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INTERPOLATION WITH MINIMUM VALUE OF L_2-NORM OF DIFFERENTIAL OPERATOR

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Abstract: For the class of bounded in l_2-norm interpolated data, we consider a problem of interpolation on a finite interval $[a,b]\subset\mathbb{R}$ with minimal value of the L_2-norm of a differential operator applied to interpolants. Interpolation is performed at knots of an arbitrary N-point mesh $\Delta_N:\ a\le x_1<x_2<\dots<x_N\le b$. The extremal function is the interpolating natural $\mathcal{L}$-spline for an arbitrary fixed set of interpolated data. For some differential operators with constant real coefficients, it is proved that on the class of bounded in l_2-norm interpolated data, the minimal value of the L_2-norm of the differential operator on the interpolants is represented through the largest eigenvalue of the matrix of a certain quadratic form.

Keywords: Interpolation, Natural $\mathcal{L}$-spline, Differential operator, Reproducing kernel, Quadratic form.

1. Introduction

Let N be any positive integer, $1\le q<\infty$, and

$$\mathfrak{M}_{N,q}=\Big\{z:\ z=\{z_j\}_{j=1}^N,\Big(\sum_{j=1}^N|z_j|^q\Big)^{1/q}\le 1\Big\}$$

be a class of interpolated values that is the unit ball in the space l_q^N.

Let $[a,b]\subset\mathbb{R}$ be an arbitrary finite interval and $W_q^m[a,b]$ be the standard Sobolev space equipped with the norm

$$\|f\|_{W_q^m[a,b]}=\|f\|_q+\sum_{j=1}^m\|f^{(j)}\|_q, \tag{1.1}$$

where $\|f\|_q$ is the usual L_q-norm of a function f on $[a,b]$.

Let $D=d/dx$ be the operator of differentiation, I be the identical operator, and

$$\mathcal{L}_m(D)=D^m+a_{m-1}D^{m-1}+\ldots+a_1D+a_0I$$

be a linear differential operator of order m with constant real coefficients. Denote by $p_m:=p_m(x)$ the characteristic polynomial of the differential operator $\mathcal{L}_m(D)$:

$$p_m(x)=x^m+a_{m-1}x^{m-1}+\dots+a_1x+a_0.$$

We restrict our attention to the case when $p_m(x)$ has only real roots $\{\beta_j\}_{j=1}^m$. This means that the differential operator $\mathcal{L}_m(D)$ has the factorization into a product of differential operators of the first order, i.e.,

$$\mathcal{L}_m(D)=(D-\beta_1I)(D-\beta_2I)\cdots(D-\beta_mI). \tag{1.2}$$

We interpolate at the knots of an arbitrary fixed mesh of N points from the interval $[a,b]$

$$\Delta_N: \ a \le x_1 < x_2 < \ldots < x_N \le b, \quad (a > -\infty, \ b < +\infty).$$

For an arbitrary fixed $z \in \mathfrak{M}_{N,q}$, we introduce the quantity

$$K_{N,q}(z) = \inf_{\substack{f \in W_q^m[a,b] \\ f(x_k)=z_k, \ k=\overline{1,N}}} \|\mathcal{L}_m(D)f\|_q. \tag{1.3}$$

The problem of finding quantity (1.3) is known as the Favard type interpolation problem (see [2, 4, 20], and the references therein).

In the paper, we study an analog of quantity (1.3) for the class $\mathfrak{M}_{N,q}$ of interpolated data, namely

$$\mathfrak{B}^q_{\mathcal{L}_m}(\Delta_N) = \sup_{z \in \mathfrak{M}_{N,q}} K_{N,q}(z). \tag{1.4}$$

Problem (1.4) can also be interpreted as the Favard type interpolation problem, but considered for the entire class of interpolated data. For the differential operator $\mathcal{L}_m(D) = D^m$, quantity (1.4) was found by the author [9] in the case of $q = 2$.

Problem (1.4) is close to extremal interpolation problems (see [15–18] and the references therein). However, the set of interpolated data in our setting (1.4) is given by the constraint imposed on the interpolated values $z = (z_1, z_2, \ldots, z_N)$, but not on their finite or divided differences.

In the present paper, we consider problems (1.3) and (1.4) only for $q = 2$. For this reason, index 2 in $\mathfrak{M}_{N,2}$, $K_{N,2}(z)$, and $\mathfrak{B}^2_{\mathcal{L}_m}(\Delta_N)$ will be omitted.

The main result of the paper is Theorem 1, in which we give the exact value of the quantity $\mathfrak{B}_{\mathcal{L}_m}(\Delta_N)$. This exact value is expressed in terms of the largest eigenvalue of the matrix of a quadratic form of interpolated data.

For $q = 2$, the extremal function in (1.3) is known. This function is a specific spline, which is called an interpolation natural $\mathcal{L}$-spline. This fact is a particular case of results of the variation spline theory.

The paper is organized as follows. Section 2 is devoted to the Favard type interpolation problem (1.3) considered from the point of view of general approaches. In Section 3, we write two representations of interpolation natural $\mathcal{L}$-splines. In Section 4, we prove the lemmas needed to prove the main result. In Section 5, we formulate and prove the main result of the paper. Section 6 is devoted to discussions and some comments.

2. On Favard-type interpolation problems

Consider problem (1.3) in the case of $q = 2$. As shown in [20], (1.3) is one of convex programming problems. In [20, p.87], it is proved that there exists a solution to (1.3), and for $N > m$, the solution is unique. As noted above, an extremal function in (1.3) is known. To write this function, we need some known results of the variation theory of splines (see for example, [1, 14], and the references therein).

We first introduce some notation. Let X be a real Hilbert space of functions with a norm $\|\cdot\|$, let $T : X \to X$ be a bounded linear operator, and let $\ker T$ be its null-space, i.e., the set of functions $\varphi \in X$ such that $T\varphi = 0$. By X^*, we denote the conjugate space of X. Let $\varphi_i \in X^*$ $(i = 1, 2, \ldots, N)$, i.e., let φ_i be bounded linear functionals on X. For every $\tau \in X$, we set $A\tau = (\varphi_1(\tau), \varphi_2(\tau), \ldots, \varphi_N(\tau))$. This means that A is a linear operator that maps a function $\tau \in X$ onto an N-dimensional vector consisting of values of the functionals $\{\varphi_i\}_{i=1}^N$ on this function.

