URAL MATHEMATICAL JOURNAL, Vol. 10, No. 2, 2024, pp. 157–173

DOI: 10.15826/umj.2024.2.014

PROPERTIES OF SOLUTIONS IN THE DUBINS CAR CONTROL PROBLEM[1]

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Abstract: This paper addresses the time-optimal control problem of the Dubins car, which is closely related to the problem of constructing the shortest curve with bounded curvature between two points in a plane. This connection allows researchers to apply both geometric methods and control theory techniques during their investigations. It is established that the time-optimal control for the Dubins car is a piecewise constant function with no more than two switchings. This characteristic enables the categorization of all such controls into several types, facilitating the examination of the solutions to the control problem for each type individually. The paper derives explicit formulas for determining the switching times of the control signal. In each case, necessary and sufficient conditions for the existence of solutions are obtained. For certain control types, the uniqueness of optimal solutions is established. Additionally, the dependence of the movement time on the initial and terminal conditions is studied.

Keywords: Dubins car, Dubins problem, Time-optimal control, Curve with bounded curvature.

1. Introduction

The Dubins car is a simple mathematical model of a car-like vehicle that moves in a plane at a constant speed and is capable of making left and right turns with a bounded turning radius. The time-optimal control problem of the Dubins car is closely related to the problem of constructing the shortest curve with bounded curvature between two points in a plane. One of the first studies on this subject was by Markov [11], in which he considered the shortest curves with a prescribed tangent at one of the endpoints. The problem of constructing the shortest curve with a constraint on average curvature and with prescribed tangents at both endpoints was later investigated by Dubins [7]. This problem later became known as the Markov–Dubins problem or simply as the Dubins problem, and the solution to this problem was referred to as the Dubins path. Moreover, as demonstrated in [8], not only geometric methods but also control theory methods can be used to study plane curves. Significant results in this direction were obtained in [5, 9, 19].

The Markov–Dubins problem and its variations have been extensively studied over the past several decades. We mention in particular the construction of the shortest bounded-curvature paths in 3-dimensional space [18], the investigation of homotopy classes of bounded-curvature paths [1], and the description of the reachable sets for the Dubins car [13, 14]. Reeds and Shepp [16] notably extended Dubins' original work by considering a vehicle capable of both forward and reverse motion, resulting in the formulation of the Reeds–Shepp car model. Numerous other extended models can also be found in [3, 4, 12]. The practical applications of the Markov–Dubins problem are widespread, impacting fields such as railroad construction [11], air traffic control [15], robotics [2], and many other domains.

[1]This research was supported by the Russian Science Foundation (grant no. 24-21-00424, https://rscf.ru/en/project/24-21-00424/).

It was shown in [7] that the shortest path with bounded curvature between two points in a plane consists of no more than three segments, each of which is either an arc of a circle or a straight line segment. The same result was obtained for the trajectories of the Dubins car [5, 19]. As a consequence, the Markov–Dubins problem can be reduced to finding the shortest path among several candidate paths. In [15], the parameters of the candidate paths were found for a fixed terminal position. The case of a moving target was investigated in [6]. Paper [17] considers the case where the starting and ending points are far apart, and provides a decision table for finding the shortest path. In [10], the endpoints of the curve segments were found by a geometric approach.

In this paper, we investigate the properties of the candidate paths and the corresponding controls in the time-optimal control problem of the Dubins car. The paper is organized as follows. Section 2 outlines the time-optimal control problem for the Dubins car and categorizes the control set into three distinct types. In Section 3, we derive formulas for calculating the switching times associated with each type of control. Section 4 identifies key properties for each control type. Finally, Section 5 illustrates how these properties can be used to solve the time-optimal control problem for the Dubins car.

2. Problem statement

Consider a vehicle that moves in a horizontal plane at a constant speed, capable of making left and right turns. The motion of the vehicle is governed by the system of ordinary differential equations

$$\begin{cases} \dot{x} = v\cos\varphi, \\ \dot{y} = v\sin\varphi, \\ \dot{\varphi} = u, \end{cases} \tag{2.1}$$

where x and y are the Cartesian coordinates of the vehicle in the xy-plane, φ is the orientation of the velocity vector, v is the speed, and u is the control variable. It is assumed that the angle φ is measured counterclockwise from the positive x axis and can take any real values. An admissible control is a Lebesgue measurable function $u(t)$ that satisfies the constraint $|u(t)| \leqslant u_m$, $u_m > 0$, on any finite time interval. The mathematical model described by (2.1) is called the “Dubins car”.

In this model, any two orientation angles φ_* and φ^* such that $\varphi_* - \varphi^* = 2\pi k$, $k \in \mathbb{Z}$, are considered equivalent. $\mathbb{Z}$ denotes the set of integers. We should note that the coefficient k will be used in this paper to define various sets. In this regard, all these coefficients should be treated independently of each other.

The time-optimal control problem of the Dubins car can be described as follows. Suppose we are given a vector of boundary conditions $\mathbf{w} = (x_0, y_0, \varphi_0, x_f, y_f, \varphi_f)$, where (x_0, y_0) and (x_f, y_f) are the initial and terminal positions of the vehicle, and φ_0 and φ_f are the initial and terminal orientations, respectively. It is required to find an admissible control that transfers system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ in the minimum possible time. Since system (2.1) is time-invariant, the initial time t_0 can be chosen arbitrarily.

In [5, 13, 19], it is shown that the time-optimal control for the Dubins car is a piecewise constant function having no more than three segments with lengths Δt_1, Δt_2, and Δt_3 and values u_1, u_2, and u_3, respectively, where $u_1 \in \{-u_m, u_m\}$, $u_2 \in \{-u_m, 0, u_m\}$, and $u_3 \in \{-u_m, u_m\}$. Let $u_* \in \{-u_m, u_m\}$. Then all such controls can be divided into the following types:

1. Control of the type $(u_*, -u_*, u_*)$, where $u_1 = u_*$, $u_2 = -u_*$, and $u_3 = u_*$.
2. Control of the type $(u_*, 0, u_*)$, where $u_1 = u_*$, $u_2 = 0$, and $u_3 = u_*$.
3. Control of the type $(u_*, 0, -u_*)$, where $u_1 = u_*$, $u_2 = 0$, and $u_3 = -u_*$.

Thus, the time-optimal control problem of the Dubins car can be solved by testing the optimal controls of several specific types. Accordingly, the purpose of this paper is to study the properties of each of these types of control. Specifically, we aim to investigate the necessary and sufficient conditions for the existence of solutions, the uniqueness of optimal solutions, and the dependence of the movement time on the initial and terminal conditions.

3. Switching times

This section provides a solution to the time-optimal control problem of the Dubins car for controls of the types $(u_*, -u_*, u_*)$, $(u_*, 0, u_*)$, and $(u_*, 0, -u_*)$. In each case, we derive explicit formulas for determining the optimal time intervals Δt_1, Δt_2, and Δt_3. The lengths of the intervals are assumed to be nonnegative. This condition, called a nonnegativity condition, can be written as

$$\begin{cases} \Delta t_1 \geqslant 0, \\ \Delta t_2 \geqslant 0, \\ \Delta t_3 \geqslant 0. \end{cases} \tag{3.1}$$

Note that the intervals are allowed to be degenerate. Knowing Δt_1, Δt_2, and Δt_3, we can find the switching times t_1 and t_2 and the terminal time t_f by the simple relations

$$t_1 = t_0 + \Delta t_1, \quad t_2 = t_1 + \Delta t_2, \quad t_f = t_2 + \Delta t_3.$$

Before proceeding to specific types of control, we introduce some notation and definitions.

