URAL MATHEMATICAL JOURNAL, Vol. 11, No. 1, 2025, pp. 63–76

DOI: 10.15826/umj.2025.1.004

A PAIR OF FOUR-ELEMENT HORIZONTAL GENERATING SETS OF A PARTITION LATTICE[1][2]

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Abstract: Let $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the lower integer part and the upper integer part of a real number x, respectively. Our main goal is to construct four partitions of a finite set A with $n \geq 7$ elements such that each of the four partitions has exactly $\lceil n/2 \rceil$ blocks and any other partition of A can be obtained from the given four by forming joins and meets in a finite number of steps. We do the same with $\lceil n/2 \rceil - 1$ instead of $\lceil n/2 \rceil$, too. To situate the paper within lattice theory, recall that the *partition lattice* $\mathrm{Eq}(A)$ of a set A consists of all partitions (equivalently, of all equivalence relations) of A. For a natural number n, $[n]$ and $\mathrm{Eq}(n)$ will stand for $\{1, 2, \ldots, n\}$ and $\mathrm{Eq}([n])$, respectively. In 1975, Heinrich Strietz proved that, for any natural number $n \geq 3$, $\mathrm{Eq}(n)$ has a four-element generating set; half a dozen papers have been devoted to four-element generating sets of partition lattices since then. We give a simple proof of his just-mentioned result. We call a generating set X of $\mathrm{Eq}(n)$ *horizontal* if each member of X has the same height, denoted by $h(X)$, in $\mathrm{Eq}(n)$; no such generating sets have been known previously. We prove that for each natural number $n \geq 4$, $\mathrm{Eq}(n)$ has two four-element horizontal generating sets X and Y such that $h(Y) = h(X) + 1$; for $n \geq 7$, $h(X) = \lfloor n/2 \rfloor$.

Keywords: Partition lattice, Equivalence lattice, Minimum-sized generating set, Horizontal generating set, Four-element generating set.

1. Notes on the dedication

Árpád Kurusa, 1961–2024, was an excellent geometer. The present paper is dedicated to his memory. In addition to his high reputation in geometry, his editorial and technical editorial work for several mathematical journals as well as his textbooks (in Hungarian) were also deeply acknowledged. From 2000 to 2018, he led the Department of Geometry at the Bolyai (Mathematical) Institute of the University of Szeged. As the title of [5] shows, our collaboration has added a piece to the traditionally strong interrelation between geometry and lattice theory. At the motivational level, the present paper has some (but very slight) connection to the just-mentioned joint paper. Indeed, partition lattices form a specific subclass of *geometric* lattices, and the term "horizontal" is rooted in a *geometric* perspective of these lattices.

2. Introduction and our theorem

Given a set A, the collection of *equivalences*, that is, the collection of reflexive, symmetric, transitive relations of A forms a lattice $\mathrm{Eq}(A)$, the *equivalence lattice* of A. In this lattice, the meet and the join are the intersection and the transitive hull of the union, respectively. By the well-known bijective correspondence between the equivalences of A and the partitions of A, $\mathrm{Eq}(A)$

[1]This research was supported by the National Research, Development and Innovation Fund of Hungary, under funding scheme K 138892.

[2]*Dedicated to the memory of my local colleague and co-author Árpád Kurusa.*

is isomorphic to the *partition lattice* of A, which consists of all partitions of A. By the just-mentioned correspondence, we make no sharp distinction between equivalences and partitions in our terminology and notations. To explain that we use the notation $\mathrm{Eq}(A)$ rather than something like $\mathrm{Part}(A)$, note that equivalences are more appropriate for performing the lattice operations and forming restrictions. For a natural number n, we let $[n] := \{1, 2, \dots, n\}$, and we usually abbreviate $\mathrm{Eq}([n])$ to $\mathrm{Eq}(n)$.

Partition lattices play an important role in lattice theory since congruence lattices, which play a central role in universal algebra, are naturally embedded in partition lattices. In fact, every lattice is embeddable into a partition lattice by Whitman [12] and each finite lattice into a finite partition lattice by Pudlák and Tůma [9]; note that these facts can be exploited in some proofs, for example, in [1]. Furthermore, every partition lattice $\mathrm{Eq}(A)$ is known to be a *geometric lattice*, that is, an atomistic semimodular lattice; see, e.g., Grätzer [7, Section IV.4] or [8, Section V.3]. Being *atomistic* means that each element x of $\mathrm{Eq}(A)$ is the join of all atoms below x. *Semimodularity* is understood as upper semimodularity, that is, for any $x, y, z \in \mathrm{Eq}(A)$, $x \preceq y$ implies that $x \vee z \preceq y \vee z$, where $\preceq$ is the "is covered by or equal to" relation.

A subset X of $\mathrm{Eq}(A)$ is a *generating set* of $\mathrm{Eq}(A)$ if X extends to no proper subset S of $\mathrm{Eq}(A)$ such that S is closed with respect to joins and meets. In the seventies, Strietz [10] and [11] proved that, for any natural number $n \geq 3$, $\mathrm{Eq}(n)$ has a four-element generating set. His result is optimal, since $\mathrm{Eq}(n)$ does not have a three-element generating set provided that $n \geq 4$. Since Strietz's pioneering work was published in [10] and [11], five additional papers have already been devoted to the four-element generating sets of equivalence lattices; see [6], the 2nd-, the 3rd-, and the 4th-item in the "References" section of [6], and Zádori [13].

For $n \geq 3$, which is always assumed, each permutation of $[n]$ extends to an automorphism of $\mathrm{Eq}(n)$, and such an automorphism sends generating sets to generating sets. We say that two generating sets of $\mathrm{Eq}(n)$ are *essentially different* if no such automorphism sends one of them to the other one. We know even from Strietz [10] and [11] that, for n large enough, $\mathrm{Eq}(n)$ has several essentially different four-element generating sets. Many more (essentially different) four-element generating sets have been given in [6]. However, it is very likely by the computer-assisted section of [6] that only an infinitesimally small percentage of the four-element generating sets of $\mathrm{Eq}(n)$ are known for n large. Exploring more such generating sets seems to be a reasonable target in its own right, and there is an additional motivation: Namely, the more small generating sets of $\mathrm{Eq}(n)$ are available, the more the cryptographic ideas of [2] can benefit from equivalence lattices. (If there are and *we know* many four-element generating sets, then we can extend them to small generating sets in very many ways.)

Before explaining what sort of new four-element generating sets of $\mathrm{Eq}(n)$ we are going to present, note that even at the very beginning of this type of research in the seventies, Strietz himself paid attention to some lattice theoretical properties of his four-element generating sets. For $n \geq 4$, he showed that a four-element generating set is either an *antichain* (that is, a subset with no comparable elements) or it is of order type $1 + 1 + 2$, that is, exactly two out of the four generators are comparable. He managed to prove that $\mathrm{Eq}(n)$ has a four-element generating set of order type $1 + 1 + 2$ for every integer $n \geq 10$. Briefly saying, $\mathrm{Eq}(n)$ is $(1 + 1 + 2)$-*generated* for $n \geq 10$. With ingenious constructions, Zádori [13] improved "$n \geq 10$" to $n \geq 7$, and he gave a visual proof of Strietz's result that $\mathrm{Eq}(n)$ has a four-element generating set; his proofs are simpler than Strietz's ones. Zádori [13] left open the problem whether $\mathrm{Eq}(5)$ and $\mathrm{Eq}(6)$ are $(1 + 1 + 2)$-generated. This problem was solved as recently as 2020 in [6], where an affirmative answer for $\mathrm{Eq}(6)$ was given but a computer-assisted negative answer for $\mathrm{Eq}(5)$ was provided.