As such functionals, we take values at the points of the mesh $\Delta_N = \{x_i\}_{i=1}^N$, i.e., we set $\varphi_i(\tau) = \tau(x_i)$, $(i = 1, 2, \dots, N)$. Thus, we have

$$A\tau = (\tau(x_1), \tau(x_2), \dots, \tau(x_N)).$$

The abstract Favard-type interpolation problem is to find the quantity

$$\mathcal{B}(T, z) = \inf_{\substack{Af=z \\ f\in X}} \|Tf\|_X^2. \tag{2.1}$$

Following [14, p. 77], a real Hilbert space X with a norm $\|\cdot\|$ and a seminorm $\rho(\,\cdot\,)$ is called an S-space if the following conditions hold:

(i) the seminorm ρ is bounded in X; i.e., for every $\tau \in X$, the inequality $\rho(\tau) \le C\|\tau\|$ is true with some constant $C > 0$ independent of τ;

(ii) X is complete with respect to the seminorm ρ.

It is proved (see [1] and [14, Sect. 5.15]) that if the space X is an S-space and is continuously embedded into the space of continuous functions, then the extremal function of (2.1) has the following form:

$$\sigma(x) = q(x) + \sum_{j=1}^{N} \lambda_j G_m(x, x_j). \tag{2.2}$$

Here $q \in \ker T$, $G_m(x, \cdot)$ is a reproducing kernel of the S-space X (see [1, 14]), and the scalars $\{\lambda_j\}_{j=1}^N$ are determined from the condition

$$\sum_{j=1}^{N} \lambda_j u(x_j) = 0 \qquad \forall u \in \ker T. \tag{2.3}$$

Problem (1.3) is a particular case (up to root-squaring) of (2.1) when $X = W_2^m[a,b]$ with norm (1.1), $T = \mathcal{L}_m(D) : \; W_2^m[a,b] \to L_2[a,b]$, and $\tau = f$. In our case, the seminorm $\rho(\cdot)$ is defined as

$$\rho(f) = \left(\int_a^b |\mathcal{L}_m(D) f(t)|^2 dt \right)^{1/2}.$$

Since the seminorm $\rho(\cdot)$ is estimated through the coefficients of the differential operator $\mathcal{L}_m(D)$ as

$$\rho(f) = \|\mathcal{L}_m(D) f\|_2 \le \max\left\{1, |a_0|, |a_1|, \dots, |a_{m-1}|\right\} \|f\|_{W_2^m[a,b]},$$

the seminorm $\rho(f)$ is bounded.

The operator $\mathcal{L}_m(D)$ acts "onto" $L_2[a,b]$, and the space $L_2[a,b]$ is complete. Therefore, $W_2^m[a,b]$ is complete with respect to the seminorm ρ. Thus, $W_2^m[a,b]$ is an S-space. In addition, the space $W_2^m[a,b]$ is continuously embedded into the space $C[a,b]$ of continuous functions (the Sobolev embedding theorem).

For finding the reproducing kernel of S-space $W_2^m[a,b]$, we introduce two subspaces

$$U_m = \left\{ f \in W_2^m[a,b] : \quad f^{(i)}(a) = 0, \; i = 0, 1, \dots, m-1 \right\}$$

and

$$V_m = \left\{ f \in W_2^m[a,b] : \quad f^{(i)}(b) = 0, \; i = 0, 1, \dots, m-1 \right\}.$$

Each of the subspaces U_m and V_m has codimension m. Also, it is easy to see that $\big(\ker \mathcal{L}_m(D)\big) \cap U_m = \{0\}$ and $\big(\ker \mathcal{L}_m(D)\big) \cap V_m = \{0\}$. From these simple facts, it follows that $W_2^m[a,b] = (\ker \mathcal{L}_m(D)) \cup U_m$ and $W_2^m[a,b] = (\ker \mathcal{L}_m(D)) \cup V_m$.

Let $\mathcal{L}_m^*(D)$ be a linear differential operator that is formal adjoint to the operator $\mathcal{L}_m(D)$, i.e., $\mathcal{L}_m^*(D) = \mathcal{L}_m(-D)$. Now, we introduce the differential operator $\mathcal{L}_{2m}(D)$ of order $2m$ as follows

$$\mathcal{L}_{2m}(D) = \mathcal{L}_m(D)\ \mathcal{L}_m^*(D).$$

From (1.2), we have

$$\mathcal{L}_{2m}(D) = (-1)^m (D^2 - \beta_1^2 I)(D^2 - \beta_2^2 I) \cdots (D^2 - \beta_m^2 I).$$

For the S-space $W_2^m[a,b]$, the reproducing kernel is coordinated with the subspaces U_m and V_m and is built through a fundamental solution of the differential operator $\mathcal{L}_{2m}(D)$ (see, e.g., [14, Ch. 5]).

As is known (see, e.g., [21, Ch. III]), the fundamental solution of a differential operator $\mathcal{L}(D)$ is a distribution $\mathcal{E}$ satisfying $\mathcal{L}(D)\mathcal{E} = \delta$, where δ is the Dirac δ-function (or δ-distribution). The fundamental solution is defined up to a summand that is an arbitrary solution of the equation $\mathcal{L}(D)y(t) = 0$. We will assume that this summand is identically equal to zero. Distributions are understood as linear continuous functionals in the space of infinitely differentiable functions with compact supports.

The following result is known.

