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TRAJECTORIES OF DYNAMIC EQUILIBRIUM AND REPLICATOR DYNAMICS IN COORDINATION GAMES

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Abstract: The paper analyzes average integral payoff indices for trajectories of the dynamic equilibrium and replicator dynamics in bimatrix coordination games. In such games, players receive large payoffs when choosing the same type of behavior. A special feature of a 2×2 coordination game is the presence of three static Nash equilibria. In the dynamic formulation, the trajectories of coordination games are estimated by the average integral payoffs for a wide range of models arising in economics and biology. In optimal control problems and dynamic games, average integral payoffs are used to synthesize guaranteed strategies, which are involved, among other things, in the constructions of the dynamic Nash equilibrium. In addition, average integral payoffs are a natural tool for assessing the quality of trajectories of replicator dynamics. In the paper, we compare values of average integral indices for trajectories of replicator dynamics and trajectories generated by guaranteed strategies in constructing the dynamic Nash equilibrium. An analysis is provided for trajectories of mixed dynamics when the first player plays a guaranteed strategy, and the behavior of replicator dynamics guides the second player.

Keywords: Dynamic bimatrix games, Coordination games, Average integral payoffs, Guaranteed strategies, Replicator dynamics, Dynamic Nash equilibrium.

1. Introduction

The paper is devoted to analyzing the behavior of equilibrium trajectories in dynamic bimatrix coordination games with average integral indices of players' payoffs. In such games, players obtain better payoffs when choosing the same type of behavior. A feature of a 2×2 coordination game is the presence of three static Nash equilibria. Players' benefits on each time interval are determined as mathematical expectations of payoffs. On the infinite time interval players' payoff functionals are defined as average integral indices (time average values), methods for whose analysis in control theory were studied in papers [1, 14].

In the first step, we consider a solution of the dynamic bimatrix game using approaches of the theory of differential–evolutionary games, ideas of N.N. Krasovskii guaranteed strategies [5, 7, 8], and constructions of L.S. Pontryagin maximum principle [12]. Based on the proposed approach we elaborate an algorithm for constructing positional strategies and equilibrium trajectories of dynamic Nash equilibrium [4, 6]. Equilibrium trajectories generated by guaranteed strategies provide payoff results not worse than those of the static Nash equilibrium [15] located inside the square of the game. In this sense, guaranteed strategies allow one to shift game solutions toward Pareto maximum points generated by cooperative constructions [9, 11].

In the second step, we consider an analysis of constructions for replicator dynamics which is widely used in the theory of evolutionary games and applications [2, 3, 10, 13, 16]. Trajectories of the replicator dynamics in coordination games converge to static Nash equilibria located at vertices of the game square and demonstrate the bifurcation behavior depending on chosen initial positions.

In the third step, we consider the so-called "mixed" dynamics, when the first player uses guaranteed strategies and equilibrium trajectories of the second player are generated by the replicator dynamics. Values of players' payoff functionals at the attraction points of the motion for equilibrium trajectories of "mixed" dynamics majorate values of payoffs at the point of the static Nash equilibrium.

A model is considered for a dynamic coordination game of two coalitions of players called "hawks" and "doves." We construct equilibrium trajectories for guaranteed strategies, replicator dynamics, and "mixed" constructions for such a game. A comparison is carried out for equilibrium trajectories of all three types of dynamics.

2. Game dynamics. Players' payoff functionals

To describe the behavior of two players, we consider the system of differential equations

$$\begin{cases} \dot{\xi}(t) = -\xi(t) + u(t), & \xi(t_0) = \xi_0, \\ \dot{\eta}(t) = -\eta(t) + v(t), & \eta(t_0) = \eta_0, \end{cases} \tag{2.1}$$

where the parameters $\xi = \xi(t)$, $0 \leq \xi \leq 1$, and $\eta = \eta(t)$, $0 \leq \eta \leq 1$ determine the probabilities of choosing strategies by players. For example, the parameter ξ stands for the probability that the first player holds to the first strategy (respectively, $(1 - \xi)$ is the probability that he holds to the second strategy). The parameter η stands for the probability of choosing the first strategy by the second player (respectively, $(1 - \eta)$ means the probability that he holds to the second strategy). The control parameters $u = u(t)$ and $v = v(t)$ satisfy the conditions $0 \leq u \leq 1$ and $0 \leq v \leq 1$ and are signals recommending players to change their strategy. For example, the value $u = 0$ ($v = 0$) corresponds to the signal: "change the first strategy to the second". The value $u = 1$ ($v = 1$) corresponds to the signal: "change the second strategy to the first". The value $u = \xi$ ($v = \eta$) corresponds to the signal: "keep the previous strategy"

The square, $(\xi, \eta) \in [0, 1] \times [0, 1]$, of the game is a strongly invariant set due to the properties of the dynamics (2.1). So, any trajectory of the dynamics (2.1), that starts in the square, survives in it on the infinite horizon of time.

Matrices A and B describe players' payoffs

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Terminal quality functionals are defined as the mathematical expectations of payoffs given by corresponding matrices A and B in a bimatrix game and can be interpreted as "local" interests of players

$$g_A(\xi(T), \eta(T)) = C_A \xi(T)\eta(T) - \alpha_1 \xi(T) - \alpha_2 \eta(T) + a_{22}$$

at given time T. Here, the parameters C_A, α_1, and α_2 are defined according to the classic theory of bimatrix games [15]

$$C_A = a_{11} - a_{12} - a_{21} + a_{22},$$
$$\alpha_1 = a_{22} - a_{12}, \quad \alpha_2 = a_{22} - a_{21}.$$

The quality functional g_B of the second player and the parameters C_B, β_1, and β_2 are defined analogously by the coefficients of the matrix B.

The "global" interests J_A^∞ of the first player are determined as limit relations for quality functionals on an infinite planning horizon

$$\begin{gathered} JI_A^\infty = [JI_A^-, JI_A^+], \\ JI_A^- = JI_A^-(\xi(\cdot),\eta(\cdot)) = \liminf_{T\to\infty} \frac{1}{(T-t_0)} \int_{t_0}^{T} g_A(\xi(t),\eta(t))dt, \\ JI_A^+ = JI_A^+(\xi(\cdot),\eta(\cdot)) = \limsup_{T\to\infty} \frac{1}{(T-t_0)} \int_{t_0}^{T} g_A(\xi(t),\eta(t))dt, \end{gathered} \tag{2.2}$$

calculated for the trajectories $(\xi(\cdot),\eta(\cdot))$ of system (2.1). For the second player, the "global" interests J_B^∞ are determined symmetrically.