As $\mathrm{Eq}(n)$ is a geometric lattice, there is a natural property of a subset, which is more restrictive than being an antichain. To introduce it, recall that the *length* of an n-element chain is $n - 1$. The least element and the largest element of $\mathrm{Eq}(n)$ or $\mathrm{Eq}(A)$ will be denoted by Δ and ∇, respectively.

If confusion threatens, we write Δ_n, ∇_A, etc. The height of an element $\mu \in \mathrm{Eq}(n)$ is the length of a maximal chain in the interval $[\Delta, \mu]$; we know from the Jordan-Hölder Chain Condition for semimodular lattices, see, e.g., Grätzer [7, Theorem IV.2.1, p. 226] or [8, Theorem 377], that no matter which maximal chain is taken. We denote the *height* of μ by $h(\mu)$. A subset X of $\mathrm{Eq}(n)$ is *horizontal* if its elements are of the same height; in this case, the common height of the elements of X is denoted by $h(X)$. A horizontal subset of $\mathrm{Eq}(n)$ is necessarily an antichain. Clearly, $\mathrm{Eq}(n)$ for $n \geq 3$ has a *horizontal generating set*, since the set of atoms is such. To get a better insight into the four-element generating sets of partition lattices, it is reasonable to determine those natural numbers n for which $\mathrm{Eq}(n)$ has a *four-element horizontal generating set*. In fact, we are going to do more by showing that whenever $\mathrm{Eq}(n)$ has a four-element antichain at all, that is, whenever $n \geq 4$, then it has two four-element horizontal generating sets of neighboring heights. To smooth our terminology, let us introduce the notation

$$\mathrm{HFHGS}(n) := \{h(X) : X \text{ is a four-element horizontal generating set of } \mathrm{Eq}(n)\};$$

the acronym above comes from the **h**eights of **f**our-element **h**orizontal **g**enerating **s**ets. For a real number r, we denote by $\lfloor r \rfloor$ and $\lceil r \rceil$ the *lower integer part* and the *upper integer part* of r; for example, $\lfloor \sqrt{2} \rfloor = 1$ and $\lceil \sqrt{2} \rceil = 2$. Let $\mathbb{N}^+$ denote the set of positive integers.

Theorem 1. *For every natural number $n \geq 4$, the partition lattice $\mathrm{Eq}(n)$ has two four-element horizontal generating sets X and Y such that $h(Y) = h(X)+1$ holds for their heights. Furthermore,*

$$\mathrm{HFHGS}(n) \supseteq \{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1\} \text{ for all integers } n \geq 7 \text{ and also for } n = 5, \text{ and} \tag{2.1}$$
$$\mathrm{HFHGS}(n) \subseteq \{k \in \mathbb{N}^+ : \lfloor (n-1)/4 \rfloor + 1 \leq k \leq n - \lceil \sqrt[4]{n} \rceil\} \text{ for all integers } n \geq 4. \tag{2.2}$$

Based on the following statement, we conjecture that "$\supseteq$" in (2.1) is never an equality for $n \geq 7$. We do not know whether $\lim_{n\to\infty} |\mathrm{HFHGS}(n)| = \infty$ and $\mathrm{HFHGS}(n)$ is always a convex subset of $\mathbb{N}^+$. We know $\mathrm{HFHGS}(n)$ only for $n \in \{4, 5, 6, 7, 8\}$. In the proposition below, each occurrence of the relation symbol $\overset{\text{comp}}{=}$ denotes an equality that we could prove only with the assistance of the brute force of a computer.

Proposition 1. *We have the following equalities and inclusions:*

$$\mathrm{HFHGS}(4) = \{1, 2\}, \tag{2.3}$$
$$\mathrm{HFHGS}(5) = \{2, 3\}, \tag{2.4}$$
$$\{2, 3\} \subseteq \mathrm{HFHGS}(6) \subseteq \{2, 3, 4\}, \quad \textit{in fact,} \quad \mathrm{HFHGS}(6) \overset{\text{comp}}{=} \{2, 3\}, \tag{2.5}$$
$$\{2, 3, 4\} \subseteq \mathrm{HFHGS}(7) \subseteq \{2, 3, 4, 5\}, \quad \textit{in fact,} \quad \mathrm{HFHGS}(7) \overset{\text{comp}}{=} \{2, 3, 4\}, \quad \textit{and} \tag{2.6}$$
$$\{3, 4, 5\} \subseteq \mathrm{HFHGS}(8) \subseteq \{2, 3, 4, 5, 6\}, \quad \textit{in fact,} \quad \mathrm{HFHGS}(8) \overset{\text{comp}}{=} \{3, 4, 5\}. \tag{2.7}$$

Remark 1. (2.3) and (2.5) witness that (2.1) fails for $n \in \{4, 6\}$. Note also that concrete four-element horizontal generating sets witnessing (2.1) and (2.3)–(2.7) are defined by Lemma 5 combined with Assertion 1, by Lemmas 6, 7 and 8 combined with both (the Key) Lemma 4 and Assertion 1, and in the rest of the lemmas presented in Section 5. For n large, the just-mentioned four-element horizontal generating sets are given only inductively; the inductive feature could be eliminated but we do not strive for non-inductive definitions of these generating sets.

The rest of the paper is devoted to proving Theorem 1 and Proposition 1. Unless explicitly stated otherwise, we assume that $4 \leq n \in \mathbb{N}^+$ for the remainder of the paper.

3. Some lemmas, the Key Lemma, and a new proof of one of Strietz's results

For a finite nonempty set A, if $\{a_{1,1},\dots,a_{1,t_1}\}$, …, $\{a_{k,1},\dots,a_{k,t_k}\}$ is a repetition-free list of the blocks of a partition $\mu\in\mathrm{Eq}(A)$, then we denote both μ and the corresponding equivalence by

$$\mathrm{eq}(a_{1,1},\dots,a_{1,t_1};\dots;a_{k,1},\dots,a_{k,t_k})\quad\text{or}\quad\mathrm{eq}(a_{1,1}\dots a_{1,t_1};\dots;a_{k,1}\dots a_{k,t_k}).$$

That is, we omit the commas when no confusion threatens but not the block-separating semicolons. Usually, the elements in a block and the blocks are listed in lexicographic order. For example,

$$\Delta_4=\mathrm{eq}(1;2;3;4),\quad \nabla_4=\mathrm{eq}(1234),\quad\text{and}\quad \nabla_{11}=\mathrm{eq}(1,2,3,4,5,6,7,8,9,10,11);$$

for more involved examples, see Lemmas 5–15. For $u,v\in A$, the least equivalence of A collapsing u and v will be denoted by $\mathrm{at}(u,v)$ or, if confusion threatens, by $\mathrm{at}_A(u,v)$. For example, in $\mathrm{Eq}(6)$, $\mathrm{at}(2,5)=\mathrm{eq}(1;25;3;4;6)$. Note that $\mathrm{at}(u,v)$ is an ***at**om* of $\mathrm{Eq}(A)$ (that is, a cover of Δ), and every atom of $\mathrm{Eq}(A)$ is of this form.

We define the *graph* $G(S)$ of a sublattice S of $\mathrm{Eq}(A)$ by letting A be the *vertex set* of $G(S)$ and letting $\{(a,b): a\neq b \text{ and } \mathrm{at}(a,b)\in S\}$ be the *edge set* of $G(S)$. (No matter if we consider (a,b) and (b,a) equal or different.) A *Hamiltonian circle* of $G(S)$ is a permutation $a_1,a_2,\dots,a_n$ of the elements of A such that $\mathrm{at}(a_{i-1},a_i)\in S$ for $i\in[n]-\{1\}$ and $\mathrm{at}(a_n,a_1)\in S$. Of course, $G(S)$ need not have a Hamiltonian circle. The following lemma occurs, explicitly or implicitly, in several papers dealing with generating sets of equivalence lattices; see, for example, Czédli and Oluoch [6, Lemma 2.5]. For the reader's convenience, we are going to outline its trivial proof.