Definition 1. *Denote by* $\operatorname{sgn}(x)$ *the function of a real variable* x *defined by*

$$\operatorname{sgn}(x) = \begin{cases} 1, & x \geqslant 0, \\ -1, & x < 0. \end{cases}$$

If $x \neq 0$, the function $\operatorname{sgn}(x)$ can be written as $\operatorname{sgn}(x) = |x|/x$.

Definition 2. *The modulo operation* $a \bmod b$ *is the binary operation that associates with each pair of real numbers* a *and* $b \neq 0$ *the nonnegative remainder after dividing* a *by* b, *that is, a number* $r \in [0, |b|)$ *such that* $a = qb + r$, *where* $q \in \mathbb{Z}$.

Definition 3. *By a multivalued function* $F\colon X \to \mathcal{P}(Y)$ *we mean a function that maps elements of* X *to subsets of* Y.

We extend standard binary operations that take two single-valued arguments to binary operations that take one single-valued argument and one multivalued argument as follows.

Definition 4. *Let* $*\colon X \times X \to X$ *be a binary operation. For each* $x \in X$ *and* $\sigma \subset X$, *define* $x * \sigma = \{x * y \mid y \in \sigma\}$ *and* $\sigma * x = \{y * x \mid y \in \sigma\}$.

Next, we proceed to prove a preliminary lemma.

Lemma 1. *Let* F *be a multivalued real function of the form* $F(x) = f(x) + G$, *where* f *is a continuous single-valued function,* $G = \{ka \mid k \in \mathbb{Z}\}$, *and* a *is a positive constant. Let a multivalued function* H *be defined as* $H(x) = F(x) \bmod a$, *and let a single-valued function* h *be defined as*

$$h(x) = \min\{y \in F(x) \mid y \geqslant 0\}.$$

Then $H(x) = \{h(x)\}$. *Moreover, if* $f(x_*) \neq ma$, $m \in \mathbb{Z}$, *then* h *is continuous at* x_*.

P r o o f. Let x^* be an arbitrary point in $\operatorname{dom} H$. We first show that $H(x^*)$ cannot contain two different elements. Suppose there are $h_1 \in H(x^*)$ and $h_2 \in H(x^*)$ such that $h_1 \neq h_2$. Then h_1 and h_2 must satisfy the system

$$\begin{cases} f(x^*) + k_1 a = q_1 a + h_1, & h_1 \in [0, a), \quad k_1 \in \mathbb{Z}, \quad q_1 \in \mathbb{Z}, \\ f(x^*) + k_2 a = q_2 a + h_2, & h_2 \in [0, a), \quad k_2 \in \mathbb{Z}, \quad q_2 \in \mathbb{Z}. \end{cases} \tag{3.2}$$

Subtracting the second equation from the first and collecting the terms that involve a factor of a on the left-hand side, we obtain

$$(k_1 - k_2 - q_1 + q_2)a = h_1 - h_2. \tag{3.3}$$

Since $h_1 \in [0, a)$ and $h_2 \in [0, a)$, we have $-a < h_1 - h_2 < a$. Hence, equality (3.3) holds only when $k_1 - k_2 - q_1 + q_2 = 0$ and $h_1 = h_2$. This is a contradiction.

From (3.2), it follows that $h_1 \in F(x^*)$. We claim that $h(x^*) = h_1$. Suppose there is $h_3 \in F(x^*)$ such that $0 \leqslant h_3 < h_1$. By definition, $h_3 = f(x^*) + k_3 a$, where $k_3 \in \mathbb{Z}$. Then

$$h_1 - h_3 = (k_1 - q_1 - k_3)a.$$

However, $0 < h_1 - h_3 < a$. This results in a contradiction. Hence, $h(x^*) = h_1$.

Next, we show that the function h is continuous at x_* if $f(x_*) \neq ma$, $m \in \mathbb{Z}$. Pick any $\varepsilon > 0$. Since f is continuous, it follows that

$$\exists \delta > 0\colon \|x - x_*\| < \delta \Rightarrow |f(x) - f(x_*)| < \varepsilon.$$

Let

$$\varepsilon_m = \min_{m \in \mathbb{Z}} |f(x_*) - ma|.$$

Then we get

$$\exists \delta_m > 0\colon \|x - x_*\| < \delta_m \Rightarrow |f(x) - f(x_*)| < \varepsilon_m.$$

Let $\delta_* = \min\{\delta, \delta_m\}$. Then we find that

$$\|x - x_*\| < \delta_* \Rightarrow |f(x) - f(x_*)| < \varepsilon.$$

Furthermore, if $\|x - x_*\| < \delta_*$, then there exists $p \in \mathbb{Z}$ such that

$$f(x) = pa + h(x), \quad f(x_*) = pa + h(x_*).$$

Finally, we obtain

$$\|x - x_*\| < \delta_* \Rightarrow |h(x) - h(x_*)| = |f(x) - f(x_*)| < \varepsilon.$$

Since ε was chosen arbitrarily, we conclude that the function h is continuous at x_*. □

3.1. Controls of the type $(u_*, -u_*, u_*)$

Given a vector of boundary conditions $\mathbf{w} = (x_0, y_0, \varphi_0, x_f, y_f, \varphi_f)$, the goal is to find a control of the type $(u_*, -u_*, u_*)$ that transfers system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ in minimum time.

For this type of control, the function $\varphi(t)$ on the time interval $[t_0, t_f]$ can be expressed as

$$\varphi(t) = \begin{cases} \varphi_0 + u_*(t - t_0), & t \in [t_0, t_1), \\ \varphi_0 + u_*\Delta t_1 - u_*(t - t_1), & t \in [t_1, t_2), \\ \varphi_0 + u_*\Delta t_1 - u_*\Delta t_2 + u_*(t - t_2), & t \in [t_2, t_f]. \end{cases} \tag{3.4}$$

Substituting (3.4) into (2.1) gives

$$\begin{aligned} x(t_f) = x_0 + \int_0^{\Delta t_1} v\cos(\varphi_0 + u_*\tau)\,d\tau + \int_0^{\Delta t_2} v\cos(\varphi_0 + u_*\Delta t_1 - u_*\tau)\,d\tau \\ + \int_0^{\Delta t_3} v\cos(\varphi_0 + u_*\Delta t_1 - u_*\Delta t_2 + u_*\tau)\,d\tau \\ = x_0 + \frac{v}{u_*}\big(2\sin(\varphi_0 + u_*\Delta t_1) - 2\sin(\varphi_0 + u_*\Delta t_1 - u_*\Delta t_2) + \sin(\varphi(t_f)) - \sin(\varphi_0)\big), \end{aligned} \tag{3.5}$$

$$\begin{aligned} y(t_f) = y_0 + \int_0^{\Delta t_1} v\sin(\varphi_0 + u_*\tau)\,d\tau + \int_0^{\Delta t_2} v\sin(\varphi_0 + u_*\Delta t_1 - u_*\tau)\,d\tau \\ + \int_0^{\Delta t_3} v\sin(\varphi_0 + u_*\Delta t_1 - u_*\Delta t_2 + u_*\tau)\,d\tau \\ = y_0 - \frac{v}{u_*}\big(2\cos(\varphi_0 + u_*\Delta t_1) - 2\cos(\varphi_0 + u_*\Delta t_1 - u_*\Delta t_2) + \cos(\varphi(t_f)) - \cos(\varphi_0)\big). \end{aligned} \tag{3.6}$$