The average integral functionals (2.2) are widely used for the problems of evolution in economics and biology. In optimal control problems, such functionals were studied in the papers [1, 14] and called time average values. Unlike the payoff functionals optimized at each time, average integral payoffs allow potential losses in some periods to win in others. Thus, they obtain the best integral result over all periods on the infinite horizon. Such property guarantees another character of switching lines in optimal closed-loop control strategies compared with the problems where payoffs are optimized at the terminal time. This construction allows the system to stay longer in advantageous domains where the local payoffs of coalitions are strictly better than the payoffs at the points of the static Nash equilibrium.

3. Dynamic Nash equilibrium

3.1. Definition of dynamic Nash equilibrium

The solution of the dynamic game is considered based on the optimal control theory [12] and differential game theory [5, 8]. Following [4], we present the definition of the dynamic Nash equilibrium in the class of positional strategies (feedbacks) $U = u(t,\xi,\eta,\varepsilon)$ and $V = v(t,\xi,\eta,\varepsilon)$.

Definition 1. *Let $\varepsilon > 0$ and $(\xi_0,\eta_0) \in [0,1]\times[0,1]$. The pair of feedbacks $U^0 = u^0(t,\xi,\eta,\varepsilon)$ and $V^0 = v^0(t,\xi,\eta,\varepsilon)$ is called the Nash equilibrium at the initial point (ξ_0,η_0) if the following conditions hold for any other feedbacks $U = u(t,\xi,\eta,\varepsilon)$ and $V = v(t,\xi,\eta,\varepsilon)$: the inequalities*

$$\begin{gathered} J_A^-(\xi^0(\cdot),\eta^0(\cdot)) \geq J_A^+(\xi_1(\cdot),\eta_1(\cdot)) - \varepsilon, \\ J_B^-(\xi^0(\cdot),\eta^0(\cdot)) \geq J_B^+(\xi_2(\cdot),\eta_2(\cdot)) - \varepsilon \end{gathered}$$

are true for any trajectories

$$\begin{gathered} (\xi^0(\cdot),\eta^0(\cdot)) \in X(\xi_0,\eta_0,U^0,V^0), \\ (\xi_1(\cdot),\eta_1(\cdot)) \in X(\xi_0,\eta_0,U,V^0), \\ (\xi_2(\cdot),\eta_2(\cdot)) \in X(\xi_0,\eta_0,U^0,V). \end{gathered}$$

Here, the symbol X stands for the set of trajectories that start from the initial point (ξ_0,η_0) and are generated by the corresponding strategies (U^0,V^0), (U,V^0), and (U^0,V) (see [8]).

3.2. Auxiliary zero-sum games

We employ the results of [4] for constructing the desired equilibrium feedbacks U^0 and V^0. Based on this approach, one can develop the notion of equilibrium using optimal feedbacks for differential games $\Gamma_A = \Gamma_A^- \cup \Gamma_A^+$ and $\Gamma_B = \Gamma_B^- \cup \Gamma_B^+$ with payoffs J_A^∞ and J_B^∞. Let us note that, in the game Γ_A, the first player aims to maximize the functional $J_A^-(\xi(\cdot),\eta(\cdot))$ with the guarantee, using the feedback $U = u(t,\xi,\eta,\varepsilon)$, and the second player, as an antagonist, intends to minimize the functional $J_A^+(\xi(\cdot),\eta(\cdot))$, using the feedback $V = v(t,\xi,\eta,\varepsilon)$. In parallel, in the game Γ_B, the second player tries to maximize the functional $J_B^-(\xi(\cdot),\eta(\cdot))$ with the guarantee, and the first player, as an opponent, wishes to minimize the functional $J_B^+(\xi(\cdot),\eta(\cdot))$.

For a description of the dynamic equilibrium, we need the following notation. Let us denote by symbols $u_A^0 = u_A^0(t,\xi,\eta,\varepsilon)$ and $v_B^0 = v_B^0(t,\xi,\eta,\varepsilon)$ the feedbacks, which solve, respectively, the problem of guaranteed maximization of the payoff functionals J_A^- and J_B^-. It is worth noting, that such feedbacks are oriented of the guaranteed maximization of players' payoffs in the long run, and can be called "positive" feedbacks. In addition, we use the symbols $u_B^0 = u_B^0(t,\xi,\eta,\varepsilon)$ and $v_A^0 = v_A^0(t,\xi,\eta,\varepsilon)$ for denoting feedbacks that work most unfavorably for the opposing players. These feedbacks aim to minimize the payoff functionals J_B^+ and J_A^+ of the opponents. We call these strategies the "penalizing" feedbacks.

According to [4], the dynamic Nash equilibrium is formed by sticking together "positive" feedbacks u_A^0, v_B^0 and "penalizing" feedbacks u_B^0 and v_A^0 by the relations

$$U^0 = \begin{cases} u_A^0, & \text{if } \|(\xi,\eta) - (\xi_\varepsilon(t),\eta_\varepsilon(t))\| < \varepsilon, \\ u_B^0, & \text{otherwise,} \end{cases}$$

$$V^0 = \begin{cases} v_B^0, & \text{if } \|(\xi,\eta) - (\xi_\varepsilon(t),\eta_\varepsilon(t))\| < \varepsilon, \\ v_A^0, & \text{otherwise.} \end{cases}$$

In the next sections, we build "positive" feedbacks u_A^0 and v_B^0 for generating trajectories $(\xi^0(\cdot),\eta^0(\cdot))$ that lead the system to more favorable positions than static Nash equilibrium located in the interior of the game square by both the criteria

$$J_A^\infty(\xi^0(\cdot),\eta^0(\cdot)) \ge v_A, \quad J_B^\infty(\xi^0(\cdot),\eta^0(\cdot)) \ge v_B.$$

4. Optimal control problems for players

To construct "positive" feedbacks $u_A^0 = u_A^0(\xi,\eta)$ and $v_B^0 = v_B^0(\xi,\eta)$, we consider in this section an auxiliary two-step optimal control problem with average integral payoff functional for the first player in the case when actions of the second player are most unfavorable. For that, we analyze an optimal control problem for the dynamic system (2.1)

$$\begin{cases} \dot\xi = -\xi + u, & \xi(0) = \xi_0, \\ \dot\eta = -\eta + v, & \eta(0) = \eta_0 \end{cases} \tag{4.1}$$

with the payoff functional

$$J_A^f = \int_0^{T_f} g_A(\xi(t),\eta(t))dt.$$

Here, without loss of generality, we assume that $t_0 = 0$, $T = T_f$, and the terminal time $T_f = T_f(\xi_0,\eta_0)$ is determined by the condition of reaching the target set.

