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STABILITY OF GENERAL QUADRATIC EULER–LAGRANGE FUNCTIONAL EQUATIONS IN MODULAR SPACES: A FIXED POINT APPROACH

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Abstract: In this paper, we establish a result on the Hyers–Ulam–Rassias stability of the Euler–Lagrange functional equation. The work presented here is in the framework of modular spaces. We obtain our results by applying a fixed point theorem. Moreover, we do not use the Δ_α-condition of modular spaces in the proofs of our theorems, which introduces additional complications in establishing stability. We also provide some corollaries and an illustrative example. Apart from its main objective of obtaining a stability result, the present paper also demonstrates how fixed point methods are applicable in modular spaces.

Keywords: Hyers–Ulam–Rassias stability, Euler–Lagrange functional equation, Modular spaces, Convexity, Fixed point method.

1. Introduction

In this paper, our main result concerns the stability property of a type of Euler–Lagrange functional equation. This type of equations was introduced by Rassias [18] in 1992. The name is derived from the Euler–Lagrange identity [19] and has several variants [12, 20, 26, 30], but our study is conducted within the framework of modular spaces.

The kind of stability investigated for the functional equation considered here is well-known as Hyers–Ulam–Rassias stability, which is very general and applicable to diverse branches of mathematics [4, 7, 25]. The concept originates from a mathematical question posed by Ulam [27] in 1940, along with its extensions and partial answers provided by Hyers [6] and Rassias [21]. In the most general terms, following Gruber [5], Hyers–Ulam–Rassias stability holds for a mathematical equation if, whenever it approximately satisfies an equation from a certain class, it admits an exact solution close to that approximate one. It involves questions such as whether a given approximately linear equation has an exact linear approximation.

Our framework of study is modular spaces [13, 16, 17, 28]. A modular space is a linear space equipped with a modular function possessing specific properties. Such a function introduces an additional structure on the linear space, thereby broadening its scope. Several studies from different domains of functional analysis have been successfully extended to this structure. References [9, 14] provide the technical details of the modular spaces mentioned above. Functional equations of various kinds have been considered in the investigation of Hyers–Ulam–Rassias stability properties [8, 23, 29]. We study the stability of such equations in modular spaces without assuming the Δ_α-condition, using a fixed point technique. It may be noted that fixed point methods have already been applied to Hyers–Ulam–Rassias stability problems in [2, 24]. Here, we apply this approach to our problems in modular spaces.

2. Preliminaries

If X and Y are assumed to be a real vector space and a Banach space, respectively, then a mapping $f : X \to Y$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad \forall x, y \in X, \tag{2.1}$$

which is known as the quadratic functional equation.

Any solution of (2.1) is called a quadratic mapping. In particular, if $X = Y = R$, the quadratic form $f(x) = ax^2$ is a solution of (2.1).

We consider here a type of Euler–Lagrange functional equation known as the general k-quadratic Euler–Lagrange functional equation:

$$q(kx+y) + q(kx-y) = 2[q(x+y) + q(x-y)] + 2(k^2-2)q(x) - 2q(y), \quad \forall x, y \in X, \tag{2.2}$$

where $k \in \mathbb{N}$, and $q : X \to Y$ is a function from a real vector space X to a Banach space Y.

Here, we recall certain definitions, theorems, and results regarding modular spaces.

Definition 1 [16, 17]. *A generalized functional $m : X \to [0, \infty]$ is called a modular if, for any two elements $x, y \in X$, where X is considered as a vector space over a field $\mathbb{K}$ (in our case $\mathbb{R}$ or $\mathbb{C}$), the following conditions hold*:

(i) $m(x) = 0$ *if and only if* $x = 0$,
(ii) $m(cx) = m(x)$ *for every scalar c with* $|c| = 1$,
(iii) $m(x') \le m(x) + m(y)$ *whenever x' is a convex combination of x and y*,
(iii)' *if $c_1, c_2 \ge 0$ and $c_1 + c_2 = 1$, then $m(c_1 x + c_2 y) \le c_1 m(x) + c_2 m(y)$, and in this case, m is said to be a convex modular.*

Definition 2. *The modular space, denoted by X_m, is defined as*

$$X_m := \{x \in X : m(\alpha x) \to 0 \text{ as } \alpha \to 0\}.$$

Example 1. If $(X, \|\cdot\|)$ is a normed space, then $\|\cdot\|$ is a convex modular on X, but the converse is not necessarily true [15].