Combining (3.4), (3.5), and (3.6) with the terminal condition, we obtain the system

$$\begin{cases} x_0 + \frac{v}{u_*}\big(2\sin(\varphi_0 + u_*\Delta t_1) - 2\sin(\varphi_0 + u_*\Delta t_1 - u_*\Delta t_2) + \sin(\varphi_f) - \sin(\varphi_0)\big) = x_f, \\ y_0 - \frac{v}{u_*}\big(2\cos(\varphi_0 + u_*\Delta t_1) - 2\cos(\varphi_0 + u_*\Delta t_1 - u_*\Delta t_2) + \cos(\varphi_f) - \cos(\varphi_0)\big) = y_f, \\ \varphi_0 + u_*\Delta t_1 - u_*\Delta t_2 + u_*\Delta t_3 = \varphi_f + 2\pi k, \quad k \in \mathbb{Z}, \end{cases} \tag{3.7}$$

where Δt_1, Δt_2, and Δt_3 are unknowns.

Thus, the problem can be formulated as follows: find a solution to system (3.7) that satisfies the nonnegativity condition (3.1) and minimizes the performance index

$$T_1 = \Delta t_1 + \Delta t_2 + \Delta t_3.$$

Introduce the notation

$$\alpha = \varphi_0 + u_*\Delta t_1, \quad \beta = u_*\Delta t_2, \quad \gamma = u_*\Delta t_3, \tag{3.8}$$

$$a_1 = \frac{u_*}{v}(x_f - x_0) - \sin(\varphi_f) + \sin(\varphi_0), \tag{3.9}$$

$$b_1 = \frac{u_*}{v}(y_f - y_0) + \cos(\varphi_f) - \cos(\varphi_0). \tag{3.10}$$

With this notation, system (3.7) may be written as

$$\begin{cases} 2\sin(\alpha) - 2\sin(\alpha - \beta) = a_1, \\ -2\cos(\alpha) + 2\cos(\alpha - \beta) = b_1, \\ \alpha - \beta + \gamma = \varphi_f + 2\pi k, \quad k \in \mathbb{Z}. \end{cases} \tag{3.11}$$

1. Suppose that $a_1^2 + b_1^2 = 0$. Then the set of solutions to (3.7) can be expressed as

$$\begin{cases} \Delta t_1 + \Delta t_3 = \frac{1}{u_*}(\varphi_f - \varphi_0 + 2\pi k), \quad k \in \mathbb{Z}, \\ \Delta t_2 = \frac{1}{u_*}2\pi k, \quad k \in \mathbb{Z}. \end{cases} \tag{3.12}$$

Applying the modulo operation to (3.12) and resolving the ambiguity, we obtain

$$\begin{cases} \Delta t_1 = \frac{1}{|u_*|}\big((\operatorname{sgn}(u_*)(\varphi_f - \varphi_0)) \mod 2\pi\big), \\ \Delta t_2 = 0, \\ \Delta t_3 = 0. \end{cases} \tag{3.13}$$

Thus, solution (3.13) defines the constant control $u(t) \equiv u_*$ over the entire time interval $[t_0, t_f]$. The trajectory of the vehicle in this case is just an arc of a circle.

2. Suppose now that $a_1^2 + b_1^2 \neq 0$. Squaring both sides of the first and second equations of system (3.11) and adding the resulting equations together, we get

$$\begin{aligned} 8 - 8\cos(\alpha)\cos(\alpha-\beta) - 8\sin(\alpha)\sin(\alpha-\beta) &= a_1^2 + b_1^2, \\ 8 - 8\cos(\beta) &= a_1^2 + b_1^2, \\ \cos(\beta) &= -\frac{a_1^2 + b_1^2 - 8}{8}. \end{aligned} \tag{3.14}$$

Let us assume that a solution to (3.14) exists. Then we can write this solution in the form

$$\beta_1 = \arccos\left(-\frac{a_1^2 + b_1^2 - 8}{8}\right) + 2\pi k, \quad k \in \mathbb{Z}, \tag{3.15}$$

$$\beta_2 = -\arccos\left(-\frac{a_1^2 + b_1^2 - 8}{8}\right) + 2\pi k, \quad k \in \mathbb{Z}. \tag{3.16}$$

So, we have expressed β in terms of a_1 and b_1. Now let us express α in terms of β. Applying the appropriate trigonometric identities to (3.11), we have

$$\begin{cases} 2\sin(\alpha) - 2\sin(\alpha)\cos(\beta) + 2\cos(\alpha)\sin(\beta) = a_1, \\ -2\cos(\alpha) + 2\cos(\alpha)\cos(\beta) + 2\sin(\alpha)\sin(\beta) = b_1; \end{cases}$$

$$\begin{cases} 2\big(1 - \cos(\beta)\big)\sin(\alpha) + 2\sin(\beta)\cos(\alpha) = a_1, \\ 2\sin(\beta)\sin(\alpha) - 2\big(1 - \cos(\beta)\big)\cos(\alpha) = b_1. \end{cases} \tag{3.17}$$

Since $a_1^2 + b_1^2 \neq 0$, then $\cos(\beta) \neq 1$ and $\sin(\beta) \neq 0$. Solving (3.17) for $\cos(\alpha)$ and $\sin(\alpha)$ yields

$$\begin{cases} \cos(\alpha) = \dfrac{a_1\sin(\beta) - b_1\big(1 - \cos(\beta)\big)}{2\big(1 - \cos(\beta)\big)^2 + 2\sin^2(\beta)}, \\ \sin(\alpha) = \dfrac{a_1\big(1 - \cos(\beta)\big) + b_1\sin(\beta)}{2\big(1 - \cos(\beta)\big)^2 + 2\sin^2(\beta)}. \end{cases} \tag{3.18}$$

Let us assume that a solution to (3.18) exists. Then we can write this solution in the form

$$\alpha = \operatorname{sgn}\left(\frac{a_1\big(1 - \cos(\beta)\big) + b_1\sin(\beta)}{2\big(1 - \cos(\beta)\big)^2 + 2\sin^2(\beta)}\right)\arccos\left(\frac{a_1\sin(\beta) - b_1\big(1 - \cos(\beta)\big)}{2\big(1 - \cos(\beta)\big)^2 + 2\sin^2(\beta)}\right) + 2\pi k, \quad k \in \mathbb{Z}. \tag{3.19}$$

Next, let us express γ in terms of α and β. From the third equation of system (3.11), we find

$$\gamma = \varphi_f - \alpha + \beta + 2\pi k, \quad k \in \mathbb{Z}. \tag{3.20}$$

Solving (3.8) for Δt_1, Δt_2, and Δt_3, and then using the modulo operation, we obtain the following result:

$$\begin{cases} \Delta t_1 = \dfrac{1}{|u_*|}\big((\operatorname{sgn}(u_*)(\alpha - \varphi_0)) \mod 2\pi\big), \\ \Delta t_2 = \dfrac{1}{|u_*|}\big((\operatorname{sgn}(u_*)\beta) \mod 2\pi\big), \\ \Delta t_3 = \dfrac{1}{|u_*|}\big((\operatorname{sgn}(u_*)\gamma) \mod 2\pi\big), \end{cases} \tag{3.21}$$

where α, β, and γ are defined by (3.19), (3.15), (3.16), and (3.20), respectively. Formula (3.21) gives us two solutions: the first solution corresponds to $\beta = \beta_1$, and the second solution corresponds to $\beta = \beta_2$. It is clear that both solutions satisfy the nonnegativity condition (3.1). Therefore, to find the minimum of the performance index, it remains necessary to compare these two solutions.