One can assume that the value of the static game equals to zero and the following conditions holds:

$$w_A = \frac{D_A}{C_A} = 0, \quad C_A > 0, \quad 0 < \xi_A = \frac{\alpha_2}{C_A} < 1, \quad 0 < \eta_A = \frac{\alpha_1}{C_A} < 1. \tag{4.2}$$

Let us consider the case when the initial conditions (ξ_0,η_0) of system (4.1) satisfy the following relations:

$$\xi_0 = \xi_A, \quad \eta_0 > \eta_A. \tag{4.3}$$

We suppose that the actions of the second player are mostly unfavorable to the first player. For trajectories of system (4.1) that start from the initial positions (ξ_0, η_0) (4.3), these actions are determined by the relation

$$v_A^0 = 0.$$

In this situation, the optimal actions u_A^0 of the first player according to the payoff functionals J_A^f can be presented as a two-step impulse control: it equals to unit from the initial time $t_0 = 0$ till the moment of optimal switch s and then equals to zero till the terminal time T_f:

$$u_A^0(t) = \begin{cases} 1 & \text{if} \quad t_0 \le t < s, \\ 0 & \text{if} \quad s \le t < T_f. \end{cases}$$

Here, the value s is the parameter of optimization. The terminal time T_f is determined from the following condition. The trajectory $(\xi(\cdot),\ \eta(\cdot))$ of system (4.1) that starts from the line on which $\xi(t_0) = \xi_A$ returns to this line when $\xi(T_f) = \xi_A$, which can be considered as the target set.

Let us consider two aggregates of characteristics. The first one is described by the system of differential equations with the value of the control parameter $u = 1$

$$\begin{cases} \dot{\xi} = -\xi + 1, \\ \dot{\eta} = -\eta, \end{cases} \tag{4.4}$$

solutions of which are determined by the Cauchy formula

$$\begin{cases} \xi(t) = (\xi_0 - 1)e^{-t} + 1, \\ \eta(t) = \eta_0 e^{-t}. \end{cases} \tag{4.5}$$

Here, the initial positions (ξ_0, η_0) satisfy conditions (4.3), and the time parameter t satisfies the inequality $0 \le t < s$.

The second aggregate of characteristics is given by the system of differential equations with the value of the control parameter $u = 0$:

$$\begin{cases} \dot{\xi} = -\xi, \\ \dot{\eta} = -\eta, \end{cases} \tag{4.6}$$

solutions of which are determined by the Cauchy formula

$$\begin{cases} \xi(t) = \xi_1 e^{-t}, \\ \eta(t) = \eta_1 e^{-t}. \end{cases} \tag{4.7}$$

Here, the initial positions $(\xi_1, \eta_1) = (\xi_1(s), \eta_1(s))$ are determined by the relations

$$\begin{cases} \xi_1 = \xi_1(s) = (\xi_0 - 1)e^{-s} + 1, \\ \eta_1 = \eta_1(s) = \eta_0 e^{-s}, \end{cases} \tag{4.8}$$

and the time parameter t satisfies the inequality $0 \le t < p$. Here, the terminal time $p = p(s)$ and the final position $(\xi_2, \eta_2) = (\xi_2(s), \eta_2(s))$ of the whole trajectory $(\xi(\cdot), \eta(\cdot))$ are given by the formulas

$$\xi_1 e^{-p} = \xi_A, \quad p = p(s) = \ln \frac{\xi_1(s)}{\xi_A}, \quad \xi_2 = \xi_A, \quad \eta_2 = \eta_1 e^{-p}. \tag{4.9}$$

The optimal control problem is to find such time s and the corresponding switching point $(\xi_1, \eta_1) = (\xi_1(s), \eta_1(s))$ on the trajectory $(\xi(\cdot), \eta(\cdot))$, where the integral $I = I(s)$ reaches its maximum,

$$\begin{gathered} I(s) = I_1(s) + I_2(s), \\ I_1(s) = \int_0^s \big(C_A((\xi_0 - 1)e^{-t} + 1)\eta_0 e^{-t} - \alpha_1((\xi_0 - 1)e^{-t} + 1) - \alpha_2 \eta_0 e^{-t} + a_{22}\big)dt, \\ I_2(s) = \int_0^{p(s)} \big(C_A \xi_1(s)\eta_1(s)e^{-2t} - \alpha_1 \xi_1(s)e^{-t} - \alpha_2 \eta_1(s)e^{-t} + a_{22}\big)dt. \end{gathered} \tag{4.10}$$

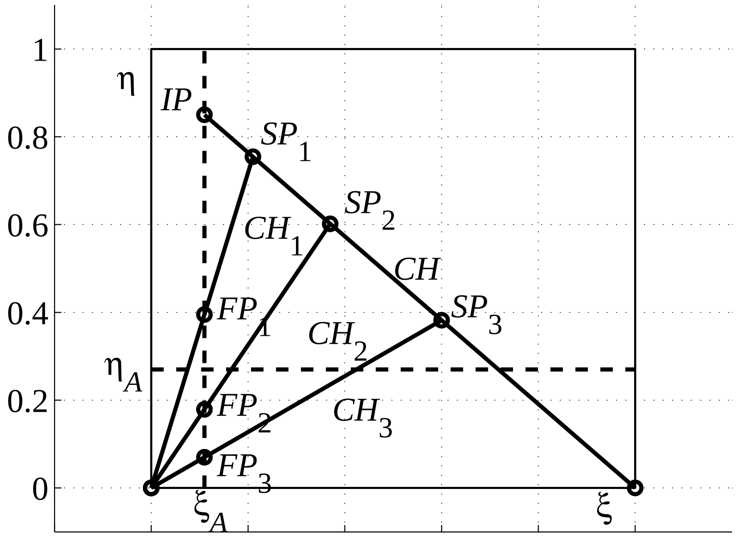

Figure 1. Characteristics of the Hamilton–Jacobi equation and the switching points.

Figure 1 shows the initial position IP chosen on the line $\xi = \xi_A$ with $\eta > \eta_A$, the characteristics CH oriented toward the vertex $(1, 0)$, the characteristics CH_1, CH_2, and CH_3 oriented toward the vertex $(0, 0)$, the switching points SP_1, SP_2, and SP_3 of the motion along the characteristics, and the endpoints FP_1, FP_2, and FP_3 of the motion located on the target line $\xi = \xi_A$.