Definition 3. *If m is a convex modular, then the norm known as the Luxemburg norm is defined as*

$$\|x\|_m := \inf\left\{\alpha > 0 : m\left(\frac{x}{\alpha}\right) \le 1\right\}.$$

Definition 4. *Consider X_m as a modular space and let $\{x_n\}$ be a sequence in X_m. Then,*

(i) *the sequence $\{x_n\}$ is called m-convergent to a point $x \in X_m$, denoted $x_n \overset{m}{\longrightarrow} x$, if $m(x_n - x) \to 0$ as $n \to \infty$* [10];

(ii) *$\{x_n\}$ is called an m-Cauchy sequence if for any $\epsilon > 0$, $m(x_n - x_p) < \epsilon$ for sufficiently large $n, p \in \mathbb{N}$* [10];

(iii) *a subset $K(\subset X_m)$ is called m-complete if every m-Cauchy sequence in X_m is m-convergent to an element in K* [10].

Note that m-convergence does not imply m-Cauchy since m does not satisfy the triangle inequality. In fact, one can show that this implication holds if and only if m satisfies the Δ_2-condition.

(iv) *The modular m is said to have the Fatou property if $m(x) \le \lim_{n\to\infty} \inf m(x_n)$ whenever the sequence $\{x_n\}$ is m-convergent to x* [10];

(v) *a modular m is said to satisfy the Δ_α-condition if there exists $\kappa \ge 0$ such that $m(\alpha x) \le \kappa m(x)$ for all $x \in X_m$ and $\alpha \in \mathbb{N}$, $\alpha \ge 2$* [3].

Observations.

(i) $m(x) \le \delta\, m((1/\delta)x)$ for all $x \in X_m$, if m is a convex modular and $0 < \delta \le 1$;

(ii) in general, the modular m does not behave like a norm or a metric since it is not subadditive [16]; however, every norm on X is a modular on X.

Definition 5. *Consider a modular space X_m, a nonempty subset $C \subset X_m$, and a mapping $D : C \to C$. The orbit of D at a point $z \in X_m$ is the set*

$$\mathbb{O}(z) := \{z, Dz, D^2z, \dots\}.$$

The quantity

$$\delta_m(z) := \sup\{m(x - y) : x, y \in \mathbb{O}(z)\}$$

is called the orbit diameter of D at z. In particular, if $\delta_m(z) < \infty$, then D has a bounded orbit at z.

Definition 6. *Let the modular m be defined on the vector space X, and let $C \subset X_m$ be nonempty. A function $D : C \to C$ is called m-Lipschitzian if there exists a constant $L \ge 0$ such that*

$$m(D(x) - D(y)) \le L\, m(x - y), \quad \forall x, y \in C.$$

If $L < 1$, then D is called an m-contraction.

Definition 7 [11]. *Let C be a subset of a modular function space X_m. A function $D : C \to C$ is called an m-strict contraction if there exists a constant $\lambda < 1$ such that*

$$m(D(x) - D(y)) \le \lambda\, m(x - y), \quad \forall x, y \in C.$$

Theorem 1 [1] (**The Banach Contraction Mapping Principle in Modular Spaces**). *Assume that X_m is m-complete. Let C be a nonempty m-closed subset of X_m, and let $T : C \to C$ be an m-contraction mapping. Then T has a fixed point z if and only if T has an m-bounded orbit. Moreover, if*

$$m(x - z) < \infty,$$

then $\{T^n(x)\}$ m-converges to z for any $x \in C$.

If x_1 and x_2 are two fixed points of T such that $m(x_1 - x_2) < \infty$, then from the above theorem we conclude that $x_1 = x_2$. Furthermore, if C is m-bounded, then T has a unique fixed point in C.