Lemma 1 (see, e.g., [21, p. 114]). *Let $\mathcal{L}_r(D)$ be an arbitrary linear differential operator of order $r \geq 2$ with constant real coefficients. Then the fundamental solution of this operator has the form*

$$\mathcal{E}_r(t) = \theta(t)\ Z_r(t),$$

where $Z_r(t)$ is a unique solution to the initial value problem

$$\begin{cases} \mathcal{L}_r(D) Z_r(t) = 0, \\ Z_r(0) = Z_r'(0) = \cdots = Z_r^{(r-2)}(0) = 0, \\ Z_r^{(r-1)}(0) = 1 \end{cases}$$

and $\theta(t)$ is the Heaviside function

$$\theta(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Now, we set $r = 2m$ and apply Lemma 1 to the differential operator $\mathcal{L}_{2m}(D)$. Since it has the leading coefficient $(-1)^m$, its fundamental solution is

$$\mathcal{E}_{2m}(t) = (-1)^m\ \theta(t)\ Z_{2m}(t), \tag{2.4}$$

where

$$\begin{cases} (D^2 - \beta_1^2 I)(D^2 - \beta_2^2 I) \cdots (D^2 - \beta_m^2 I) Z_{2m}(t) = 0, \\ Z_{2m}(0) = Z_{2m}'(0) = \cdots = Z_{2m}^{(2m-2)}(0) = 0, \\ Z_{2m}^{(2m-1)}(0) = 1. \end{cases}$$

Based on [14, Sect. 5.13], we will prove that the function $\mathcal{E}_{2m}(x-t)$ is the reproducing kernel of the S-space $W_2^m[a,b]$.

Lemma 2. $\mathcal{E}_{2m}(x-t) \in U_m$ *for any fixed* $t \in [a,b]$.

P r o o f. From (2.4), we have

$$\mathcal{E}_{2m}^{(i)}(u) = (-1)^m\ \theta(\tau)\ Z_{2m}^{(i)}(u) \quad (i = 0, 1, \ldots, 2m-1). \tag{2.5}$$

Now, we set $u = x - t$. From the definition of the Heaviside function, we see that

$$\mathcal{E}_{2m}^{(i)}(a-t) = 0, \qquad (i = 0, 1, \ldots, 2m-1),$$

i.e., $\mathcal{E}_{2m}(x-t) \in U_m$ for any fixed $t \in [a, b]$. □

Lemma 3. *The following equality holds for any fixed $t \in [a, b]$ and any function $f \in V_m$:*

$$\int_a^b \big(\mathcal{L}_m(D)\mathcal{E}_{2m}(x-t)\big)\ \mathcal{L}_m(D) f(x) dx = f(t).$$

P r o o f. Let $a < t < b$. We write the integral on the left-hand side as the sum of two integrals

$$\int_a^b \big(\mathcal{L}_m(D)\mathcal{E}_{2m}(x-t)\big)\ \mathcal{L}_m(D) f(x)\ dx = I_1 + I_2,$$

where

$$I_1 = \int_a^t \big(\mathcal{L}_m(D)\mathcal{E}_{2m}(x-t)\big)\ \mathcal{L}_m(D) f(x) dx, \quad I_2 = \int_t^b \big(\mathcal{L}_m(D)\mathcal{E}_{2m}(x-t)\big)\ \mathcal{L}_m(D) f(x) dx.$$

Changing the variable $u = x - t$ and noting that $\mathcal{E}_{2m}(u) = 0$ for all $u \le 0$, we have

$$I_1 = \int_{a-t}^0 \big(\mathcal{L}_m(D)\mathcal{E}_{2m}(u)\big)\ \mathcal{L}_m(D) f(u+t)\ du = 0.$$

Integrating I_2 by parts, we obtain

$$I_2 = w(b) - w(t) + \int_t^b \big(\mathcal{L}_{2m}(D)\mathcal{E}_{2m}(x-t)\big) f(x) dx,$$

where

$$w(x) = f^{(m-1)}(x)\ \mathcal{L}_m(D)\mathcal{E}_{2m}(x-t) + f^{(m-2)}(x)\ \Big[a_{m-1}\ \mathcal{L}_m(D)\mathcal{E}_{2m}(x-t) - \big(\mathcal{L}_m(D)\mathcal{E}_{2m}(x-t)\big)'_x\Big]$$
$$+ \cdots + f'(x)\ \sum_{\nu=0}^{m-2} (-1)^\nu a_{\nu+2} \big(\mathcal{L}_m(D)\mathcal{E}_{2m}(x-t)\big)_x^{(\nu)} + f(x)\ \sum_{\nu=0}^{m-1} (-1)^\nu a_{\nu+1} \big(\mathcal{L}_m(D)\mathcal{E}_{2m}(x-t)\big)_x^{(\nu)},$$

and $\{a_\nu\}_{\nu=0}^m$, $a_m = 1$, are the constant real coefficients in the standard representation of the differential operator $\mathcal{L}_m(D)$.

By the definition of the set V_m, we have $w(b) = 0$.

From (2.5) for $i = 0, 1, \ldots, m-1$, it follows that

$$\mathcal{L}_m(D)\mathcal{E}_{2m}(x-t)|_{x=t} = 0.$$

From (2.5) for $i = m, m+1, \ldots, 2m-1$, we conclude that all derivatives of $\mathcal{L}_m(D)\mathcal{E}_{2m}(x-t)$ in $w(x)$ are equal to zero when $x = t$. Therefore, $w(t) = 0$.

Using the definition of the fundamental solution and one of the known properties of the Dirac δ-function (see, for example, [21, p. 134]), we finally have

$$I_2 = \int_t^b \big(\mathcal{L}_{2m}(D)\mathcal{E}_{2m}(x-t)\big) f(x) dx = \int_t^b \delta(x-t) f(x) dx = f(t).$$

The cases $t = a$ and $t = b$ are easily checked. Lemma 3 is proved. □

From [14, Sect. 5.13] and Lemmas 2 and 3, we obtain the following statement.

Lemma 4. $G_m(x,t) = \mathcal{E}_{2m}(x-t)$.

Lemma 4 together with (2.2) and (2.3) give the following expression of the extremal function $\sigma(x)$ in problem (1.3):

$$\sigma(x) = q(x) + \sum_{j=1}^{N} \lambda_j \mathcal{E}_{2m}(x - x_j), \tag{2.6}$$

where $q \in \ker \mathcal{L}_m(D)$ and there are additional conditions

$$\sum_{j=1}^{N} \lambda_j g_\nu(x_j) = 0 \quad (\nu = 1, 2, \ldots, m), \tag{2.7}$$

for finding parameters $\{\lambda_j\}_{j=1}^N$. Here, the set of functions $\{g_\nu(x)\}_{\nu=1}^m$ is a basis in $\ker \mathcal{L}_m(D)$.