5. Construction of switching lines

To solve the optimal control problem (4.4)–(4.10), we are based on the following algorithm. We use the necessary optimality conditions and calculate the derivative dI/ds, derive it as the function of optimal switching points $(\xi, \eta) = (\xi_1, \eta_1)$, equate this derivative to zero $dI/ds = 0$, and obtain the equation $F(\xi, \eta) = 0$ for the curve that consists of optimal switching points (ξ, η). This curve is called the switching line.

In the first stage, let us calculate the integrals I_1 and I_2:

$$\begin{aligned} I_1 = I_1(s) = C_A(\xi_0 - 1)\eta_0 \frac{(1 - e^{-2s})}{2} + C_A\eta_0(1 - e^{-s}) - \alpha_1((\xi_0 - 1)(1 - e^{-s}) + s) \\ -\alpha_2\eta_0(1 - e^{-s}) + a_{22}s, \\ I_2 = I_2(s) = C_A\xi_1(s)\eta_1(s)\frac{(1 - e^{-2p(s)})}{2} - \alpha_1\xi_1(s)(1 - e^{-p(s)}) - \alpha_2\eta_1(s)(1 - e^{-p(s)}) + a_{22}p(s). \end{aligned}$$

Next, we calculate the derivatives dI_1/ds and dI_2/ds and represent them as functions of optimal switching points $(\xi, \eta) = (\xi_1, \eta_1)$

$$\begin{aligned}\frac{dI_1}{ds} &= C_A(\xi_0 - 1)\eta_0 e^{-2s} + C_A \eta_0 e^{-s} - \alpha_1\big((\xi_0 - 1)e^{-s} + 1\big) - \alpha_2 \eta_0 e^{-s} + a_{22}\\ &= C_A \xi\eta - \alpha_1 \xi - \alpha_2 \eta + a_{22},\\ \frac{dI_2}{ds} &= C_A\Big(\frac{d\xi}{ds}\eta\frac{(1-e^{-2p})}{2} + \xi\frac{d\eta}{ds}\frac{(1-e^{-2p})}{2} + \xi\eta e^{-2p}\frac{dp}{ds}\Big) - \alpha_1\frac{d\xi}{ds}(1-e^{-p})\\ &\quad -\alpha_1 \xi e^{-p}\frac{dp}{ds} - \alpha_2\frac{d\eta}{ds}(1-e^{-p}) - \alpha_2 \eta e^{-p}\frac{dp}{ds} + a_{22}\frac{dp}{ds}\end{aligned}$$
$$= (C_A^2\xi^2\eta - \alpha_2^2\eta - 2C_A^2\xi^3\eta - 2\alpha_1 C_A\xi^2 + 2\alpha_1 C_A \xi^3 + 2\alpha_2 C_A\xi^2\eta + 2C_A a_{22}\xi - 2C_A a_{22}\xi^2)/(2C_A\xi^2).$$

In the latter equation, we use the following expressions for the derivatives $d\xi/ds$, $d\eta/ds$, and dp/ds and the exponents e^{-p}, e^{-2p}, $(1-e^{-p})$, and $(1-e^{-2p})$ as functions of the variables (ξ, η):

$$\frac{d\xi}{ds} = 1 - \xi, \quad \frac{d\eta}{ds} = -\eta, \quad \frac{dp}{ds} = \frac{1-\xi}{\xi},$$
$$e^{-p} = \frac{\alpha_2}{C_A\xi}, \quad e^{-2p} = \frac{\alpha_2^2}{C_A^2\xi^2}, \quad 1 - e^{-p} = \frac{C_A\xi - \alpha_2}{C_A\xi}, \quad 1 - e^{-2p} = \frac{C_A^2\xi^2 - \alpha_2^2}{C_A^2\xi^2}.$$

Transforming the derivatives dI_1/ds and dI_2/ds, we obtain the following equation for the switching line:

$$\frac{C_A^2\xi^2\eta - 2\alpha_1 C_A\xi^2 - \alpha_2^2\eta + 2C_A a_{22}\xi}{2C_A\xi^2} = 0.$$

Using the assumption that $w_A = 0$ (see (4.2)), we get the final expression for the switching line M_A^1:

$$\eta = \frac{2\alpha_1\xi}{C_A\xi + \alpha_2}.$$

The curve M_A^1 is a hyperbola that passes through the points $(0, 0)$, (ξ_A, η_A) and possesses the horizontal asymptote

$$\eta = \frac{2\alpha_1}{C_A}.$$

To complete the construction of the switching line M_A in the case when $C_A > 0$, we add a similar line M_A^2 to the line M_A^1 in the domain when $\eta \leq \eta_A$:

$$M_A = M_A^1 \cup M_A^2, \tag{5.11}$$
$$M_A^1 = \Big\{(\xi, \eta) \in [0,1]\times[0,1] \colon \eta = \frac{2\alpha_1\xi}{C_A\xi + \alpha_2},\ \eta \geq \frac{\alpha_1}{C_A}\Big\},$$
$$M_A^2 = \Big\{(\xi, \eta) \in [0,1]\times[0,1] \colon \eta = -\frac{2(C_A - \alpha_1)(1-\xi)}{C_A(1-\xi) + (C_A - \alpha_2)} + 1,\ \eta \leq \frac{\alpha_1}{C_A}\Big\}.$$

Let us note that, in the case when $C_A < 0$, the lines M_A, M_A^1, and M_A^2 are described by the formulas

$$M_A = M_A^1 \cup M_A^2, \tag{5.12}$$
$$M_A^1 = \Big\{(\xi, \eta) \in [0,1]\times[0,1] \colon \eta = \frac{2\alpha_1(1-\xi)}{C_A(1-\xi) + (C_A - \alpha_2)},\ \eta \geq \frac{\alpha_1}{C_A}\Big\},$$
$$M_A^2 = \Big\{(\xi, \eta) \in [0,1]\times[0,1] \colon \eta = -\frac{2(C_A - \alpha_1)\xi}{C_A\xi + \alpha_2} + 1,\ \eta \leq \frac{\alpha_1}{C_A}\Big\}.$$

