \\ \left\lceil \frac{n}{3} \right\rceil, & n \equiv 1 \pmod 3, \\ \left\lfloor \frac{n}{3} \right\rfloor + 2, & n \equiv 2 \pmod 3. \end{cases}$$

Theorem 5 [6]. *Let G be a connected graph with $\gamma_I^p(G) > 2$. Then $\gamma_I^p(G) = 3$ if and only if $\overline{G}$ has a perfect dominating set of size 3.*

Theorem 6. *For a path P_n, $\gamma_I^p(P_n) = \lceil (n+1)/2 \rceil$ and*

$$\gamma_I^p(\overline{P}_n) = \begin{cases} 1, & n = 1, \\ 2, & n = 2, \\ 3, & 3 \le n \le 9, \\ n, & \textit{otherwise.} \end{cases}$$

P r o o f. For a path P_n, $\gamma_I^p(P_n) = \lceil (n+1)/2 \rceil$ [6].

1. For $n \ge 10$: The two end vertices of P_n are adjacent vertices of degree $(n-2)$ in $\overline{P}_n$ and the remaining vertices which are of degree 2 in P_n are of degree $n-3$ in $\overline{P}_n$. This implies that $\gamma_I^p(\overline{P}_n) > 2$.
   (a) If a vertex of degree $(n-2)$, say u_i, is labelled 0, then u_{i+1} can be non-zero labelled and a vertex x in the neighbourhood of u_i is labelled 2 (or two vertices x, y in its neighbourhood are labelled 1 each). This implies that all the remaining vertices are labelled 0. Since $n \ge 10$, and vertices are of degree at least $n-3$ there exists a zero labelled vertex adjacent to the vertices x, y, u_{i+1}. This is a contradiction to the PID condition. Hence u_{i+1} is not labelled zero but then this is a contradiction to $\gamma_I^p(\overline{P}_n) > 2$.
   (b) If a vertex of degree $(n-3)$, say u_i, is labelled 0, then at most two of its adjacent vertices say a, b are non zero labelled and at least $n-5$ vertices are labelled 0. In the previous case we proved that the vertices of degree $(n-2)$ cannot be labelled 0, since $n \ge 10$ there exists at least one vertex of degree $(n-2)$ in the neighbourhood of u_i. This implies that at least one among a, b say a is of degree $(n-2)$. Let u_{i-1}, u_{i+1} be the vertices not adjacent to u_i and if one among them say u_{i-1} is non zero labelled, then u_{i-1} is not adjacent to u_i and at most one more vertex. a is not adjacent to one vertex and b is not adjacent to at most two vertices. This implies that there exists at least $n-5-(1+1+2) = n-9$ vertices labelled 0 adjacent to a, b and u_{i-1}. This is a contradiction to the perfect Italian domination condition. This implies that neither u_{i-1} nor u_{i+1} can be non-zero labelled.
   This is a contradiction to $\gamma_I^p(\overline{P}_n) > 2$. Hence no vertex of degree $(n-3)$ can be labelled 0.

   Thus no vertex in $\overline{P}_n$ where $n \ge 10$ can be labelled by 0. This implies that $\gamma_I^p(\overline{P}_n) = n$.
2. For $n = 1$, the complement is a K_1. Hence $\gamma_I^p(\overline{P}_1) = 1$.
3. For $n = 2$, $\overline{P}_2$ is two isolated vertices and $\gamma_I^p(\overline{P}_2) = 2$.
4. Assume $3 \le n \le 9$. The graph $\overline{P}_3$ is $K_1 \cup K_2$ and the PID number is 3. The graph $\overline{P}_4$ is P_4 and the PID number is 3. Let $u_1 u_2 ... u_5$ be a P_5. Then $\{u_1, u_4, u_5\}$ is a perfect dominating set of size 3 and from the Theorem 5 we can conclude that $\gamma_I^p(\overline{P}_5) = 3$. Similarly the vertices $\{u_2, u_4, u_5\}$ is a perfect dominating set of a P_6, $u_1, u_2 ... u_6$. This implies that $\gamma_I^p(\overline{P}_6) = 3$ (from Theorem 5). For $n = 7, 8, 9$, $\gamma_p(P_n) = 3$ (from Theorem: 3), this implies that $\gamma_I^p(\overline{P}_n) = 3$ (from Theorem 5). Hence for $3 \le n \le 9$, $\gamma_I^p(\overline{P}_n) = 3$.