Remark 1. When substituting (3.19), (3.15), (3.16), and (3.20) into (3.21), one can assume $k = 0$ in all these formulas.

Remark 2. From [5, Theorem 12], it follows that if the time-optimal control of the Dubins car is of the type $(u_*, -u_*, u_*)$ with nondegenerate Δt_1, Δt_2, and Δt_3, then $\Delta t_2 > \pi/|u_*|$. This condition implies that

$$\beta = -\operatorname{sgn}(u_*) \arccos\left(-\frac{a_1^2 + b_1^2 - 8}{8}\right) + 2\pi k, \quad k \in \mathbb{Z}. \tag{3.22}$$

When substituting (3.22) into (3.21), one can assume $k = 0$.

3.2. Controls of the type $(u_*, 0, u_*)$

Given a vector of boundary conditions $\mathbf{w} = (x_0, y_0, \varphi_0, x_f, y_f, \varphi_f)$, the goal is to find a control of the type $(u_*, 0, u_*)$ that transfers system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ in minimum time.

For this type of control, the function $\varphi(t)$ on the time interval $[t_0, t_f]$ can be expressed as

$$\varphi(t) = \begin{cases} \varphi_0 + u_*(t - t_0), & t \in [t_0, t_1), \\ \varphi_0 + u_*\Delta t_1, & t \in [t_1, t_2), \\ \varphi_0 + u_*\Delta t_1 + u_*(t - t_2), & t \in [t_2, t_f]. \end{cases} \tag{3.23}$$

Substituting (3.23) into (2.1) gives

$$\begin{aligned} x(t_f) = x_0 &+ \int_0^{\Delta t_1} v\cos\big(\varphi_0 + u_*\tau\big)\,d\tau + \int_0^{\Delta t_2} v\cos\big(\varphi_0 + u_*\Delta t_1\big)\,d\tau \\ &+ \int_0^{\Delta t_3} v\cos\big(\varphi_0 + u_*\Delta t_1 + u_*\tau\big)\,d\tau \\ = x_0 &+ \frac{v}{u_*}\Big(\sin\big(\varphi(t_f)\big) - \sin\big(\varphi_0\big)\Big) + v\Delta t_2\cos\big(\varphi_0 + u_*\Delta t_1\big), \end{aligned} \tag{3.24}$$

$$\begin{aligned} y(t_f) = y_0 &+ \int_0^{\Delta t_1} v\sin\big(\varphi_0 + u_*\tau\big)\,d\tau + \int_0^{\Delta t_2} v\sin\big(\varphi_0 + u_*\Delta t_1\big)\,d\tau \\ &+ \int_0^{\Delta t_3} v\sin\big(\varphi_0 + u_*\Delta t_1 + u_*\tau\big)\,d\tau \\ = y_0 &- \frac{v}{u_*}\Big(\cos\big(\varphi(t_f)\big) - \cos\big(\varphi_0\big)\Big) + v\Delta t_2\sin\big(\varphi_0 + u_*\Delta t_1\big). \end{aligned} \tag{3.25}$$

Combining (3.23), (3.24), and (3.25) with the terminal condition, we obtain the system

$$\begin{cases} x_0 + \dfrac{v}{u_*}(\sin(\varphi_f) - \sin(\varphi_0)) + v\Delta t_2\cos(\varphi_0 + u_*\Delta t_1) = x_f, \\ y_0 - \dfrac{v}{u_*}(\cos(\varphi_f) - \cos(\varphi_0)) + v\Delta t_2\sin(\varphi_0 + u_*\Delta t_1) = y_f, \\ \varphi_0 + u_*\Delta t_1 + u_*\Delta t_3 = \varphi_f + 2\pi k, \quad k \in \mathbb{Z}, \end{cases} \tag{3.26}$$

where Δt_1, Δt_2, and Δt_3 are unknowns.

Thus, the problem can be formulated as follows: find a solution to system (3.26) that satisfies the nonnegativity condition (3.1) and minimizes the performance index

$$T_2 = \Delta t_1 + \Delta t_2 + \Delta t_3.$$

Introduce the notation

$$\begin{aligned} \alpha = \varphi_0 + u_*\Delta t_1, \quad \gamma = u_*\Delta t_3, \\ a_1 = \frac{u_*}{v}(x_f - x_0) - \sin(\varphi_f) + \sin(\varphi_0), \\ b_1 = \frac{u_*}{v}(y_f - y_0) + \cos(\varphi_f) - \cos(\varphi_0). \end{aligned} \tag{3.27}$$

With this notation, system (3.26) may be written as

$$\begin{cases} u_*\Delta t_2 \cos(\alpha) = a_1, \\ u_*\Delta t_2 \sin(\alpha) = b_1, \\ \alpha + \gamma = \varphi_f + 2\pi k, \quad k \in \mathbb{Z}. \end{cases} \tag{3.28}$$

1. Suppose that $a_1^2 + b_1^2 = 0$. It is easy to show that, in this case, the solution is (3.13).

2. Suppose now that $a_1^2 + b_1^2 \neq 0$. Squaring both sides of the first and second equations of system (3.28) and adding the resulting equations together, we get

$$\begin{aligned} u_*^2 \Delta t_2^2 = a_1^2 + b_1^2, \\ \Delta t_2 = \frac{1}{|u_*|}\sqrt{a_1^2 + b_1^2}. \end{aligned} \tag{3.29}$$

So, we have expressed Δt_2 in terms of a_1 and b_1. Now we can find α by substituting (3.29) into (3.28). We have

$$\alpha = \operatorname{sgn}\left(\frac{|u_*|b_1}{u_*\sqrt{a_1^2 + b_1^2}}\right) \arccos\left(\frac{|u_*|a_1}{u_*\sqrt{a_1^2 + b_1^2}}\right) + 2\pi k, \quad k \in \mathbb{Z}. \tag{3.30}$$

Next, let us express γ in terms of α. From the third equation of system (3.28), we find

$$\gamma = \varphi_f - \alpha + 2\pi k, \quad k \in \mathbb{Z}. \tag{3.31}$$

Solving (3.27) for Δt_1 and Δt_3, and then using the modulo operation, we obtain the following result:

$$\begin{cases} \Delta t_1 = \dfrac{1}{|u_*|}\big((\operatorname{sgn}(u_*)(\alpha - \varphi_0)) \mod 2\pi\big), \\ \Delta t_2 = \dfrac{1}{|u_*|}\sqrt{a_1^2 + b_1^2}, \\ \Delta t_3 = \dfrac{1}{|u_*|}\big((\operatorname{sgn}(u_*)\gamma) \mod 2\pi\big), \end{cases} \tag{3.32}$$

where α and γ are defined by (3.30) and (3.31), respectively.

Remark 3. When substituting (3.30) and (3.31) into (3.32), one can assume $k = 0$ in all these formulas.

3.3. Controls of the type $(u_*, 0, -u_*)$

Given a vector of boundary conditions $\mathbf{w} = (x_0, y_0, \varphi_0, x_f, y_f, \varphi_f)$, the goal is to find a control of the type $(u_*, 0, -u_*)$ that transfers system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ in minimum time.