One can see that the line M_A divides the unit square $[0,1]\times[0,1]$ into two parts: the upper part
$$D_A^u \supset \big\{(\xi,\eta)\colon \xi=\xi_A,\ \eta>\eta_A\big\}$$
and the lower part
$$D_A^l \supset \big\{(\xi,\eta)\colon \xi=\xi_A,\ \eta<\eta_A\big\}.$$
The "positive" feedback u_A^0 has the following structure:
$$u_A^0 \;=\; u_A^0(\xi,\eta)=\left\{\begin{array}{lll} \max\{0,-\operatorname{sgn}(C_A)\} & \text{if} & (\xi,\eta)\in D_A^u,\\ \max\{0,\operatorname{sgn}(C_A)\} & \text{if} & (\xi,\eta)\in D_A^l,\\ [0,1] & \text{if} & (\xi,\eta)\in M_A. \end{array}\right. \tag{5.13}$$

One can obtain the similar switching lines M_B for the second player whose profit is oriented on the payoff matrix B. For example, in the case when $C_B>0$, the switching line M_B is presented by the relations
$$M_B = M_B^1\cup M_B^2, \tag{5.14}$$
$$M_B^1=\Big\{(\xi,\eta)\in[0,1]\times[0,1]\colon \eta=\frac{\beta_1\xi}{2\beta_2-C_B\xi},\ \xi\geq\frac{\beta_2}{C_B}\Big\},$$
$$M_B^2=\Big\{(\xi,\eta)\in[0,1]\times[0,1]\colon \eta=-\frac{(C_B-\beta_1)(1-\xi)}{2(C_B-\beta_2)-C_B(1-\xi)}+1,\ \xi\leq\frac{\beta_2}{C_B}\Big\}.$$

When the parameter C_B is negative, $C_B<0$, the lines M_B, M_B^1, and M_B^2 are constructed by the formulas
$$M_B = M_B^1\cup M_B^2, \tag{5.15}$$
$$M_B^1=\Big\{(\xi,\eta)\in[0,1]\times[0,1]\colon \eta=-\frac{(C_B-\beta_1)\xi}{2\beta_2-C_B\xi}+1,\ \xi\geq\frac{\beta_2}{C_B}\Big\},$$
$$M_B^2=\Big\{(\xi,\eta)\in[0,1]\times[0,1]\colon \eta=\frac{\beta_1(1-\xi)}{2(C_B-\beta_2)-C_B(1-\xi)},\ \xi\leq\frac{\beta_2}{C_B}\Big\}.$$

Similarly, the line M_B divides the unit square $[0,1]\times[0,1]$ into two parts: the left part
$$D_B^l \supset \big\{(\xi,\eta)\colon \xi<\xi_B,\ \eta=\eta_B\big\}$$
and the right part
$$D_B^r \supset \big\{(\xi,\eta)\colon \xi>\xi_B,\ \eta=\eta_B\big\}.$$
The "positive" feedback v_B^0 has the following structure:
$$v_B^0=v_B^0(\xi,\eta)=\left\{\begin{array}{lll} \max\{0,-\operatorname{sgn}(C_B)\} & \text{if} & (\xi,\eta)\in D_B^l,\\ \max\{0,\operatorname{sgn}(C_B)\} & \text{if} & (\xi,\eta)\in D_B^r,\\ [0,1] & \text{if} & (\xi,\eta)\in M_B. \end{array}\right. \tag{5.16}$$

6. Models of coordination games

Let us consider two different examples of coordination games.

The first example is the following. Two individuals (two species) compete for territory or a useful resource. Each player can choose one of the strategies: "hawk" or "dove" (see, for example, [16]). The names of the strategies are conditional, denoting only two types of behavior: enter into an aggressive conflict or retreat. In the asymmetric form of the game, we will consider the damage to the players to be different if they choose different strategies.

Let the first player be “the owner” and the second be “the invader” in the game of competing for territory. If both choose aggressive behavior, the damage will be considered the same and equal to 1, if both retreated — 0. In the event of an attack by “an invader”, the damage is equal to 4 and 3, respectively. In the aggressive behavior of “the owner”, the damage equals 3 and 5, respectively.

The matrix A reflects the damage of the first player, and the matrix B stands for the damage of the second player:

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 5 \\ 3 & 0 \end{pmatrix}. \tag{6.1}$$

Let us present the main “game” parameters with the matrices A and B [15]:

$$\begin{gathered} C_A = a_{11} - a_{12} - a_{21} + a_{22} = -6, \\ \alpha_1 = a_{22} - a_{12} = -3, \quad \alpha_2 = a_{22} - a_{21} = -4, \\ \xi_A = \frac{\alpha_2}{C_A} = 0.67, \quad \eta_A = \frac{\alpha_1}{C_A} = 0.5, \end{gathered} \tag{6.2}$$

$$\begin{gathered} C_B = b_{11} - b_{12} - b_{21} + b_{22} = -7, \\ \beta_1 = b_{22} - b_{12} = -5, \quad \beta_2 = b_{22} - b_{21} = -3, \\ \xi_B = \frac{\beta_2}{C_B} = 0.43, \quad \eta_B = \frac{\beta_1}{C_B} = 0.71. \end{gathered} \tag{6.3}$$

In parallel, we consider the second example where we construct a modification of the previous coordination game “hawk” and “dove” with the following payoff matrices:

$$A = \begin{pmatrix} 10 & 0 \\ 7 & 23 \end{pmatrix}, \quad B = \begin{pmatrix} 19 & 0 \\ 4 & 11 \end{pmatrix}, \tag{6.4}$$

$$\begin{gathered} C_A = a_{11} - a_{12} - a_{21} + a_{22} = 26, \\ \alpha_1 = a_{22} - a_{12} = 23, \quad \alpha_2 = a_{22} - a_{21} = 16, \\ \xi_A = \frac{\alpha_2}{C_A} = 0.62, \quad \eta_A = \frac{\alpha_1}{C_A} = 0.88, \end{gathered} \tag{6.5}$$

$$\begin{gathered} C_B = b_{11} - b_{12} - b_{21} + b_{22} = 26, \\ \beta_1 = b_{22} - b_{12} = 11, \quad \beta_2 = b_{22} - b_{21} = 7, \\ \xi_B = \frac{\beta_2}{C_B} = 0.27, \quad \eta_B = \frac{\beta_1}{C_B} = 0.42. \end{gathered} \tag{6.6}$$

7. Feedback strategies and equilibrium trajectories

In this section, we provide feedback strategies and equilibrium trajectories for the given examples of the “hawk”–“dove” game based on the solution constructions given in formulas (5.11)–(5.16).

