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GENERALIZED ASYMPTOTIC NOTATIONS VIA FILTERS: A NEW FRAMEWORK FOR ALGORITHM ANALYSIS

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Abstract: This paper introduces a generalization of asymptotic notation using filters, a topological structure. We present key properties of this generalization, including reflexivity, symmetry, and transitivity, supported by illustrative examples. Our research demonstrates that classical asymptotic notations imply their filter-generalized counterparts, but we provide examples showing the converse is not universally true. We also propose a characterization of traditional asymptotic notations using filters, offering a new perspective on these fundamental concepts. Furthermore, we establish relationships between bounded or vanishing sequences and filter-based asymptotic notations, enabling the determination of properties central to this study. This generalization provides a more nuanced framework for analyzing algorithmic complexity, potentially capturing behaviors overlooked by classical notations and opening new avenues for theoretical computer science research.

Keywords: Asymptotic notation, Filters, Sequences.

1. Introduction

In computer science, finding efficient algorithms is key for developing new technologies [2, 14]. Therefore, there is a need for tools to measure the efficiency in a code. This process is widely known as *Asymptotic Notation* whose main objective is to estimate mathematically the amount of resources employed during the execution of a computer program [11]. Currently, many notations are implemented and commonly used in the field. The most well-known are *big* O and *little* o [13]. From these notions, other ideas can be derived, such as, *big* Ω, *little* ω, *theta* Θ [9] and *weak theta* $\overline{\overline{\Theta}}$ [10] and recently some generalizations of aforementioned notations are provided [4].

A well known research topic in computer science is Computational Complexity [7], which is deeply related to asymptotic notation. Another issue closely linked to asymptotic notation is Artificial Intelligence (AI) [12], more specifically in Machine Learning where this notion can be used to evaluate the complexity of different algorithms [8]. On the other hand, the pure concept of an asymptotic notation can be described by a sequence of real positive numbers, which we will denote by

$$F^{+} = \{(x_n) \subset \mathbb{R} : x_n > 0\},$$

these sequences can be considered bounded or convergent to zero, for instance, let $(y_n) \in F^{+}$, and $x_n = O(y_n)$, this implies that there are constants $c > 0$ and $n_0 \in \mathbb{N}$ such that $x_n \leq c \cdot y_n$ for all $n \geq n_0$. Since (y_n) is a positive sequence, then $x_n/y_n \leq c$ for all $n \geq n_0$ and by definition the sequence $\big(x_n/y_n\big)$ is bounded. The same deduction can be established for $x_n = o(y_n)$, with the distinction that $\big(x_n/y_n\big)$ is a vanishing sequence.

Since convergent sequences, and bounded sequences, are sensitive to generalization using topological structures such as filters [1, 3, 5, 6], thereby, it is mathematically appealing to generalize the asymptotic notation using these structures. We recall that a filter over a non-empty set X is a collection of subsets of X, which can be assessed as the biggest set that fulfills certain statements. Formally, $\mathcal{F}$ is said a filter if:

1) $\varnothing \notin \mathcal{F}$,
2) $A, B \in \mathcal{F}$ implies that $A \cap B \in \mathcal{F}$,
3) $A \in \mathcal{F}$ and $A \subset B$ implies that $B \in \mathcal{F}$.

Given an infinite set X, the collection

$$\mathcal{F}_r = \big\{F \subset X : X - F \text{ is finite}\big\}$$

is a filter known as the *Fréchet filter*. Another example of this concept for a non-empty set X and given a non-empty subset $A \subset X$ is the *Principal filter*, defined as follows

$$\mathcal{F}_A = \big\{F \in \wp(X) : A \subseteq F\big\},$$

where $\wp(X)$ is the family of all subsets of X. In particular, if $A = \{x\}$, then this filter is denoted as $\mathcal{F}_x$.

In this paper we will only consider the filters on the set of positive integers. In addition, we provide another example of filter defined as follows: given a subset A of $\mathbb{N}$, the density of A, characterized by $d(A)$, given by:

$$d(A) = \lim_{n\to\infty} \frac{\text{Card}\,(A \cap \{1, 2, ..., n\ \})}{n},$$

the collection

$$\mathcal{F}_d = \{A \subseteq \mathbb{N} : d(A) = 1\}$$

is a filter on $\mathbb{N}$, known as the *Density filter*. A *Free filter* stands for a filter that satisfies the inclusion $\mathcal{F}_r \subset \mathcal{F}$. If a filter does not fulfill the previous statement then it is known as a *Fixed filter*. For example, the Density filter $\mathcal{F}_d$ is a free filter and the Principal filter $\mathcal{F}_A$ is a fixed filter.

This paper is organized as follows: Section 3 introduces the concept of generalized asymptotic notation using filters and presents key properties of this generalization, including reflexivity and transitivity. Section 4 explores the relationships between different F-asymptotic notations, establishing important theorems about their intersections and subset properties. Section 5 concludes the paper with a discussion of the implications of this generalization and potential avenues for future research.

2. Classical asymptotic notation: properties

In this section we record the standard properties of the classical asymptotic relations $O(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$, $o(\cdot)$, $\omega(\cdot)$, and $\overline{\overline{\Theta}}(\cdot)$ for positive sequences. These facts are textbook material and are included to bridge our filter-based framework with the traditional viewpoint; see, e.g., [9, 10, 13]. Throughout, (x_n), (y_n), (z_n), (w_n) denote positive sequences.