Lemma 1 (Hamiltonian Cycle Lemma). *For a finite set A with at least three elements and a sublattice S of $\mathrm{Eq}(A)$, we have that $S=\mathrm{Eq}(A)$ if and only if $G(S)$ has a Hamiltonian circle.*

P r o o f. The "only if" part is trivial. To prove the "if" part, let $a_1,\dots,a_n$ be a Hamiltonian circle of $G(S)$. As each element of the atomistic lattice $\mathrm{Eq}(A)$ is the join of some atoms, it suffices to show that for all $i\neq j$, $i,j\in[n]$, we have that $\mathrm{at}(a_i,a_j)\in S$. This membership follows from

$$\begin{aligned}\mathrm{at}(a_i,a_j)=&\big(\mathrm{at}(a_i,a_{i+1})\vee\mathrm{at}(a_{i+1},a_{i+2})\vee\dots\vee\mathrm{at}(a_{j-1},a_j)\big)\\ &\wedge\big(\mathrm{at}(a_i,a_{i-1})\vee\mathrm{at}(a_{i-1},a_{i-2})\vee\dots\vee\mathrm{at}(a_2,a_1)\\ &\vee\mathrm{at}(a_1,a_n)\vee\mathrm{at}(a_n,a_{n-1})\vee\mathrm{at}(a_{n-1},a_{n-2})\vee\dots\vee\mathrm{at}(a_{j+1},a_j)\big)\end{aligned}$$

and the "commutativity" $\mathrm{at}(x,y)=\mathrm{at}(y,x)$. □

Let $\mathbb{Z}_4:=(\{0,1,2,3\},+)$ denote the cyclic group of order 4; the addition in it is performed modulo 4. To give the lion's share of the proof of (2.3) and also to present an easy consequence of Lemma 1, we present the following lemma, in which the addition is understood in $\mathbb{Z}_4$.

Lemma 2. *Both*

$$X:=\{\mathrm{at}(i,i+1): i\in\mathbb{Z}_4\}$$

and

$$Y:=\{\mathrm{at}(i,i+1)\vee\mathrm{at}(i+1,i+2): i\in\mathbb{Z}_4\}$$

are four-element horizontal generating sets of $\mathrm{Eq}(\mathbb{Z}_4)\cong\mathrm{Eq}(4)$.

P r o o f. By Lemma 1, X generates $\mathrm{Eq}(\mathbb{Z}_4)$. Since

$$\mathrm{at}(i,i+1)=\big(\mathrm{at}(i,i+1)\vee\mathrm{at}(i+1,i+2)\big)\wedge\big(\mathrm{at}(i-1,i)\vee\mathrm{at}(i,i+1)\big)\quad\text{for}\quad i\in\mathbb{Z}_4,$$

it follows that X is contained in the sublattice of $\mathrm{Eq}(\mathbb{Z}_4)$ generated by Y, whence Y also generates $\mathrm{Eq}(\mathbb{Z}_4)$. □

Next, we introduce a concept that is crucial in the proof of Theorem 1. By an n-element *eligible structure* we mean a 7-tuple

$$\mathcal{A} = (A, \alpha, \beta, \gamma, \delta, u, v) \tag{3.1}$$

such that A is an n-element finite set, u and v are distinct elements of A, $\{\alpha, \beta, \gamma, \delta\}$ is a four-element generating set of $\mathrm{Eq}(A)$, and

$$\alpha \vee \delta = \nabla, \quad \alpha \wedge \delta = \Delta, \tag{3.2}$$

$$\beta \wedge \big(\gamma \vee \mathrm{at}(u, v)\big) = \Delta, \quad \gamma \wedge \big(\beta \vee \mathrm{at}(u, v)\big) = \Delta, \tag{3.3}$$

$$\text{and} \ \ \beta \vee \gamma \vee \mathrm{at}(u, v) = \nabla. \tag{3.4}$$

To present an example and also for a later reference, we formulate the following statement.

Lemma 3. *With* $\alpha = \mathrm{eq}(123; 4)$*,* $\beta = \mathrm{eq}(14; 2; 3)$*,* $\gamma = \mathrm{eq}(1; 2; 34)$*, and* $\delta = \mathrm{eq}(1; 24; 3)$*,*

$$\mathcal{A} := ([4], \alpha, \beta, \gamma, \delta, 1, 2) \tag{3.5}$$

is an eligible structure.

P r o o f. Let S be the sublattice of $\mathrm{Eq}(4)$ generated by $\{\alpha, \beta, \gamma, \delta\}$. Since

$$\mathrm{at}(1, 2) = \mathrm{eq}(12; 3; 4) = \alpha \wedge (\beta \vee \delta) \in S, \quad \mathrm{at}(2, 3) = \alpha \wedge (\gamma \vee \delta) \in S, \quad \mathrm{at}(3, 4) = \gamma \in S,$$

and $\mathrm{at}(4, 1) = \beta \in S$, the sequence 1, 2, 3, 4 is a Hamiltonian cycle in $G(S)$. Thus, Lemma 1 implies that $\{\alpha, \beta, \gamma, \delta\}$ generates $\mathrm{Eq}(4)$. Since (3.2), (3.3), and (3.4) are trivially satisfied, the proof of Lemma 3 is complete. □

For $A \subseteq B$ and $\mu \in \mathrm{Eq}(A)$, the smallest equivalence of B that includes μ will be denoted by μ_B^{ext}. The superscript in the notation comes from "**ext**ension". As a partition, μ_B^{ext} consists of the blocks of μ and the singleton blocks $\{b\}$ for $b \in B - A$.

Lemma 4 (Key Lemma). *Assume that* $(A, \alpha, \beta, \gamma, \delta, u, v)$ *is an eligible structure,* $|A| \geq 4$*,* $w \notin A$*, and* $B = A \cup \{w\}$*. Let*

$$\begin{gathered} \alpha' := \beta_B^{\mathrm{ext}} \vee \mathrm{at}_B(u, w), \quad \beta' := \alpha_B^{\mathrm{ext}}, \quad \gamma' := \delta_B^{\mathrm{ext}}, \quad \delta' := \gamma_B^{\mathrm{ext}} \vee \mathrm{at}_B(v, w), \\ u' := u, \quad v' := w. \end{gathered} \tag{3.6}$$

Then the **e**xtended **s**tructure

$$\mathrm{ES}(\mathcal{A}) := \mathcal{B} = \big(B, \alpha', \beta', \gamma', \delta', u', v'\big) \tag{3.7}$$

is also an eligible structure. The heights of the partitions occurring in (3.6)–(3.7) *satisfy that*

$$h(\alpha') = h(\beta) + 1, \quad h(\beta') = h(\alpha), \quad h(\gamma') = h(\delta), \quad h(\delta') = h(\gamma) + 1. \tag{3.8}$$

