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ON λ-WEAK CONVERGENCE OF SEQUENCES DEFINED BY AN ORLICZ FUNCTION

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Abstract: In this article, we introduce and rigorously analyze the concept of difference λ-weak convergence for sequences defined by an Orlicz function. This notion generalizes the classical weak convergence by incorporating a λ-density framework and an Orlicz function, providing a more flexible tool for analyzing convergence behavior in sequence spaces. We systematically investigate the algebraic and topological properties of these newly defined sequence spaces, establishing that they form linear and normed spaces under suitable conditions. Our results include demonstrating the convexity of these spaces and identifying several important inclusion relationships among them, such as strict inclusions between spaces involving different orders of difference operators and Orlicz functions satisfying the Δ_2-condition.

Keywords: Weak convergence, Orlicz function, λ convergence.

1. Introduction and preliminaries

The concept of weak convergence, first introduced by Banach [1], is central to functional analysis, providing a foundation for evaluating how sequences converge in infinite-dimensional spaces. While important, weak convergence has its limitations, especially when applied to complex sequence structures or when more precise convergence criteria are required.

Recently, Mahanta and Tripathy [21] made important advances in the study of vector-valued sequence spaces by investigating novel types of convergence and their repercussions. Their contributions have improved our understanding of the algebraic and topological properties of these spaces, enabling the development of new tools and approaches for investigating convergence in broader contexts. This growing field of study emphasizes the continual growth and improvement of sequence space theory, overcoming the limitations of traditional weak convergence while responding to the demands of more complex mathematical analysis.

The concept of natural density for subsets of $\mathbb{N}$ was extended by Mursaleen [13] to what is known as λ-density, which enabled a further generalization of the statistical convergence of real sequences, leading to the concept of λ-statistical convergence. If $\lambda = \{\lambda_s\}_{s\in\mathbb{N}}$ represents a nondecreasing sequence of positive real numbers tending to infinity, satisfying $\lambda_1 = 1$ and $\lambda_{s+1} \leq \lambda_s + 1$, $s \in \mathbb{N}$, then for any subset $T \subset \mathbb{N}$, the λ-density $d_\lambda(T)$ is defined as

$$d_\lambda(T) = \lim_{s\to\infty} \frac{|\{k \in I_s : k \in T\}|}{\lambda_s},$$

where $I_s = [s - \lambda_s + 1, s]$.

A sequence $t = \{t_\alpha\}_{\alpha \in \mathbb{N}}$ of real numbers is called λ-statistically convergent or S_λ-convergent to $t_0 \in \mathbb{R}$ if, for every $\epsilon > 0$, $d_\lambda(T(\epsilon)) = 0$, where

$$T(\epsilon) = \{\alpha \in \mathbb{N} : |t_\alpha - t_0| \geq \epsilon\}.$$

The generalized de la Vallée-Poussin mean is defined by

$$q_s(t) = \frac{1}{\lambda_s} \sum_{\alpha \in I_s} t_\alpha$$

where $I_s = [s - \lambda_s + 1, s]$. A sequence is called (V, λ)-summable to a number t_0 if $q_s(t) \to t_0$ as $s \to \infty$.

If $\lambda_s = s$ for all $s \in \mathbb{N}$, then the notions of λ-density and λ-statistical convergence coincide with the notions of natural density and statistical convergence, respectively. Some discussions and applications related to λ-statistical convergence can be found in [2, 4, 5, 12, 14, 15, 17–20].

Let X be a normed space. The concept of the difference sequence space $Z(\Delta)$ was first introduced by Kizmaz [10] and is defined as follows:

$$Z(\Delta) = \{t = (t_\alpha) : (\Delta t_\alpha) \in X\},$$

where $\Delta t = (\Delta t_\alpha) = (t_\alpha - t_{\alpha+1})$ for all $\alpha \in \mathbb{N}$. Later, Et and Çolak [3] extended this idea by defining generalized difference sequence spaces, expressed as

$$Z(\Delta^p) = \{t = (t_\alpha) : (\Delta^p t_\alpha) \in X\}$$

for $Z = \ell_\infty, c$, and c_0, where $\Delta^p t_\alpha = \Delta^{p-1} t_\alpha - \Delta^{p-1} t_{\alpha+1}$ and $\Delta^0 t_\alpha = t_\alpha$ for all $\alpha \in \mathbb{N}$.