Definition 1. *Let* (x_n), (y_n), (z_n) *be positive sequences. The classical asymptotic notations are understood as follows*:

1) $x_n = O(y_n)$ *if there exist* $c > 0$ *and* $n_0 \in \mathbf{N}$, *such that* $x_n \le c\,y_n$ *for all* $n > n_0$;
2) $x_n = \Omega(y_n)$ *if there exist* $c > 0$ *and* $n_0 \in \mathbf{N}$ *such that* $c\,y_n \le x_n$ *for all* $n \ge n_0$;
3) $x_n = \Theta(y_n)$ *if there exist* $c_1, c_2 > 0$ *and* $n_0 \in \mathbf{N}$ *such that* $c_1\,y_n \le x_n \le c_2\,y_n$ *for all* $n \ge n_0$;
4) $x_n = o(y_n)$ *if for every* $c > 0$ *there exists* $n_0 \in \mathbf{N}$ *such that* $x_n < c\,y_n$ *for all* $n \ge n_0$;
5) $x_n = \omega(y_n)$ *if for every* $c > 0$ *there exists* $n_0 \in \mathbf{N}$ *such that* $c\,y_n < x_n$ *for all* $n \ge n_0$;
6) $x_n = \overline{\overline{\Theta}}(y_n, z_n)$ *if there exist* $c_1, c_2 > 0$ *and* $n_0 \in \mathbf{N}$ *such that* $c_1\,y_n \le x_n \le c_2\,z_n$ *for all* $n \ge n_0$.

The next statements collect the usual transitivity and duality rules, together with basic set-theoretic relations among the associated classes. Proofs follow the standard ε–C arguments and can be found, for example, in [9, 10, 13].

Theorem 1 (Transitivity). *Let* (x_n), (y_n), (z_n) *be positive sequences. Then*:

1) *If* $x_n = O(y_n)$ *and* $y_n = O(z_n)$, *then* $x_n = O(z_n)$;
2) *If* $x_n = \Omega(y_n)$ *and* $y_n = \Omega(z_n)$, *then* $x_n = \Omega(z_n)$;
3) *If* $x_n = \Theta(y_n)$ *and* $y_n = \Theta(z_n)$, *then* $x_n = \Theta(z_n)$;
4) *If* $x_n = o(y_n)$ *and* $y_n = o(z_n)$, *then* $x_n = o(z_n)$;
5) *If* $x_n = \omega(y_n)$ *and* $y_n = \omega(z_n)$, *then* $x_n = \omega(z_n)$;
6) *If* $x_n \in \overline{\overline{\Theta}}(y_n, z_n)$ *and* $y_n \in \overline{\overline{\Theta}}(z_n, w_n)$, *then* $x_n \in \overline{\overline{\Theta}}(z_n, w_n)$;
7) *If* $y_n = O(z_n)$, *then* $O(y_n) \subset O(z_n)$;
8) *If* $y_n = \Omega(z_n)$, *then* $\Omega(y_n) \subset \Omega(z_n)$;
9) *If* $y_n = \Theta(z_n)$, *then* $\Theta(y_n) \subset \Theta(z_n)$.

The next result records the usual dualities between the big/little notations.

Theorem 2 (Dualities). *Let* (x_n), (y_n) *be positive sequences. Then*:

1) $x_n = O(y_n)$ *if and only if* $y_n = \Omega(x_n)$;
2) $x_n = o(y_n)$ *if and only if* $y_n = \omega(x_n)$;
3) $x_n = \Theta(y_n)$ *if and only if* $y_n = \Theta(x_n)$.

Finally, we summarize the basic set-theoretic structure of the classes $O(y_n)$, $\Omega(y_n)$, $\Theta(y_n)$, $o(y_n)$, and $\omega(y_n)$.

Theorem 3 (Set relations among classes). *Let* (y_n) *be a positive sequence. Then*:

1) $o(y_n) \cap \omega(y_n) = \varnothing$;
2) $O(y_n) \cap \Omega(y_n) = \Theta(y_n)$;
3) $o(y_n) \subset O(y_n)$;
4) $\Theta(y_n) \subset O(y_n)$;
5) $\omega(y_n) \subset \Omega(y_n)$ *and* $\Theta(y_n) \subset \Omega(y_n)$;
6) $o(y_n) \cap \Omega(y_n) = \varnothing$ *and* $O(y_n) \cap \omega(y_n) = \varnothing$;
7) $o(y_n) \cap \Theta(y_n) = \varnothing$ *and* $\omega(y_n) \cap \Theta(y_n) = \varnothing$.

The above properties are the classical backbone of asymptotic analysis and will serve as the reference point for our filter-based generalization developed in the subsequent sections; compare with [9, 10, 13].

3. Generalized notation asymptotic and some properties

In this section, we going to introduce the notion of asymptotic notation generalized by filters, and we going to study some of its properties.

Definition 2. *Let* $(x_n), (y_n), (z_n) \in F^+$ *and* $\mathcal{F}$ *be a filter on* $\mathbb{N}$. *The* $\mathcal{F}$*-asymptotic notations are defined as follows*:

1. *If* $\{n \in \mathbb{N} : x_n \leq c \cdot y_n\} \in \mathcal{F}$, *for some* $c > 0$, *then* $x_n = O_{\mathcal{F}}(y_n)$.
2. *If* $\{n \in \mathbb{N} : c \cdot y_n \leq x_n\} \in \mathcal{F}$, *for some* $c > 0$, *then* $x_n = \Omega_{\mathcal{F}}(y_n)$.
3. *If* $\{n \in \mathbb{N} : c_1 \cdot y_n \leq x_n \leq c_2 \cdot y_n\} \in \mathcal{F}$, *for some* $c_1, c_2 > 0$, *then* $x_n = \Theta_{\mathcal{F}}(y_n)$.
4. *If* $\{n \in \mathbb{N} : x_n < c \cdot y_n\} \in \mathcal{F}$, *for all* $c > 0$, *then* $x_n = o_{\mathcal{F}}(y_n)$.
5. *If* $\{n \in \mathbb{N} : c \cdot y_n \leq x_n\} \in \mathcal{F}$, *for all* $c > 0$, *then* $x_n = \omega_{\mathcal{F}}(y_n)$.
6. *If* $\{n \in \mathbb{N} : c_1 \cdot y_n \leq x_n \leq c_2 \cdot z_n\} \in \mathcal{F}$, *for some* $c_1, c_2 > 0$, *then* $x_n = \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n)$.