P r o o f. Assume that $\mathcal{A}$ is an eligible structure and $\mathcal{B} = \mathrm{ES}(\mathcal{A})$ is as in (3.7). We will frequently but mostly implicitly use the obvious fact that the function $f\colon \mathrm{Eq}(A) \to \mathrm{Eq}(B)$ defined by $\mu \mapsto \mu_B^{\mathrm{ext}}$ is a lattice embedding and, for any $\mu \in \mathrm{Eq}(A)$, $h(f(\mu)) = h(\mu)$. Denote by S the sublattice generated by $\{\alpha', \beta', \gamma', \delta'\}$ in $\mathrm{Eq}(B)$. For $\mu \in \mathrm{Eq}(B)$, let $\mu\restriction_A$ denote the *restriction* of μ to A. That is, as an equivalence, $\mu\restriction_A = \mu \cap (A \times A)$. E.g.,

$$\big((\Delta_A)^{\mathrm{ext}}_B\big)\restriction_A = \Delta_A.$$

Note the obvious rule:

$$(\rho^{\mathrm{ext}}_B)\restriction_A = \rho \quad \text{and} \quad (\mu\restriction_A)^{\mathrm{ext}}_B = \mu \wedge (\nabla_A)^{\mathrm{ext}}_B \quad \text{for every} \quad \rho \in \mathrm{Eq}(A) \quad \text{and} \quad \mu \in \mathrm{Eq}(B). \tag{3.9}$$

Let us agree that, for $x, y \in B$, $\mathrm{at}(x, y)$ is understood as $\mathrm{at}_B(x, y)$ even when $x, y \in A$. We claim that for any $\mu \in \mathrm{Eq}(A)$ and for any $d \in A$,

$$\big(\mu^{\mathrm{ext}}_B \vee \mathrm{at}_B(d, w)\big)\restriction_A = \mu; \tag{3.10}$$

and, in particular,

$$\alpha'\restriction_A = \beta \quad \text{and} \quad \delta'\restriction_A = \gamma. \tag{3.11}$$

The inequality

$$\big(\mu^{\mathrm{ext}}_B \vee \mathrm{at}_B(d, w)\big)\restriction_A \geq \mu$$

is clear. To show the converse inequality, assume that $a \neq b$ and (a, b) belongs to $\big(\mu^{\mathrm{ext}}_B \vee \mathrm{at}_B(d, w)\big)\restriction_A$. Then $a, b \in A$ and, by the description of the join in equivalence lattices, there exists a *shortest* sequence $x_0 = a$, x_1, …, x_{t-1}, $x_t = b$ of elements of B such that, for each $i \in [t]$,

$$\text{either } (x_{i-1}, x_i) \in \mu^{\mathrm{ext}}_B \quad \text{or} \quad (x_{i-1}, x_i) \in \{(d, w), (w, d)\}. \tag{3.12}$$

Since this sequence is repetition-free, the first alternative in (3.12) means that $(x_{i-1}, x_i) \in \mu$. By way of contradiction, suppose that not all elements of the sequence are in A. Let j be the smallest subscript such that $x_j \notin A$. As $x_0 = a \in A$ and $x_t = b \in A$, we have that $0 < j < t$. By the choice of j, $x_{j-1} \in A$. This rules out that $(x_{j-1}, x_j) = (w, d)$. Since $x_j \notin A$, $(x_{j-1}, x_j) \in \mu$ cannot occur either. Hence, $(x_{j-1}, x_j) = (d, w)$. However, then the only possibility to continue the sequence is that $(x_j, x_{j+1}) = (w, d)$. So d occurs in the sequence at least twice, which contradicts the fact that our sequence is repetition-free. Therefore, all elements of the sequence are in A, whereby the first alternative of (3.12) holds for all i. Thus, $(x_{i-1}, x_i) \in \mu$ for $i \in [t]$, and we obtain the required membership $(a, b) = (x_0, x_t) \in \mu$ by transitivity. We have shown (3.10). Letting $(\mu, d) := (\beta, u)$ and $(\mu, d) := (\gamma, v)$, (3.10) implies (3.11).

Next, using the first half of (3.2) (and the fact that f is an embedding), we obtain that

$$(\nabla_A)^{\mathrm{ext}}_B = (\alpha \vee \delta)^{\mathrm{ext}}_B = \alpha^{\mathrm{ext}}_B \vee \delta^{\mathrm{ext}}_B = \beta' \vee \gamma'$$

belongs to S. Hence, so does $\alpha' \wedge (\nabla_A)^{\mathrm{ext}}_B$. By the second half of (3.9) applied to $\mu := \alpha'$, this equivalence is $(\alpha'\restriction_A)^{\mathrm{ext}}_B$, whence $(\alpha'\restriction_A)^{\mathrm{ext}}_B \in S$. Therefore, applying (3.11), $\beta^{\mathrm{ext}}_B \in S$. As β and γ play a symmetric role, γ^{ext}_B is also in S. By (3.6), S contains $\alpha^{\mathrm{ext}}_B = \beta'$ and $\delta^{\mathrm{ext}}_B = \gamma'$. So $f(\mu) = \mu^{\mathrm{ext}}_B \in S$ for every $\mu \in \{\alpha, \beta, \gamma, \delta\}$. Since f is an embedding and $\{\alpha, \beta, \gamma, \delta\}$ generates $\mathrm{Eq}(A)$, we conclude that $f(\mathrm{Eq}(A)) \subseteq S$. In particular, $\mathrm{at}_B(u, v) = f(\mathrm{at}_A(u, v)) \in S$. Based on this containment, we claim that

$$\mathrm{at}_B(u, w) = \alpha' \wedge \big(\mathrm{at}_B(u, v) \vee \delta'\big) \in S. \tag{3.13}$$

As $\mathrm{at}_B(u, v), \alpha', \delta' \in S$, it suffices to show the equality in (3.13). The inequality "$\leq$" in place of the equality is clear by the definition of α' given in (3.6). To show the converse inequality, assume that $a \neq b$ and (a, b) belongs to the right-hand side of the equality in (3.13). Let

$$\nu := \mathrm{at}_A(u, v) \vee \gamma.$$

Observe that

$$(a, b) \in \alpha' \wedge \big(\nu_B^{\rm ext} \vee {\rm at}_B(v, w)\big), \tag{3.14}$$

since

$$\begin{aligned}\alpha' \wedge \big(\nu_B^{\rm ext} \vee {\rm at}_B(v, w)\big) &= \alpha' \wedge \big(({\rm at}_A(u, v) \vee \gamma)_B^{\rm ext} \vee {\rm at}_B(v, w)\big)\\ &= \alpha' \wedge \big(({\rm at}_A(u, v))_B^{\rm ext} \vee \gamma_B^{\rm ext} \vee {\rm at}_B(v, w)\big)\end{aligned} \tag{3.15}$$

$$= \alpha' \wedge \big({\rm at}_B(u, v) \vee \gamma_B^{\rm ext} \vee {\rm at}_B(v, w)\big) \overset{(3.6)}{=} \alpha' \wedge \big({\rm at}_B(u, v) \vee \delta'\big). \tag{3.16}$$

As $a \neq b$ and $|B - A| = |\{w\}| = 1$, at least one of a and b is in A. By symmetry, we can assume that $a \in A$. Depending on the position of b, there are two cases.

First, assume that b is also in A. Then $(a, b) \in \alpha'$ and (3.11) give that $(a, b) \in \beta$. As (a, b) is in the second meetand in (3.14) and $a, b \in A$, we have that

$$(a, b) \in \big(\nu_B^{\rm ext} \vee {\rm at}_B(v, w)\big)\rceil_A.$$

Hence, (3.10) applied to $(\mu, d) := (\nu, v)$ yields that $(a, b) \in \nu$. Thus, (a, b) belongs to

$$\beta \wedge \nu = \beta \wedge \big({\rm at}_A(u, v) \vee \gamma\big),$$

which is Δ_A by (3.3). Since $(a, b) \in \Delta_A$ contradicts the assumption $a \neq b$, the first case cannot occur.