For this type of control, the function $\varphi(t)$ on the time interval $[t_0, t_f]$ can be expressed as

$$\varphi(t) = \begin{cases} \varphi_0 + u_*(t - t_0), & t \in [t_0, t_1), \\ \varphi_0 + u_*\Delta t_1, & t \in [t_1, t_2), \\ \varphi_0 + u_*\Delta t_1 - u_*(t - t_2), & t \in [t_2, t_f]. \end{cases} \tag{3.33}$$

Substituting (3.33) into (2.1) gives

$$\begin{aligned} x(t_f) = x_0 + \int_0^{\Delta t_1} v\cos(\varphi_0 + u_*\tau)\, d\tau + \int_0^{\Delta t_2} v\cos(\varphi_0 + u_*\Delta t_1)\, d\tau \\ + \int_0^{\Delta t_3} v\cos(\varphi_0 + u_*\Delta t_1 - u_*\tau)\, d\tau \\ = x_0 + \frac{v}{u_*}\big(2\sin(\varphi_0 + u_*\Delta t_1) - \sin(\varphi(t_f)) - \sin(\varphi_0)\big) + v\Delta t_2\cos(\varphi_0 + u_*\Delta t_1), \end{aligned} \tag{3.34}$$

$$\begin{aligned} y(t_f) = y_0 + \int_0^{\Delta t_1} v\sin(\varphi_0 + u_*\tau)\, d\tau + \int_0^{\Delta t_2} v\sin(\varphi_0 + u_*\Delta t_1)\, d\tau \\ + \int_0^{\Delta t_3} v\sin(\varphi_0 + u_*\Delta t_1 - u_*\tau)\, d\tau \\ = y_0 - \frac{v}{u_*}\big(2\cos(\varphi_0 + u_*\Delta t_1) - \cos(\varphi(t_f)) - \cos(\varphi_0)\big) + v\Delta t_2\sin(\varphi_0 + u_*\Delta t_1). \end{aligned} \tag{3.35}$$

Combining (3.33), (3.34), and (3.35) with the terminal condition, we obtain the system

$$\begin{cases} x_0 + \frac{v}{u_*}\big(2\sin(\varphi_0 + u_*\Delta t_1) - \sin(\varphi_f) - \sin(\varphi_0)\big) + v\Delta t_2\cos(\varphi_0 + u_*\Delta t_1) = x_f, \\ y_0 - \frac{v}{u_*}\big(2\cos(\varphi_0 + u_*\Delta t_1) - \cos(\varphi_f) - \cos(\varphi_0)\big) + v\Delta t_2\sin(\varphi_0 + u_*\Delta t_1) = y_f, \\ \varphi_0 + u_*\Delta t_1 - u_*\Delta t_3 = \varphi_f + 2\pi k, \quad k \in \mathbb{Z}, \end{cases} \tag{3.36}$$

where Δt_1, Δt_2, and Δt_3 are unknowns.

Thus, the problem can be formulated as follows: find a solution to system (3.36) that satisfies the nonnegativity condition (3.1) and minimizes the performance index

$$T_3 = \Delta t_1 + \Delta t_2 + \Delta t_3.$$

Introduce the notation

$$\begin{aligned} \alpha = \varphi_0 + u_*\Delta t_1, \quad \gamma = u_*\Delta t_3, \\ a_2 = \frac{u_*}{v}(x_f - x_0) + \sin(\varphi_f) + \sin(\varphi_0), \\ b_2 = \frac{u_*}{v}(y_f - y_0) - \cos(\varphi_f) - \cos(\varphi_0). \end{aligned} \tag{3.37}$$

With this notation, system (3.7) may be written as

$$\begin{cases} 2\sin(\alpha) + u_*\Delta t_2\cos(\alpha) = a_2, \\ -2\cos(\alpha) + u_*\Delta t_2\sin(\alpha) = b_2, \\ \alpha - \gamma = \varphi_f + 2\pi k, \quad k \in \mathbb{Z}. \end{cases} \tag{3.38}$$

Squaring both sides of the first and second equations of system (3.38) and adding the resulting equations together, we get

$$4 + u_*^2 \Delta t_2^2 = a_2^2 + b_2^2. \tag{3.39}$$

Let us assume that a solution to (3.39) exists. Then, we have

$$\Delta t_2 = \frac{1}{|u_*|} \sqrt{a_2^2 + b_2^2 - 4}. \tag{3.40}$$

So, we have expressed Δt_2 in terms of a_2 and b_2. Now let us express α in terms of Δt_2. Solving the first two equations of system (3.38) for $\cos(\alpha)$ and $\sin(\alpha)$ yields

$$\begin{cases} \cos(\alpha) = \dfrac{a_2 u_* \Delta t_2 - 2b_2}{4 + u_*^2 \Delta t_2^2}, \\ \sin(\alpha) = \dfrac{b_2 u_* \Delta t_2 + 2a_2}{4 + u_*^2 \Delta t_2^2}. \end{cases} \tag{3.41}$$

Let us assume that a solution to (3.41) exists. Then, after substituting (3.40) into (3.41), we can write this solution in the form

$$\begin{aligned} \alpha = \operatorname{sgn} &\left(\frac{(b_2 u_*/|u_*|)\sqrt{a_2^2 + b_2^2 - 4} + 2a_2}{a_2^2 + b_2^2} \right) \\ \times \arccos &\left(\frac{(a_2 u_*/|u_*|)\sqrt{a_2^2 + b_2^2 - 4} - 2b_2}{a_2^2 + b_2^2} \right) + 2\pi k, \quad k \in \mathbb{Z}, \end{aligned} \tag{3.42}$$

where (3.40) guarantees that $a_2^2 + b_2^2 \neq 0$.

Next, let us express γ in terms of α. From the third equation of system (3.38), we find

$$\gamma = \alpha - \varphi_f + 2\pi k, \quad k \in \mathbb{Z}. \tag{3.43}$$

Solving (3.37) for Δt_1 and Δt_3, and then using the modulo operation, we obtain the following result:

$$\begin{cases} \Delta t_1 = \dfrac{1}{|u_*|}\big((\operatorname{sgn}(u_*)(\alpha - \varphi_0)r) \mod 2\pi\big), \\ \Delta t_2 = \dfrac{1}{|u_*|}\sqrt{a_2^2 + b_2^2 - 4}, \\ \Delta t_3 = \dfrac{1}{|u_*|}\big((\operatorname{sgn}(u_*)\gamma) \mod 2\pi\big), \end{cases} \tag{3.44}$$

where α and γ are defined by (3.42) and (3.43), respectively.

Remark 4. When substituting (3.42) and (3.43) into (3.44), one can assume $k = 0$ in all these formulas.

4. Analysis of solutions

Let us introduce some additional definitions.

Definition 5. *An open (closed) disc of radius r and center (x_*, y_*) is the set of points (x, y) such that*

$$(x - x_*)^2 + (y - y_*)^2 < r^2 \quad ((x - x_*)^2 + (y - y_*)^2 \leqslant r^2).$$

Definition 6. *We say that a vector of boundary conditions $\mathbf{w} = (x_0, y_0, \varphi_0, x_f, y_f, \varphi_f)$ is feasible for controls of the type (u_1, u_2, u_3) if there exists a control of the type (u_1, u_2, u_3) that transfers system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states*

$$\big\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\big\}.$$

We now proceed to investigate the properties of solutions.