From (2.4), it follows that

$$\mathcal{E}_{2m}(x - x_j) \in C^{2m-2}(\mathbb{R}) \quad (j = 1, 2, \ldots, N).$$

Therefore, the function $\sigma(x)$ has the following properties:

(1) $\sigma \in C^{2m-2}(\mathbb{R})$;
(2) $\mathcal{L}_{2m}(D)\sigma(x) = 0$ for all $x \in (x_j, x_{j+1})$ $(j = 1, 2, \ldots, N-1)$;
(3) $\sigma(x_i) = z_i$ $(i = 1, 2, \ldots, N)$;
(4) $\sigma \in \ker \mathcal{L}_m(D)$ for $x \le x_1$ and $x \ge x_N$.

Properties (1)–(3) mean that $\sigma(x)$ is an interpolating $\mathcal{L}$-spline corresponding to the differential operator $\mathcal{L}_{2m}(D)$ with knots at the points of the mesh Δ_N and has the minimal defect. Due to property (4), the $\mathcal{L}$-spline $\sigma(x)$ can be extended beyond the interval $[x_1, x_N]$ with maintaining of the smoothness. By analogy with polynomial splines, such splines are called *natural* $\mathcal{L}$-splines.

Thus, $\sigma(x)$ is a natural $\mathcal{L}$-spline corresponding to the differential operator

$$\mathcal{L}_{2m}(D) = \mathcal{L}_m(D)\ \mathcal{L}_m^*(D)$$

with knots at the points of the mesh $\Delta_N = \{x_j\}_{j=1}^N$. If $N > m$, then this solution is unique [14, Sect. 5.21].

Remark 1. If $\mathcal{L}_m(D) = D^m$, $m \ge 2$, then it is to see from Lemma 1 that the fundamental solution of the operator $\mathcal{L}_{2m}(D) = (-1)^m D^{2m}$ is

$$\mathcal{E}_{2m}(x-t) = \frac{(-1)^m (x-t)_+^{2m-1}}{(2m-1)!},$$

where $(x-t)_+ = \max\{x-t, 0\}$ is the truncated function, which is traditionally widely used in the spline theory. By choosing $g_\nu(x) = x^\nu$ $(\nu = 0, 1, \ldots, m)$, from (2.6) and (2.7), we arrive at the well-known polynomial natural splines. More information about these splines can be found, for example, in [5, 8, 14].

3. On natural $\mathcal{L}$-splines

First, we get an explicit expression of the fundamental solution for the differential operator $\mathcal{L}_{2m}(D) = \mathcal{L}_m(D)\mathcal{L}_m^*(D)$, where $\mathcal{L}_m(D)$ is given in (1.2). For simplicity, we impose additional restrictions on the roots of the characteristic polynomial of the differential operator (1.2). We will assume that $\beta_j \in \mathbb{R}\backslash\{0\}$ and $\beta_i \ne \pm\beta_j$, $i \ne j$, for all $i, j = 1, 2, \ldots, r$.

Lemma 5. *Let $\mathcal{L}_r(D)$ be a linear differential operator of order $r \geq 2$ of the form*

$$\mathcal{L}_r(D) = (D - b_1 I)(D - b_2 I) \cdots (D - b_r I).$$

If $b_j \in \mathbb{R}\backslash\{0\}$ and $b_i \neq \pm b_j$, $i \neq j$, for all $i, j = 1, 2, \ldots, r$. Then

$$\mathcal{E}_r(t) = (-1)^{r-1}\theta(t) \sum_{s=1}^{r} \frac{e^{b_s t}}{\prod_{\nu=1,\,\nu\neq s}^{r}(b_\nu - b_s)},$$

where $\theta(t)$ is the Heaviside function.

P r o o f. Find a solution to the initial value problem from Lemma 1. The assumptions about the numbers $\{b_j\}_{j=1}^m$ mean that all roots of the characteristic polynomial of the differential operator $\mathcal{L}_r(D)$ are simple and nonzero.

The general solution to the differential equation $\mathcal{L}_r(D)Z_r(t) = 0$ is written as

$$Z_r(t) = C_1 e^{b_1 t} + C_2 e^{b_2 t} + \ldots + C_r e^{b_r t},$$

where $\{C_s : s = 1, 2, \ldots, r\}$ are some real numbers. To find the function $Z_r(t)$, we have initial conditions, which lead to a system of linear algebraic equations with respect to $\{C_s\}_{s=1}^r$. The system has the Vandermonde matrix. We solve the system using Cramer's rule and obtain

$$C_s = \frac{(-1)^{r-1}}{\prod_{\nu=1,\,\nu\neq s}^{r}(b_\nu - b_s)}, \quad s = 1, 2, \ldots, r.$$

It remains to use Lemma 1. □

Now, we apply Lemma 5 to the differential operator

$$\mathcal{L}_{2m}(D) = \mathcal{L}_m(D)\mathcal{L}_m^*(D) = (-1)^m (D^2 - \beta_1^2 I)(D^2 - \beta_2^2 I)\ldots(D^2 - \beta_m^2 I)$$

with the restrictions $\beta_j \in \mathbb{R}\backslash\{0\}$ and $\beta_i \neq \pm\beta_j$, $i \neq j$, for all $i, j = 1, 2, \ldots, m$.

