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EXACT CONTROLLABILITY OF FRACTIONAL IMPULSIVE SYSTEM

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Abstract: This paper discusses the exact controllability of linear and nonlinear impulsive Caputo fractional systems. The exact controllability of a linear impulsive system is studied using the concept of generators and functional analysis. In contrast, the controllability of a nonlinear system is discussed using nonlinear functional analysis. An example is provided in the paper to support the results.

Keywords: Fractional system, Exact controllability, Functional analysis, Fixed-point theorem.

1. Introduction

Fractional calculus, which generalizes differentiation and integration to noninteger orders, has revolutionized the modeling of complex dynamical systems over the past three decades. Unlike classical integer-order models, which assume instantaneous responses, fractional-order models capture memory and hereditary effects, making them ideal for describing viscoelastic materials, anomalous diffusion, signal processing, fluid dynamics, and control systems [1, 15, 19, 25–27]. By providing a continuum of operators between local and global dynamics, fractional calculus bridges traditional differentiation and integration, offering unparalleled flexibility in modeling real-world phenomena.

Many physical systems exhibit nonlocal behavior in time or space, where responses depend on the entire history of states. This property has made fractional differential equations indispensable in fields such as fluid mechanics (e.g., the Basset problem), viscoelasticity (e.g., the Bagley–Torvik equation), and biological systems involving diffusion or wave propagation [2]. The theoretical and numerical frameworks for these systems have advanced significantly to address diverse engineering and scientific challenges.

Fractional dynamical systems have gained prominence in control theory due to their enhanced design flexibility. Fractional-order controllers, such as the fractional PID (a generalization of the classical Proportional–Integral–Derivative controller using noninteger derivatives), introduce additional parameters that improve transient and steady-state performance, robustness, and noise immunity [16, 28]. Despite these advances, the controllability of fractional systems – i.e., the ability to steer a system from any initial state to any desired state in finite time – remains less developed than that of integer-order systems, largely due to the complexity of fractional operators.

Significant progress has been made in understanding controllability for fractional systems. D. Matignon and B. D'Andréa-Novel [16] explored linear fractional systems, while B. M. Vinagre et al. [28] established foundational frameworks. M. Bettayeb and S. Djennoune [8] used rank conditions to assess controllability, and Y. Q. Chen et al. [9] investigated robust controllability for uncertain linear systems. S. Guermah et al. [11] extended these ideas to discrete-time systems, and D. Mozyrska and D. F. M. Torres [17, 18] introduced energy-based methods for systems with Riemann—Liouville and Caputo derivatives. K. Balachandran et al. [3–6] derived conditions for exact controllability of linear and nonlinear systems using fixed-point theorems, while V. Govindraj and R. K. George [10] applied functional analytic techniques to semilinear systems. These studies highlight the richer controllability structures of fractional systems compared to integer-order counterparts.

Impulsive fractional systems, which combine memory effects with abrupt state changes, are particularly relevant for modeling phenomena such as population dynamics with sudden harvesting, mechanical impacts, or event-triggered controls [22–24]. Such systems pose unique mathematical challenges but are critical for applications requiring intermittent or discrete control actions.

This article investigates the exact controllability of a generalized fractional impulsive system:

$$\begin{aligned} {}^c\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{F}_k(t, x(t), u(t)) + \mathcal{B}_k u(t), \quad t \in [t_{k-1}, t_k), \quad k = 1, 2, \dots, \rho, \\ x(0) &= x_0, \\ \Delta x(t_k) &= \mathcal{M}_k x(t_k) + \mathcal{N}_k u(t_k), \quad t = t_k, \quad k = 1, 2, \dots, \rho, \end{aligned} \tag{1.1}$$

where $0 < \alpha \le 1$, $x(t)$ evolves in a Hilbert space $\mathbb{X}$ over $J_0 = [0, T_0]$, $\mathcal{A}$ and $\mathcal{M}_k$ are linear operators, $u \in L^2([0, T_0], \mathbb{U})$, and $\mathcal{B}_k, \mathcal{N}_k : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ are bounded linear operators. The nonlinear mappings $\mathcal{F}_k : [0, T_0] \times \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ model varying nonlinearities across the subintervals.

Unlike prior studies focusing on linear or specific nonlinear systems [3, 16], this work develops a unified analytical framework for the exact controllability of fractional impulsive systems with general nonlinearities. By designing control inputs tailored to each subinterval, we extend existing results to address complex dynamics arising from both hereditary effects and instantaneous perturbations, with applications in hybrid control and event-driven systems.

2. Mathematical preliminaries

This section introduces the necessary mathematical preliminaries, including the Banach Fixed Point theorem, Caputo fractional derivatives, Mittag-Leffler functions, and operator families generated by linear operators.

Definition 1 [20]. *The Caputo fractional derivative of order $\alpha > 0$, with $n - 1 < \alpha < n$, $n \in \mathbb{N}$, is defined as*

$${}^cD^\alpha_{t_0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n f(s)}{ds^n}\, ds, \quad n = \lceil \alpha \rceil,$$

provided the integral on the right-hand side exists.

Definition 2. *The one-parameter and two-parameter Mittag-Leffler functions are defined by*

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},$$

for all $\alpha, \beta > 0$ and $z \in \mathbb{C}$, where $\Gamma(\cdot)$ denotes the gamma function.

Definition 3 [21]. *The families of operators* $\mathcal{T}_\alpha(t), \mathcal{T}_{\alpha,\beta}(t) : \mathbb{X} \to \mathbb{X}$, $t \geq 0$ *are generated by a linear operator* $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ *and satisfy the following properties*:

(1) $\mathcal{T}_\alpha(0) = \mathcal{I}$ *and* $\mathcal{T}_{\alpha,\beta}(0) = \mathcal{I}$, *where* $\mathcal{I}$ *is the identity operator*;
(2) $\mathcal{T}_\alpha(t)$ *satisfies the linear fractional equation* ${}^cD^\alpha x(t) = \mathcal{A}x(t)$ *in the Banach space* $\mathbb{X}$;
(3) $\lim\limits_{\beta\to 1} \mathcal{T}_{\alpha,\beta}(t) = \mathcal{T}_\alpha(t)$.