3. The generalized Hyers–Ulam stability of (2.2) in modular spaces

Lemma 1. *Assume that X is a linear space, and let X_m be an m-complete convex modular space. Consider the set*

$$\mathbb{M} = \{h : X \to X_m : h(0) = 0\}$$

and define a mapping $\tilde{m}$ on $\mathbb{M}$ by

$$\tilde{m}(h) = \inf\{c > 0 : m(h(x)) \leq c\psi(x, x)\}, \quad h \in \mathbb{M},$$

where $\psi : X^2 \to [0, \infty)$. Then $M_{\tilde{m}}$ is a complete convex modular space.

P r o o f. It is easy to prove that $\tilde{m}$ is a convex modular on $\mathbb{M}$ [22].

For completeness, let $\{h_n\}$ be an $\tilde{m}$-Cauchy sequence in $\mathbb{M}_{\tilde{m}}$, and let $\epsilon > 0$ be given. Then there exists $k \in \mathbb{N}$ such that $\tilde{m}(h_n - h_p) \leq \epsilon$ for all $p, n \geq k$. Therefore,

$$m\left(h_n(x) - h_p(x)\right) \leq \epsilon\psi(x, x) \quad \text{for all} \quad x \in X \quad \text{and} \quad p, n \geq k. \tag{3.1}$$

This shows that $\{h_n(x)\}$ is an m-Cauchy sequence in X_m for each fixed $x \in X_m$. Since X_m is m-complete, it follows that $\{h_n(x)\}$ is m-convergent in X_m for each fixed $x \in X$. Thus, we can define $h : X \to X_m$ by

$$h(x) = \lim_{n\to\infty} h_n(x), \quad \text{for any} \quad x \in X.$$

Clearly, $h \in \mathbb{M}_{\tilde{m}}$. Since m has the Fatou property, taking the limit as $m \to \infty$ in (3.1), we obtain

$$m\left(h_n(x) - h(x)\right) \leq \epsilon\psi(x, x) \quad \text{for all} \quad x \in X \quad \text{and} \quad n \geq k.$$

Thus, $\tilde{m}(h_n - h) \leq \epsilon$ for all $n \geq k$, and therefore $\{h_n\}$ is an $\tilde{m}$-convergent sequence in $\mathbb{M}_{\tilde{m}}$. Hence, $\mathbb{M}_{\tilde{m}}$ is complete. □

Theorem 2. *Let X be a linear space, and X_m be an m-complete convex modular space. Suppose that $q : X \to X_m$ is a function with $q(0) = 0$ satisfying the inequality*

$$m\left(q(kx + y) + q(kx - y) - 2[q(x + y) + q(x - y)] - 2(k^2 - 2)q(x) + 2q(y)\right) \leq \psi(x, y) \tag{3.2}$$

for all $x, y \in X$ and some $k \in \mathbb{N}$, where $\psi : X^2 \to [0, \infty)$ is a function satisfying

$$\psi\left(kx, ky\right) \leq k^2 L\psi(x, y)$$

for all $x, y \in X$ and some L with $0 < L < 1$. Then there exists a unique mapping $P : X \to X_m$ satisfying (2.2) such that

$$m\left(2P(x) - q(x)\right) \leq \frac{1}{2k^2(1 - L)}\psi(x, 0). \tag{3.3}$$

P r o o f. Putting $y = 0$ in (3.2), we obtain

$$m\left(2q(kx) - 2k^2q(x)\right) \leq \psi(x, 0) \tag{3.4}$$

or equivalently,

$$m\left(q(kx) - k^2q(x)\right) \leq \frac{1}{2}\psi(x, 0). \tag{3.5}$$

Now,

$$m\Big(q(x) - \frac{q(kx)}{k^2}\Big) = m\Big(\frac{1}{2k^2}(2q(k\,x) - 2k^2 q(x))\Big) \le \frac{1}{2k^2}\psi(x,0).$$

Consider the set

$$\mathbb{M} = \{h : X \to X_m : h(0) = 0\}$$

and define a function $\tilde{m}$ on $\mathbb{M}$ by

$$\tilde{m}(h) = \inf\{c > 0 : m(h(x)) \le c\psi(x,x)\}, \quad h \in \mathbb{M}.$$

By Lemma 1, $M_{\tilde{m}}$ is a complete convex modular space.