The structure of the dynamic Nash equilibrium of the first example (6.1)–(6.3) is presented in Figure 2. Here we depict the saddle points S_A and S_B of the static game, points of the static Nash equilibria NE_1, NE_2, and NE_3, and the switching lines $M_A = M_A^1 \cup M_A^2$ and $M_B = M_B^1 \cup M_B^2$. The equilibrium trajectories start from the initial points IP_1, IP_2, and IP_3, then move along characteristics of the Hamilton–Jacobi equations, meet the switching lines where they change orientation, and converge to the final points FP_1, FP_2, and FP_3.

The values of players’ payoff functionals at the final points of the motion of the equilibrium trajectories are the following: $g_A(FP_1) = g_A(FP_2) = 3$, $g_B(FP_1) = g_B(FP_2) = 5$, $g_A(FP_3) = 4$,

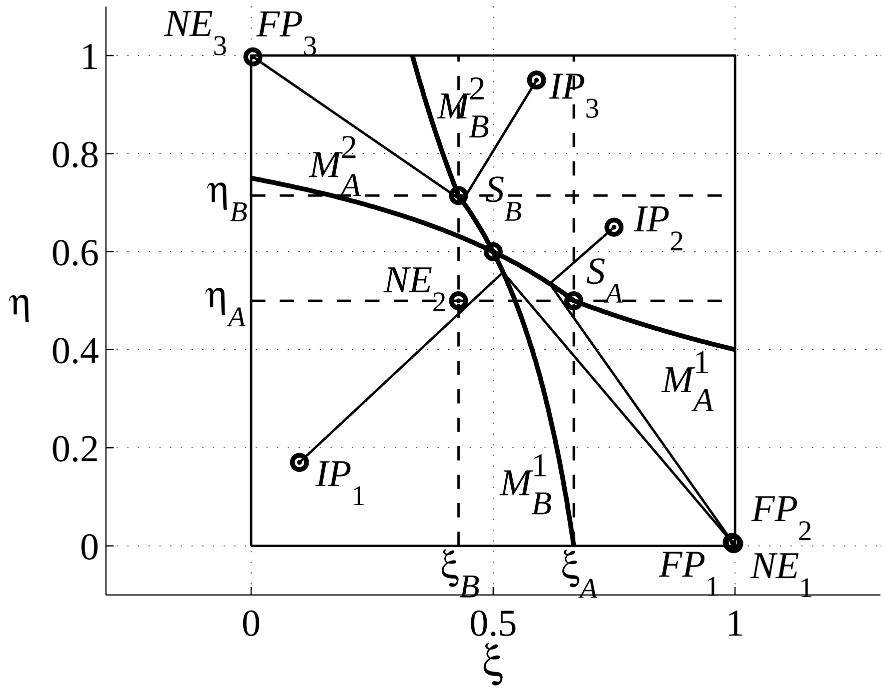

Figure 2. Equilibrium trajectories in the game with average integral payoffs (Example 1).

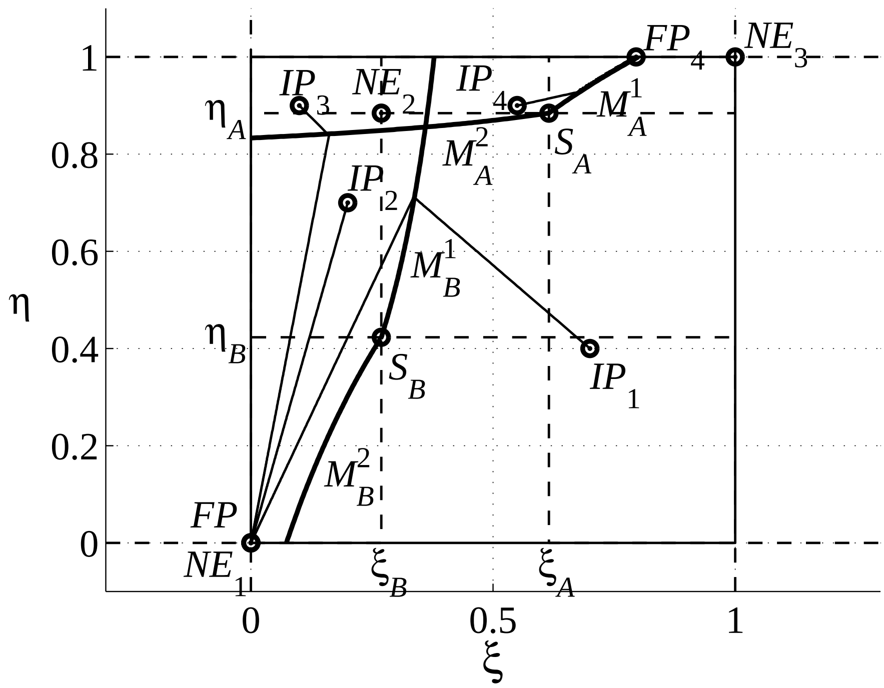

Figure 3. Equilibrium trajectories in the game with average integral payoffs (Example 2).

and $g_B(FP_3) = 3$. Let us note that these values majorate the payoffs at the point of the static Nash equilibrium NE_2: $g_A(NE_2) = 2$ and $g_B(NE_2) = 2.14$.

The structure of the dynamic Nash equilibrium of the second example (6.4)–(6.6) is presented in Figure 3. Here we depict the saddle points S_A and S_B of the static game, points of the static Nash equilibria NE_1, NE_2, and NE_3, and the switching lines $M_A = M_A^1 \cup M_A^2$ and $M_B = M_B^1 \cup M_A^B$. Equilibrium trajectories start from the initial points IP_1, IP_2, IP_3, and IP_4, then move along characteristics of the Hamilton–Jacobi equations, meet the switching lines where they change orientation, and converge to the final points FP and FP_4. Let us note that the final point FP_4 does not coincide with the Nash equilibrium NE_3.

The values of players' payoff functionals at the final points of the motion of the equilibrium trajectories are the following: $g_A(FP) = 23, g_B(FP) = 11$, $g_A(FP_4) = 9.39$, and $g_B(FP_4) = 15.93$. Let us note that these values majorate the payoffs at the point of the static Nash equilibrium NE_2: $g_A(NE_2) = 6.91$ and $g_B(NE_2) = 7.11$. At the point FP_4 located on the boundary of the game square, guaranteed strategies provide a result that brings closer the interests of the players.

8. Replicator dynamics

In this section, we present the structure of the replicator dynamics.