The binomial expansion for this generalized difference operator is given by

$$\Delta^p t_\alpha = \sum_{d=0}^{p} (-1)^d \binom{p}{d} t_{\alpha+d}, \quad \text{for all } \alpha \in \mathbb{N}. \tag{1.1}$$

These generalized difference sequence spaces have been further studied by researchers such as Tripathy [22, 23], Tripathy and Esi [24], among others.

Definition 1. *Let V be a real vector space and let $u, v \in V$. Then, the set of all convex combinations of u and v is the set of points*

$$\{w_\varrho \in V : w_\varrho = (1 - \varrho) u + \varrho v, \ 0 \leq \varrho \leq 1\}. \tag{1.2}$$

In, say, $\mathbb{R}^2$, this set is exactly the line segment joining the two points u and v. We now introduce the concept of a convex set.

Definition 2. *Let $M \subset V$. Then the set M is said to be convex if, for any two points $u, v \in M$, the set defined in (1.2) is a subset of M.*

An Orlicz function $\mathcal{U} : [0, \infty) \to [0, \infty)$ is defined such that $\mathcal{U}(0) = 0$, $\mathcal{U}(t) > 0$ for $t > 0$, and $\mathcal{U}(t) \to \infty$ as $t \to \infty$. This function is continuous, nondecreasing, and convex.

Lindenstrauss and Tzafriri [11] introduced the concept of an Orlicz function to define the sequence space

$$\ell_{\mathcal{U}} = \left\{ (t_i) \in \omega : \sum_{i=1}^{\infty} \mathcal{U}\left(\frac{|t_i|}{v}\right) < \infty \text{ for some } v > 0 \right\},$$

where ω denotes the class of all sequences. The norm on the sequence space $\ell_{\mathcal{U}}$ is defined by

$$\|t\| = \inf\left\{v > 0 : \sum_{i=1}^{\infty} \mathcal{U}\left(\frac{|t_i|}{v}\right) \leq 1\right\},$$

which turns $\ell_{\mathcal{U}}$ into a Banach space, commonly referred to as an Orlicz sequence space. Various researchers, including Khan [6], Khan et al. [7–9], Parashar and Choudhury [16], and Tripathy and Mahanta [21], have explored different forms of Orlicz sequence spaces.

Definition 3. *A sequence (t_i) in a normed linear space X is called weakly convergent to an element $t_0 \in X$ if*

$$\lim_{i\to\infty} f\,(t_i - t_0) = 0 \quad \textit{for all } f \in X',$$

where X' denotes the continuous dual space of X.

Definition 4. *A sequence (t_i) in a normed linear space X is said to be λ-weakly convergent to $t_0 \in X$ if*

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{k\in I_s} f\,(t_k - t_0) = 0$$

for every $f \in X'$, where X' is the continuous dual space of X. In this context, the notation $\mathcal{D}_\lambda^w$ is used to denote the set of all λ-weakly convergent sequences.

Definition 5. *A sequence space E is called solid if, for any scalar sequence (β_i) with $|\beta_i| \leq 1$ for all $i \in \mathbb{N}$, the condition $(t_i) \in E$ implies that $(\beta_i t_i) \in E$.*

Definition 6. *A sequence space $E \subset \omega$ is called monotone if it contains all preimages of its step spaces.*

Definition 7. *A sequence space $E \subset \omega$ is called symmetric if, whenever $(t_i) \in E$, the permuted sequence $\left(t_{\pi(i)}\right)$ also belongs to E, where π is a permutation of $\mathbb{N}$.*

Lemma 1. *A sequence space E being solid does not necessarily mean that E is monotone.*

Definition 8. *An Orlicz function $\mathcal{U}$ satisfies the Δ_2-condition if there exists a constant $T > 0$ such that for all $u \geq 0$,*

$$\mathcal{U}(2u) \leq T\mathcal{U}(u).$$

2. Main result

This section presents the following classes of sequences and establishes results related to them:

$$[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0 = \left\{t = (t_\alpha) : \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) = 0 \text{ for some } v > 0\right\},$$

$$[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_1 = \left\{t = (t_\alpha) : \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\frac{|f(\Delta^p t_\alpha - t_0)|}{v} \text{ for some } t_0 \text{ and } v > 0\right\},$$

$$[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\infty = \left\{t = (t_\alpha) : \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) < \infty \text{ for some } v > 0\right\}.$$

The following result is presented here with a sketch of the proof.