Theorem 1 (Banach Fixed Point Theorem), [7]. *Let* $(\mathbb{X}, d)$ *be a complete metric space, and let* $\mathcal{T} : \mathbb{X} \to \mathbb{X}$ *be a mapping such that* $\mathcal{T}^n$ *is a contraction mapping for some* $n \geq 1$. *Then* $\mathcal{T}$ *has a unique fixed point in* $\mathbb{X}$.

3. Controllability of a linear system

The controllability of a nonlinear system depends on the controllability of its corresponding linear system, which for fractional linear systems has been extensively studied in the literature [12]. Therefore, we first discuss the controllability of the linear system associated with (1.1):

$$\begin{aligned} {}^c\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{B}_k u(t), \quad t \in [t_{k-1}, t_k), \quad k = 1, 2, \ldots, \rho, \\ x(0) &= x_0, \\ \Delta x(t_k) &= \mathcal{M}_k x(t_k) + \mathcal{N}_k u(t_k), \quad t = t_k, \quad k = 1, 2, \ldots, \rho, \end{aligned} \tag{3.1}$$

over the interval J_0.

The solution of system (3.1) is given by

$$\begin{gathered} x(t) = \mathcal{T}_\alpha(t - t_{k-1})x(t_{k-1}^+) + \int_{t_{k-1}}^t (t-s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t-s)\mathcal{B}_k u(s)\,\mathrm{d}s, \\ t \in [t_{k-1}, t_k), \end{gathered} \tag{3.2}$$

where $\mathcal{T}_\alpha$ and $\mathcal{T}_{\alpha,\beta}$ denote the solution operators generated by $\mathcal{A}$. Here $t_0 = 0$, and

$$x(t_{k-1}^+) = x(t_{k-1}^-) + \mathcal{M}_{k-1}x(t_{k-1}^-) + \mathcal{N}_{k-1}u(t_{k-1}).$$

Definition 4 (Exact Controllability), [14]. *System* (3.1) *is said to be* exactly controllable *over* $[0, T_0]$ *if, for every* $x_0, x_1 \in \mathbb{X}$, *there exists a control* $u \in L^2(J_0, \mathbb{X})$ *such that the mild solution* $x(t)$ *of* (3.1) *satisfies* $x(T_0) = x_1$.

Exact controllability refers to the controllability of infinite-dimensional systems [13, 16].
Since $\mathcal{B}_k$ may change after each impulse, we first study the controllability of the system

$${}^c\mathcal{D}^\alpha x(t) = \mathcal{A}x(t) + \mathcal{B}_k u(t), \quad x(t_{k-1}^+) = q_{k-1}, \tag{3.3}$$

over the subinterval $[t_{k-1}, t_k]$ for $k = 1, 2, \ldots, \rho$. Its solution is

$$x(t) = \mathcal{T}_\alpha(t - t_{k-1})q_{k-1} + \int_{t_{k-1}}^t (t-s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t-s)\mathcal{B}_k u(s)\,\mathrm{d}s.$$

System (3.3) is exactly controllable over $[t_{k-1}, t_k]$ if there exists a control $u(t)$ that steers the initial state q_{k-1} to a desired final state x_1 at $t = t_k$, i.e.,

$$x_1 = x(t_k) = \mathcal{T}_\alpha(t_k - t_{k-1})q_{k-1} + \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t_k - s)\mathcal{B}_k u(s)\,\mathrm{d}s.$$

Define the operator $\mathcal{C}_k : L^2([t_{k-1}, t_k], \mathbb{U}) \to \mathbb{X}$ by

$$\mathcal{C}_k u(t) = \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \mathcal{T}_{\alpha,\alpha}(t_k - s) \mathcal{B}_k u(s) \, \mathrm{d}s, \tag{3.4}$$

whose adjoint $\mathcal{C}_k^* : \mathbb{X} \to L^2([t_{k-1}, t_k], \mathbb{U})$ is

$$\mathcal{C}_k^* \omega = (t_k - t)^{1-\alpha} \mathcal{B}_k^* \mathcal{T}_{\alpha,\alpha}^*(t_k - t)\omega,$$

and define the operator $\mathcal{W}_k : \mathbb{X} \to \mathbb{X}$ as

$$\mathcal{W}_k \omega = \int_{t_{k-1}}^{t_k} \mathcal{T}_{\alpha,\alpha}(t_k - s) \mathcal{B}_k \mathcal{B}_k^* \mathcal{T}_{\alpha,\alpha}^*(t_k - s)\omega \, \mathrm{d}s.$$

Impulses in system (3.1) introduce discontinuities at times t_k, which require the control $u(t)$ to compensate for abrupt state changes. While impulses do not alter the fundamental controllability framework, they necessitate piecewise analysis over subintervals $[t_{k-1}, t_k]$, increasing the complexity of the control design compared to continuous systems [4].