Second, assume that $b \notin A$. Then

$$(a, w) = (a, b) \in \alpha' \wedge \big({\rm at}_B(u, v) \vee \delta'\big)$$

and $a \in A$. By (3.6), $(w, u) \in \alpha'$. As both (w, v) and (v, u) belong to the second meetand of (3.15), (w, u) belongs to this meetand, too. These facts, (3.15), and (3.16) give that $\alpha' \wedge \big({\rm at}_B(u, v) \vee \delta'\big)$ contains (w, u). By transitivity, it contains (a, u), too. If we had that $a \neq u$, then (a, u) (with u playing the role of b) would be a contradiction by the first case. Thus, $a = u$, that is, $(a, b) = (u, w) \in {\rm at}_B(u, w)$, as required. We have shown the validity of (3.13).

We obtain the following fact analogously; we can derive it also from (3.13) by symmetry, since $(A, \delta, \gamma, \beta, \alpha, v, u)$ is also an eligible structure:

$${\rm at}_B(v, w) = \delta' \wedge \big({\rm at}_B(u, v) \vee \alpha'\big) \in S. \tag{3.17}$$

With $n := |A|$, list the elements of B as follows:

$$c_1 := u, \quad c_2, \ \ldots, \ c_{n-1}, \ c_n := v, \quad c_{n+1} := w.$$

Since $f({\rm Eq}(A)) \subseteq S$ and $c_1, \ldots, c_n \in A$, we have that

$${\rm at}_B(c_i, c_{i+1}) = f\big({\rm at}_A(c_i, c_{i+1})\big) \in S,$$

that is, (c_i, c_{i+1}) is an edge of $G(S)$ for $i \in [n-1]$. So are $(c_n, c_{n+1}) = (v, w)$ and $(c_{n+1}, c_1) = (w, u)$ by (3.17) and by (3.13), respectively. Therefore, our list is a Hamiltonian cycle, and Lemma 1 implies that $\{\alpha', \beta', \gamma', \delta'\}$ is a generating set of ${\rm Eq}(B)$. This set is four-element since $|B| \geq 4$ and so we know from Strietz [10] or [11] that ${\rm Eq}(B)$ cannot be generated by less than four elements.

Clearly, $u' = u \in A$ is distinct from $v' = w \in B - A$. Since

$$\begin{aligned}\alpha' \vee \delta' &\overset{(3.6)}{=} \beta_B^{\rm ext} \vee {\rm at}_B(u, w) \vee \gamma_B^{\rm ext} \vee {\rm at}_B(v, w) = \beta_B^{\rm ext} \vee \gamma_B^{\rm ext} \vee {\rm at}_B(u, v) \vee {\rm at}_B(v, w)\\ &= \big(\beta \vee \gamma \vee {\rm at}_A(u, v)\big)_B^{\rm ext} \vee {\rm at}_B(v, w) \overset{(3.4)}{=} (\nabla_A)_B^{\rm ext} \vee {\rm at}_B(v, w) = \nabla_B,\end{aligned}$$

$\mathcal{B}$ satisfies the first half of (3.2). To show by way of contradiction that $\mathcal{B}$ fulfills the second half, suppose that $a \neq b$ and $(a,b) \in \alpha' \wedge \delta'$. If $a,b \in A$, then (3.11) leads to $(a,b) \in \beta \wedge \gamma = \Delta_A$, contradicting that $a \neq b$. So one of a and b is w, and we can assume that $a \in A$ and $b = w$. As $(a,w) = (a,b) \in \alpha'$ and $(w,u) \in \alpha'$, we have that $(a,u) \in \alpha'$. Hence, $(a,u) \in \beta$ by (3.11). Similarly, $(a,w),(w,v) \in \delta'$ and (3.11) imply that $(a,v) \in \gamma$. The just-obtained memberships and relations give that

$$(a,u) \in \beta \wedge \big(\gamma \vee \mathrm{at}_A(u,v)\big) \quad \text{and} \quad (a,v) \in \gamma \wedge \big(\beta \vee \mathrm{at}_A(u,v)\big).$$

Combining this with (3.3), we obtain that $a = u$ and $a = v$, contradicting $u \neq v$. So we have proved that $\mathcal{B}$ fulfills (3.2).

By symmetry, to show that $\mathcal{B}$ satisfies (3.3), it suffices to deal with its first half. For the sake of contradiction, suppose that

$$\beta' \wedge \big(\gamma' \vee \mathrm{at}_B(u',v')\big) \neq \Delta_B.$$

Then we can pick $a,b \in B$ such that $a \neq b$ and

$$(a,b) \in \beta' \wedge \big(\gamma' \vee \mathrm{at}_B(u',v')\big) \overset{(3.6)}{=} \alpha_B^{\mathrm{ext}} \wedge \big(\delta_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w)\big). \tag{3.18}$$

The containment $(a,b) \in \alpha_B^{\mathrm{ext}}$ gives that $a,b \in A$. The meet in $\mathrm{Eq}(B)$ is the set-theoretic intersection, so it commutes with the restriction map. Hence, applying the first equality of (3.9) with $\rho := \alpha$ and (3.10) with $(\mu,d) := (\delta,u)$ at $\overset{*}{=}$, (3.18) leads to

$$(a,b) \in \big(\alpha_B^{\mathrm{ext}} \wedge (\delta_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w))\big)\rceil_A = \alpha_B^{\mathrm{ext}}\rceil_A \wedge \big(\delta_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w)\big)\rceil_A \overset{*}{=} \alpha \wedge \delta \overset{(3.2)}{=} \Delta_A \subseteq \Delta_B,$$

which contradicts the assumption $a \neq b$ and proves that $\mathcal{B}$ satisfies (3.3). Since

$$\beta' \vee \gamma' \vee \mathrm{at}_B(u',v') \overset{(3.6)}{=} \alpha_B^{\mathrm{ext}} \vee \delta_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w) = (\alpha \vee \delta)_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w)$$
$$\overset{(3.2)}{=} (\nabla_A)_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w) = \nabla_B,$$

$\mathcal{B}$ satisfies (3.4), too. We have proved that $\mathcal{B}$ is an eligible structure, as required.

For a finite nonempty set H and μ in $\mathrm{Eq}(H)$, let $\mathrm{NumB}(\mu)$ denote the ***num**ber of **b**locks* of μ. For example, if $\mu = \mathrm{eq}(14;25;3) \in \mathrm{Eq}(5)$, then $\mathrm{NumB}(\mu) = 3$. The following folkloric fact is trivial:

$$\text{For any } \mu \in \mathrm{Eq}(H), \quad h(\mu) + \mathrm{NumB}(\mu) = |H|. \tag{3.19}$$

Clearly, (3.6) leads to

$$\mathrm{NumB}(\alpha') = \mathrm{NumB}(\beta), \quad \mathrm{NumB}(\beta') = \mathrm{NumB}(\alpha) + 1,$$
$$\mathrm{NumB}(\gamma') = \mathrm{NumB}(\delta) + 1, \quad \mathrm{NumB}(\delta') = \mathrm{NumB}(\gamma).$$

These equalities and (3.19) imply (3.8), completing the proof of the Key Lemma. □

Now we are in the position to give a new proof of Strietz's result stating that $\mathrm{Eq}(n)$ is four-generated. For those who prefer theoretical arguments rather than long and tedious computations with concrete partitions, the proof below is presumably simpler than the earlier ones.