Theorem 1. *The classes of sequences* $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$, $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_1$, *and* $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\infty$ *are linear spaces.*

P r o o f. The proof is provided only for the class $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$; the other cases can be established using a similar approach. Let $(t_\alpha), (q_\alpha) \in [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$, and let $\mathfrak{y}, \mathfrak{z} \in \mathbb{C}$. To prove the result, we need to find some $v_3 > 0$ such that

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\mathfrak{y}\Delta^p t_\alpha + \mathfrak{z}\Delta^p q_\alpha)|}{v_3}\right) = 0.$$

Since $(t_\alpha), (q_\alpha) \in [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$, there exist $v_1, v_2 > 0$ such that

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v_1}\right) = 0$$

and

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p q_\alpha)|}{v_2}\right) = 0.$$

We set $v_3 = \max(2|\mathfrak{y}|v_1, 2|\mathfrak{z}|v_2)$. Suppose that $\mathcal{U}$ is both convex and nondecreasing; it follows that

$$\begin{aligned}\frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\mathfrak{y}\Delta^p t_\alpha + \mathfrak{z}\Delta^p q_\alpha)|}{v_3}\right) &\le \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\mathfrak{y}\Delta^p t_\alpha)|}{v_3} + \frac{|f(\mathfrak{z}\Delta^p q_\alpha)|}{v_3}\right)\\ &\le \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \frac{1}{2}\left[\mathcal{U}\left(\frac{f(\mathfrak{y}\Delta^p t_\alpha)}{v_1} + \frac{f(\mathfrak{z}\Delta^p q_\alpha)}{v_2}\right)\right] \to 0 \quad \text{as} \quad s\to\infty.\end{aligned}$$

This proves that $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$ is a linear space over the field $\mathbb{C}$ of complex numbers.

Theorem 2. *For any Orlicz function* $\mathcal{U}$*, the space* $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\infty$ *forms a normed linear space with respect to the norm*

$$\varkappa_{\Delta^p}(t) = \sum_{i=1}^{p} |f(x_i)| + \inf\left\{v > 0 : \sup_s \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) \le 1\right\}.$$

P r o o f. To prove the theorem, we begin by examining the implications of $\varkappa_{\Delta^p}(t) = \varkappa_{\Delta^p}(-t)$ and $t = \theta$, which leads to $\Delta^p t_\alpha = 0$. As a result, we find $\mathcal{U}(\theta) = 0$, which consequently yields $\varkappa_{\Delta^p}(\theta) = 0$. Conversely, suppose $\varkappa_{\Delta^p}(t) = 0$, which implies that

$$\sum_{i=1}^{p} |f(t_i)| + \inf\left\{v > 0 : \sup_s \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) \le 1\right\} = 0.$$

Thus, we conclude that

$$\sum_{i=1}^{p} |f(t_i)| = 0 \quad \text{and} \quad \inf\left\{v > 0 : \sup_s \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) \le 1\right\} = 0.$$

From the first part, it follows that

$$t_i = \bar{\theta} \quad \text{for} \quad i = 1, 2, 3, \ldots, m, \tag{2.1}$$

where $\bar{\theta}$ denotes the zero element. For the second part, for any $\sigma > 0$, there exists some v_σ with $0 < v_\sigma < \sigma$ such that

$$\sup_s \frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v_\sigma}\right) \leq 1 \Rightarrow \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v_\sigma}\right) \leq 1.$$

Therefore,

$$\sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{\sigma}\right) \leq \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v_\sigma}\right) \leq 1.$$

Suppose that $\Delta^p t_{q_i} \neq \bar{\theta}$ for each $i \in \mathbb{N}$. As $\sigma \to 0$, it follows that

$$\frac{|f(\Delta^p t_{q_i})|}{\sigma} \to \infty.$$

Thus,

$$\frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{\sigma}\right) \to \infty$$

as $\sigma \to 0$, where $q_i \in I_s$, which leads to a contradiction. Hence, $\Delta^p t_{q_i} = \bar{\theta}$ for each $i \in \mathbb{N}$, and consequently $\Delta t_\alpha = \bar{\theta}$ for all $\alpha \in \mathbb{N}$. Therefore, it follows from (1.1) and (2.1) that $t_\alpha = \bar{\theta}$ for all $\alpha \in \mathbb{N}$, implying that $t = \theta$.