The general view of replicator dynamics for the dynamic bimatrix game can be presented as follows (see, for example, [2, 3, 16]):

$$\begin{cases} \dot{u_i} = u_i\big((\mathbf{Av})_i - (\mathbf{u}, \mathbf{Av})\big), \\ \dot{v_j} = v_j\big((\mathbf{Bu})_j - (\mathbf{v}, \mathbf{Bu})\big), \quad 1 \le i \le n, \quad 1 \le j \le n. \end{cases} \tag{8.1}$$

Here, the vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ describe the system state. The symbols $(\mathbf{Av})_i$ and $(\mathbf{Bu})_j$ stand for the fitness of the corresponding type. An average fitness is defined as follows:

$$(\mathbf{u}, \mathbf{Av}) = \sum_{i=1}^{n} u_i(\mathbf{Av})_i, \quad (\mathbf{v}, \mathbf{Bu}) = \sum_{i=1}^{n} v_i(\mathbf{Bu})_i.$$

System (8.1) is consistent with one of the basic principles of Darwinism: the reproductive success of an individual or a group depends on the advantage of one's fitness over the population's average fitness.

Let us present the main characteristics of the replicator systems of the type (8.1).

The Jacobi matrix at the stationary point (the static Nash equilibrium) in the general case has the form [3]:

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{C} \\ \mathbf{D} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{0}$ is the zero submatrix of size $(n-1) \times (n-1)$, and $\mathbf{C}$ and $\mathbf{D}$ are submatrices formed by some constant coefficients.

The characteristic polynomial of the system has the form

$$p(\lambda) = \det(\lambda^2 \mathbf{I} - \mathbf{DC}).$$

From the structure of the characteristic polynomial, it follows that, in the two-dimensional case, the system cannot have a stationary point of the focus or node type.

For the dynamic bimatrix 2×2 game, the replicator dynamics can be written in the form of the system of differential equations of the second order:

$$\begin{cases} \dot{\xi}(t) = \xi(t)\big(1 - \xi(t)\big)\big(C_A\eta(t) - \alpha_1\big), \quad \xi(t_0) = \xi_0, \\ \dot{\eta}(t) = \eta(t)\big(1 - \eta(t)\big)\big(C_B\xi(t) - \beta_2\big), \quad \eta(t_0) = \eta_0. \end{cases} \tag{8.2}$$

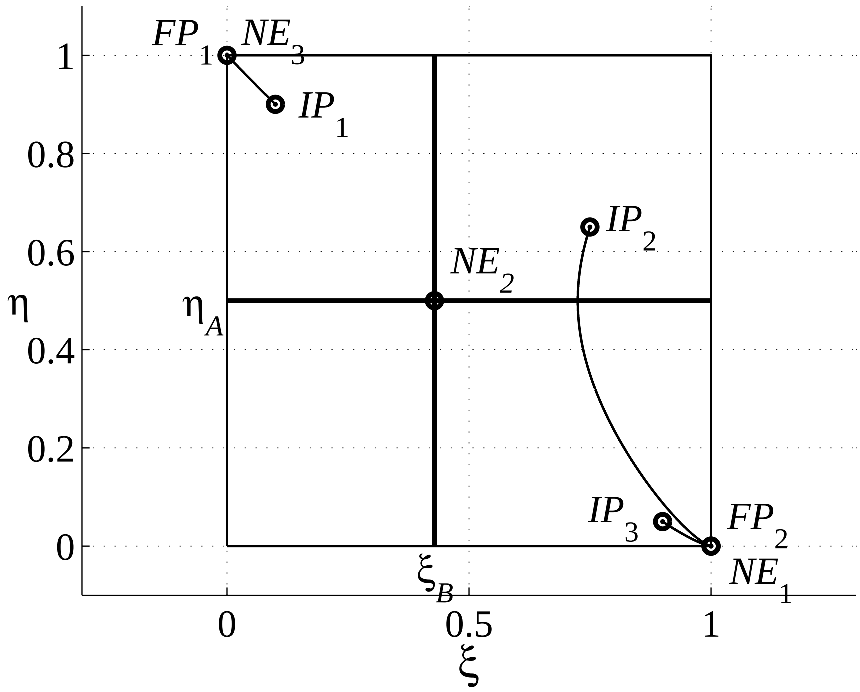

Figure 4. Trajectories of replicator dynamics (Example 1).

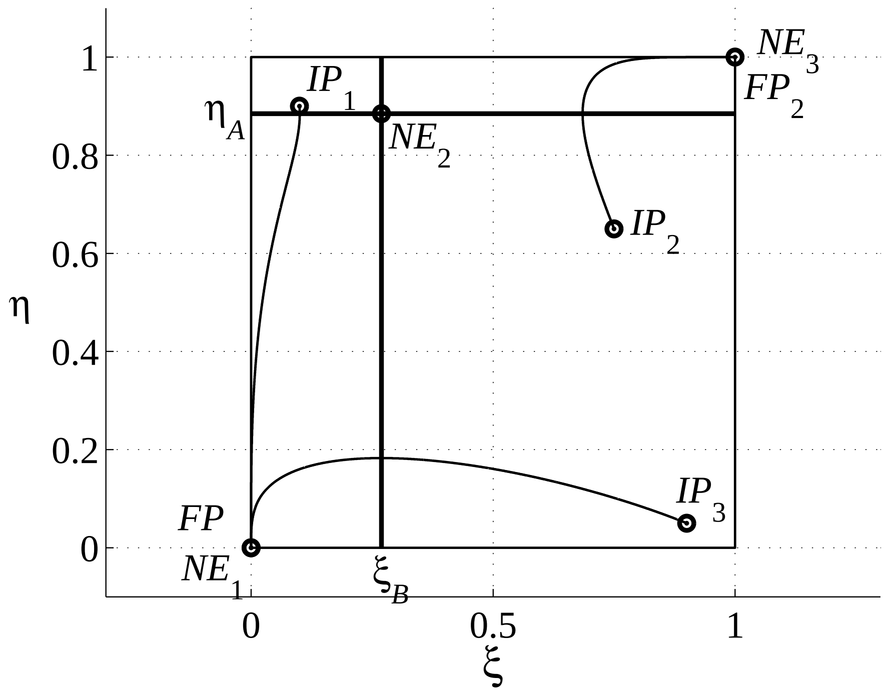

Figure 5. Trajectories of replicator dynamics (Example 2).

Figure 4 presents the trajectories of the replicator dynamics for the first example. They start from the initial points IP_1, IP_2, and IP_3 and tend to the final points FP_1 and FP_2, which coincide with the Nash equilibria NE_1 and NE_3.

Figure 5 presents the trajectories of the replicator dynamics for the second example. They start from the initial points IP_1, IP_2, and IP_3 and terminate their motion at the points FP and FP_2 matching with the Nash equilibria NE_1 and NE_3.