□

Theorem 7. *For a cycle C_n, $\gamma_I^p(C_n) = \lceil n/2 \rceil$ and*

$$\gamma_I^p(\overline{C}_n) = \begin{cases} 3, & n = 3, 5, 7, 9, \\ 4, & n = 4, 6, 8, \\ n, & \textit{otherwise.} \end{cases}$$

P r o o f. For a cycle C_n, $\gamma_I^p(C_n) = \lceil n/2 \rceil$ [6]. Since each vertex in C_n is of degree 2, the vertices of $\overline{C}_n$ are of degree $n-3$. This implies $\overline{C}_n$ is a $(n-3)$ regular graph and $\gamma_I^p(\overline{C}_n) > 2$.

1. Assume $n \geq 10$. If a vertex, v is labelled 0, then v is adjacent to $n-3$ vertices, say $u_1, u_2, u_3 ... u_{n-3}$, and is not adjacent to w_1, w_2. Among the $u_i's$ two vertices are labelled 1, say u_1, u_2 (or one vertex u_1 is labelled 2) and the remaining $(n-5)$ (or $(n-4)$) $u_i's$ are labelled 0. The vertex v is not adjacent to w_1, w_2, as $\gamma_I^p(\overline{C}_n) > 2$, at least one of them, say w_1, should be non-zero labelled.
   (a) If both w_1, w_2 are non-zero labelled, then at least $(n-6)$ zero labelled vertices are adjacent to each of them. Vertices u_1, u_2 are adjacent to at least $n-7$ vertices. Since $n \geq 10$, there exists at least one vertex adjacent to three non-zero labelled vertices. This is a contradiction to the PID condition.
   (b) If w_1 is non zero labelled and w_2 is zero labelled, then w_2 is adjacent to at least $n-5$ zero labelled vertices (as w_1 should be adjacent to w_2, it cannot be adjacent to one of the u_1, u_2, say u_2.) This implies that w_1 is adjacent to at least $n-6$ zero labelled vertices, u_1 is adjacent to $n-7$ vertices labelled 0 and u_2 is adjacent to $n-6$ zero labelled vertices. This means that there exists at least one zero labelled vertex adjacent to all the three non-zero labelled vertices. This is a contradiction to the PID condition.

   Thus no vertex in $\overline{C}_n$ can be labelled 0.
2. Assume $n = 3, 5, 7, 9$. The graph $\overline{C}_3$ is $3K_1$ and the PID number is 3. Perfect domination number of cycles C_n, where $n = 5, 7, 9$ is 3 (from the Theorem 4). This implies that $\gamma_I^p(\overline{C}_n) = 3$ (from the Theorem 5).
3. Assume $n = 4, 6, 8$. The graph $\overline{C}_4$ is $2K_2$ and the PID number is 4. When $\gamma_p(C_6) = 2$, it cannot have a perfect dominating set of size 3. This implies that $\gamma_I^p(\overline{C}_6) \neq 3$. Hence, $\gamma_p(C_8) = 4 \implies \gamma_I^p(\overline{C}_8) \neq 3$ (from the Theorems 4, 5). The Fig. 2 shows a PID labelling with γ_I^p value equals to 4. Hence, for $n = 4, 6, 8$, $\gamma_I^p(\overline{C}_n) = 4$.

□

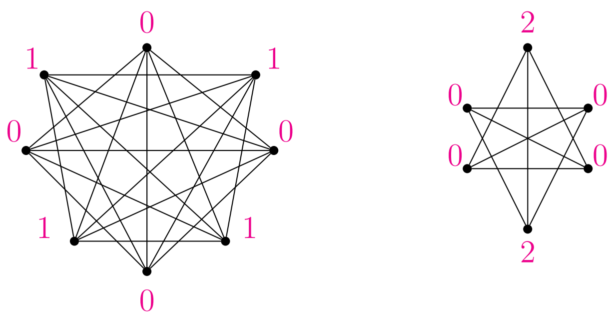

Figure 2. PID labelling of $\overline{C}_8, \overline{C}_6$.

Theorem 8. *Let G be a connected graph of order $n/2$. Then,*

$$\gamma_I^p(\overline{G \circ K_1}) = \begin{cases} 3, & G \cong C_3 \text{ or } P_3, \\ n, & \text{otherwise.} \end{cases}$$

P r o o f. Let the vertices of G be $u_1, u_2 ... u_{n/2}$ and the corresponding $K_1's$ be $v_1, v_2 ... v_{n/2}$. The $v_i's$ form a clique $K_{n/2}$ and each of these $v_i's$ will be adjacent to all the $u_j's$ such that $j \neq i$ for $i, j = 1, 2, 3, ..., n/2$.

Since G is a connected graph, $G \circ K_1$ has neither an isolated vertex nor a K_2. This implies that there exists neither a universal vertex nor two non-adjacent vertices adjacent to all the remaining vertices in $\overline{G \circ K_1}$. Thus, $\gamma_I^p(\overline{G \circ K_1}) > 2$ and degree of each vertex v_i belonging to the clique $K_{n/2}$ is $(n-1)$.