The above definitions extend the classical asymptotic notations to the context of filters. This generalization allows for a more nuanced analysis of growth rates, capturing behaviors that might be overlooked by traditional notations. In the following theorem, we establish the relationship between these new $\mathcal{F}$-asymptotic notations and their classical counterparts.

Theorem 4. *Let* $\mathcal{F}$ *be a free filter and* $(y_n), (z_n) \in F^+$.

1. *If* $x_n = O(y_n)$, *then* $x_n = O_{\mathcal{F}}(y_n)$.
2. *If* $x_n = \Omega(y_n)$, *then* $x_n = \Omega_{\mathcal{F}}(y_n)$.
3. *If* $x_n = \Theta(y_n)$, *then* $x_n = \Theta_{\mathcal{F}}(y_n)$.
4. *If* $x_n = o(y_n)$, *then* $x_n = o_{\mathcal{F}}(y_n)$.
5. *If* $x_n = \omega(y_n)$, *then* $x_n = \omega_{\mathcal{F}}(y_n)$.
6. *If* $x_n = \overline{\overline{\Theta}}(y_n, z_n)$, *then* $x_n = \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n)$.

The reciprocal propositions of the aforementioned statements are all true when the Fréchet filter $\mathcal{F}_r$ *is considered.*

Proof. Let $(y_n), (z_n) \in F^+$ and $\mathcal{F}$ be a filter on $\mathbb{N}$. Suppose that $x_n = O(y_n)$. By hypothesis, there exist $c > 0$ and $n_0 \in \mathbb{N}$ such that $x_n \leq c \cdot y_n$, for all $n \geq n_0$. Consider the set

$$A = \{n \in \mathbb{N} : x_n \leq c \cdot y_n\},$$

take $N \in \mathbb{N} - A$, then $x_N > c \cdot y_N$, this means that $N < n_0$, then, $N \in \{1, \ldots, n_0 - 1\}$. Thus, the complement of A is a subset of a finite set, so it must be finite. Also, by definition $A \in \mathcal{F}_r \subset \mathcal{F}$ and therefore $x_n = O_{\mathcal{F}}(y_n)$.

On the other hand, suppose that $x_n = O_{\mathcal{F}_r}(y_n)$. By hypothesis, there exists $c \geq 0$ such that the set

$$A = \{n \in \mathbb{N} : x_n \leq c \cdot y_n\} \in \mathcal{F}_r,$$

so the complement of A is finite , that is, $\mathbb{N} - A = \{n_1, \ldots, n_k\}$ for some $k \in \mathbb{N}$. Take $m = \max\{n_1, \ldots, n_k\}$, so that, for all $n \geq m + 1$ it is true that $x_n \leq c \cdot y_n$ and therefore, $x_n = O(y_n)$. Each one of the other propositions can be demonstrated by a similar argument. □

It follows that $x_n = O(y_n)$ if and only if $x_n = O_{\mathcal{F}_r}(y_n)$ since the Frechét filter $\mathcal{F}_r$ is a free filter. Now, through the next example we going to show that there exist sequences that satisfy the definition generalized by filters but not the usual one.

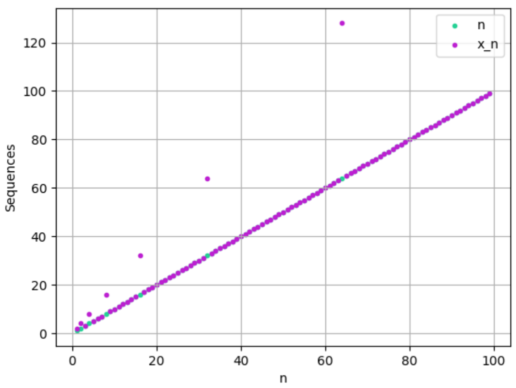

Figure 1. $x_n = O_{\mathcal{F}_d}(n)$ but $x_n \neq O(n)$.

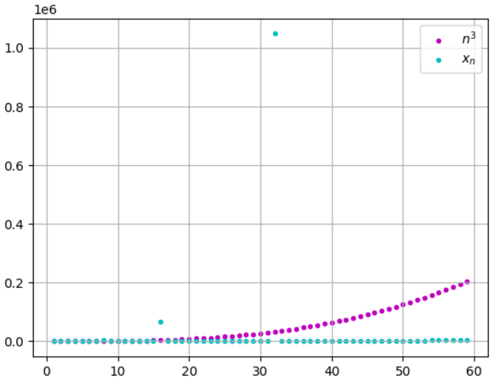

Figure 2. $x_n = O_{\mathcal{F}_d}(n)$ but $x_n \neq O(n)$.

Example 1. Consider the sequence (x_n) defined as follows:

$$x_n = \begin{cases} n, & n \neq 2^k \quad \text{for all} \quad k \in \mathbb{N}, \\ 2n, & n = 2^k \quad \text{for some} \quad k \in \mathbb{N}. \end{cases}$$

Note that:

$$\{n \in \mathbb{N} : x_n \leq n\} = \mathbb{N} - \{2^n : n \in \mathbb{N}\} \in \mathcal{F}_d$$

So, $x_n = O_{\mathcal{F}_d}(n)$, however, $x_n \neq O(n)$ (see Fig. 1).