Theorem 2. *System* (3.2) *is exactly controllable over the subinterval* $[t_{k-1}, t_k]$ *if one of the following conditions holds*:

(1) Range$(\mathcal{C}_k) = \mathbb{X}$;
(2) *there exists* $\gamma_k > 0$ *such that* $||\mathcal{C}_k^* \omega||^2 \geq \gamma_k^2 ||\omega||^2$ *for all* $\omega \in \mathbb{X}$;
(3) *there exists* $\gamma_k > 0$ *such that* $\langle \mathcal{W}_k \omega, \omega \rangle \geq \gamma_k^2 ||\omega||^2$ *for all* $\omega \in \mathbb{X}$;
(4) Ker$(\mathcal{C}_k^*) = \{0\}$ *and* Range$(\mathcal{C}_k^*)$ *is closed.*

A controller that steers a given initial state q_{k-1} *to a desired final state* x_1 *is given by*

$$u(t) = \mathcal{B}_k^* \mathcal{T}_{\alpha,\alpha}^*(t_k - t_{k-1}) \mathcal{W}_k^{-1} \big[x_1 - \mathcal{T}_{\alpha,\alpha}(t_k - t_{k-1}) q_{k-1}\big].$$

P r o o f. System (3.3) is exactly controllable over the subinterval $[t_{k-1}, t_k]$ if there exists a steering function u that steers the initial state q_{k-1} to any desired state x_1 at time t_k. This means

$$x_1 = \mathcal{T}_\alpha(t_k - t_{k-1}) q_{k-1} + \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \mathcal{T}_{\alpha,\alpha}(t_k - s) \mathcal{B}_k u(s) \, ds.$$

Define

$$\omega = x_1 - \mathcal{T}_\alpha(t_k - t_{k-1}) q_{k-1}.$$

Then $\mathcal{C}_k u = \omega$. Since $x_1 \in \mathbb{X}$ is arbitrary, it follows that Range$(\mathcal{C}_k) = \mathbb{X}$.

Conversely, let $x_1 \in \mathbb{X}$. Since Range$(\mathcal{C}_k) = \mathbb{X}$, for every $\omega \in \mathbb{X}$ there exists u such that $\mathcal{C}_k u = \omega$. In particular, if we select

$$\omega = x_1 - \mathcal{T}_\alpha(t_k - t_{k-1}) q_{k-1},$$

then

$$\mathcal{C}_k u = x_1 - \mathcal{T}_\alpha(t_k - t_{k-1}) q_{k-1},$$

and therefore,

$$x_1 = \mathcal{T}_\alpha(t_k - t_{k-1}) q_{k-1} + \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \mathcal{T}_{\alpha,\alpha}(t_k - s) \mathcal{B}_k u(s) \, ds.$$

Hence, system (3.3) is controllable. This proves (1) $\Longleftrightarrow$ (2).

For all $\omega \in \mathbb{X}$, the equality

$$\|\mathcal{C}_k^*\omega\|^2 = \langle \mathcal{C}_k^*\omega, \mathcal{C}_k^*\omega\rangle = \langle \mathcal{C}_k\mathcal{C}_k^*\omega, \omega\rangle = \langle \mathcal{W}_k\omega, \omega\rangle$$

establishes the equivalence of (2), (3), and (4).

Under condition (2), $\mathcal{C}_k^*$ is an injective operator; therefore, $\mathrm{Ker}\,(\mathcal{C}_k^*) = \{0\}$. To prove that $\mathrm{Range}\,(\mathcal{C}_k^*)$ is closed, consider a Cauchy sequence $\{\mathcal{C}_k^*\omega_n\}$ in $\mathrm{Range}\,(\mathcal{C}_k^*)$. Then, by condition (2), ω_n is a Cauchy sequence in $\mathbb{X}$; therefore, there exists ω such that $\omega_n \to \omega$. Thus, $\mathcal{C}_k^*\omega_n \to \mathcal{C}_k^*\omega$, which proves that $\mathrm{Range}\,(\mathcal{C}_k^*)$ is closed.

Conversely, if condition (4) holds, then $\mathcal{C}_k^*$ is bijective onto $\mathrm{Range}\,(\mathcal{C}_k^*)$, which is a subspace of $\mathbb{L}^2([t_0, t_1], \mathbb{U})$, and

$$\mathrm{Domain}\,\left((\mathcal{C}_k^*)^{-1}\right) = \mathrm{Range}\,(\mathcal{C}_k^*).$$

Therefore, there exists $\gamma_k > 0$ such that

$$\|(\mathcal{C}_k^*)^{-1}u\|^2 \le \gamma_k^2\|u\|^2 \quad \forall\, u \in \mathrm{Range}(\mathcal{C}_k^*),$$

and this implies

$$\|\mathcal{C}_k^*\omega\|^2 \ge \gamma_k^2\|\omega\|^2.$$

This shows the equivalence between (2) and (4).

This completes the proof. □

Corollary 1. *If system (3.3) is controllable over $[t_{k-1}, t_k]$, then the operator $\mathcal{S}_k : \mathbb{X} \to L^2([t_{k-1}, t_k], \mathbb{U})$ defined by $\mathcal{S}_k\omega = \mathcal{C}_k^*\mathcal{W}_k^{-1}\omega$ is the right inverse of $\mathcal{C}_k$, i.e., $\mathcal{C}_k \circ \mathcal{S}_k = \mathcal{I}$.*

The following theorem establishes the controllability of the impulsive system (3.1) over the full interval $[0, T_0]$.

Theorem 3. *If the following conditions hold:*

(A1) the system is controllable over each subinterval $[t_{k-1}, t_k)$,
(A2) the operators $(I + \mathcal{M}_k)$ are nonsingular for all k,

then system (3.1) is exactly controllable over the interval $[0, T_0]$.

P r o o f. On the subinterval $[t_0, t_1)$, the system becomes

$$\begin{aligned} {}^c\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{B}_1u(t), \\ x(t_0) &= x_0, \end{aligned} \tag{3.5}$$

and the solution of system (3.5) over this interval is

$$x(t) = \mathcal{T}_\alpha(t - t_0)x_0 + \int_{t_0}^{t} (t - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t - s)\mathcal{B}_1u(s)ds.$$

By condition (A1), the system is exactly controllable over the interval $[t_0, t_1)$. The control $u(t)$ that steers system (3.4) from initial state x_0 to x_1 at $t = t_1$ is given by

$$u(t) = (t_1 - t)^{1-\alpha}\mathcal{B}_1^*\mathcal{T}_{\alpha,\alpha}^*(t_1 - t)\mathcal{W}_1^{-1}[x_1 - \mathcal{T}(t_1 - t_0)x_0].$$

When the control u is applied, the state at $t = t_1$ becomes $x(t_1^-) = x_1$.