1. Assume any connected graph $G \ncong C_3$ or P_3, i.e., $n/2 \geq 4$.
   (a) If any vertex belonging to the clique $K_{n/2}$, say v_1, is labelled 0, then u_1 which is not adjacent to v_1 can be non-zero labelled and two vertices belonging to the neighbourhood of v_1 are labelled 1 each (or a vertex is labelled 2). This implies that all the remaining vertices of the graph is labelled 0. Since $n/2 \geq 4$, there exists a vertex belonging to the clique adjacent to all the three non-zero labelled vertices. This violates the PID condition, i.e., u_1 cannot be non-zero labelled. But this is a contradiction to $\gamma_I^p(\overline{G \circ K_1}) > 2$.
   (b) If a vertex u_i belonging to G is labelled 0, then it is adjacent to at least $n/2-1$ vertices belonging to the clique. From the above case it is clear that no vertex of K_k can be labelled 0, i.e., they are all non-zero labelled. A vertex u_i belonging to G is adjacent to at least $n/2-1$ vertices belonging to K_k. Hence, no vertex u_i belonging to G can be labelled 0.

   This implies that no vertex in $\overline{G \circ K_1}$ can be labelled 0. Hence, $\gamma_I^p(\overline{G \circ K_1}) = 2 \times n/2 = n$.
2. Assume $G \cong C_3$ or P_3. Labelling all the three vertices $v_i's$ 1 and all the $u_i's$ 0 gives a PID labelling, i.e., $\gamma_I^p(\overline{G \circ K_1}) \leq 3$. Since $\gamma_I^p(\overline{G \circ K_1}) > 2$, we can conclude that $\gamma_I^p(\overline{G \circ K_1}) = 3$.

□

Remark 6. Let G be a graph with an isolated vertex v. Then $\gamma_I^p(\overline{G \circ K_1}) = 2$ since $v \in G$ and its corresponding pendant vertices in $G \circ K_1$ are non-adjacent vertices of degree $n-2$ in $\overline{G \circ K_1}$.

Remark 7. Let G be a complete bipartite graph. Then $\gamma_I^p(G) = \gamma_I^p(\overline{G}) = 4$.

4. A unique family $\mathcal{G}$ of graphs G

Theorem 9. *For any positive integer $n \geq 20$ there exists a graph G of order n such that G, $\overline{G}$ are both connected and $\gamma_I^p(G) = \gamma_I^p(\overline{G}) = n$.*

P r o o f. Let $\mathcal{G}$ be a collection of graphs G each of order n. Then each graph G in $\mathcal{G}$ is constructed as follows.

Construction of the graph G in $\mathcal{G}$. Let $\{v_1, v_2, ...v_{n/2}\}$, $\{u_1, u_2, ...u_{n/2}\}$ be the vertices of two paths $P_{n/2}$ each of order $n/2$ and $P_{n/2} + P_{n/2}$ be the graph obtained by taking join of these two paths. Then G is a graph of order n obtained by removing the edge v_1u_1 from $P_{n/2} + P_{n/2}$.

Any vertex in G is of degree $n/2+2$, $n/2+1$ or $n/2$. This implies that there exists no universal vertex or two non-adjacent vertices of degree $n-2$. Hence $\gamma_I^p(G) > 2$. Let $A = \{u_1, u_2, ...u_{n/2}\}$ and $B = \{v_1, v_2, ...v_{n/2}\}$. Then the following are the possible labellings for the vertices of the graph G.

1. If two vertices belonging to the set A are labelled 1 each or one vertex in the set A is labelled 2, then labelling a vertex belonging to the set A makes all the vertices belonging to the set B labelled 0. (If the vertex labelled 0 is u_1, then all the vertices in B except v_1.) Since there exist vertices in B which are PI dominated by the non-zero labelled vertices in

A, all the remaining vertices in A should be labelled 0. (Since v_1 is adjacent to v_2 which is zero labelled and is PI dominated by the vertices of A, v_1 is also labelled 0). Similarly, if a vertex in B is labelled 0, then all the remaining vertices in A are labelled 0. (If v_1 is the vertex labelled zero, then all the remaining vertices except u_1 is labelled 0.) There exists at least one vertex x belonging to B adjacent to the zero labelled vertex which implies that x also should be labelled 0 and is PI dominated by the vertices of the set A. Since B is a connected graph, this continues and all the vertices of B are labelled 0. This forces u_1 also is to be labelled 0.