Example 2. Consider the sequence (x_n) defined as follows:

$$x_n = \begin{cases} n^2 & n \neq 2^k \quad \text{for all} \quad k \in \mathbb{N}, \\ n^4 & n = 2^k \quad \text{for some} \quad k \in \mathbb{N}. \end{cases}$$

Given $c > 0$ there exists $n_0 \in \mathbb{N}$ such that:

$$\begin{aligned} A = \{n \in \mathbb{N} : x_n \leq n\} &= \left\{n \in \mathbb{N} : n^2 \leq cn^3\right\} \cup \left\{n \in \mathbb{N} : n^4 \leq cn^3\right\} \\ &= \left\{n \neq 2^k \text{ and } n \geq n_0 : \frac{1}{n} \leq c\right\} \cup \left\{n = 2^k : n \leq c\right\} \\ &= \mathbb{N} - \{2^n : n \geq n_0\} \in \mathcal{F}_d \end{aligned}$$

So, $x_n = o_{\mathcal{F}_d}(n^3)$, but $x_n \neq o(n^3)$ (see Fig. 2).

Having established the connection between F-asymptotic notations and classical asymptotic notations, we now turn our attention to some fundamental properties of these generalized notations. The following theorems demonstrate that $\mathcal{F}$-asymptotic notations preserve many of the essential characteristics of their classical counterparts, such as reflexivity and transitivity.

Theorem 5. *The $\mathcal{F}$-asymptotic notation $O_{\mathcal{F}}$, $\Omega_{\mathcal{F}}$ and $\Theta_{\mathcal{F}}$ are non-empty for any filter $\mathcal{F}$.*

Proof. Let $(y_n) \in F^+$ note that $y_n = O_{\mathcal{F}}(y_n)$, $y_n = \Omega_{\mathcal{F}}(y_n)$ and $y_n = \Theta_{\mathcal{F}}(y_n)$. □

The above theorem implies that the $\mathcal{F}$-asymptotic notations $O_{\mathcal{F}}$, $\Omega_{\mathcal{F}}$ and $\Theta_{\mathcal{F}}$ satisfies notions of reflexivity as well.

Theorem 6. *Let* $(y_n), (z_n) \in F^+$. *Then, for any filter* $\mathcal{F}$*,*

1) $\overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n) \neq \emptyset$ *if and only if* $y_n = O_{\mathcal{F}}(z_n)$;
2) $\overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n) \neq \emptyset$ *if and only if* $z_n = \Omega_{\mathcal{F}}(y_n)$.

Proof. Suppose that there exists $(x_n) \in F^+$ such that

$$x_n = \overline{\overline{\Theta}}\mathcal{F}(y_n, z_n).$$

Then there exist $c_1, c_2 > 0$ such that

$$B = \big\{n \in \mathbb{N} : c_1 \cdot y_n \leq x_n \leq c_2 \cdot z_n\big\} \in \mathcal{F}.$$

Note that

$$B \subset A = \Big\{n \in \mathbb{N} : y_n \leq \frac{c_2}{c_1} \cdot z_n\Big\}.$$

Hence, $A \in \mathcal{F}$ and $y_n = O\mathcal{F}(z_n)$. Also note that

$$B \subset C = \Big\{n \in \mathbb{N} : \frac{c_1}{c_2} \cdot y_n \leq z_n\Big\}.$$

Hence, $C \in \mathcal{F}$ and $z_n = \Omega_{\mathcal{F}}(y_n)$.

Reciprocally, suppose that $y_n = O_{\mathcal{F}}(z_n)$ and $z_n = \Omega_{\mathcal{F}}(y_n)$, then there exist $c_1, c_2 > 0$ such that:

$$A = \big\{n \in \mathbb{N} : y_n \leq y_n \leq c_1 \cdot z_n\big\} \in \mathcal{F}, \quad B = \big\{n \in \mathbb{N} : c_2 \cdot y_n \leq z_n \leq z_n\big\} \in \mathcal{F}.$$

Hence, $y_n = \overline{\overline{\Theta}}\mathcal{F}(y_n, z_n)$ and $z_n = \overline{\overline{\Theta}}\mathcal{F}(y_n, z_n)$. ☐

Theorem 7. *The* $\mathcal{F}$*-asymptotic notations* $o_{\mathcal{F}}$ *and* $\omega_{\mathcal{F}}$ *are non-empty for any filter free* $\mathcal{F}$.

Proof. Let $(y_n) \in F^+$. Then, $z_n = o_{\mathcal{F}}(y_n)$, where $z_n = y_n/n$. Also, let $w_n = \omega_{\mathcal{F}}(y_n)$, where $w_n = y_n \cdot n$. ☐

The previous result suggests that the $\mathcal{F}$-asymptotic notations $o_{\mathcal{F}}$ and $\omega_{\mathcal{F}}$ do not satisfy the properties of reflexivity. On the other hand, the following example illustrates that the requirement of a free filter is essential to ensure that $o_{\mathcal{F}}$ and $\omega_{\mathcal{F}}$ are nonempty.

Example 3. Let $(y_n) \in F^+$ and $m \in \mathbb{N}$. Suppose there is a sequence $(z_n) \in F^+$ which satisfies $z_n = o_{\mathcal{F}_m}(y_n)$. Then, for all $c > 0$, it follows that

$$\big\{n \in \mathbb{N} : z_n < c \cdot y_n\big\} \in \mathcal{F}_m.$$

In other words, $z_m/y_m < c$ for all $c > 0$. Since $c > 0$ is arbitrary, then $z_m/y_m = 0$, which contradicts the fact that $(z_n) \in F^+$. Following the same argument, it can be proved that $\omega\mathcal{F}_m(y_n) = \emptyset$.