Over the subinterval $[t_1, t_2)$, the system becomes

$$x'(t) = \mathcal{A}x(t) + \mathcal{B}_2 u(t),$$
$$x(t_1^+) = (I + \mathcal{M}_1)x_1 + \mathcal{N}_1 u(t_1).$$

To derive a sufficient condition, assume that $N_1 u(t_1) = 0$. Then the system becomes

$$^c\mathcal{D}^\alpha x(t) = \mathcal{A}x(t) + \mathcal{B}_2 u(t),$$
$$x(t_1^+) = (I + \mathcal{M}_1)x_1,$$

and the solution over this interval is

$$x(t) = \mathcal{T}_\alpha(t - t_1)(I + M_1)x_1 + \int_{t_1}^{t} (t - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t - s)\mathcal{B}_1 u(s)ds.$$

By conditions (A1) and (A2), the evolution system is controllable on the subinterval $[t_1, t_2)$. The controller $u(t)$ that steers the state $(I + \mathcal{M}_1)x_1$ to the desired state x_1 at time $t = t_2$ is

$$u(t) = (t_2 - t)^{1-\alpha}\mathcal{B}_2^*\mathcal{T}_{\alpha,\alpha}^*(t_1 - t)\mathcal{W}_2^{-1}[x_1 - \mathcal{T}_\alpha(t_2 - t_1)(I + \mathcal{M}_1)x_1].$$

Continuing this process for all $k = 3, 4, \dots, \rho$ and assuming $\mathcal{N}_k u(t_k) = 0$, we find that the system over the subinterval $[t_{k-1}, t_k)$ becomes

$$\begin{aligned} ^c\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{B}_k u(t), \\ x(t_{k-1}^+) &= (I + \mathcal{M}_{k-1})x_1. \end{aligned} \tag{3.6}$$

The mild solution of the system is

$$x(t) = \mathcal{T}_\alpha(t - t_{k-1})(I + \mathcal{M}_{k-1})x_1 + \int_{t_{k-1}}^{t} (t - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t - t_{k-1})\mathcal{B}_k u(s)ds.$$

Under the hypothesis, the evolution system (3.6) is exactly controllable over the subinterval $[t_{k-1}, t_k)$. The controller $u(t)$ that steers the state $(I + \mathcal{M}_{k-1})x_1$ to the desired final state x_1 at time $t = t_k$ is

$$u(t) = (t_k - t)^{1-\alpha}\mathcal{B}_k^*\mathcal{T}_{\alpha,\alpha}^*(t_k - t)\mathcal{W}_k^{-1}[x_1 - \mathcal{T}(t_k - t_{k-1})(I + \mathcal{M}_k)x_1].$$

Finally, on the subinterval $[t_\rho, T_0]$, with $\mathcal{N}_\rho u(t_p) = 0$, the evolution system becomes

$$\begin{aligned} ^c\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{B}_{\rho+1} u(t), \\ x(t_p^+) &= (I + \mathcal{M}_\rho)x_1 \end{aligned} \tag{3.7}$$

and the solution is

$$x(t) = \mathcal{T}_\alpha(t - t_\rho)(I + \mathcal{M}_\rho)x_1 + \int_{t_\rho}^{t} (t - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t - s)\mathcal{B}_{\rho+1} u(s)ds.$$

Under conditions (A1) and (A2), the evolution system (3.7) is exactly controllable over the interval $[t_p, T_0]$ and the control $u(t)$ that steers the evolution system from the state $(I + \mathcal{M}_\rho)x_1$ to the desired final state x_1 at time $t = T_0$ is given by

$$u(t) = (t_\rho - t)^{1-\alpha}\mathcal{B}_{\rho+1}^*\mathcal{T}_{\alpha,\alpha}^*(t_\rho - t)\mathcal{W}_\rho^{-1}[x_1 - \mathcal{T}(T_0 - t_\rho)(I + \mathcal{M}_\rho)x_1].$$

Therefore, the system is steered from the given initial state x_0 to the desired final state x_1 at $t = T_0$. Hence, system (3.1) is exactly controllable over the interval $[0, T_0]$. □

4. Controllability of the nonlinear system

This section discusses the exact controllability of the nonlinear system (1.1) over the interval $[0, T_0]$.

Definition 5. *System (1.1) is said to be exactly controllable over the interval $[0, T_0]$ if, for all $x_0, x_1 \in \mathbb{X}$, there exists a control $u \in L^2(J_0, \mathbb{U})$ such that the corresponding mild solution $x(t)$ of (1.1) satisfies $x(T_0) = x_1$.*

Since the system is such that perturbing forces $\mathcal{F}_k$ change after every time moment, to control system (1.1) over the entire interval $[0, T_0]$, one should apply a controller for every subinterval $[t_{k-1}, t_k]$ for each $k = 1, 2, \ldots, \rho + 1$.