Next, let $v_1, v_2 > 0$ be such that

$$\sup_s \frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v_1}\right) \leq 1$$

and

$$\sup_s \frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p \varpi_\alpha)|}{v_2}\right) \leq 1.$$

Let $v = v_1 + v_2$, then we have

$$\sup_s \frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p (t_\alpha + \varpi_\alpha))|}{v}\right) \leq 1.$$

Since v is nonnegative, we have

$$\varkappa_{\Delta^p} f(t+\varpi) = \sum_{i=1}^{p} |f(t_i+\varpi_i)| + \inf\left\{v>0 : \sup_s \frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p (t_\alpha+\varpi_\alpha))|}{v}\right) \leq 1\right\}$$

$$\Rightarrow \varkappa_{\Delta^p} f(t+\varpi) \leq \varkappa_{\Delta^p} f(t) + \varkappa_{\Delta^p} f(\varpi).$$

Let $\vartheta \neq 0$ and $\vartheta \in \mathbb{C}$. Then

$$\varkappa_{\Delta^p}(\vartheta t) = \sum_{i=1}^{p} |\vartheta t_i| + \inf\left\{v>0 : \sup_s \frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p (\vartheta t_\alpha))|}{v}\right) \leq 1\right\} \leq |\vartheta| \varkappa_{\Delta^p} f(t).$$

This completes the proof. □

Every normed space is convex. In fact, we will show that the space $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\infty$, defined in this work, is convex, as stated in the following result.

Corollary 1. *The sequence space* $[\mathcal{D}_{\lambda}^{w}, \mathcal{U}, \Delta^{p}]_{\infty}$ *is convex.*

P r o o f. Let (t_α), $(\varpi_\alpha) \in [\mathcal{D}_{\lambda}^{w}, \mathcal{U}, \Delta^{p}]_{\infty}$. Then, from the definition of the space, we can write

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p(t_\alpha))|}{v_t}\right) < \infty \quad \text{for some} \quad v_t > 0,$$

and

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p(\varpi_\alpha))|}{v_\varpi}\right) < \infty \quad \text{for some} \quad v_\varpi > 0.$$

For $\varrho = \mu t + (1-\mu)\varpi$, we have to show that

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p(\mu t_\alpha + (1-\mu)\varpi_\alpha))|}{v_\varrho}\right) < \infty \quad \text{for some} \quad v_\varrho > 0.$$

Since $\mathcal{U}$ is a convex function, we have

$$\mathcal{U}\left(\frac{|f(\Delta^p(\mu t_\alpha + (1-\mu)\varpi_\alpha))|}{v_\varrho}\right) \le \mu\mathcal{U}\left(\frac{|f(\Delta^p(t_\alpha))|}{v_t}\right) + (1-\mu)\mathcal{U}\left(\frac{|f(\Delta^p(\varpi_\alpha))|}{v_\varpi}\right),$$

where $v_\varrho = \mu v_t + (1-\mu)v_\varpi$.

Now, taking the limit as $s \to \infty$, we have

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p \varrho_\alpha)|}{v_\varrho}\right) \le \mu \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p(t_\alpha))|}{v_t}\right) + (1-\mu) \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p \varpi_\alpha)|}{v_\varpi}\right).$$

Therefore,

$$\varrho = \mu t + (1-\mu)\varpi \in [\mathcal{D}_{\lambda}^{w}, \mathcal{U}, \Delta^{p}]_{\infty}.$$

Hence, the space $[\mathcal{D}_{\lambda}^{w}, \mathcal{U}, \Delta^{p}]_{\infty}$ is convex. □

Theorem 3. *Let* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *be Orlicz functions satisfying the* Δ_2*-condition. Then the following strict inclusions hold*:

(i) $[\mathcal{D}_{\lambda}^{w}, \mathcal{U}_1, \Delta^{p}]_{\mathcal{K}} \subseteq [\mathcal{D}_{\lambda}^{w}, \mathcal{U}_2 \cdot \mathcal{U}_1, \Delta^{p}]_{\mathcal{K}}$;

(ii) $[\mathcal{D}_{\lambda}^{w}, \mathcal{U}_1, \Delta^{p}]_{\mathcal{K}} \cap [\mathcal{D}_{\lambda}^{w}, \mathcal{U}_2, \Delta^{p}]_{\mathcal{K}} \subseteq [\mathcal{D}_{\lambda}^{w}, \mathcal{U}_1 + \mathcal{U}_2, \Delta^{p}]_{\mathcal{K}}$, *where* $\mathcal{K} = 0, 1,$ *and* ∞.

P r o o f. We first prove the statement in the case $\mathcal{K} = 0$. The same methods can then be applied to the remaining cases.

(i) Let $(t_\alpha) \in [\mathcal{D}_{\lambda}^{w}, \mathcal{U}_1, \Delta^{p}]_{0}$. Then there exists $v > 0$ such that

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}_1\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) = 0.$$

Let $0 < \sigma < 1$ and $0 < \beta < 1$ be such that $\mathcal{U}_2(m) < \sigma$ for $0 \le m < \beta$.

Define

$$\varpi_\alpha = \mathcal{U}_1\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right).$$

Consider the expression

$$\sum_{\alpha\in I_s} \mathcal{U}_2(\varpi_\alpha) = \sum_{1} \mathcal{U}_2(\varpi_\alpha) + \sum_{2} \mathcal{U}_2(\varpi_\alpha),$$

where the first summation runs over terms with $\varpi_\alpha > \beta$ and the second summation includes terms with $\varpi_\alpha \le \beta$. Since

$$\frac{1}{\lambda_s}\sum_{1}\mathcal{U}_2\left(\varpi_\alpha\right) < \mathcal{U}_2(2)\frac{1}{\lambda_s}\sum_{1}\left(\varpi_\alpha\right) \tag{2.2}$$

for $\varpi_\alpha > \beta$, we have

$$\varpi_\alpha < 1 + \frac{\varpi_\alpha}{\beta}.$$

Since $\mathcal{U}_2$ is nondecreasing and convex, it follows that

$$\mathcal{U}_2\left(\varpi_\alpha\right) < \frac{1}{2}\mathcal{U}_2(2) + \frac{1}{2}\mathcal{U}_2\left(\frac{2\varpi_\alpha}{\beta}\right).$$

Since $\mathcal{U}_2$ satisfies the Δ_2-conditions, we have

$$\mathcal{U}_2\left(\varpi_\alpha\right) = T\frac{\varpi_\alpha}{\beta}\mathcal{U}_2(2).$$

Hence,

$$\frac{1}{\lambda_s}\sum_{2}\mathcal{U}_2\left(\varpi_\alpha\right) \le \max\left(1, T\beta^{-1}\mathcal{U}_2(2)\right)\frac{1}{\lambda_s}\sum_{2}\varpi_\alpha. \tag{2.3}$$

Taking the limit as $s \to \infty$, from (2.2) and (2.3), we obtain

$$(t_\alpha) \in \left[\mathcal{D}_\lambda^w, \mathcal{U}_2 \cdot \mathcal{U}_1, \Delta^p\right]_0.$$

A similar approach can be applied to demonstrate the result for the remaining cases.

(ii) The proof is standard and is omitted. □

By taking $\mathcal{U}_1(t) = t$ and $\mathcal{U}_2 = \mathcal{U}(t)$ in Theorem 3 (i), we obtain the following particular case.

Corollary 2. *The inclusion* $\left[\mathcal{D}_\lambda^w, \Delta^p\right]_0 \subseteq \left[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p\right]_0$ *is strict.*

Here, the space $\left[\mathcal{D}_\lambda^w, \Delta^p\right]_0$ is defined by

$$\left[\mathcal{D}_\lambda^w, \Delta^p\right]_0 = \left\{t = (t_\alpha) : \lim_{s\to\infty}\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) = 0 \text{ for some } v > 0\right\}.$$

Theorem 4. *Let* $p \ge 1$ *and* $\mathcal{K} = 1, 2, \infty$. *Then, the inclusion* $\left[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^{p-1}\right]_{\mathcal{K}} \subset \left[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p\right]_{\mathcal{K}}$ *is strict. In general,* $\left[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^i\right]_{\mathcal{K}} \subset \left[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p\right]_{\mathcal{K}}$ *for* $i = 0, 1, 2, \ldots, p-1$.