2. Let a vertex x from set A and a vertex y from a set B be labelled 1 each. Then a vertex in the neighbourhood of x and y belonging to the set A or B, is labelled zero forces all the remaining vertices in the other set are to be labelled 0. There exists at least one zero labelled vertex adjacent to the y in B. This implies that all the remaining vertices in A should be labelled 0.

Both the cases are contradictions to $\gamma_I^p(G) > 2$. This implies that no vertex in G is labelled 0. Hence

$$\gamma_I^p(G) = \frac{n}{2} + \frac{n}{2} = n.$$

The complement $\overline{G}$ is $\overline{P}_{n/2} \cup \overline{P}_{n/2}$ with an edge between v_1 and u_1. The vertex v_1 belonging to a path complement is adjacent to vertex u_1 belonging to another path complement. Hence, the adjacency between any two vertices of $\overline{G}$ other than $\{v_1, u_1\}$ is same as its adjacency in $\overline{P}_{n/2}$. This implies that as given in the proof of Theorem 6, if any vertex in the graph is labelled 0, then at most two vertices can only be non-zero labelled and they are labelled 1 each. Since $n \geq 20$ and v_1, u_1 are of degree $n/2-1+1 = n/2$ each, $\gamma_I^p(\overline{G}) > 2$. This implies that no vertex can be labelled 0 and

$$\gamma_I^p(\overline{G}) = \frac{n}{2} + \frac{n}{2} = n.$$

□

This theorem proves that there exists a family of graphs in which each of them and its corresponding complement are connected as well as have their PID number same as its order. This shows that the upper bound of *Nordhaus–Gaddum inequalities* for the Perfect Italian Domination is tight.

Thus, $\gamma_I^p(G) + \gamma_I^p(\overline{G}) = 2n$ if and only if $\gamma_I^p(G) = \gamma_I^p(\overline{G}) = n$. Since there is no complete characterization of graphs satisfying $\gamma_I^p(G) = n$, characterizing the graphs such that

$$\gamma_I^p(G) + \gamma_I^p(\overline{G}) = 2n$$

remains an open problem.

5. Conclusion

The lower and upper bounds in the Nordhaus–Gaddum type inequalities for the Perfect Italian domination number of an arbitrary graph G are way apart. Hence, particular cases of the graphs are considered to find the Nordhaus–Gaddum type inequalities. We have constructed different graph classes to show that the bounds are tight since there is no complete characterization of graphs satisfying $\gamma_I^p(G) = n$. Thus characterizing the graphs such that $\gamma_I^p(G) + \gamma_I^p(\overline{G}) = 2n$ remains an open problem.

REFERENCES

1. Anto A.Ṁ., Hawkins P. P., Mary T. S. I. Perfect dominating sets and perfect dominating polynomial of a cycle. *Adv. Math.: Sci. J.*, 2019. Vol. 8, No. 3. P. 538–543.
2. Bhatt T. J., Bhimani G. C. Perfect domination number of path graph P_n and its Corona product with another path graph P_{n-1}. *Malaya J. Mat.*, 2021. Vol. 9, No. 1. P. 118–123. DOI: 10.26637/MJM0901/0020
3. Chellali M., Rad N. J., Sheikholeslami S. M., Volkmann L. Varieties of Roman domination. In: *Structures of domination in graphs. Haynes T.W., Hedetniemi S.T., Henning M.A. (eds.).* Ser. Dev. Math., vol. 66. Cham: Springer, 2021. P. 273–307. DOI: 10.1007/978-3-030-58892-2_10
4. Fellows M. R., Hoover M. N. Perfect domination. *Australas. J. Combin.*, 1991. Vol. 3. P. 141–150.
5. Haynes T. W., Henning M. A. Perfect Italian domination in trees. *Discrete Appl. Math.*, 2019. Vol. 260. P. 164–177. DOI: 10.1016/j.dam.2019.01.038
6. Lauri J., Mitillos C. Perfect Italian domination on planar and regular graphs. *Discrete Appl. Math.*, 2020. Vol. 285. P. 676–687. DOI: 10.1016/j.dam.2020.05.024
7. Nordhaus E. A., Gaddum J. W. On complementary graphs. *Amer. Math. Monthly*, 1956. Vol. 63, No. 3. P. 175–177. DOI: 10.2307/2306658
8. Poovathingal A., Kureethara J. V. Modelling networks with attached storage using perfect Italian domination. In: *Machine Intelligence for Research and Innovations: Proc. MAiTRI 2023, vol. 1, Verma O.P., Wang L., Kumar R., Yadav A. (eds).* Ser. Lect. Notes Netw. Syst., vol 832. Singapore: Springer, 2023. P. 23–33. DOI: 10.1007/978-981-99-8129-8_3