Consider an arbitrary subinterval $[t_{k-1}, t_k)$. Over this subinterval, consider the system of the form

$$\begin{aligned} &{}^c\mathcal{D}^\alpha x(t) = \mathcal{A}x(t) + \mathcal{F}_k(t, x(t), u(t)) + \mathcal{B}_k u(t), \quad t \in [t_{k-1}, t_k), \\ &x(t_{k-1}) = z_{k-1}. \end{aligned} \tag{4.1}$$

If $\mathcal{F}_k$ are sufficiently regular, then system (4.1) has a unique mild solution

$$\begin{aligned} x(t) = \mathcal{T}_\alpha(t - t_{k-1})z_{k-1} &+ \int_{t_{k-1}}^{t} (t-s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t-s)\mathcal{B}_k u(s)\, ds \\ &+ \int_{t_{k-1}}^{t} \mathcal{T}_{\alpha,\alpha}(t-s)\mathcal{F}_k(s, x(s), u(s))\, ds, \end{aligned} \tag{4.2}$$

for all $u \in L^2([0, T_0], \mathbb{U})$. A unique mild solution exists if $\mathcal{F}_k$ are measurable in t and Lipschitz continuous in x and u.

Define the operator $\mathcal{G}_k : L^2([t_{k-1}, t_k], \mathbb{U}) \to \mathbb{X}$ by

$$\mathcal{G}_k u = \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t_k - s)\mathcal{B}_k u(s)\, ds + \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}\mathcal{T}_{\alpha,\alpha}(t_k - s)\mathcal{F}_k(s, x(s), u(s))\, ds.$$

Then system (4.1) is controllable if $\mathcal{G}_k$ is surjective.

Define the operator $\bar{\mathcal{G}}_k : \mathbb{X} \to \mathbb{X}$ by

$$\bar{\mathcal{G}}_k \zeta = (\mathcal{G}_k \circ \mathcal{S}_k)\zeta.$$

Assuming the corresponding linear system is controllable over $[t_{k-1}, t_k]$ and using Corollary 1, we define

$$\bar{\mathcal{G}}_k \zeta = \zeta + \int_{t_{k-1}}^{t_k} (t_k - s)\mathcal{T}_{\alpha,\alpha}(t_k - s)\mathcal{F}_k\big(s, x_\zeta(s), (\mathcal{S}_k\zeta)(s)\big)\, ds = (\mathcal{I} + \mathcal{H}_k)\zeta,$$

where $\mathcal{I}$ is the identity operator. Therefore, system (4.1) is controllable if $\bar{\mathcal{G}}_k$ is invertible.

Lemma 1. *If the operator $\mathcal{H}_k^{(N)}$ on $\mathbb{X}$ for some $N \geq 1$ is a contraction, then $(\mathcal{I} + \mathcal{H}_k)$ is invertible.*

P r o o f. Since $\mathcal{H}_k^{(N)}$ is a contraction for some $N \geq 1$, by the Banach fixed point theorem, the equation $\zeta = -\mathcal{H}_k\zeta$ has a unique solution on $\mathbb{X}$. This implies the invertibility of the operator $(\mathcal{I} + \mathcal{H}_k)$. □

Theorem 4. *If the corresponding linear system of (4.1) is controllable over $[t_{k-1}, t_k]$ and $(\mathcal{I} + \mathcal{H}_k)$ is invertible, then the nonlinear system (4.1) is exactly controllable over $[t_{k-1}, t_k]$. The controller*

$$u(t) = (t_k - t)^{1-\alpha} \mathcal{B}_k^* \mathcal{T}_{\alpha,\alpha}^*(t_k - t) \mathcal{W}_k^{-1} (\mathcal{I} + \mathcal{H}_k)^{-1} \big(x_1 - \mathcal{T}_\alpha(t_k - t_{k-1}) z_{k-1} \big)$$

steers the initial state z_{k-1} to the desired state x_1 at $t = t_k$.

P r o o f. Plugging $u(t)$ into (4.2) at $t = t_k$ and using the nonsingularity of $(\mathcal{I} + \mathcal{H}_k)$, we find that the state of system (4.1) becomes

$$\begin{aligned} x(t_k) &= \mathcal{T}_\alpha(t_k - t_{k-1}) z_{k-1} + \mathcal{G}_k \big[(t_k - s)^{1-\alpha} \mathcal{B}_k^* \mathcal{T}_{\alpha,\alpha}^*(t_k - s) \mathcal{W}_k^{-1} (\mathcal{I} + \mathcal{H}_k)^{-1} (x_1 - \mathcal{T}_\alpha(t_k - t_{k-1}) z_{k-1}) \big] \\ &= \mathcal{T}_\alpha(t_k - t_{k-1}) z_{k-1} + \big[\mathcal{G}_k \circ \mathcal{S}_k \big] \circ \bar{\mathcal{G}_k}^{-1} \big(x_1 - \mathcal{T}_\alpha(t_k - t_{k-1}) z_{k-1} \big) \\ &= \mathcal{T}_\alpha(t_k - t_{k-1}) z_{k-1} + x_1 - \mathcal{T}_\alpha(t_k - t_{k-1}) z_{k-1} = x_1. \end{aligned}$$

Hence, the evolution system (4.1) is exactly controllable over the subinterval $[t_{k-1}, t_k]$. □

The following assumptions are made to discuss the controllability of system (4.1) over the subinterval $[t_{k-1}, t_k]$.

($B1$) Let $\|\mathcal{T}_\alpha(t)\| \leq M_1$ and $\|\mathcal{T}_{\alpha,\beta}(t)\| \leq M_2$ be operators generated by $\mathcal{A}$ for all $t \in J_0$, and $\|\mathcal{B}_k\| \leq b_k^*$.

($B2$) The perturbations f_k are measurable with respect to the first argument and Lipschitz continuous with respect to x and u with Lipschitz constants f_{1k}^* and f_{2k}^*, respectively.