Corollary 1 (Strietz [10] and [11]). *For any natural number $n \geq 3$, $\mathrm{Eq}(n)$ has a four-element generating set.*

P r o o f. As the case $n = 3$ is trivial, we assume that $n \geq 4$. Let $\mathcal{A}_4$ be the eligible structure given in (3.5); see (3.1). For $n > 4$, define $\mathcal{A}_n$ as $\mathrm{ES}(\mathcal{A}_{n-1})$. Then, for each $n \geq 4$, $\mathcal{A}_n$ is an n-element eligible structure by Lemmas 3 and (the Key) Lemma 4. Thus, by the definition of eligible structures, $\mathrm{Eq}(n)$ is four-generated, completing the proof of Corollary 1. □

4. A tediously provable lemma

The n-th *Bell number* $B(n)$ is defined to be the number of elements of $\mathrm{Eq}(n)$, that is, $B(n) := |\mathrm{Eq}(n)|$. As n grows, $B(n)$ grows very fast; see https://oeis.org/A000110 of N. J. A. Sloan's Online Encyclopedia of Integer Sequences. For example,

$$|\mathrm{Eq}(6)| = B(6) = 203, \quad |\mathrm{Eq}(8)| = 4\,140, \quad |\mathrm{Eq}(9)| = 21\,147, \quad \text{and}$$
$$|\mathrm{Eq}(20)| = 51\,724\,158\,235\,372 \approx 5.17 \cdot 10^{13}.$$

These large numbers explain our experience that even when it is feasible to prove that a four-element subset X of $\mathrm{Eq}(n)$ generates $\mathrm{Eq}(n)$, this task requires straightforward but tedious computations in general. Each of Lemmas 5–15 belongs to this category by stating that a subset X of $\mathrm{Eq}(n)$ generates $\mathrm{Eq}(n)$; some of these lemmas state slightly more, but these surpluses are trivial to verify. We offer two ways to verify these lemmas.

First, one can read their proofs based on Lemma 1. One of these proofs is given in this section. As the rest of these proofs are long without containing a single new idea, the proofs of Lemmas 6–15 are given only in Appendix 1 of the extended version of the paper. At the time of writing, this extended version is at https://tinyurl.com/czg-h4ge (and also at the author's website[3] http://tinyurl.com/g-czedli/), and it will be available at www.arxiv.org soon.

Second, the author has developed three closely related computer programs in Dev-Pascal 1.9.2 under Windows 10. These programs, available at https://tinyurl.com/czg-equ2024p or at the author's website given in the previous paragraph, form a mini-package. The main program and its auxiliary program are also given in Appendices 2 and 3 of the extended version of the paper. The third program performs the same tasks as the first one and also uses the auxiliary program. Despite being slower, it is more cross-platform because it requires less computer memory. For $n \le 9$, the auxiliary program lists the elements of $\mathrm{Eq}(n)$; the other two programs rely on this list. In what follows, by a program, we mean the main program. The program can "prove" Lemmas 5–15, and it can also "prove" the $\overset{\text{comp}}{=}$ parts of (2.5)–(2.7). In fact, the program has been designed to perform the following two tasks.

First, the program can take an $n \in \{4, 5, \dots, 9\}$ and a four-element subset X of $\mathrm{Eq}(n)$ as inputs. After enlarging X by adding the join and the meet of any two of its elements as long as the enlargement is proper, the program computes the sublattice S generated by X. Then the program displays the size $|S|$ of S on the screen and tells whether X generates $\mathrm{Eq}(n)$. The program can prove Lemma 8, where $n = 9$, in about fifteen minutes. For Lemma 14, where $n = 8$, 25 seconds suffice. Note that for just one four-element subset X of $\mathrm{Eq}(n)$, it is not worthwhile to create and the program does not create the operation tables of $\mathrm{Eq}(n)$. For this (the first) task, there is no difference between the main program and its slower variant.

Second, for a given $n \in \{4, 5, \dots, 9\}$ and a $k \in [n-1]$ as inputs, the program decides whether $\mathrm{Eq}(n)$ has a four-element horizontal generating set of height k. For $(n, k) = (8, 2)$, this takes about three and a half minutes, provided the program runs on a desktop computer with AMD Ryzen 7 2700X Eight-Core Processor and 3.70 GHz with 16 GB memory. For $(n, k) = (9, 3)$, if $\mathrm{Eq}(9)$ has no four-element horizontal generating set of height 3, which we do not know, the program would need about a month; partially because there is not enough computer memory to store the operation tables of $\mathrm{Eq}(9)$ and also because there are significantly more cases.

The quotation marks around "proved" in a paragraph above indicate that the author believes but cannot prove that the program itself is error-free. The source code of the program and that of its auxiliary program are 24 and 8 kilobytes, respectively, totaling 32 kilobytes. Proving *exactly* that the program is perfect would probably be harder than verifying all proofs in Appendix 1.

[3]This standard "tiny" short link redirects us to the real URL https://www.math.u-szeged.hu/~czedli/ .

Lemma 5. *With*

$$\alpha := \mathrm{eq}(123;4;5), \tag{4.1}$$
$$\beta := \mathrm{eq}(1;23;45), \tag{4.2}$$
$$\gamma := \mathrm{eq}(13;25;4), \text{ and} \tag{4.3}$$
$$\delta := \mathrm{eq}(15;2;34), \tag{4.4}$$

$([5], \alpha, \beta, \gamma, \delta, 1, 4)$ *is an eligible structure and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 2$.

P r o o f. Let S denote the sublattice of $\mathrm{Eq}(5)$ generated by $\{\alpha, \beta, \gamma, \delta\}$. We will list some members of S; each of them belongs to S by earlier containments as indicated.

$$\mathrm{eq}(1;23;4;5) = \mathrm{eq}(123;4;5) \wedge \mathrm{eq}(1;23;45) \in S \quad \text{by (4.1) and (4.2)}, \tag{4.5}$$
$$\mathrm{eq}(13;2;4;5) = \mathrm{eq}(123;4;5) \wedge \mathrm{eq}(13;25;4) \in S \quad \text{by (4.1) and (4.3)}, \tag{4.6}$$
$$\mathrm{eq}(1235;4) = \mathrm{eq}(123;4;5) \vee \mathrm{eq}(13;25;4) \in S \quad \text{by (4.1) and (4.3)}, \tag{4.7}$$
$$\mathrm{eq}(15;234) = \mathrm{eq}(15;2;34) \vee \mathrm{eq}(1;23;4;5) \in S \quad \text{by (4.4) and (4.5)}, \tag{4.8}$$
$$\mathrm{eq}(1345;2) = \mathrm{eq}(15;2;34) \vee \mathrm{eq}(13;2;4;5) \in S \quad \text{by (4.4) and (4.6)}, \tag{4.9}$$
$$\mathrm{eq}(15;2;3;4) = \mathrm{eq}(15;2;34) \wedge \mathrm{eq}(1235;4) \in S \quad \text{by (4.4) and (4.7)}, \tag{4.10}$$
$$\mathrm{eq}(1;2;3;45) = \mathrm{eq}(1;23;45) \wedge \mathrm{eq}(1345;2) \in S \quad \text{by (4.2) and (4.9)}, \tag{4.11}$$
$$\mathrm{eq}(13;245) = \mathrm{eq}(13;25;4) \vee \mathrm{eq}(1;2;3;45) \in S \quad \text{by (4.3) and (4.11)}, \tag{4.12}$$
$$\mathrm{eq}(1;24;3;5) = \mathrm{eq}(15;234) \wedge \mathrm{eq}(13;245) \in S \quad \text{by (4.8) and (4.12)}. \tag{4.13}$$

Let $E(S)$ denote the edge set of the graph $G(S)$; it is defined in the paragraph preceding Lemma 1. Since $(1,3) \in E(S)$ by (4.6), $(3,2) \in E(S)$ by (4.5), $(2,4) \in E(S)$ by (4.13), $(4,5) \in E(S)$ by (4.11), and $(5,1) \in E(S)$ by (4.10), the sequence $1,3,2,4,5$ is a Hamiltonian cycle of $G(S)$. Hence, $\{\alpha, \beta, \gamma, \delta\}$ is a generating set of $\mathrm{Eq}(5)$ by Lemma 1. Armed with this fact, now it is a trivial task to verify that $([5], \alpha, \beta, \gamma, \delta, 1, 4)$ satisfies (3.2), (3.3), and (3.4), whereby it is an eligible structure. Thus, (3.19) completes the proof Lemma 5. □

5. The rest of tediously provable lemmas

We need the following ten lemmas, too. As indicated in the second paragraph of Section 4, their proofs are given only in Appendix 1 of the extended version of the paper.