Also, consider the operator $S : \mathbb{M}_{\tilde{m}} \to \mathbb{M}_{\tilde{m}}$ defined by

$$Sh(x) = \frac{1}{k^2}h(kx) \quad \forall h \in \mathbb{M}_{\tilde{m}}, \quad x \in X \quad \text{and} \quad k \in \mathbb{N}.$$

Thus,

$$S^n\,h(x) = \frac{1}{k^{2n}}h(k^n\,x) \quad \forall h \in \mathbb{M}_{\tilde{m}}, \quad x \in X \quad \text{and} \quad k \in \mathbb{N}.$$

Let us show that S is an $\tilde{m}$-strictly contractive mapping. Let $h, z \in \mathbb{M}_{\tilde{m}}$, and suppose there exists a constant $c \in [0,\infty)$ such that

$$\tilde{m}(h - z) \le c.$$

Then,

$$m(h(x) - z(x)) \le c\psi(x,x) \quad \forall x \in X.$$

Now,

$$\begin{aligned} m(Sh(x) - Sz(x)) &= m\Big(\frac{1}{k^2}h(kx) - \frac{1}{k^2}z(kx)\Big) \le \frac{1}{k^2}m(h(kx) - z(kx)) \\ &\le \frac{1}{k^2}c\psi(kx,kx) \le cL\psi(x,x) \quad \forall x \in X. \end{aligned}$$

Therefore,

$$\tilde{m}(Sh - Sz) \le \, c\,L.$$

Hence,

$$\tilde{m}(Sh - Sz) \le L\,\tilde{m}(h - z) \quad \text{for all} \quad g, h \in \mathbb{M}_{\tilde{m}}.$$

That is, S is an $\tilde{m}$-strict contraction.

Now, we prove

$$\delta_{\tilde{m}} = \sup\{\tilde{m}(S^n\,(f) - S^m\,(f)) : m, n \in \mathbb{N}\} < \infty.$$

From (3.5), we have

$$m\left(q(k^2x) - k^2q(kx)\right) \le \frac{1}{2}\psi(kx,0). \tag{3.6}$$

Thus,

$$\begin{aligned} m\left(\frac{q(k^2x)}{(k^2)^2} - q(x)\right) &= m\left(\frac{1}{(k^2)^2}(q(k^2x) - k^2q(kx)) + \frac{1}{k^2}(q(kx) - k^2q(x))\right) \\ &\le \frac{1}{(k^2)^2}\,m(q(k^2x) - k^2q(kx)) + \frac{1}{k^2}\,m(q(kx) - k^2q(x)) \end{aligned}$$

$$\le \frac{1}{2(k^2)^2}\psi(kx,0) + \frac{1}{2k^2}\psi(x,0) \overset{(3.5),(3.6)}{=} \frac{1}{2}\sum_{i=0}^{1}\frac{1}{k^{2(i+1)}}\psi(k^ix,0) \quad \text{for all} \quad x \in X.$$

Since

$$\frac{1}{2}\sum_{i=0}^{n-1}\frac{1}{k^{2(i+1)}} \leq 1,$$

for all $n \geq 0$, we have

$$m\Big(\frac{q(k^n x)}{k^{2n}} - q(x)\Big) = m\Big[\sum_{i=0}^{n-1}\Big(\frac{q(k^{i+1}x)}{k^{2(i+1)}} - \frac{q(k^i x)}{k^{2i}}\Big)\Big]$$

$$= \sum_{i=0}^{n-1}\frac{1}{2\,k^{2(i+1)}}\, m\left(2\,q(k^{i+1}x) - 2\,k^2\,q(k^i x)\right) = \sum_{i=0}^{n-1}\frac{1}{2\,k^{2(i+1)}}\psi(k^i x,\,0)$$

$$\overset{(3.4)}{\leq} \frac{\psi(x,0)}{2k^2}\sum_{i=0}^{n-1}L^i \leq \frac{\psi(x,0)}{2k^2(1-L)} \quad \text{since} \quad 0<L<1.$$