4.1. Controls of the type $(u_*, -u_*, u_*)$

Let the notation be as in Section 3.1.

We first obtain necessary and sufficient conditions for the existence of a solution to the time-optimal control problem of the Dubins car for controls of the type $(u_*, -u_*, u_*)$. To do this, we prove the following proposition.

Proposition 1. *System (2.1) can be transferred from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ by a control of the type $(u_*, -u_*, u_*)$ if and only if the point (x_f, y_f) belongs to a closed disc $\mathbb{B}_1$ of radius $4v/|u_*|$ centered at the point (x_*, y_*) defined by*

$$(x_*, y_*) = \Big(x_0 + \frac{v}{u_*}\sin(\varphi_f) - \frac{v}{u_*}\sin(\varphi_0),\ y_0 - \frac{v}{u_*}\cos(\varphi_f) + \frac{v}{u_*}\cos(\varphi_0)\Big).$$

P r o o f. **1.** First, we show that if $(x_f, y_f) \in \mathbb{B}_1$, then there exists a control of the type $(u_*, -u_*, u_*)$ that transfers system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$.

It is easy to see that (x_f, y_f) is the center of the closed disc $\mathbb{B}_1$ if and only if $a_1^2 + b_1^2 = 0$. This follows immediately from (3.9) and (3.10). In this case, system (3.7) has solution (3.13).

Let the point (x_f, y_f) belong to the closed disc $\mathbb{B}_1$, but it is not the center of this disc. In this case, equation (3.14) has a solution if and only if

$$-1 \leqslant -\frac{a_1^2 + b_1^2 - 8}{8} < 1. \tag{4.1}$$

We write (4.1) as

$$0 < a_1^2 + b_1^2 \leqslant 16. \tag{4.2}$$

Multiplying all parts of (4.2) by $(v/u_*)^2$ gives

$$0 < \left(\frac{v}{u_*}a_1\right)^2 + \left(\frac{v}{u_*}b_1\right)^2 \leqslant \left(4\frac{v}{u_*}\right)^2. \tag{4.3}$$

Thinking of x_f and y_f as variables, it is easy to see that expression (4.3) defines all the points of the closed disc $\mathbb{B}_1$ except for the center point. Since we assumed that (x_f, y_f) belongs to the closed disc $\mathbb{B}_1$, but it is not the center of this disc, conditions (4.1)–(4.3) are met, which implies that a solution to equation (3.14) exists. Let us check that a solution to system (3.18) also exists. For this, we find the sum of the squares of the right-hand sides of the equations of this system. Substituting (3.14) into (3.18), we have

$$\frac{a_1^2\sin^2(\beta) - 2a_1b_1\sin(\beta)(1-\cos(\beta)) + b_1^2(1-\cos(\beta))^2}{4\big((1-\cos(\beta))^2 + \sin^2(\beta)\big)^2}$$
$$+\frac{a_1^2(1-\cos(\beta))^2 + 2a_1b_1\sin(\beta)(1-\cos(\beta)) + b_1^2\sin^2(\beta)}{4\big((1-\cos(\beta))^2 + \sin^2(\beta)\big)^2} = \frac{(a_1^2+b_1^2)\big((1-\cos(\beta))^2 + \sin^2(\beta)\big)}{4\big((1-\cos(\beta))^2 + \sin^2(\beta)\big)^2}$$
$$= \frac{a_1^2+b_1^2}{4 - 8\cos(\beta) + 4\cos^2(\beta) + 4\sin^2(\beta)} = \frac{a_1^2+b_1^2}{8 - 8\cos(\beta)} = 1.$$

Thus, we see that, for any β satisfying (3.14), under the condition $a_1^2 + b_1^2 \neq 0$, the equations of system (3.18) indeed represent the sine and cosine of some angle α. Consequently, system (3.7) has solution (3.21).

2. Now we show that if there exists a control of the type $(u_*, -u_*, u_*)$ that transfers system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$, then $(x_f, y_f) \in \mathbb{B}_1$.

Suppose that $(x_f, y_f) \notin \mathbb{B}_1$. Then (4.1)–(4.3) are not met. Hence, equation (3.14) has no solution, and therefore system (3.7) also has no solution. This is a contradiction. □

Corollary 1. *System (2.1) can be transferred from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ by a control of the type $(u_*, -u_*, u_*)$ if and only if the point (x_0, y_0) belongs to a closed disc $\mathbb{B}_1^*$ of radius $4v/|u_*|$ centered at the point (x^*, y^*) defined by*

$$(x^*, y^*) = \Big(x_f - \frac{v}{u_*}\sin(\varphi_f) + \frac{v}{u_*}\sin(\varphi_0),\ y_f + \frac{v}{u_*}\cos(\varphi_f) - \frac{v}{u_*}\cos(\varphi_0)\Big).$$

Corollary 2. *If the point (x_f, y_f) belongs to the closed disc $\mathbb{B}_1$, but it is not the center of this disc, then solutions to equations (3.14) and (3.18) exist. In this case, system (3.7) will have solution (3.21).*

Corollary 3. *If the point (x_f, y_f) is the center of the closed disc $\mathbb{B}_1$, then system (3.7) will have solution (3.13).*

Next, we turn to the question of the uniqueness of the time-optimal control.

Proposition 2. *Let W_1 be the set of all feasible vectors of boundary conditions for controls of the type $(u_*, -u_*, u_*)$. For any $\mathbf{w} \in W_1$, there are at most two different time-optimal controls of the type $(u_*, -u_*, u_*)$.*

P r o o f. **1.** Assume that $a_1^2 + b_1^2 = 0$. Then the set of all solutions to system (3.7) will be determined by expression (3.12). The right-hand sides of the equations of (3.12) can be expressed in the form $\lambda\big(f(x) + G\big)$, where λ is a positive real number, $f(x)$ is a constant function, and $G = \{2\pi k \mid k \in \mathbb{Z}\}$. By the first part of Lemma 1, it follows that the modulo operation allows us to extract the smallest nonnegative value from this sets of values. After doing this, we can see that the middle segment of the optimal control is degenerate, and since $u_1 = u_3$, we infer that all optimal solutions to system (3.7) generate the same optimal control. So, in this case, the optimal control is unique.

2. Assume that $a_1^2 + b_1^2 \neq 0$. Using Lemma 1, we see that expression (3.21) determines at most two different solutions to system (3.7). The first of them corresponds to the case $\beta = \beta_1$, and the second corresponds to the case $\beta = \beta_2$, where β_1 and β_2 are defined by (3.15) and (3.16). It also follows from Lemma 1 that one of these solutions will be optimal. If the values of the performance index are the same for both the solutions, then both the solutions will be optimal. □

Corollary 4. *Under condition (3.22), expression (3.21) defines the unique control that transfers system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states*

$$\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}.$$

Finally, we study the dependence of the movement time on the initial and terminal conditions.

Proposition 3. *Let W_1 be the set of all feasible vectors of boundary conditions for controls of the type $(u_*, -u_*, u_*)$, and let T_1^{opt} be a function that assigns to each $\mathbf{w} = (x_0, y_0, \varphi_0, x_f, y_f, \varphi_f)$ in W_1 the minimum time required to transfer system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ by a control of the type $(u_*, -u_*, u_*)$ under condition (3.22). If $\mathbf{w}_*$ is a point of discontinuity of T_1^{opt}, then at least one of the following conditions holds at $\mathbf{w}_*$:*

1. $a_1^2 + b_1^2 = 0$;
2. $\Delta t_1 = 0$;
3. $\Delta t_3 = 0$.