By B_{2m}, we denote the set of roots of the characteristic polynomial of the operator $\mathcal{L}_{2m}(D)$:

$$B_{2m} = \{b_1, b_2, \ldots, b_{2m}\}.$$

Each of the numbers b_j $(j = 1, 2, \ldots, 2m)$ is either a root of the characteristic polynomial p_m of the differential operator $\mathcal{L}_m(D)$ or a root of the characteristic polynomial p_m^* of the formal adjoint operator $\mathcal{L}_m^*(D)$. All numbers b_j $(j = 1, 2, \ldots, 2m)$ are different. Therefore, Lemma 5 gives

$$\mathcal{E}_{2m}(t) = (-1)^{m-1}\ \theta(t) \sum_{s=1}^{2m} \frac{e^{b_s t}}{\prod_{\nu=1,\,\nu\neq s}^{2m}(b_\nu - b_s)}. \tag{3.1}$$

Under imposed constraints on $\{\beta_j\}_{j=1}^m$, the system of functions $\{e^{\beta_1 x}, e^{\beta_2 x}, \ldots, e^{\beta_m x}\}$ is a basis in $\ker \mathcal{L}_m(D)$. Therefore, from (2.6) and (2.7) for an arbitrary set of interpolated data $z = \{z_i\}_{i=1}^N$, we have the system of $N + m$ linear equations with respect to $N + m$ unknowns $\{\lambda_j\}_{j=1}^N$ and $\{c_k\}_{k=1}^m$:

$$\begin{cases} \sum_{j=1}^N \lambda_j \mathcal{E}_{2m}(x_i - x_j) + \sum_{k=1}^m c_k e^{\beta_k x_i} = z_i \quad (i = 1, 2, \ldots, N), \\ \sum_{j=1}^N \lambda_j e^{\beta_1 x_j} = 0, \ldots, \sum_{j=1}^N \lambda_j e^{\beta_m x_j} = 0. \end{cases}$$

If $N > m$, then the system has a unique solution.

Taking into account the properties of the function $\mathcal{E}_{2m}(x)$, we study the matrix $\mathcal{A}$ of the system. Since $x_1 < x_2 < \cdots < x_{N-1} < x_N$, we have $\mathcal{E}_{2m}(x_i - x_j) = 0$ for $i \le j$. Therefore, the matrix $\mathcal{A}$ consists of four blocks

$$\mathcal{A} = \left(\begin{array}{c|c} E_{N\times N} & B_{N\times m} \\ \hline V_{m\times N} & O_{m\times m} \end{array} \right),$$

where

$$B_{N\times m} = \begin{pmatrix} e^{\beta_1 x_1} & e^{\beta_2 x_1} & \dots & e^{\beta_m x_1} \\ e^{\beta_1 x_2} & e^{\beta_2 x_2} & \dots & e^{\beta_m x_2} \\ \dots & \dots & \dots & \dots \\ e^{\beta_1 x_N} & e^{\beta_2 x_N} & \dots & e^{\beta_m x_N} \end{pmatrix},$$

$$E_{N\times N} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ \mathcal{E}_{2m}(x_2 - x_1) & 0 & \cdots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mathcal{E}_{2m}(x_{N-1} - x_1) & \mathcal{E}_{2m}(x_{N-1} - x_2) & \cdots & \mathcal{E}_{2m}(x_{N-1} - x_{N-2}) & 0 & 0 \\ \mathcal{E}_{2m}(x_N - x_1) & \mathcal{E}_{2m}(x_N - x_2) & \cdots & \mathcal{E}_{2m}(x_N - x_{N-2}) & \mathcal{E}_{2m}(x_N - x_{N-1}) & 0 \end{pmatrix},$$

$O_{m\times m}$ is the zero block of size $m \times m$, and $V_{m\times N} = B^T_{N\times m}$. Here, the upper index T means the transpose of the matrix. The first row of the matrix $\mathcal{A}$ has N zeros, the second row has $(N-1)$ zeros, etc.; and finally, the $(N-1)$th row has two zeros, and the Nth row has only one zero.

From the existence and uniqueness of the natural $\mathcal{L}$-spline, it follows that $\det \mathcal{A} \ne 0$.

Now, we will find a representation of the natural $\mathcal{L}$-spline by fundamental natural $\mathcal{L}$-spline interpolants. Let

$$\{\mathcal{E}_{2m}(x - x_1), \dots, \mathcal{E}_{2m}(x - x_N), e^{\beta_1 x}, e^{\beta_2 x}, \dots, e^{\beta_m x}\}.$$

One by one, we replace the kth $(k = 1, 2, \dots, N)$ row of the matrix $\mathcal{A}$ with this row. The obtained matrices are denoted by $\mathcal{A}_k(x)$. Now, we set

$$Q_k(x) = \frac{\det \mathcal{A}_k(x)}{\det \mathcal{A}}, \quad (k = 1, 2, \dots, N).$$

The functions $\{Q_k(x)\}_{k=1}^N$ are the fundamental natural $\mathcal{L}$-spline interpolants, since it is easy to see that $\det \mathcal{A}_k(x_j) = \delta_{kj} \det \mathcal{A}$ $(k, j = 1, 2, \dots, N)$, where δ_{kj} is the Kronecker symbol.

The natural $\mathcal{L}$-spline $\sigma(x)$ (see (2.6)) is a linear combination of the fundamental natural $\mathcal{L}$-splines $\{Q_k(x)\}_{k=1}^N$; i.e.,

$$\sigma(x) = \sum_{k=1}^N z_k Q_k(x), \tag{3.2}$$

where $\{z_k\}_{k=1}^N$ are interpolated data.

Thus, we have two representations (2.6) and (3.2) for the extremal function of the Favard-type interpolation problem.

4. Lemmas

In this section, we establish several lemmas needed to prove our main result.

Lemma 6. *Let $N > m$, and let $\sigma(x)$ be the natural $\mathcal{L}$-spline that is the extremal function in the Favard-type interpolation problem (1.3). Let $z = \{z_k\}_{k=1}^N$ be an arbitrary set of interpolated data. Then*

$$\mathcal{L}_m(D)\sigma(x) = \frac{1}{\det \mathcal{A}} \left(\sum_{\mu=1}^N \left(\sum_{k=1}^N z_k \alpha_{\mu k} \right) \mathcal{L}_m(D) \mathcal{E}_{2m}(x - x_\mu) \right),$$

where $\{\alpha_{\mu k}\}$ are certain values independent of x.