Hence,

$$m\Big(\frac{q(k^n x)}{k^{2n}} - q(x)\Big) \leq \frac{\psi(x,0)}{2k^2(1-L)} \quad \text{since} \quad 0<L<1 \tag{3.7}$$

$\forall x \in X$ and $n \in \mathbb{N}$. Thus, from (3.7) it follows that for any $n, p \in \mathbb{N}$,

$$m\Big(\frac{q(k^n x)}{2k^{2n}} - \frac{q(k^p x)}{2k^{2p}}\Big) \leq \frac{1}{2}m\Big(\frac{q(k^n x)}{k^{2n}} - q(x)\Big) + \frac{1}{2}m\Big(\frac{q(k^p x)}{k^{2p}} - q(x)\Big)$$

$$\leq \frac{1}{2}\cdot\frac{\psi(x,0)}{2k^2(1-L)} + \frac{1}{2}\cdot\frac{\psi(x,0)}{2k^2(1-L)} \leq \frac{\psi(x,0)}{2k^2(1-L)} \quad \text{for all} \quad x\in X \quad [\text{by } (3.7)].$$

This implies that

$$\tilde{m}\Big(S^n\Big(\frac{1}{2}q\Big) - S^p\Big(\frac{1}{2}q\Big)\Big) \leq \frac{1}{2K^2(1-L)} < \infty$$

for all $p, n \in \mathbb{N}$.

This shows that S has a bounded orbit at $1/2q$. Then,

$$m\Big(S^n(\frac{1}{2}q(x)) - \frac{1}{2}q(x)\Big) = m\Big(\frac{q(k^n x)}{2k^{2n}} - \frac{1}{2}q(x)\Big)$$

$$\leq \frac{1}{2}m\Big(\frac{q(k^n x)}{k^{2n}} - q(x)\Big) \leq \frac{1}{2}\cdot\frac{\psi(x,0)}{2k^2(1-L)} < \text{finite} \quad \forall x \in X \quad \text{and} \quad \forall k \in \mathbb{N} \quad [\text{by } (3.7)].$$

Thus, by applying Theorem 1,

(i) S has a fixed point $P \in \mathbb{M}$ at $1/2q$, that is, $SP = P$, or equivalently,

$$P(x) = \frac{1}{k^2}P(kx) \quad \text{for all} \quad x \in X;$$

(ii) the sequence $\{S^n\,(1/2q)\}$ $\tilde{m}$-converges to P.

Therefore,

$$\lim_{n\to\infty} m\Big(\Big(\frac{1}{2k^{2n}}q(k^n x)\Big) - P(x)\Big) = 0.$$

Thus, we can define

$$P(x) := \frac{1}{2}\lim_{n\to\infty}\frac{q(k^n x)}{k^{2n}}.$$

Again, replacing x and y by $k^n x$ and $k^n y$, respectively, in (3.2), we obtain

$$m\Big(\frac{1}{2k^{2n}}q(k^n(kx+y)) + q(k^n(kx-y)) - 2[q(k^n(x+y)) + q(k^n(x-y))]$$
$$-2(k^2-2)q(k^n x) + 2q(k^n y)\Big) \le \frac{1}{2k^{2n}}\psi(k^n x, k^n y) \le \frac{1}{2}L^n\psi(x,y) \quad \forall x \in X, \quad n \in \mathbb{N}.$$

Now, taking the limit as $n \to \infty$ and applying the Fatou property, where $0 < L < 1$, we get

$$P(kx+y) + P(kx-y) = 2[P(x+y) + P(x-y)] + 2(k^2-2)P(x) - 2P(y).$$

Thus, P is a k-quadratic Euler–Lagrange mapping.