P r o o f. We will prove this proposition by contradiction. Suppose that none of conditions 1–3 holds at $\mathbf{w}_*$. Observe that $T_1^{opt}(\mathbf{w})$, $\mathbf{w} \in W_1$, represents the optimal value of the performance index T_1 for $\mathbf{w}$ under condition (3.22). Therefore, according to Corollaries 2 and 3, the value of $T_1^{opt}(\mathbf{w})$, $\mathbf{w} \in W_1$, is determined by either (3.13) or (3.21). Since we assumed that condition 1 does not hold at $\mathbf{w}_*$, (3.13) can be ruled out. So, it remains to consider only (3.21). By Corollary 4, under condition (3.22), the value of $T_1^{opt}(\mathbf{w}_*)$ is unique. We need to prove that T_1^{opt} is continuous at $\mathbf{w}_*$. To do this, we will consider Δt_1, Δt_2, Δt_3, α, β, γ, a_1, and b_1 as functions of the vector of boundary conditions $\mathbf{w}$.

It is obvious that a_1 and b_1 are continuous on W_1.

Let us consider expression (3.22). We see that the function β is of the form $\beta(\mathbf{w}) = f_\beta(\mathbf{w}) + G$, where f_β is a continuous single-valued function and $G = \{2\pi k \mid k \in \mathbb{Z}\}$. Since we assumed that condition 1 does not hold at $\mathbf{w}_*$, we have $f_\beta(\mathbf{w}_*) \neq 2\pi k$, $k \in \mathbb{Z}$. Hence, by Lemma 1, the function Δt_2 is continuous at $\mathbf{w}_*$.

Let us consider expression (3.19). This expression is a solution of system (3.18). The values of the numerators and denominators of the fractions in this expression continuously depend on $\mathbf{w}$, and the denominators cannot vanish unless $a_1^2(\mathbf{w}) + b_1^2(\mathbf{w}) \neq 0$. Consequently, $\cos\big(\alpha(\mathbf{w})\big)$ and $\sin\big(\alpha(\mathbf{w})\big)$ are continuous at $\mathbf{w}_*$. It can be shown that, in a neighborhood of $\mathbf{w}_*$, the function α is of the form $\alpha(\mathbf{w}) = f_\alpha(\mathbf{w}) + G$, where f_α is a continuous single-valued function. Therefore, the expression $\operatorname{sgn}(u_*)(\alpha - \varphi_0)$ in the first equation of formula (3.21) can also be represented in the same form. Since we assumed that condition 2 does not hold at $\mathbf{w}_*$, we have

$$\operatorname{sgn}(u_*)\big(\alpha(\mathbf{w}_*) - \varphi_0\big) \neq 2\pi k, \quad k \in \mathbb{Z}.$$

Hence, by Lemma 1, the function Δt_1 is continuous at $\mathbf{w}_*$.

Let us consider expression (3.20). It can be shown that, in a neighborhood of $\mathbf{w}_*$, the function γ is of the form $\gamma(\mathbf{w}) = f_\gamma(\mathbf{w}) + G$, where f_γ is a continuous single-valued function. Since we assumed that condition 3 does not hold at $\mathbf{w}_*$, we have $f_\gamma(\mathbf{w}_*) \neq 2\pi k$, $k \in \mathbb{Z}$. Hence, by Lemma 1, the function Δt_3 is continuous at $\mathbf{w}_*$.

Thus, we have shown that each of the functions Δt_1, Δt_2, and Δt_3 is continuous at $\mathbf{w}_*$. So, T_1^{opt} is also continuous at $\mathbf{w}_*$. □

4.2. Controls of the type $(u_*, 0, u_*)$

Let the notation be as in Section 3.2.

We first obtain necessary and sufficient conditions for the existence of a solution to the time-optimal control problem of the Dubins car for controls of the type $(u_*, 0, u_*)$. To do this, we prove the following proposition.

Proposition 4. *For any vector of boundary conditions $\mathbf{w} = (x_0, y_0, \varphi_0, x_f, y_f, \varphi_f)$, system (2.1) can be transferred from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ by a control of the type $(u_*, 0, u_*)$.*

P r o o f. It is obvious that system (3.28) is solvable for any a_1, b_1, and φ_f; so, the switching times can be easily found by using (3.13) and (3.32). ☐

Corollary 5. *If the point (x_f, y_f) is not the center of the closed disc $\mathbb{B}_1$ defined in Proposition 1, then system (3.26) will have solution (3.32).*

Corollary 6. *If the point (x_f, y_f) is the center of the closed disc $\mathbb{B}_1$ defined in Proposition 1, then system (3.26) will have solution (3.13).*

Next, we turn to the question of the uniqueness of the time-optimal control.

Proposition 5. *For any vector of boundary conditions, the time-optimal control of the type $(u_*, 0, u_*)$ is unique.*

P r o o f is similar to that of Proposition 2. ☐

Finally, we study the dependence of the movement time on the initial and terminal conditions.

Proposition 6. *Let T_2^{opt} be a function that assigns to each $\mathbf{w} = (x_0, y_0, \varphi_0, x_f, y_f, \varphi_f)$ in $\mathbb{R}^6$ the minimum time required to transfer system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ by a control of the type $(u_*, 0, u_*)$. If $\mathbf{w}_*$ is a point of discontinuity of T_2^{opt}, then at least one of the following conditions holds at $\mathbf{w}_*$:*
1. $a_1^2 + b_1^2 = 0$;
2. $\Delta t_1 = 0$;
3. $\Delta t_3 = 0$.

P r o o f is similar to that of Proposition 3. ☐

4.3. Controls of the type $(u_*, 0, -u_*)$

Let the notation be as in Section 3.3.

We first obtain necessary and sufficient conditions for the existence of a solution to the time-optimal control problem of the Dubins car for controls of the type $(u_*, 0, -u_*)$. To do this, we prove the following proposition.

Proposition 7. *System (2.1) can be transferred from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ by a control of the type $(u_*, 0, -u_*)$ if and only if the point (x_f, y_f) does not belong to an open disc $\mathbb{B}_2$ of radius $2v/|u_*|$ centered at the point (x_*, y_*) defined by*

$$(x_*, y_*) = \Big(x_0 - \frac{v}{u_*}\sin(\varphi_f) - \frac{v}{u_*}\sin(\varphi_0),\ y_0 + \frac{v}{u_*}\cos(\varphi_f) + \frac{v}{u_*}\cos(\varphi_0)\Big).$$

P r o o f. **1.** First, we show that if $(x_f, y_f) \notin \mathbb{B}_2$, then there exists a control of the type $(u_*, 0, -u_*)$ that transfers system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$.