Lemma 6. *With*

$$\alpha := \mathrm{eq}(134;256;7), \quad \beta := \mathrm{eq}(146;27;3;5), \quad \gamma := \mathrm{eq}(135;2;4;67), \quad \text{and} \quad \delta := \mathrm{eq}(12;357;46),$$

$([7], \alpha, \beta, \gamma, \delta, 2, 3)$ *is an eligible structure,* $h(\alpha) = h(\delta) = 4$*, and* $h(\beta) = h(\gamma) = 3$.

Lemma 7. *With*

$$\alpha := \mathrm{eq}(134;258;67), \quad \beta := \mathrm{eq}(14;2;36;578),$$
$$\gamma := \mathrm{eq}(17;25;348;6), \quad \text{and} \quad \delta := \mathrm{eq}(12;378;456),$$

$([8], \alpha, \beta, \gamma, \delta, 2, 6)$ *is an eligible structure,* $h(\alpha) = h(\delta) = 5$*, and* $h(\beta) = h(\gamma) = 4$.

Lemma 8. *With*

$$\alpha := \mathrm{eq}(178; 249; 356), \quad \beta := \mathrm{eq}(19; 26; 378; 45),$$
$$\gamma := \mathrm{eq}(1; 28; 359; 467), \quad \textit{and} \quad \delta := \mathrm{eq}(169; 258; 347),$$

$([9], \alpha, \beta, \gamma, \delta, 1, 2)$ *is an eligible structure,* $h(\alpha) = h(\delta) = 6$*, and* $h(\beta) = h(\gamma) = 5$.

Lemma 9. *With*

$$\alpha := \mathrm{eq}(134; 25), \quad \beta := \mathrm{eq}(13; 245), \quad \gamma := \mathrm{eq}(12; 345), \quad \textit{and} \quad \delta := \mathrm{eq}(124; 35),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(5)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 3$.

Lemma 10. *With*

$$\alpha := \mathrm{eq}(12; 34; 5; 6), \quad \beta := \mathrm{eq}(1; 2; 35; 46), \quad \gamma := \mathrm{eq}(1; 25; 36; 4), \quad \textit{and} \quad \delta := \mathrm{eq}(15; 24; 3; 6),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(6)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 2$.

Lemma 11. *With*

$$\alpha := \mathrm{eq}(13; 256; 4), \quad \beta := \mathrm{eq}(156; 2; 34), \quad \gamma := \mathrm{eq}(12; 35; 46), \quad \textit{and} \quad \delta := \mathrm{eq}(13; 246; 5),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(6)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 3$.

Lemma 12. *With*

$$\alpha := \mathrm{eq}(1; 24; 35; 6; 7), \quad \beta := \mathrm{eq}(14; 26; 3; 5; 7),$$
$$\gamma := \mathrm{eq}(1; 2; 34; 5; 67), \quad \textit{and} \quad \delta := \mathrm{eq}(17; 2; 3; 4; 56),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(7)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 2$.

Lemma 13. *With*

$$\alpha := \mathrm{eq}(13; 24; 567), \quad \beta := \mathrm{eq}(125; 3; 467) \quad \gamma := \mathrm{eq}(1357; 26; 4), \quad \textit{and} \quad \delta := \mathrm{eq}(126; 35; 47),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(7)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 4$.

Lemma 14. *With*

$$\alpha := \mathrm{eq}(18; 2; 35; 4; 67), \quad \beta := \mathrm{eq}(1; 24; 37; 5; 68),$$
$$\gamma := \mathrm{eq}(16; 2; 34; 57; 8), \quad \textit{and} \quad \delta := \mathrm{eq}(12; 3; 45; 6; 78),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(8)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 3$.

Lemma 15. *With*

$$\alpha := \mathrm{eq}(137; 246; 58), \quad \beta := \mathrm{eq}(146; 257; 38), \quad \gamma := \mathrm{eq}(136; 2; 4578), \quad \textit{and} \quad \delta := \mathrm{eq}(1245; 37; 68),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(8)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 5$.

6. Proving Theorem 1 and Proposition 1 with our lemmas

Since the proof of Theorem 1 relies on parts of Proposition 1 and the proof of Proposition 1 uses (2.2) from Theorem 1, we present a combined proof of both the theorem and the proposition.

P r o o f (Proving Theorem 1 and Proposition 1). First, we deal with (2.2). Assume that $\{\alpha_1, \dots, \alpha_4\}$ is a four-element horizontal generating set of $\mathrm{Eq}(n)$ with height k. That is, $k = h(\alpha_i)$ for $i \in [4]$. We need to prove that

$$\lfloor (n-1)/4 \rfloor + 1 \le k \le n - \lceil \sqrt[4]{n}\, \rceil. \tag{6.1}$$

By semimodularity, see Grätzer [7, Theorem IV.2.2, p. 226], the height of $\alpha_1 \vee \dots \vee \alpha_4$ is at most $h(\alpha_1) + \dots + h(\alpha_4) = 4k$. The just-mentioned join is the largest element of the sublattice S generated by $\{\alpha_1, \dots, \alpha_4\}$. But this sublattice is $\mathrm{Eq}(n)$, so this join is ∇_n, whereby $h(\nabla_n) \le 4k$. We know from, say, (3.19) that $h(\nabla_n) = n - 1$. Thus, the previous inequality turns into $(n-1)/4 \le k$. If $(n-1)/4 < k$, then $\lfloor (n-1)/4 \rfloor < k$ and we obtain the first inequality of (6.1) since k is an integer. Hence, it suffices to exclude that $(n-1)/4 = k$. To obtain a contradiction, suppose that $(n-1)/4 = k$, that is, $n - 1 = h(\nabla_n) = 4k$. Let $i \in [4]$. As $h(\alpha_i) = k$, we can find k atoms $\beta_{k(i-1)+1}$, $\beta_{k(i-1)+2}$, ..., β_{ki} in $\mathrm{Eq}(n)$ such that α_i is the join of these atoms; the existence of such atoms is clear in $\mathrm{Eq}(n)$ and it is true even in any geometric lattice by Grätzer [7, Theorems IV.2.4–IV.2.5, p. 228–229] or [8, Theorems 380–381]. As $\{\alpha_1, \dots, \alpha_4\}$ generates $\mathrm{Eq}(n)$, $\alpha_1 \vee \dots \vee \alpha_4 = \nabla_n$. Hence,

$$h\Bigl(\bigvee_{j=1}^{4k} \beta_j\Bigr) = h(\alpha_1 \vee \dots \vee \alpha_4) = h(\nabla_n) = n - 1 = 4k.$$

Therefore, Grätzer [7, Theorem IV.2.4, p. 228] or [8, Theorem 380] yields that $\{\beta_1, \dots, \beta_{4k}\}$ is an independent set of atoms; this means that $\{\beta_1, \dots, \beta_{4k}\}$ generates a Boolean sublattice T of $\mathrm{Eq}(n)$. In particular, T is a distributive. As $\alpha_1, \dots, \alpha_4$ are in T, they generate a sublattice of T, which is distributive, too. This means that $\mathrm{Eq}(n)$ is distributive, which contradicts the assumption that $n \ge 4$. Therefore, $(n-1)/4 = k$ cannot occur and we have proved the first inequality in (6.1).