Under assumptions ($B1$) and ($B2$),

$$\begin{aligned} ||\mathcal{H}_{k,n}^{(m)} x_1 - \mathcal{H}_{k,n}^{(m)} x_2|| &\leq \frac{M^m \big(f_{1k}^* + b^* M \|W_k\| f_{2k}^* \big)^m}{(m-1)!} \int_{t_{k-1}}^{t_k} (t - s)^{\alpha(m-1)} ds \|x_1 - x_2\| \\ &= \frac{M^m \big(f_{1k}^* + b^* M \|W_k\| f_{2k}^* \big)^m (t_k - t_{k-1})^{\alpha m}}{m!} \|x_1 - x_2\|. \end{aligned}$$

The value

$$c^* = \frac{M^m \big(f_{1k}^* + b^* M \|W_k\| f_{2k}^* \big)^m (t_k - t_{k-1})^{\alpha m}}{m!} \to 0 \quad \text{as} \quad m \to \infty.$$

Therefore, there exists at least one N such that $\mathcal{H}_{k,n}^{(N)}$ is a contraction.

The following theorem discusses the controllability of system (4.1) over the subinterval $[t_{k-1}, t_k]$.

Theorem 5. *If ($B1$) and ($B2$) are satisfied, then the evolution system (4.1) is exactly controllable over the subinterval $[t_{k-1}, t_k]$ with the controller*

$$u(t) = (t_k - t)^{1-\alpha} \mathcal{B}_k^* \mathcal{T}_{\alpha,\alpha}^*(t_k - t) \mathcal{W}_k^{-1} (\mathcal{I} + \mathcal{H}_k)^{-1} \big(x_1 - \mathcal{T}_\alpha(t_k - t_{k-1}) z_{k-1} \big)$$

that steers the system to the desired final state x_1 at $t = t_k$.

The following theorem discusses the controllability of the nonlinear impulsive system (1.1).

Theorem 6. *If the corresponding linear system (3.1) is controllable and assumptions ($B1$) and ($B2$) hold, then the nonlinear impulsive system (1.1) is exactly controllable over J_0.*

P r o o f. To discuss the exact controllability of system (1.1), we assume that the corresponding linear system is exactly controllable and that $\mathcal{N}_k u(t_k) = 0$ for all $k = 1, 2, \ldots, \rho$.

On the subinterval $[0, t_1)$, the evolution system becomes

$$\begin{aligned} {}^c\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{F}_1(t, x(t), u(t)) + \mathcal{B}_k u(t), \\ x(0) &= x_0. \end{aligned}$$

By assumptions $(B1)$ and $(B2)$ and Theorem 5, the evolution system is exactly controllable on the subinterval $[0, t_1)$ with the control

$$u(t) = (t_1 - t)^{1-\alpha} \mathcal{B}_1^* \mathcal{T}_{\alpha,\alpha}^*(t_1 - t) \mathcal{W}_1^{-1} (\mathcal{I} + \mathcal{H}_1)^{-1} (x_1 - \mathcal{T}_\alpha(t_1) x_0),$$

and the system state at $t = t_1$ is steered to x_1.

Over the subinterval $[t_1, t_2)$, the evolution system becomes

$$\begin{aligned} {}^c\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{F}_2(t, x(t), u(t)) + \mathcal{B}_k u(t), \\ x(t_1) &= (\mathcal{I} + \mathcal{M}_1) x_1. \end{aligned}$$

By assumptions $(B1)$ and $(B2)$ and Theorem 5, the system is exactly controllable over the interval $[t_1, t_2)$ with the controller

$$u(t) = (t_1 - t)^{1-\alpha} \mathcal{B}_2^* \mathcal{T}_{\alpha,\alpha}^*(t_2 - t) \mathcal{W}_2^{-1} (\mathcal{I} + \mathcal{H}_2)^{-1} (x_1 - \mathcal{T}_\alpha(t_2 - t_1)(\mathcal{I} + \mathcal{M}_1) x_1),$$

and the system state at $t = t_2$ is steered to x_1.

If we continue this process up to the final interval $[t_\rho, T_0]$, the system becomes

$$\begin{aligned} {}^c\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{F}_{\rho+1}(t, x(t), u(t)) + \mathcal{B}_{\rho+1} u(t), \\ x(t_\rho) &= (\mathcal{I} + \mathcal{M}_\rho) x_1. \end{aligned} \tag{4.3}$$

By assumptions $(B1)$ and $(B2)$ and Theorem 5, the evolution system (4.3) is exactly controllable over the subinterval $[t_\rho, T_0]$. The controller

$$u(t) = (T_0 - t)^{1-\alpha} \mathcal{B}_{\rho+1}^* \mathcal{T}_{\alpha,\alpha}^*(T_0 - s) \mathcal{W}_{\rho+1}^{-1} (\mathcal{I} + \mathcal{H}_{\rho+1})^{-1} (x_1 - \mathcal{T}_\alpha(T_0 - t_\rho)(\mathcal{I} + \mathcal{M}_\rho) x_1)$$

steers the state of system (4.3) to x_1 at time $t = T_0$.

This completes the proof of the theorem. □

5. Example

Consider the Caputo fractional-order partial differential equation of order $0 < \alpha \leq 1$

$$\begin{aligned} {}^c\mathcal{D}_{t,0+}^\alpha w(\zeta, t) + \frac{\partial^3 w}{\partial \zeta^3}(\zeta, t) &= \mathcal{B}_1 u(\zeta, t) + F_1(t, \zeta, w, u), \quad t \in [0, t_1), \\ {}^c\mathcal{D}_{t,t_1+}^\alpha w(\zeta, t) + \frac{\partial^3 w}{\partial \zeta^3}(\zeta, t) &= \mathcal{B}_2 u(\zeta, t) + F_2(t, \zeta, w, u), \quad t \in [t_1, T_0], \\ \Delta w(x, t_1) &= 2w(x, t_1) + 3u(x, t_1), \end{aligned} \tag{5.1}$$

with periodic boundary conditions

$$\frac{\partial^i w}{\partial \zeta^i}(0, t) = \frac{\partial^i w}{\partial \zeta^i}(2\pi, t) \quad \forall i = 0, 1, 2$$