Also, since m has the Fatou property, it follows from (3.7) that

$$m(2P(x) - q(x)) \le \frac{1}{2k^2(1-L)}\psi(x,0) \quad \forall x \in X.$$

To prove uniqueness, let $P' : X \to X_m$ be another k-quadratic Euler–Lagrange functional mapping satisfying inequality (3.3). Then we have

$$m\left(P(x) - P'(x)\right) \le \frac{1}{2}m\left(2P(x) - q(x)\right) + \frac{1}{2}m\left(2P'(x) - q(x)\right) \le \frac{\psi(x,0)}{2k^2(1-L)} < \infty$$

for all $x \in X$ and $k \in \mathbb{N}$.

Again, let P and P' be two fixed points of S such that

$$m\left(P(x)) - P'(x)\right) < \infty.$$

Then, by Theorem 1, we conclude that $P(x) = P'(x)$ for all $x \in X$.

This completes the proof of the theorem. □

Corollary 1. *Let X be a normed linear space, and let X_m be an m-complete convex modular space. Suppose $\theta \ge 0$. Let $q : X \to X_m$ be a function with $q(0) = 0$ satisfying*

$$m\big(q(kx+y) + q(kx-y) - 2[q(x+y) + q(x-y)] - 2(k^2-2)q(x) + 2q(y)\big) \le \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in X$, $k \in \mathbb{N}$, and $0 \le p < 1$. Then there exists a unique k-quadratic mapping $P : X \to X_m$ such that

$$m\big(2P(x) - q(x)\big) \le \frac{\theta}{k^2(2-2^p)}\|x\|^p$$

for all $x \in X$.

P r o o f. Define

$$\psi(x,y) = \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in X$ and take $L = 2^{p-1}$. Then the proof of the result follows similarly to Theorem 2. □

Corollary 2. *Let $\epsilon \ge 0$, X be a normed linear space, and X_m be an m-complete convex modular spaces. Suppose a function $q : X \to X_m$ with $q(0) = 0$ satisfies*

$$m(q(kx+y) + q(kx-y) - 2[q(x+y) + q(x-y)] - 2(k^2-2)q(x) + 2q(y)\,) \le \epsilon$$

for all $x, y \in X$ and $k \in \mathbb{N}$. Then there exists a unique k-quadratic mapping $P : X \to X_m$ such that

$$m(2P(x) - q(x)) \le \frac{\epsilon}{k^2}$$

for all $x \in X$.

P r o o f. Define $\psi(x, y) = \epsilon$ for all $x, y \in X$ and take $L = 1/2$. Then the proof of the result follows similarly to Theorem 2. □

Corollary 3. *Let $\theta, \epsilon \geq 0$, X be a normed linear space, and let Y be a Banach space. Suppose that a mapping $q : X \to Y$ with $q(0) = 0$ satisfies the inequality*

$$\big\|q(kx + y) + q(kx - y) - 2[q(x + y) + q(x - y)] - 2(k^2 - 2)q(x) + 2q(y)\big\| \leq \epsilon + \theta(\|x\| + \|y\|)$$

for all $x, y \in X$ and $k \in \mathbb{N}$. Then there exists a unique k-quadratic mapping $P : X \to Y$ such that

$$\|(2P(x) - q(x))\| \leq \frac{\epsilon}{k^2(2 - 2^p)} + \frac{\theta}{k^2(2 - 2^p)}\|x\|^p$$

for all $x \in X$ and $0 \leq p < 1$.

P r o o f. Since every normed linear space is a modular space, we define $m(x) = \|x\|$ and

$$\psi(x, y) = \epsilon + \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ and take $L = 2^{p-1}$. Then the proof follows from Theorem 2. □

Example 2. Let $(X, \|\cdot\|)$ be a commutative Banach algebra, and let X_m be an m-complete convex modular space, where $m(x) = \|x\|$.