Transitivity is a crucial property in asymptotic analysis, allowing us to chain comparisons and build more complex relationships between functions. The following theorem establishes that our $\mathcal{F}$-asymptotic notations maintain this important property, further validating their utility in algorithmic analysis.

Theorem 8. *Let* $(x_n), (y_n), (z_n) \in F^+$. *The following properties are satisfied, for any filter* $\mathcal{F}$.

1. *If* $x_n = O_{\mathcal{F}}(y_n)$ *and* $y_n = O_{\mathcal{F}}(z_n)$, *then* $x_n = O_{\mathcal{F}}(z_n)$.

2. *If* $x_n = \Omega_{\mathcal{F}}(y_n)$ *and* $y_n = \Omega_{\mathcal{F}}(z_n)$*, then* $x_n = \Omega_{\mathcal{F}}(z_n)$*.*
3. *If* $x_n = \Theta_{\mathcal{F}}(y_n)$ *and* $y_n = \Theta_{\mathcal{F}}(z_n)$*, then* $x_n = \Theta_{\mathcal{F}}(z_n)$*.*
4. *If* $x_n = o_{\mathcal{F}}(y_n)$ *and* $y_n = o_{\mathcal{F}}(z_n)$*, then* $x_n = o_{\mathcal{F}}(z_n)$*.*
5. *If* $x_n = \omega_{\mathcal{F}}(y_n)$ *and* $y_n = \omega_{\mathcal{F}}(z_n)$*, then* $x_n = \omega_{\mathcal{F}}(z_n)$*.*
6. *If* $x_n = \overline{\overline{\overline{\Theta}}}_{\mathcal{F}}(y_n)$ *and* $y_n = \overline{\overline{\overline{\Theta}}}_{\mathcal{F}}(z_n)$*, then* $x_n = \overline{\overline{\overline{\Theta}}}_{\mathcal{F}}(z_n)$*.*

Proof. Let $(x_n), (y_n), (z_n) \in F^+$.

(1) Let $x_n = O_{\mathcal{F}}(y_n)$ and $y_n = O_{\mathcal{F}}(z_n)$, then there exists $c_1, c_2 > 0$ such that

$$A = \{n \in \mathbb{N} : x \leq c_1 \cdot y_n\} \in \mathcal{F} \quad \text{and} \quad B = \{n \in \mathbb{N} : y_n \leq c_2 \cdot z_n\} \in \mathcal{F}.$$

Note that

$$A \cap B \subset \{n \in \mathbb{N} : x_n \leq c_1 \cdot c_2 \cdot z_n\}.$$

Therefore,

$$\{n \in \mathbb{N} : x_n \leq c_1 \cdot c_2 \cdot z_n\} \in \mathcal{F}.$$

Hence, $x_n = O_{\mathcal{F}}(z_n)$.

(2) and (3) could be shown by applying a similar argument to the previous one.

(4) Let $x_n = o_{\mathcal{F}}(y_n)$ and $y_n = o_{\mathcal{F}}(z_n)$, and let $d > 0$, Then

$$A = \left\{n \in \mathbb{N} : x_n < \frac{d}{2} y_n\right\} \in \mathcal{F} \quad \text{and} \quad B = \{n \in \mathbb{N} : y_n < 2z_n\} \in \mathcal{F}.$$

Note that

$$A \cap B \subset \{n \in \mathbb{N} : x_n < d \cdot z_n\}.$$

Thus,

$$\{n \in \mathbb{N} : x_n \leq c_1 \cdot c_2 \cdot z_n\} \in \mathcal{F}$$

and $x_n = o_{\mathcal{F}}(z_n)$.

(5) and (6) could be shown by applying a similar argument to the previous one.

□

As a direct result of the transitivity properties, we have the follow result.

Corollary 1. *Let* $(y_n), (z_n) \in F^+$*. The following properties are fulfilled, for any filter* $\mathcal{F}$*.*

1. *If* $y_n = O_{\mathcal{F}}(z_n)$*, then* $O_{\mathcal{F}}(y_n) \subset O_{\mathcal{F}}(z_n)$*.*
2. *If* $y_n = \Omega_{\mathcal{F}}(z_n)$*, then* $\Omega_{\mathcal{F}}(y_n) \subset \Omega_{\mathcal{F}}(z_n)$*.*
3. *If* $y_n = \Theta_{\mathcal{F}}(z_n)$*, then* $\Theta_{\mathcal{F}}(y_n) \subset \Theta_{\mathcal{F}}(z_n)$*.*

4. Properties between $\mathcal{F}$-asymptotic notations

In this section, we explore the relationships between different $\mathcal{F}$-asymptotic notations. These relationships are fundamental to understanding how these notations can be used in practice and how they compare to one another. We begin by examining the symmetry properties of these notations, which reveal interesting duality relationships, due to $O_{\mathcal{F}}$ and $\Omega_{\mathcal{F}}$ are related to bounded sequences and $o_{\mathcal{F}}$ and $\omega_{\mathcal{F}}$ are relate to convergent sequences.