9. Mixed dynamics

In this section, we consider mixed dynamics when the first player uses the guaranteed strategy with switching line M_A (5.11), (5.12) that has the form $u_A^0 = u_A^0(\xi(t), \eta(t))$ (5.13), and the strategy of the second player is formed by the replicator dynamics (8.2):

$$\begin{cases} \dot{\xi}(t) = -\xi(t) + u_A^0(\xi(t), \eta(t)), & \xi(t_0) = \xi_0, \\ \dot{\eta}(t) = \eta(t)\big(1 - \eta(t)\big)\big(C_B \xi(t) - \beta_2\big), & \eta(t_0) = \eta_0. \end{cases}$$

Figure 6 presents the mixed dynamics for the first example. Here we show the switching line $M_A = M_A^1 \cup M_A^2$ for the control of the first player and the switching line $\xi = \xi_B$ for the control of the second player related to the replicator dynamics. The trajectories of the mixed dynamics start from the initial points IP_1, IP_2, and IP_3, switch control on the line M_A, and converge to the final points FP_1 and FP_2.

Figure 7 presents the mixed dynamics for the second example. Here we show the switching line $M_A = M_A^1 \cup M_A^2$ for the control of the first player and the switching line $\xi = \xi_B$ for the control of the second player formed by the replicator dynamics. The trajectories of the mixed dynamics start from the initial points IP_1, IP_2, IP_3, and IP_4, have a control switch on the line M_A, and converge to the final points FP_1 and FP_2.

In the second example, the mixed dynamics demonstrate that guaranteed strategies can provide convergence to the final points, for instance, to the final point FP_1, which differs from the Nash equilibrium NE_3 and gives the payoff results with closer interests of the players.

10. Conclusion

An analysis of the behavior of equilibrium trajectories is provided for the 2×2 dynamic bimatrix coordination game. First, trajectories of the dynamic Nash equilibrium are constructed within the approach of guaranteed strategies in the sense of N.N. Krasovskii in combination with the L.S. Pontryagin maximum principle. Second, an analysis is provided for the replicator dynamics whose trajectories converge to the static Nash equilibrium points located in the vertices of the game square. Third, computational experiments are carried out for the mixed dynamics in which we couple the strategies of the considered dynamics: the strategies of the dynamic Nash equilibrium and the replicator dynamics. Finally, the comparison results are presented for equilibrium trajectories of the considered dynamics.

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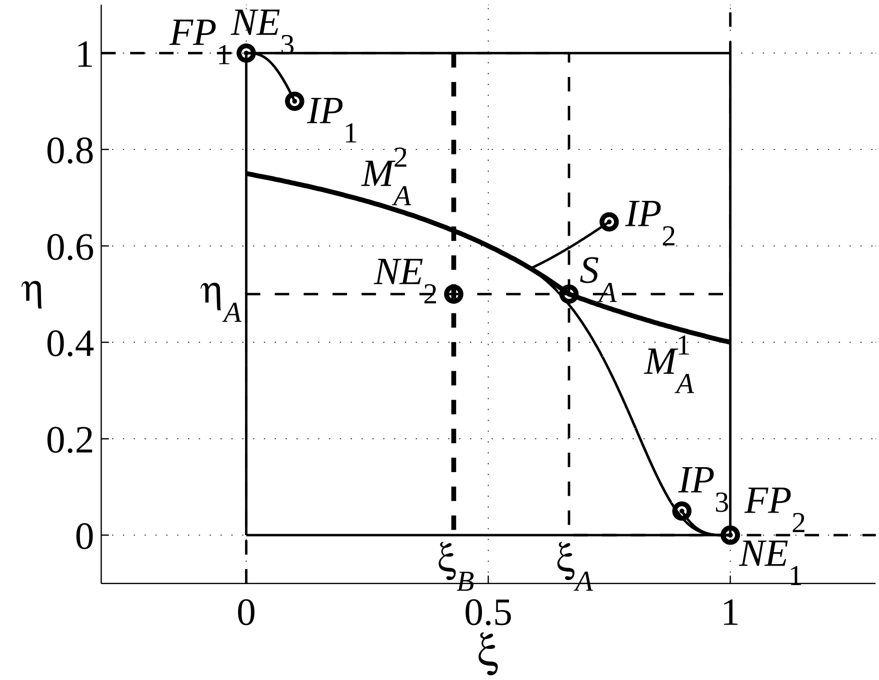

Figure 6. Equilibrium trajectories of mixed dynamics (Example 1).

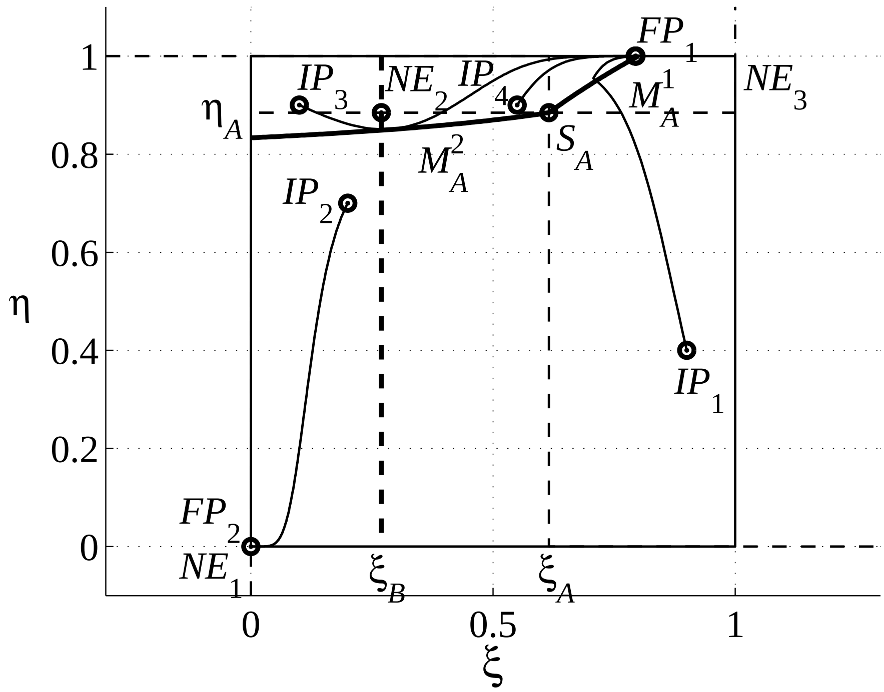

Figure 7. Equilibrium trajectories of mixed dynamics (Example 2).

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