and the initial condition

$$w(\zeta, 0) = w_0(\zeta).$$

Here, u is the control function and the linear operators $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined by

$$\mathcal{B}_1 u(\zeta, t) = g_1(\zeta)\left[u(\zeta, t) - \int_0^{2\pi} g_1(\psi) u(\psi, t) d\psi\right],$$
$$\mathcal{B}_2 u(\zeta, t) = g_2(\zeta)\left[u(\zeta, t) - \int_0^{2\pi} g_2(\psi) u(\psi, t) d\psi\right],$$

where g_1 and g_2 are continuous functions over the interval $[0, 2\pi]$. Fix $\mathbb{X} = L^2([0, 2\pi], \mathbb{R})$ and define an operator $\mathcal{A}$ with domain

$$\mathcal{D}(\mathcal{A}) = \left\{\eta \in \mathbb{H}^{(2)}[0, 2\pi] \colon \eta^i(0) = \eta^i(2\pi)\right\} \quad \text{by} \quad \mathcal{A}w = -\frac{\partial^3 w}{\partial \zeta^3}.$$

Define

$$\mathcal{T}_\alpha(t) w_0 = \sum_{k \in \mathbb{Z}} E_\alpha(ik^3 t^\alpha) < w_0, \quad e^{ikx} > e^{ikx}$$

and

$$\mathcal{T}_{\alpha,\beta}(t) = \sum_{k \in \mathbb{Z}} E_{\alpha,\beta}(ik^3 t^\alpha) < w_0, \quad e^{ikx} > e^{ikx}$$

as linear operators that meet the conditions of Definition 3. Thus, $\mathcal{T}_\alpha(t)$ and $\mathcal{T}_{\alpha,\beta}(t)$ are operators generated by $\mathcal{A}$ that satisfy $||\mathcal{T}_\alpha(t)|| \leq M_1$ and $||\mathcal{T}_{\alpha,\alpha}(t)|| \leq M_2$ for some $M_1, M_2 \geq 0$ and for all $t \in [0, T_0]$.

Let $x(t) = w(\cdot, t)$ and $u(t) = u(\cdot, t)$. Then the system transforms into an abstract system of the form

$$\begin{aligned} {}^c\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{B}_k u(t) + \mathcal{F}_k(t, x(t), u(t)), \quad t \in [0, t_1) \cup [t_1, T_0), \\ \Delta x(t_1) &= 2x(t_1) + 3u(t_1), \end{aligned}$$

over the space $\mathbb{X}$.

The jump $\mathcal{M}_1 x(t_1) = 2x(t_1)$ is such that $(\mathcal{I} + \mathcal{M}_1)$ is invertible and $\mathcal{B}_1$ and $\mathcal{B}_2$ are bounded. Thus, the corresponding linear system

$$\begin{aligned} {}^c\mathcal{D}^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{B}_k u(t), \quad t \in [0, t_1) \cup [t_1, T_0], \\ \Delta x(t_1) &= 2x(t_1) + 3u(t_1) \end{aligned}$$

is exactly controllable, and the controller is defined by

$$u(t) = \begin{cases} (t_1 - t)^{1-\alpha} \mathcal{B}_1^* \mathcal{T}_{\alpha,\alpha}^*(t_1 - t) \mathcal{W}_1^{-1}[x_1 - \mathcal{T}_\alpha(t_1) x_0], & t \in [0, t_1), \\ (T_0 - t)^{1-\alpha} \mathcal{B}_2^* \mathcal{T}_{\alpha,\alpha}^*(T_0 - t) \mathcal{W}_2^{-1}[x_1 - \mathcal{T}_\alpha(t_1)(I + \mathcal{M}_1) x_1], & t \in (t_1, T_0]. \end{cases}$$

Also, the following conditions are assumed:

1) $\mathcal{F}_1$, and $\mathcal{F}_2$ are measurable with respect to argument t;
2) $\mathcal{F}_i$'s are continuous with respect to x and u there exist constants $\mathcal{F}_i^*$ that satisfy

$$\|\mathcal{F}_i(t, x_1, u_1) - \mathcal{F}_i(t, x_2, u_2)\| \leq f_{1i}^* \|x_1 - x_2\| + f_{2i}^* \|u_1 - u_2\|.$$

Hence, the evolution system (5.1) is exactly controllable over the entire interval $[0, T_0]$.

For example, if $\mathcal{F}_1(t, x, u) = \sin(t)x + \tanh(u)$ and $\mathcal{F}_2(t, x, u) = \sin(t)\tanh(x)\tanh(u)$, then $\mathcal{F}_i$ are Lipchitz continuous with respect to x and u, respectively. Thus the system is exactly controllable over the interval $[0, T_0]$.

6. Conclusion

In this paper, we have investigated the exact controllability of nonlinear impulsive fractional systems. One possible approach is to observe the system until the final impulse moment and then apply control only over the last interval $[t_\rho, T_0]$. However, this method requires a large control effort within a very short time, which may lead to system instability.

To overcome this issue, we have proposed applying the controller over every subinterval $[t_{k-1}, t_k]$, $k = 1, 2, \ldots, \rho + 1$. This strategy distributes the control action more evenly across the entire time horizon, thereby improving the stability of the system while ensuring exact controllability.