P r o o f. Applying the differential operator $\mathcal{L}_m(D)$ to (3.2), we obtain

$$\mathcal{L}_m(D)\sigma(x) = \sum_{k=1}^{N} z_k \mathcal{L}_m(D) Q_k(x) = \frac{1}{\det \mathcal{A}} \sum_{k=1}^{N} z_k \,\big(\mathcal{L}_m(D) \det \mathcal{A}_k(x)\big).$$

Using the well-known rule of differentiation of determinants and taking into account that only one row in $\mathcal{A}_k(x)$ depends on the variable x, we find that $\mathcal{L}_m(D)\det \mathcal{A}_k(x)$ is the determinant in which the kth row has the form

$$\mathcal{L}_m(D)\mathcal{E}_{2m}(x - x_1) \dots \mathcal{L}_m(D)\mathcal{E}_{2m}(x - x_N) \underbrace{0 \dots 0}_{m \text{ times}}.$$

We expand each determinant $\mathcal{L}_m(D)\det \mathcal{A}_k(x)$ $(k = 1, 2, \dots, N)$ according to the elements of kth row and obtain the required equality, in which $\alpha_{\mu k}$ are the corresponding minors taken with their signs. Lemma 6 is proved. □

Lemma 7. *Let x_μ be an arbitrary point of the mesh $\Delta_N : x_i < x_2 < \dots < x_N$, and let the differential operator $\mathcal{L}_m(D)$ be such that*

$$\mathcal{L}_m(D) = (D - \beta_1 I)(D - \beta_2 I) \dots (D - \beta_m I)$$

with $\beta_j \in \mathbb{R}\backslash\{0\}$ and $\beta_i \neq \pm\beta_j$, $i \neq j$, for all $i, j = 1, 2, \dots, m$. Then

$$\mathcal{L}_m(D)\mathcal{E}_{2m}(x - x_\mu) = (-1)^m \sum_{i=1}^{m} \frac{e^{-\beta_i (x - x_\mu)}}{\prod_{\nu=1,\, \nu\neq i}^{m} (\beta_\nu - \beta_i)} \tag{4.1}$$

for $x > x_\mu$ and $\mathcal{L}_m(D)\mathcal{E}_{2m}(x - x_\mu) = 0$ for $x \le x_\mu$.

P r o o f. We write (3.1) as

$$\mathcal{E}_{2m}(x - x_\mu) = (-1)^{m-1}\, \theta(x - x_\mu) \sum_{s=1}^{2m} \omega_s e^{b_s (x - x_\mu)}, \tag{4.2}$$

where $b_s = \beta_s$, $b_{s+m} = -\beta_s$ $(s = 1, 2, \dots, m)$, and

$$\omega_s = \Big(\prod_{\nu=1,\, \nu\neq s}^{2m} (b_\nu - b_s)\Big)^{-1} \quad (s = 1, 2, \dots, 2m).$$

Hence, $\mathcal{L}_m(D)\mathcal{E}_{2m}(x - x_\mu) = 0$ for $x \le x_\mu$.

Let $x > x_\mu$. From (4.2), we have

$$\mathcal{L}_m(D)\mathcal{E}_{2m}(x - x_\mu) = (-1)^{m-1} \Bigg(\sum_{s:\ b_s = \beta_s} \omega_s\, \mathcal{L}_m(D) e^{b_s (x - x_\mu)} + \sum_{s:\ b_s = -\beta_s} \omega_s\, \mathcal{L}_m(D) e^{b_s (x - x_\mu)}\Bigg).$$

The former sum on the right-hand side vanishes because

$$\mathcal{L}_m(D) e^{\beta_s x} = 0 \quad (s = 1, 2, \dots, m).$$

By simple calculation, we obtain

$$\mathcal{L}_m(D) e^{-\beta x} = e^{-\beta x}\, p_m^*(\beta),$$

where p_m^* is the characteristic polynomial of the formal adjoint operator $\mathcal{L}_m^*(D)$. By using this relation, we arrive at the equality

$$\mathcal{L}_m(D)\mathcal{E}_{2m}(x-x_\mu) = (-1)^{m-1} \sum_{i=1}^{m} \omega_i \, e^{-\beta_i(x-x_\mu)} \, p_m^*(\beta_i).$$

Since

$$\omega_i^{-1} = 2(-1)^{m-1}\beta_i \prod_{\nu=1,\,\nu\neq i}^{m} (\beta_\nu^2 - \beta_i^2)$$

and

$$p_m^*(\beta_i) = 2(-1)^{m-1}\beta_i \prod_{\nu=1,\,\nu\neq i}^{m} (\beta_\nu + \beta_i),$$

we come to (4.1). Lemma 7 is proved. □

5. The main result

Finally, we directly turn to the problem of finding quantity (1.4).

Theorem 1. *Assume that*

$$\mathcal{L}_m(D) = (D-\beta_1 I)(D-\beta_2 I)\ldots(D-\beta_m I)$$

is a linear differential operator of order $m \geq 2$ such that $\beta_j \in \mathbb{R}\backslash\{0\}$ and $\beta_i \neq \pm\beta_j$, $i \neq j$ $(i,j = 1,2,\ldots,m)$.

If $N > m$, then

$$\mathfrak{B}_{\mathcal{L}_m}(\Delta_N) = \frac{1}{|\det \mathcal{A}|} \sqrt{\lambda_{max}}\,,$$

where λ_{max} is the largest eigenvalue of the matrix $Q = (a_{ij})_{i,j=1}^N$ whose entries $\{a_{ij}\}$ are such that

$$a_{kk} = \sum_{\mu=1}^{N} R_{\mu\mu}\alpha_{\mu k}^2 + 2\sum_{\mu>n}^{N} R_{\mu n}\alpha_{\mu k}\alpha_{nk},$$

$$a_{kj} = \sum_{\mu=1}^{N} R_{\mu\mu}\alpha_{\mu k}\alpha_{\mu j} + 2\sum_{\mu>n}^{N} R_{\mu n}\alpha_{\mu k}\alpha_{nj} \quad (k \neq j),$$

$$R_{\mu n} = \sum_{i,\nu=1}^{m} \omega_i\omega_\nu \, \frac{e^{-\beta_\nu(x_\mu - x_n)} - e^{-\beta_i(x_N - x_\mu)} e^{-\beta_\nu(x_N - x_n)}}{\beta_i + \beta_\nu} \quad (\mu \geq n),$$

$$\omega_i = \Big(\prod_{s=1,\,s\neq i}^{m} (\beta_s - \beta_i) \Big)^{-1} \quad (i = 1,2,\ldots,m),$$

and $\{\alpha_{s\mu}\}_{s,\mu=1}^N$ are the minors taken with their signs in the decompositions of the determinants $\mathcal{L}_m(D)\det\mathcal{A}_k(x)$ $(k = 1,2,\ldots,N)$ according to the elements of the kth row.