Define $q : X \to X_m$ by

$$q(x) = ax^2 + A\|x\|x_0$$

for all $x \in X$, where a, $A \in \mathbb{R}^+$ and x_0 is a unit vector in X. Then

$$\begin{aligned} m\big(q(kx + y) + q(kx - y) - 2[q(x + y) + q(x - y)] - 2(k^2 - 2)q(x) + 2q(y)\big) \\ \leq 2A\big[(k^2 - k - 2)\|x\| + 4\|y\|\big] \end{aligned}$$

for all $x, y \in X$.

Define

$$\psi(x, y) = 2A[(k^2 - k - 2)\|x\| + 4\|y\|]$$

for all $x, y \in X$ and take $L = 1/2$. Thus, all the conditions of Theorem 2 are satisfied. Then there exists a unique k-quadratic Euler–Lagrange function $P : X \to X_m$ such that

$$m(2P(x) - q(x) \leq \frac{2A(k^2 - k - 2)}{k^2}\|x\| \quad \forall\, x \in X.$$

Remark 1. Many of the Hyers–Ulam–Rassias stability results rely on the Δ_α-condition stated in part (v) of Definition 4 for various values of $\alpha \geq 2$. Our theorems are established without assuming this condition on the modular space. Omitting this condition makes the proof more involved. Furthermore, we have employed fixed point methods within the framework of modular spaces. Such an approach to stability problems in modular spaces has previously appeared in [22]. This methodology can also be adapted to other functional equations, potentially serving as a foundation for future research.