Clearly, $\alpha_1 \wedge \dots \wedge \alpha_4$, which is the smallest element of S, is Δ_n. Let $b := \mathrm{NumB}(\alpha_i)$; by (3.19), $b = n - k$ does not depend on $i \in [4]$. The largest block C_1 of α_1 has at least n/b elements. When we form the meet $\alpha_1 \wedge \alpha_2$, then C_1 splits into at most b blocks of $\alpha_1 \wedge \alpha_2$ and the largest one of these blocks has at least $(n/b)/b$ elements. So $\alpha_1 \wedge \alpha_2$ has a block C_2 with at least n/b^2 elements. And so on; finally, $\Delta_n = \alpha_1 \wedge \dots \wedge \alpha_4$ has a block with at least n/b^4 elements. But Δ_n has only one-element blocks, whereby $n/b^4 \le 1$, that is, $b \ge \sqrt[4]{n}$. Thus $b \ge \lceil \sqrt[4]{n}\, \rceil$, since $b \in \mathbb{N}^+$. Therefore, as we know from (3.19) that $b = n - k$, we obtain that $k \le n - \lceil \sqrt[4]{n}\, \rceil$. This completes the proof of (6.1) and that of (2.2).

Next, assume that $\mathcal{A} = (A, \alpha, \beta, \gamma, \delta, u, v)$. With the "extended structure operator" introduced in (3.7), we use the notation $(C, \alpha'', \beta'', \gamma'', \delta'', u'', v'')$ for $\mathrm{ES}^2(\mathcal{A}) := \mathrm{ES}(\mathrm{ES}(\mathcal{A}))$. Clearly, (the Key) Lemma 4 implies the following assertion.

Assertion 1. *If $\mathcal{A} = (A, \alpha, \beta, \gamma, \delta, u, v)$ is an eligible structure and $\mathcal{C} = (C, \alpha'', \beta'', \gamma'', \delta'', u'', v'')$ is $\mathrm{ES}^2(\mathcal{A})$, then $\mathcal{C}$ is also an eligible structure,*

$$h(\alpha'') = h(\alpha) + 1, \quad h(\beta'') = h(\beta) + 1, \quad h(\gamma'') = h(\gamma) + 1, \quad \textit{and} \quad h(\delta'') = h(\delta) + 1.$$

Resuming the proof, let us agree that, for any meaningful x, $\mathcal{A}_{\mathrm{L}x}$ denotes the eligible structure defined in Lemma x. For example, $\mathcal{A}_{\mathrm{L}5}$ is defined in Lemma 5. We call an eligible structure *horizontal* if its four partitions have the same height; this common height is the *height* of the structure.

By Lemma 5, $\mathcal{A}_{L5}$ is a 5-element horizontal eligible structure of height 2. Applying Assertion 1 repeatedly, we obtain a 7-element horizontal eligible structure, a 9-element horizontal eligible structure, etc. of heights 3, 4, ..., respectively. Thus,

$$\text{for } n \geq 5 \text{ odd, Eq}(n) \text{ has a four-element horizontal generating set of height } \lfloor n/2 \rfloor. \tag{6.2}$$

By Lemma 7 and (the Key) Lemma 4, $\text{ES}(\mathcal{A}_{L7})$ is a 9-element horizontal eligible structure of height 5. Applying Assertion 1 repeatedly, we obtain an 11-element horizontal eligible structure, a 13-element horizontal eligible structure, etc. of heights 6, 7, ..., respectively. Hence,

$$\text{for } n \geq 9 \text{ odd, Eq}(n) \text{ has a four-element horizontal generating set of height } \lfloor n/2 \rfloor + 1. \tag{6.3}$$

By Lemma 6 and (the Key) Lemma 4, $\text{ES}(\mathcal{A}_{L6})$ is an 8-element horizontal eligible structure of height 4. Hence, the repeated use of Assertion 1 yields that

$$\text{for } n \geq 8 \text{ even, Eq}(n) \text{ has a four-element horizontal generating set of height } \lfloor n/2 \rfloor. \tag{6.4}$$

By Lemma 8 and (the Key) Lemma 4, $\text{ES}(\mathcal{A}_{L8})$ is a 10-element horizontal eligible structure of height 6. Hence, the repeated use of Assertion 1 yields that

$$\text{for } n \geq 10 \text{ even, Eq}(n) \text{ has a four-element horizontal generating set of height } \lfloor n/2 \rfloor + 1. \tag{6.5}$$

We know from Lemma 9 that Eq(5) is generated by a four-element horizontal generating set of height $\lceil 5/2 \rceil + 1$. By Lemma 13, Eq(7) has four-element horizontal generating set of height $(\lfloor 7/2 \rfloor + 1)$. For Eq(8), a four-element horizontal generating set of height $(\lfloor 8/2 \rfloor + 1)$ is provided by Lemma 15. These three facts, (6.2), (6.3), (6.4), and (6.5) imply (2.1).

In what follows, we will implicitly use that Eq(n) has no four-element horizontal subset of height 0 or $n - 1$. Since there is no four-element subset of height 0 or 3 in Eq(4), Lemma 2 implies (2.3).

Since $\{2, 3\} \subseteq \text{HFHGS}(5)$ by (2.2), (2.1) implies (2.4).

We obtain from (2.2) and Lemmas 10–11 that $\{2, 3\} \subseteq \text{HFHGS}(6) \subseteq \{2, 3, 4\}$. As the already mentioned computer program yields that $4 \notin \text{HFHGS}(6)$ in less than a second[4], (2.5) holds.

Lemma 12, (2.1), and (2.2) imply that $\{2, 3, 4\} \subseteq \text{HFHGS}(7) \subseteq \{2, 3, 4, 5\}$. In 2 seconds, the program excludes that $5 \in \text{HFHGS}(7)$. Thus, we have shown (2.6).

Lemma 14, (2.1) and (2.2) yield that $\{3, 4, 5\} \subseteq \text{HFHGS}(8) \subseteq \{2, 3, 4, 5, 6\}$, as required. The program excludes 2 and 6 from HFHGS(8) in three and a half minutes and in one minute, respectively. Thus, we proved the validity of (2.7) and that of Proposition 1.

Finally, the first sentence of Theorem 1 follows from (2.3), (2.4) or (2.1), the first inclusion in (2.5), and from (2.1). The combined proof of Theorem 1 and Proposition 1 is complete. □

7. Conclusion

Motivated by earlier results on four-element generating sets of finite equivalence lattices and their link to cryptography, we have proved the existence of two four-element horizontal generating sets of consecutive heights in these lattices. After the first submission of the paper, this result—and the method behind it—motivated two subsequent papers on four-element generating sets of equivalence lattices with other special properties (see [3] and [4]). We anticipate similar results in the future.

[4]The auxiliary program creates the auxiliary files containing the lists of partitions of $[n]$ for $n \leq 9$ in 4 seconds, but this has to be done only once. Thus, here and later, even though the program needs these files, the just-mentioned 4 seconds are not counted. The time for entering n and k are not counted either.

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