P r o o f. Since $N > m$, the natural $\mathcal{L}$-spline $\sigma(x)$ is the unique extremal function in (1.3). After applying the differential operator $\mathcal{L}_m(D)$ to (2.6), we have

$$\mathcal{L}_m(D)\sigma(x) = \sum_{\mu=1}^{N} \lambda_\mu \, \mathcal{L}_m(D)\mathcal{E}_{2m}(x-x_\mu).$$

Comparing this result with Lemma 6 yields

$$\lambda_\mu = \frac{1}{\det \mathcal{A}} \sum_{k=1}^{N} z_k \alpha_{\mu k} \quad (\mu = 1, 2, \ldots, N). \tag{5.1}$$

Using Lemma 7, we obtain

$$\mathcal{L}_m(D)\sigma(x) = (-1)^m \sum_{\mu=1}^{N} \lambda_\mu F_\mu(x),$$

where

$$F_\mu(x) = \sum_{i=1}^{m} \omega_i \begin{cases} e^{-\beta_i(x-x_\mu)}, & x > x_\mu, \\ 0, & x \le x_\mu, \end{cases}$$

and

$$\omega_i = \Big(\prod_{\nu=1,\, \nu\neq i}^{m} (\beta_\nu - \beta_i) \Big)^{-1} \quad (i = 1, 2, \ldots, m).$$

Define

$$R_{\mu n} = \int\limits_{x_1}^{x_N} F_\mu(x) F_n(x) dx \quad (\mu, n = 1, 2, \ldots, N,\ \ \mu \ge n). \tag{5.2}$$

For the natural $\mathcal{L}$-spline $\sigma(x)$, we have

$$\int_a^b \big|\mathcal{L}_m(D)\sigma(x)\big|^2 dx = \int_{x_1}^{x_N} \big|\mathcal{L}_m(D)\sigma(x)\big|^2 dx = \int_{x_1}^{x_N} \Big|\sum_{\mu=1}^{N} \lambda_\mu F_\mu(x)\Big|^2 dx$$

$$= \int_{x_1}^{x_N} \Big(\sum_{\mu=1}^{N} \lambda_\mu^2 F_\mu^2(x) + 2\sum_{\mu>n}^{N} \lambda_\mu \lambda_n F_\mu(x) F_n(x) \Big) dx = \sum_{\mu=1}^{N} \lambda_\mu^2 R_{\mu\mu} + 2\sum_{\mu>n}^{N} \lambda_\mu \lambda_n R_{\mu n}.$$

Now, we substitute λ_μ from (5.1) into the latter expression. After simple transformations, we obtain

$$\big(K_N(z)\big)^2 = \Big(\frac{1}{\det \mathcal{A}}\Big)^2 \Big\{ \sum_{\mu=1}^{N} \Big(\sum_{k=1}^{N} z_k \alpha_{\mu k} \Big)^2 R_{\mu\mu} + 2\sum_{\mu>n}^{N} R_{\mu n} \Big(\sum_{k=1}^{N} z_k \alpha_{\mu k} \Big) \Big(\sum_{j=1}^{N} z_j \alpha_{nj} \Big) \Big\}$$

$$= \frac{1}{(\det \mathcal{A})^2} \Big(\sum_{k=1}^{N} a_{kk} z_k^2 + \sum_{k,j=1,\ k\neq j}^{N} a_{kj} z_k z_j \Big),$$

where

$$a_{kk} = \sum_{\mu=1}^{N} R_{\mu\mu} \alpha_{\mu k}^2 + 2\sum_{\mu>n}^{N} R_{\mu n} \alpha_{\mu k} \alpha_{nk},$$

$$a_{kj} = \sum_{\mu=1}^{N} R_{\mu\mu} \alpha_{\mu k} \alpha_{\mu j} + 2\sum_{\mu>n}^{N} R_{\mu n} \alpha_{\mu k} \alpha_{nj} \quad (k \neq j).$$

It remains to calculate the integrals (5.2). Suppose that $\mu \ge n$.Then $x_\mu \ge x_n$, and we have

$$R_{\mu n} = \int_{x_\mu}^{x_N} \Big(\sum_{i=1}^{m} \omega_i e^{-\beta_i(x-x_\mu)} \Big) \Big(\sum_{\nu=1}^{m} \omega_\nu e^{-\beta_\nu(x-x_n)} \Big) dx = \sum_{i,\nu=1}^{m} \omega_i \omega_\nu \Big(\int_{x_\mu}^{x_N} e^{-(\beta_i+\beta_\nu)x} dx \Big) e^{\beta_i x_\mu + \beta_\nu x_n}$$

$$= \sum_{i,\nu=1}^{m} \omega_i \omega_\nu \; \frac{e^{-\beta_\nu(x_\mu - x_n)} - e^{-\beta_i(x_N - x_\mu)} e^{-\beta_\nu(x_N - x_n)}}{\beta_i + \beta_\nu}.$$

In our calculations, we used the fact that the functions $F_\mu(x)$ are equal to zero for $x \le x_\mu$. Note that all the denominators do not vanish due to the restrictions imposed on the differential operator $\mathcal{L}_m(D)$.

Thus, to find quantity (1.4), it is sufficient to solve the following extremal problem:

$$\begin{cases} V(z_1, z_2, \dots, z_N) = \sum\limits_{k=1}^{N} a_{kk} z_k^2 \; + \sum\limits_{k,j=1,\; k\neq j}^{N} a_{kj} z_k z_j \;\to\; \max, \\ \sum\limits_{k=1}^{N} |z_k|^2 \le 1. \end{cases}$$

As in [9, p. 224–225], it is proved that the maximum is attained at the boundary of the set $\mathfrak{M}_N$, i.e., at some points $z = (z_1, z_2, \dots, z_N)$ with $\sum_{k=1}^{N} |z_k|^2 = 1$. As a result, we arrive at the problem of maximizing the quadratic form on the unit sphere in the space l_2^N.