Theorem 9. *Let* $(x_n), (y_n) \in F^+$*. The following statements are true, for any filter* $\mathcal{F}$:

1) $x_n = O_{\mathcal{F}}(y_n)$ *if and only if* $y_n = \Omega_{\mathcal{F}}(x_n)$;
2) $x_n = o_{\mathcal{F}}(y_n)$ *if and only if* $y_n = \omega_{\mathcal{F}}(x_n)$;
3) $x_n = \Theta_{\mathcal{F}}(y_n)$ *if and only if* $y_n = \Theta_{\mathcal{F}}(x_n)$.

Proof. Let $(x_n), (y_n) \in F^+$. It is a trivial consequence that for $c > 0$;

$$\{n \in \mathbb{N} : x_n \le c \cdot y_n\} = \{n \in \mathbb{N} : c^{-1} \cdot x_n \le y_n\}.$$

□

In addition to the symmetric property, a few other statements involving intersections and subset properties can also be established. Where we can find how some $\mathcal{F}$-asymptotic notations do not have elements in common and others, on the contrary can be defined in terms of one another.

Theorem 10. *Let $(y_n) \in F^+$. The following properties are true, for any filter $\mathcal{F}$:*

1) $o_{\mathcal{F}}(y_n) \cap \omega_{\mathcal{F}}(y_n) = \varnothing$.
2) $O_{\mathcal{F}}(y_n) \cap \Omega_{\mathcal{F}}(y_n) = \Theta_{\mathcal{F}}(y_n)$;
3) $o_{\mathcal{F}}(y_n) \subset O_{\mathcal{F}}(y_n)$;
4) $\Theta_{\mathcal{F}}(y_n) \subset O_{\mathcal{F}}(y_n)$;
5) $\omega_{\mathcal{F}}(y_n) \subset \Omega_{\mathcal{F}}(y_n)$ *and* $\Theta_{\mathcal{F}}(y_n) \subset \Omega_{\mathcal{F}}(y_n)$;
6) $o_{\mathcal{F}}(y_n) \cap \Omega_{\mathcal{F}}(y_n) = \varnothing$ *and* $O_{\mathcal{F}}(y_n) \cap \omega_{\mathcal{F}}(y_n) = \varnothing$;
7) $o_{\mathcal{F}}(y_n) \cap \Theta_{\mathcal{F}}(y_n) = \varnothing$ *and* $\omega_{\mathcal{F}}(y_n) \cap \Theta_{\mathcal{F}}(y_n) = \varnothing$.

Proof. Let $(y_n) \in F^+$.

1) Suppose that $x_n = o_{\mathcal{F}}(y_n) \cap \omega_{\mathcal{F}}(y_n)$, then

$$\{n \in \mathbb{N} : x_n < y_n\} \cap \{n \in \mathbb{N} : y_n < x_n\} = \varnothing \in \mathcal{F}.$$

Therefore, $o_{\mathcal{F}}(y_n) \cap \omega_{\mathcal{F}}(y_n) = \varnothing$.

2) Let $x_n = O_{\mathcal{F}}(y_n) \cap \Omega_{\mathcal{F}}(y_n)$, then there exist $c_1, c_2 > 0$ such that

$$A = \{n \in \mathbb{N} : x_n \le c_2 \cdot y_n\} \in \mathcal{F}$$

and

$$B = \{n \in \mathbb{N} : c_1 \cdot y_n \le x_n\} \in \mathcal{F}.$$

Then:

$$\begin{aligned} A \cap B &= \{n \in \mathbb{N} : x_n \le c_2 \cdot y_n\} \cap \{n \in \mathbb{N} : c_1 \cdot y_n \le x_n\} \\ &= \{n \in \mathbb{N} : c_1 \cdot y_n \le x_n \le c_2 \cdot y_n\} \in \mathcal{F}. \end{aligned}$$

Hence, $x_n = \Theta_{\mathcal{F}}(y_n)$. Now, suppose that $x_n = \Theta_{\mathcal{F}}(y_n)$, then there exist $c_1, c_2 > 0$ such that

$$C = \{n \in \mathbb{N} : c_1 \cdot y_n \le x_n \le c_2 \cdot y_n\} \in \mathcal{F}.$$

Note that $C = A \cap B$, where

$$A = \{n \in \mathbb{N} : x_n \le c_2 \cdot y_n\} \quad \text{and} \quad B = \{n \in \mathbb{N} : c_1 \cdot y_n \le x_n\},$$

it follow that $A \in \mathcal{F}$ and $B \in \mathcal{F}$. Therefore $x_n = O_{\mathcal{F}}(y_n)$ and $x_n = \Omega_{\mathcal{F}}(y_n)$.

3) Let $x_n = o_{\mathcal{F}}(y_n)$, then

$$\{n \in \mathbb{N} : x_n < c \cdot y_n\} \in \mathcal{F}$$

for all $c > 0$. Thus, $x_n = O_{\mathcal{F}}(y_n)$.

4) Let $x_n = \Theta_{\mathcal{F}}(y_n)$, there exist $c_1, c_2 > 0$ such that

$$\big\{n \in \mathbb{N} : c_1 \cdot y_n \leq x_n \leq c_2 \cdot y_n\big\}.$$

Note that

$$\big\{n \in \mathbb{N} : c_1 \cdot y_n \leq x_n \leq c_2 \cdot y_n\big\}\big\{n \in \mathbb{N} : x_n \leq c_2 \cdot y_n\big\}.$$

Therefore

$$\big\{n \in \mathbb{N} : x_n \leq c_2 \cdot y_n\big\} \in \mathcal{F},$$

and thus, $x_n = O_{\mathcal{F}}(y_n)$.