Observe that equation (3.39) has a solution if and only if

$$a_2^2 + b_2^2 \geqslant 4. \tag{4.4}$$

Multiplying both sides of (4.4) by $(v/u_*)^2$ gives

$$\left(\frac{v}{u_*}a_2\right)^2 + \left(\frac{v}{u_*}b_2\right)^2 \geqslant \left(2\frac{v}{u_*}\right)^2. \tag{4.5}$$

Thinking of x_f and y_f as variables, it is easy to see that expression (4.5) defines the points that do not belong to the open disc $\mathbb{B}_2$. Since we assumed that (x_f, y_f) does not belong to the open disc $\mathbb{B}_2$, conditions (4.4) and (4.5) are met, which implies that a solution to equation (3.39) exists. Let us check that a solution to system (3.41) also exists. For this, we find the sum of the squares of the right-hand sides of the equations of this system. Substituting (3.39) into (3.41), we have

$$\frac{a_2^2u_*^2\Delta t_2^2 - 4a_2b_2u_*\Delta t_2 + 4b_2^2}{\left(4+u_*^2\Delta t_2^2\right)^2} + \frac{b_2^2u_*^2\Delta t_2^2 + 4a_2b_2u_*\Delta t_2 + 4a_2^2}{\left(4+u_*^2\Delta t_2^2\right)^2}$$
$$= \frac{a_2^2u_*^2\Delta t_2^2 + 4b_2^2 + b_2^2u_*^2\Delta t_2^2 + 4a_2^2}{\left(4+u_*^2\Delta t_2^2\right)^2} = \frac{\left(a_2^2+b_2^2\right)\left(4+u_*^2\Delta t_2^2\right)}{\left(4+u_*^2\Delta t_2^2\right)^2} = \frac{a_2^2+b_2^2}{4+u_*^2\Delta t_2^2} = 1.$$

Thus, we see that, for any Δt_2 satisfying (3.39), the equations of system (3.41) indeed represent the sine and cosine of some angle α. Consequently, system (3.36) has solution (3.44).

2. Now we show that if there exists a control of the type $(u_*, 0, -u_*)$ that transfers the system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$, then $(x_f, y_f) \notin \mathbb{B}_2$.

Suppose that $(x_f, y_f) \in \mathbb{B}_2$. Then (4.4), (4.5) are not met. Hence, equation (3.39) has no solution, and therefore system (3.36) also has no solution. This is a contradiction. □

Corollary 7. *System (2.1) can be transferred from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ by a control of the type $(u_*, 0, -u_*)$ if and only if the point (x_0, y_0) does not belong to an open disc $\mathbb{B}_2^*$ of radius $2v/|u_*|$ centered at the point (x^*, y^*) defined by*

$$(x^*, y^*) = \Big(x_f + \frac{v}{u_*}\sin(\varphi_f) + \frac{v}{u_*}\sin(\varphi_0),\ y_f - \frac{v}{u_*}\cos(\varphi_f) - \frac{v}{u_*}\cos(\varphi_0)\Big).$$

Corollary 8. *If the point (x_f, y_f) does not belong to the open disc $\mathbb{B}_2$, then solutions to equations (3.39) and (3.41) exist. In this case, system (3.36) will have solution (3.44).*

Next, we turn to the question of the uniqueness of the time-optimal control.

Proposition 8. *Let W_3 be the set of all feasible vectors of boundary conditions for controls of the type $(u_*, 0, -u_*)$. For any $\mathbf{w} \in W_3$, the time-optimal control of the type $(u_*, 0, -u_*)$ is unique.*

P r o o f is similar to that of Proposition 2. □

Finally, we study the dependence of the movement time on the initial and terminal conditions.

Proposition 9. *Let W_3 be the set of all feasible vectors of boundary conditions for controls of the type $(u_*, 0, -u_*)$, and let T_3^{opt} be a function that assigns to each $\mathbf{w} = (x_0, y_0, \varphi_0, x_f, y_f, \varphi_f)$ in W_3 the minimum time required to transfer system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ by a control of the type $(u_*, 0, -u_*)$. If $\mathbf{w}_*$ is a point of discontinuity of T_3^{opt}, then at least one of the following conditions holds at $\mathbf{w}_*$:*

1. $\Delta t_1 = 0$;
2. $\Delta t_3 = 0$.

P r o o f is similar to that of Proposition 3. □

5. Example

As an example, we will demonstrate how the properties of solutions deduced in Section 4 can reduce the computational effort required to solve the time-optimal control problem of the Dubins car. This issue was previously addressed in [17] for scenarios where the starting and ending points are far apart. However, the results from Section 4 are applicable to any configuration of the points.

Suppose that

$$x_0 = 0, \quad y_0 = 0, \quad \varphi_0 = \pi/2, \quad x_f = 3, \quad y_f = 0, \quad \varphi_f = 3\pi/2, \quad u_m = 1, \quad v = 1,$$

and it is desired to find a control that transfers system (2.1) from the initial state (x_0, y_0, φ_0) to one of the terminal states $\{(x_f, y_f, \varphi_f + 2\pi k) \mid k \in \mathbb{Z}\}$ in minimum time.

To solve this problem, we must find the values of Δt_1, Δt_2, and Δt_3 for controls of the types $(1, -1, 1)$, $(-1, 1, -1)$, $(1, 0, 1)$, $(-1, 0, -1)$, $(1, 0, -1)$, $(-1, 0, 1)$ using the corresponding formulas from Section 3, and then choose the one of these controls that transfers system (2.1) from the initial state to the terminal state in minimum time. Observe that Propositions 1 and 7 allow us to rule out some cases. Namely, we can exclude controls of the type $(1, -1, 1)$ since the point $(3, 0)$ does not belong to a closed disc of radius 4 centered at $(-2, 0)$. Let us calculate the values of Δt_1, Δt_2, and Δt_3 for the remaining types of controls. Calculations for controls of the type $(-1, 1, -1)$ will be carried out taking into account Remark 2. The results of these calculations are given in Table 1, where $T = \Delta t_1 + \Delta t_2 + \Delta t_3$.

Table 1. Time intervals for different control types.

Control types	Δt_1	Δt_2	Δt_3	T
$(-1, 1, -1)$	4.46	5.78	4.46	14.7
$(1, 0, 1)$	4.71	5.0	4.71	14.42
$(-1, 0, -1)$	1.57	1.0	1.57	4.14
$(1, 0, -1)$	5.44	2.24	2.3	9.98
$(-1, 0, 1)$	2.3	2.24	5.44	9.98

Comparing the total movement times T of each type of control, we see that the time-optimal control is a control of the type $(-1, 0, -1)$. Figure 1 shows the trajectory of the vehicle in the xy-plane, generated by this control. The arrow indicates the direction of the movement.

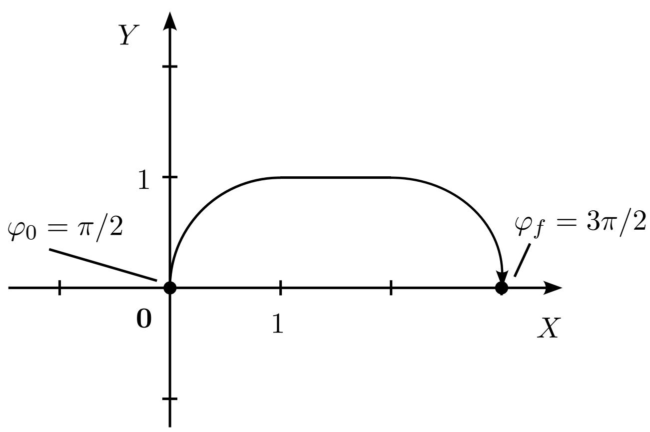

Figure 1. The optimal trajectory of the vehicle in the xy-plane.

6. Conclusion

In this paper, we have developed several fundamental properties for each type of controls in the time-optimal control problem of the Dubins car. The necessary and sufficient conditions for the existence of solutions determine the shape of the regions in the plane to which the vehicle can be driven by a control of the corresponding type. Since the regions are circular in shape, checking whether points belong to these regions can be done quite simply, and so this reduces the computational effort in solving the Dubins car control problem.

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