It is known (see, for example, [3, p. 476–477]) that the unique solution to this problem is the largest eigenvalue $\lambda = \lambda_{max}$ of the matrix $Q = (a_{ij})_{i,j=1}^{N}$ of the quadratic form $V(z_1, z_2, \dots, z_N)$, i.e., the maximum root of the equation

$$\det(Q - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1N} \\ a_{12} & a_{22} - \lambda & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1N} & a_{2N} & \cdots & a_{NN} - \lambda \end{vmatrix} = 0.$$

The matrix Q is symmetric. Therefore, all its eigenvalues are real. It is also not difficult to see that $\lambda_{max} > 0$. Theorem 1 is proved. □

6. Discussion and comments

(1) The constraints imposed on the roots of the characteristic polynomial $p_m(t)$ of the differential operator $\mathcal{L}_m(D)$ in Theorem 1 can be partially weakened. In particular, one can exclude the constraints $\beta_i \neq 0$ and $\beta_i \neq \pm\beta_j$, $i \neq j$ $(i, j = 1, 2, \dots, m)$. In this case, the characteristic polynomials of the differential operators $\mathcal{L}_m(D)$ and $\mathcal{L}_m^*(D)$ will have common roots. The approach used in the paper can be implemented with minor modifications for such differential operators. In so doing, the explicit expressions for $\mathcal{E}_{2m}(t)$ (see Lemma 5), the entries of the matrix $\mathcal{A}$, and the numbers $R_{\mu n}$ (see Theorem 1) will be different from those given in the paper. However, in the framework of the applied approach, it is impossible to discard the constraint $\beta_i \in \mathbb{R}$, $i = 1, 2, \dots, m$. The point is that if the polynomial p_m has nonreal complex roots (two or more), then for an arbitrary interval $[a, b]$, it fails to prove that a solution to the Favard-type interpolation problem is a natural $\mathcal{L}$-spline. Apparently, in this case, we need to introduce some additional restrictions that would associate the segment $[a, b]$ with oscillation properties of the differential operator $\mathcal{L}_m(D)$.

(2) Another approach to the study of problems (1.3) and (1.4) is known. This approach is based on the concept of Chebyshevian splines (see e.g. [6, Chap. 10, Sect. 3] or [7, Chap. 11]). Since $\ker \mathcal{L}_m(D) = \operatorname{span}\{e^{\beta_1 x}, e^{\beta_2 x}, \dots, e^{\beta_m x}\}$, this set of functions is an ECT-system on any finite interval $[a, b]$. Due to this, Theorem 1 can be proved by using of this approach.

(3) For any given N and prescribed knots of the mesh Δ_N, all coefficients of the quadratic form $V(z_1, z_2, \dots, z_N)$ are calculated. This can be made directly or by using symbolic computation systems (Maple and others). Numerical algorithms allow one to find the largest eigenvalue of

the symmetric matrix Q approximately. One can also estimate λ_{max} from above and below (see, for example, [12, 19] and many other publications). Specifically, from [19], we can write the estimate

$$\frac{|\mathrm{tr}\ Q|}{N} \le |\lambda_{max}| \le \frac{1}{N}\Big(|\mathrm{tr}\ Q| + \sqrt{N-1}\ \big(N\ \mathrm{tr}\ Q^2 - (\mathrm{tr}\ Q)^2\big)^{1/2}\Big),$$

where tr Q is the sum of elements located on the main diagonal of the matrix Q.

(4) The periodical analogs of quantities (1.3) and (1.4) for the differential operator $\mathcal{L}_m(D) = D^m$ were studied in the author's paper [10]. Periodicity requirements for both interpolated values $z = \{z_k\}_{k=0}^{2N-1}$ and interpolating functions have been added to problem statements. However, the results in [10] were obtained only for the mesh with equidistant knots $\Delta_{2N} = \{j\pi/N\}_{j=0}^{2N-1}$ that was 2π-periodically extended into $\mathbb{R}$. For the class of interpolated values

$$\widetilde{\mathfrak{M}_{2N}} = \Big\{z:\ z = \{z_j\}_{j=0}^{2N-1}, \Big(\sum_{j=0}^{2N-1} |z_j|^2\Big)^{1/2} \le 1\Big\},$$

it was proved that if $N > m,\ m \ge 2$, then

$$\widetilde{\mathfrak{B}_m}(\Delta_{2N}) = \sup_{z\in\widetilde{\mathfrak{M}_{2N}}} \inf_{\substack{f^{(m-1)}\in\widetilde{AC} \\ f(j\pi/N)=z_j}} \Big(\int_0^{2\pi} |f^{(m)}(t)|^2 dt\Big)^{1/2} = \Big(\frac{\pi}{N}\Big)^{1/2}\Bigg(\sum_{l\in\mathbb{Z}} \frac{1}{(2Nl+(N-1))^{2m}}\Bigg)^{-1/2},$$

where $\widetilde{AC}$ is the class of 2π-periodic absolutely continuous functions.

Note that the series on the right hand side of the last equality is convergent. It is easy to see using the comparison test for number series.

While proving this result, the largest eigenvalue of an analog of the matrix Q also arose. However, due to the specificity of the periodic case and the uniform grid, the analog of the matrix Q is the circulant, and its eigenvalues are known in an explicit form.

(5) Along with our settings of problems (1.3) and (1.4), one can consider the corresponding multivariate settings. For the case of two variables, analogs of natural splines are known [13]. Problem (1.4) was considered in the case of the Laplace operator. This was done in the author's recent paper [11].

7. Conclusion

We have considered the problem of finding quantity (1.4). In a certain sense, this quantity is the value of the L_2-norm of the differential operator applied to the "best" interpolant under interpolating the "worst" data from the given class. In this paper, we found the exact value of the studied quantity.

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