The rest of the propositions can be proven using arguments similar to the previous ones. □

These propositions are a logical outcome of the initial definition, since every convergent sequence is bounded then it is expected that an inclusion relation is presented between $o_{\mathcal{F}}$ and $O_{\mathcal{F}}$, and for $\omega_{\mathcal{F}}$ and $\Omega_{\mathcal{F}}$ as well. Finally, we are listing a few properties in regard to the relationship between the $\overline{\overline{\Theta}}_{\mathcal{F}}$ and the other $\mathcal{F}$-asymptotic notations. Since the $\overline{\overline{\Theta}}_{\mathcal{F}}$, in terms of complexity, is related to the worst case (a lower bound) and best case (an upper bound), then it is meant to be connected to the other $\mathcal{F}$-asymptotic notations that are associated with bounded sequences.

Theorem 11. *Let $(x_n), (y_n), (z_n) \in F^+$. The following properties are satisfied, for any filter $\mathcal{F}$.*

1) $\overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n) = \Omega_{\mathcal{F}}(y_n) \cap O_{\mathcal{F}}(z_n)$ *and* $\overline{\overline{\Theta}}_{\mathcal{F}}(y_n, y_n) = \Theta_{\mathcal{F}}(y_n)$;
2) $\Theta_{\mathcal{F}}(y_n) = \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n) \cap O_{\mathcal{F}}(y_n)$ *and* $\Theta_{\mathcal{F}}(z_n) = \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n) \cap \Omega_{\mathcal{F}}(z_n)$;
3) $\Theta_{\mathcal{F}}(y_n) \subset \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n)$ *and* $\Theta_{\mathcal{F}}(z_n) \subset \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n)$.

Proof. Let $(x_n), (y_n), (z_n) \in F^+$.

1. Let $x_n = O_{\mathcal{F}}(z_n) \cap \Omega_{\mathcal{F}}(y_n)$, then there exist $c_1, c_2 > 0$ such

$$A = \big\{n \in \mathbb{N} : x_n \leq c_2 \cdot z_n\big\} \in \mathcal{F} \quad \text{and} \quad B = \big\{n \in \mathbb{N} : c_1 \cdot y_n \leq x_n\big\} \in \mathcal{F}.$$

Therefore:

$$\begin{aligned} A \cap B &= \big\{n \in \mathbb{N} : x_n \leq c_2 \cdot z_n\big\} \cap \big\{n \in \mathbb{N} : c_1 \cdot y_n \leq x_n\big\} \\ &= \big\{n \in \mathbb{N} : c_1 \cdot y_n \leq x_n \leq c_2 \cdot z_n\big\} \in \mathcal{F}. \end{aligned}$$

Thus, $x_n = \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n)$. Let $x_n = \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n)$, then there exist $c_1, c_2 > 0$ such that

$$\big\{n \in \mathbb{N} : c_1 \cdot y_n \leq x_n \leq c_2 \cdot z_n\big\} \in \mathcal{F}.$$

Note that

$$\big\{n \in \mathbb{N} : c_1 \cdot y_n \leq x_n \leq c_2 \cdot z_n\big\} = \big\{n \in \mathbb{N} : x_n \leq c_2 \cdot z_n\big\} \cap \big\{n \in \mathbb{N} : c_1 \cdot y_n \leq x_n\big\}.$$

Then

$$\big\{n \in \mathbb{N} : x_n \leq c_2 \cdot z_n\big\} \in \mathcal{F} \quad \text{and} \quad \big\{n \in \mathbb{N} : c_1 \cdot y_n \leq x_n\big\} \in \mathcal{F}.$$

Therefore $x_n = O_{\mathcal{F}}(y_n)$ and $x_n = \Omega_{\mathcal{F}}(y_n)$. Additionally, due to the previous proof, $\overline{\overline{\Theta}}_{\mathcal{F}}(y_n, y_n) = \Omega_{\mathcal{F}}(y_n) \cap O_{\mathcal{F}}(y_n)$ and as a consequence of the Theorem 10 we have that $O_{\mathcal{F}}(y_n) \cap \Omega_{\mathcal{F}}(y_n) = \Theta_{\mathcal{F}}(y_n)$, thus $\overline{\overline{\Theta}}_{\mathcal{F}}(y_n, y_n) = \Theta_{\mathcal{F}}(y_n)$.

2. Suppose that $\overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n) \neq \emptyset$, because if it is empty then $\Theta_{\mathcal{F}}(y_n) = \emptyset$ and this contradicts the reflexivity property that $\Theta_{\mathcal{F}}$ satisfies. Then due to the Theorem 6 it follows that $y_n = O_{\mathcal{F}}(z_n)$. Consequently, we have that $O_{\mathcal{F}}(y_n) \subset O_{\mathcal{F}}(z_n)$ because of Theorem 1. On account of the previous proof, we have the following:

$$\begin{aligned}\Theta_{\mathcal{F}}(y_n) &= \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n) \cap O_{\mathcal{F}}(y_n) = (\Omega_{\mathcal{F}}(y_n) \cap O_{\mathcal{F}}(z_n)) \cap O_{\mathcal{F}}(y_n) \\ &= \Omega_{\mathcal{F}}(y_n) \cap O_{\mathcal{F}}(y_n) = \Theta_{\mathcal{F}}(y_n).\end{aligned}$$

   Additionally, by a similar argument it follows that $\Theta_{\mathcal{F}}(z_n) = \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n) \cap \Omega_{\mathcal{F}}(z_n)$.

3. As a result of the previous proposition, it follows that

$$\Theta_{\mathcal{F}}(y_n) = \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n) \cap O_{\mathcal{F}}(y_n) \quad \text{and} \quad \Theta_{\mathcal{F}}(z_n) = \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n) \cap \Omega_{\mathcal{F}}(z_n).$$

   Then, by basic set properties we have that $\Theta_{\mathcal{F}}(y_n) \subset \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n)$ and $\Theta_{\mathcal{F}}(z_n) \subset \overline{\overline{\Theta}}_{\mathcal{F}}(y_n, z_n)$.