REFERENCES

1. Axtell M., Bise M. E. Fractional calculus applications in control systems. In: *Proc. IEEE Conf. Aerosp. Electr., 21–25 May 1990.* New York, 1990. P. 563–566. DOI: 10.1109/NAECON.1990.112826
2. Bagley R. L., Calico R. A. Fractional order state equations for the control of viscoelastically damped structure. *J. Guid. Control Dyn.*, 1991. Vol. 14, No. 2. P. 304–311. DOI: 10.2514/3.20641
3. Balachandran K., Park J. Y., Trujillo J. J. Controllability of nonlinear fractional dynamical systems. *Nonlinear Anal.*, 2012. Vol. 75, No. 4. P. 1919–1926. DOI: 10.1016/j.na.2011.09.042
4. Balachandran K., Kokila J., Trujillo J. J. Relative controllability of fractional dynamical systems with multiple delays in control. *Comput. Math. Appl.*, 2012. Vol. 64, No. 10. P. 3037–3045. DOI: 10.1016/j.camwa.2012.01.071
5. Balachandran K., Govindaraj V., Rodríguez-Germá L., Trujillo J. J. Controllability of nonlinear higher order fractional dynamical systems. *Nonlinear Dyn.*, 2013. Vol. 71. P. 605–612. DOI: 10.1007/s11071-012-0612-y
6. Balachandran K., Govindaraj V., Rodríguez-Germá L., Trujillo J. J. Controllability results for nonlinear fractional-order dynamical systems. *J. Optim. Theory Appl.*, 2013. Vol. 156. P. 33–44. DOI: 10.1007/s10957-012-0212-5
7. Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.*, 1922. Vol. 3 (1). P. 133–181. DOI: 10.4064/FM-3-1-133-181 (in French)
8. Bettayeb M., Djennoune S. New results on the controllability and observability of fractional dynamical systems. *J. Vib. Control,* 2008. Vol. 14, No. 9–10. P. 1531–1541. DOI: 10.1177/1077546307087432
9. Chen Y. Q., Ahn H.-S., Xue D. Robust controllability of interval fractional order linear time invariant systems. *Signal Process.,* 2006. Vol. 86, No. 10. P. 2794–2802. DOI: 10.1016/j.sigpro.2006.02.021
10. Govindaraj V., George R. K. Controllability of fractional dynamical systems: A functional analytic approach. *Math. Control Relat. Fields*, 2017. Vol. 7, No. 4. P. 537–562. DOI: 10.3934/mcrf.2017020
11. Guermah S., Djennoune S., Bettayeb M. Controllability and observability of linear discrete-time fractional-order systems. *Int. J. Appl. Math. Comput.,* 2008. Vol. 18, No. 2. P. 213–222. DOI: 10.2478/v10006-008-0019-6
12. Kaczorek T. *Selected Problems of Fractional Systems Theory.* Berlin: Springer-Verlag, 2011. 346 p. DOI: 10.1007/978-3-642-20502-6
13. Klamka J. Controllability of dynamical systems – a survey. *Arch. Control Sci.,* 1993. Vol. 2. P. 283–310. DOI: 10.2478/bpasts-2013-0031
14. Leiva H. Exact controllability of semilinear evolution equation and applications. *Int. J. Syst., Control Commun.*, 2008. Vol. 1, No. 1. P. 1–12.
15. Magin R. *Fractional Calculus in Bioengineering.* Redding: Begell House Inc., 2006. 703 p.
16. Matignon D., d'Andréa-Novel B. Some results on controllability and observability of finite-dimensional fractional differential systems. In: *Computational Engineering in Systems Applications, vol. 2.* Citeseer, 1996. P. 952–956.
17. Mozyrska D., Torres D. F. M. *Minimal Modified Energy Control for Fractional Linear Control Systems with the Caputo Derivative.* arXiv:1004.3113 [math.OC]
18. Mozyrska D., Torres D. F. M. Modified optimal energy and initial memory of fractional continuous-time linear systems. *Signal Process.,* 2011. Vol. 91, No. 3. P. 379–385. DOI: 10.1016/j.sigpro.2010.07.016
19. Petráš I. Control of fractional order Chua's system. *J. Electr. Eng.,* 2002. Vol. 53, No. 7–8. P. 219–222.

20. Podlubny I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications.* San Diego: Academic Press, 1998. 340 p.
21. Shah V., Sharma J., George R. K. Existence and uniqueness of classical and mild solutions of fractional Cauchy problem with impulses. *Malaya J. Mat.,* 2023. Vol. 11, No. 01. P. 66–79. DOI: 10.26637/mjm1101/005
22. Shah V., Sharma J., Samei M. E., Trivedi G. The trajectory controllability of second order semi-linear systems using a functional analytic approach. *J. Comput. Eng. Math.,* 2024. Vol. 11, No. 4. P. 22–32. DOI: 10.14529/jcem240403
23. Shah V., Anukiruthika K., Muthukumar P., Sharma J., Trivedi G. Existence and trajectory controllability for the conformable fractional evolution systems. *Malaya J. Mat.*, 2025. Vol. 13, No. 01. P. 63–74.
24. Shah V., Trivedi G., Sharma J., Patel P. On controllability of semilinear generalized impulsive systems on finite dimensional space. *Differ. Equ. Appl.*, 2025. Vol. 17, No. 2. P. 65–84. DOI: 10.7153/dea-2025-17-05
25. Torvik P. J., Bagley R. L. On the appearance of the fractional derivative in the behavior of real materials. *J. Appl. Mech.*, 1984. Vol. 51, No. 2. P. 294–298. DOI: 10.1115/1.3167615
26. Trivedi G. Mathematical modeling and numerical analysis of diffusion processes in image processing. *J. Applied Math.*, 2025. Vol. 3, No. 2. Art. no. 2736. DOI: 10.59400/jam2736
27. Trivedi G., Shah V., Sharma J., Sanghvi R. C. On solution of non-instantaneous impulsive Hilfer fractional integro-differential evolution system. *Math. Appl. (Warsaw)*, 2023. Vol. 51, No. 1. P. 33–50. DOI: 10.14708/ma.v51i1.7167
28. Vinagre B. M., Monje C. A., Calderón A. J. Fractional order systems and fractional order control actions. In: *IEEE Conference on Decision and Control: Fractional Calculus Applications in Automatic Control and Robotics.* Las Vegas, 2002. P. 2550–2554.