REFERENCES

1. Abdou A. A. N., Khamsi M. A. Fixed point theorems in modular vector spaces. *J. Nonlinear Sci. Appl.*, 2017. Vol. 10, No. 8. P. 4046–4057. DOI: 10.22436/jnsa.010.08.01
2. Cădariu L., Radu V. Fixed points and the stability of Jensen's functional equation. *J. Inequal. Pure Appl. Math.*, 2003. Vol. 4, No. 1. Art. no. 4.
3. Eskandani G. Z., Rassias J. M. Stability of general A-cubic functional equations in modular spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 2018. Vol. 112. P. 425–435. DOI: 10.1007/s13398-017-0388-5
4. Gevirtz J. Stability of isometries on Banach spaces. *Proc. Amer. Math. Soc.*, 1983. Vol. 89, No. 4. P. 633–636. DOI: 10.2307/2044596
5. Gruber P. M. Stability of isometries. *Trans. Amer. Math. Soc.*, 1978. Vol. 245. P. 263–277. DOI: 10.2307/1998866
6. Hyers D. H. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. USA*, 1941. Vol. 27, No. 4. P. 222–224. DOI: 10.1073/pnas.27.4.222
7. Jung S.-M. Hyers–Ulam stability of linear differential equations of first order, II. *App. Math. Lett.*, 2006. Vol. 19, No. 9. P. 854–858. DOI: 10.1016/j.aml.2005.11.004
8. Kayal N. C., Mondal P., Samanta T. K. The fuzzy stability of a Pexiderized functional equation. *Math. Morav.*, 2014. Vol. 18, No. 2. P. 1–14. DOI: 10.5937/MatMor1402001K
9. Khamsi M. A., Kozlowski W. M. *Fixed Point Theory in Modular Function Spaces.* Cham: Birkhäuser, 2015. 245 p. DOI: 10.1007/978-3-319-14051-3
10. Khamsi M. A., Kozlowski W. M., Reich S. Fixed point theory in modular function spaces. *Nonlinear Anal.*, 1990. Vol. 14, No. 11. P. 935–953. DOI: 10.1016/0362-546X(90)90111-S
11. Khamsi M. A. A convexity property in modular function spaces. *Math. Japonica*, 1996. Vol. 44, No. 2. P. 269–279.
12. Kim H.-M., Kim M.-Y. Generalized stability of Euler–Lagrange quadratic functional equation. *Abstr. Appl. Anal.*, 2012. Vol. 2012. Art. no. 219435. DOI: 10.1155/2012/219435
13. Koh H. A new generalized cubic functional equation and its stability problems. *J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math.*, 2021. Vol. 28, No. 1. P. 15–26. DOI: 10.7468/jksmeb.2021.28.1.15
14. Kozłowski W. M. *Modular Function Spaces.* Ser. Monogr. Textb. Pure Appl. Math., vol. 122. New York: Marcel Dekker, 1988.
15. Musielak J., Orlicz W. On modular spaces. *Studia Math.*, 1959. Vol. 18, No. 1. P. 49–65.
16. Musielak J. *Orlicz Spaces and Modular Spaces.* Lecture Notes in Math., vol. 1034. Berlin, Heidelberg: Springer, 1983. 226 p. DOI: 10.1007/BFb0072210
17. Nakano H. *Modular Semi-Ordered Spaces.* Tokyo, Japan: Maruzen Co., Ltd., 1950. 288 p.
18. Rassias J. M. On the stability of the Euler–Lagrange functional equation. *Chinese J. Math.*, 1992. Vol. 20, No. 2. P. 185–190. URL: https://www.jstor.org/stable/43836466
19. Rassias J. M. On the stability of the Euler–Lagrange functional equation. *C. R. Acad. Bulgare Sci.*, 1992. Vol. 45. P. 17–20.
20. Rassias J. M. Solution of the Ulam stability problem for Euler–Lagrange quadratic mappings. *J. Math. Anal. Appl.*, 1998. Vol. 220, No. 2. P. 613–639. DOI: 10.1006/jmaa.1997.5856
21. Rassias Th. M. On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soci.*, 1978. Vol. 72, No. 2. P. 297–300. DOI: 10.2307/2042795
22. Sadeghi G. A fixed point approach to stability of functional equations in modular spaces. *Bull. Malays. Math. Sci. Soc.*, 2014. Vol. 37, No. 2. P. 333–344.
23. Saha P., Samanta T. K., Mondal P., Choudhury B. S. Stability of two variable pexiderized quadratic functional equation in intuitionistic fuzzy Banach spaces. *Proyecciones J. Math.*, 2019. Vol. 38, No. 3. P. 447–467. DOI: 10.22199/issn.0717-6279-2019-03-0029
24. Saha P., Samanta T. K., Mondal P., Choudhury B. S. Stability of a two-variable pexiderized additive functional equation in intuitionistic fuzzy Banach spaces: A fixed point approach. *Tamsui Oxf. J. Inf. Math. Sci.*, 2019. Vol. 33, No. 1. P. 30–46.
25. Saha P., Samanta T. K., Mondal P., Choudhury B. S., Sen M. D. L. Applying fixed point techniques to stability problems in intuitionistic fuzzy Banach spaces. *Mathematics*, 2020. Vol. 8, No. 6. Art. no. 974. DOI: 10.3390/math8060974

26. Tamilvanan K., Alkhaldi A. H., Jakhar J., Chugh R., Jakhar J., Rassias J. M. Ulam stability results of functional equations in modular spaces and 2-Banach spaces. *Mathematics*, 2023. Vol. 11, No. 2. Art. no. 371. DOI: 10.3390/math11020371
27. Ulam S. M. *Problems in Modern Mathematics*. New York: J. Wiley & Sons, 1964. 150 p.
28. Uthirasamy N., Tamilvanan K., Nashine H. K., George R. Solution and stability of quartic functional equations in modular spaces by using Fatou property. *J. Funct. Spaces*, 2022. Vol. 2022. Art. no. 5965628. DOI: 10.1155/2022/5965628
29. Wongkum K., Kumam P., Cho Y. J., Thounthong Ph., Chaipunya P. On the generalized Ulam–Hyers–Rassias stability for quartic functional equation in modular spaces. *J. Nonlinear Sci. Appl.*, 10 (2017), 1399–1406. DOI: 10.22436/jnsa.010.04.10
30. Zivari-Kazempour A., Gordji M. E. Generalized Hyers-Ulam stabilities of an Euler–Lagrange–Rassias quadratic functional equation. *Asian-Eur. J. Math.*, 2012. Vol. 5, No. 1. Art. no. 1250014. DOI: 10.1142/S1793557112500143