□

5. Conclusion

In this paper, we have introduced a new generalization of asymptotic notation using filters, extending the classical concepts to a broader mathematical framework. Our work demonstrates that these $\mathcal{F}$-asymptotic notations preserve many of the fundamental properties of their classical counterparts while offering enhanced analytical capabilities.

The key contributions of this study include:

- The formal definition of $\mathcal{F}$-asymptotic notations ($O_{\mathcal{F}}$, $\Omega_{\mathcal{F}}$, $\Theta_{\mathcal{F}}$, $o_{\mathcal{F}}$, and $\omega_{\mathcal{F}}$) using filters.
- Proof of the relationship between classical asymptotic notations and their $\mathcal{F}$-asymptotic counterparts.
- Establishment of essential properties such as reflexivity, symmetry, and transitivity for $\mathcal{F}$-asymptotic notations.
- Exploration of set-theoretic relationships between different $\mathcal{F}$-asymptotic notations, providing insights into their behavior and interactions.
- Characterization of the $\Theta_{\mathcal{F}}$ notation and its relationships with other $\mathcal{F}$-asymptotic notations.

These results have significant implications for the field of algorithmic analysis. The generalization to filters allows for more nuanced comparisons of growth rates, potentially capturing behaviors that might be overlooked by traditional asymptotic notation. This could lead to more precise complexity analyses, especially in cases where traditional asymptotic notations fall short.

Moreover, this work opens up several promising avenues for future research:

1. Investigation of specific types of filters and their implications for asymptotic analysis.
2. Application of $\mathcal{F}$-asymptotic notations to concrete algorithmic problems, particularly those where traditional asymptotic analysis provides insufficient insight.
3. Development of computational tools and techniques for working with $\mathcal{F}$-asymptotic notations in practice.
4. Extension of this framework to multivariate functions and its potential applications in the analysis of multi-parameter algorithms.

The generalization of asymptotic notation using filters presented in this paper provides a robust theoretical foundation for more refined algorithmic analysis. As computational problems grow in complexity, such tools may become in.

Finally, at the referee's request, we note that our filter–based framework is not restricted to sequences. All definitions and results for positive sequences $(O_{\mathcal{F}}, \Omega_{\mathcal{F}}, \Theta_{\mathcal{F}}, o_{\mathcal{F}}, \omega_{\mathcal{F}})$ can be extend verbatim to positive functions $f, g, h : [0, +\infty) \to (0, \infty)$, and the proofs remain valid without modification, as they rely on set–theoretic inclusions.

REFERENCES

1. Akin E. *Recurrence in Topological Dynamics. Furstenberg Families and Ellis actions.* Univ. Ser. Math. New York: Plenum Press, 1997. 266 p. DOI: 10.1007/978-1-4757-2668-8
2. Akomolafe D. T., Nwanze N. M. Deployment of an efficient algorithm for searching motor vehicle database. *Asian J. Adv. Res.*, 2021. Vol. 10, No. 4. P. 18–26.
3. Bernstein A. R. A new kind of compactness for topological spaces. *Fund. Math.*, 1970. Vol. 66, No. 2. P. 185–193. DOI: 10.4064/fm-66-2-185-193
4. Folea R., Slusanschi E. A new metric for evaluating the performance and complexity of computer programs: A new approach to the traditional ways of measuring the complexity of algorithms and estimating running times. In: *23rd Int. Conf. Control Systems and Computer Science (CSCS), 26–28 May 2021, Bucharest, Romania.* IEEE Xplore, 2021. P. 157–164. DOI: 10.1109/CSCS52396.2021.00033
5. Frolík Z. Sums of ultrafilters. *Bull. Amer. Math. Soc.*, 1967. Vol. 73. P. 87–91.
6. Furstenberg H. *Recurrence in Ergodic Theory and Combinatorial Number Theory.* Princeton: Princeton Univ. Press, 1981. 202 p.
7. Goldreich O. Computational complexity: a conceptual perspective. *ACM Sigact News*, 2008. Vol. 39, No. 3. P. 35–39. DOI: 10.1145/1412700.1412710
8. Lim T. S., Loh W. Y., Shih Y. S. A comparison of prediction accuracy, complexity, and training time of thirty-three old and new classification algorithms. *Mach. Learn.*, 2000. Vol. 40, No. 3. P. 203–228. DOI: 10.1023/A:1007608224229
9. Mogoş A.-H., Florea A. M. A method to compare two complexity functions using complexity classes. *U.P.B. Sci. Bull., Series A*, 2010. Vol. 72, No. 2. P. 69–84.
10. Mogoş A., Mogoş B., Florea A. A new asymptotic notation: weak theta. *Math. Probl. Eng.*, 2015. Vol. 2015. Art. no. 912962. 15 p. DOI: 10.1155/2015/912962
11. Ramesh V. P., Gowtham R. Asymptotic notations and its applications. *Math. Newsl.*, 2017. Vol. 28, No. 4. P. 10–16.
12. Russell S., Norvig P. *Artificial Intelligence: a Modern Approach.* 3rd ed. Upper Saddle River: Prentice–Hall, 2010. 1152 p.
13. Thomas C., Leiserson C. E., Stein C. *Introduction to Algorithms*, 3rd. 2009. 1312 p.
14. van Leeuwen J. (ed.) *Handbook of Theoretical Computer Science. Volume A: Algorithms and Complexity.* Cambridge: MIT Press, 1990. 1016 p.