P r o o f. Let $(t_\alpha) \in \left[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^{p-1}\right]_0$. Then we have

$$\lim_{s\to\infty}\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\Delta^{p-1}t_\alpha)|}{v}\right) = 0 \quad \text{for some } v > 0. \tag{2.4}$$

Since $\mathcal{U}$ is both convex and nondecreasing, we can deduce that

$$\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{2v}\right) = \frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\Delta^{p-1}t_\alpha - \Delta^{p-1}t_{\alpha+1})|}{2v}\right)$$

$$\le \left(\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\Delta^{p-1}t_\alpha)|}{v}\right) - \frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\Delta^{p-1}t_{\alpha+1})|}{v}\right)\right).$$

As $s \to \infty$, we have

$$\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{2v}\right)=0$$

by (2.4), which implies $(t_\alpha) \in \left[\mathcal{D}^w_\lambda,\mathcal{U},\Delta^{p-1}\right]_0$.

The other cases will follow by a similar approach. Using induction, we can establish that

$$\left[\mathcal{D}^w_\lambda,\mathcal{U},\Delta^{i}\right]_{\mathcal{K}} \subset \left[\mathcal{D}^w_\lambda,\mathcal{U},\Delta^{p}\right]_{\mathcal{K}}$$

and $i = 0, 1, \ldots, p-1$. □

The following example directly illustrates this inclusion.

Example 1. Let $\lambda_s = (s)$ be a sequence and $\mathcal{U}(t) = t$. Consider the sequence $(t_\alpha) = \left(\alpha^{p-1}\right)$. Then

$$\Delta^p t_\alpha = 0, \quad \Delta^{p-1} t_\alpha = (-1)^{p-1}(p-1)!$$

for all $\alpha \in \mathbb{N}$. Therefore, $(t_\alpha) \in [\mathcal{D}^w_\lambda,\mathcal{U},\Delta^p]_0$ but $(t_\alpha) \notin \left[\mathcal{D}^w_\lambda,\mathcal{U},\Delta^{p-1}\right]_0$.

Theorem 5. *The space $[\mathcal{D}^w_\lambda,\mathcal{U},\Delta^p]_{\mathcal{K}}$, where $\mathcal{K} = 0, 1, \infty$, is generally not solid. The spaces $[\mathcal{D}^w_\lambda,\mathcal{U}]_0$ and $[\mathcal{D}^w_\lambda,\mathcal{U}]_\infty$ are solid.*

P r o o f. Let $(t_\alpha) \in [\mathcal{D}^w_\lambda,\mathcal{U}]_0$. Then there exists $v > 0$ such that

$$\lim_{s\to\infty}\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f\,(t_\alpha)|}{v}\right)=0.$$

Let (δ_α) be a sequence of scalars such that $|\delta_\alpha| \le 1$. Then, for each s, we can write

$$\begin{aligned}\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\delta_\alpha t_\alpha)|}{v}\right) &\le \frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(t_\alpha)|}{v}\right)\\ \Rightarrow \lim_{s\to\infty}\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f\,(\delta_\alpha t_\alpha)|}{v}\right)&=0\\ \Rightarrow (\delta_\alpha t_\alpha) \in [\mathcal{D}^w_\lambda,\mathcal{U}]_0\,.&\end{aligned} \tag{2.5}$$

From the inequality presented in (2.5), it follows that $[\mathcal{D}^w_\lambda,\mathcal{U}]_\infty$ is solid. □

To demonstrate that the spaces $[\mathcal{D}^w_\lambda,\mathcal{U},\Delta^p]_1$ and $[\mathcal{D}^w_\lambda,\mathcal{U},\Delta^p]_\infty$ are generally not solid, we provide the following example.

Example 2. Consider the function $f(t) = t$ for all $t \in \mathbb{R}$. Let $X = \mathbb{R}$ with $p = 1$. Let the sequence (t_α) be defined by $t_\alpha = \alpha$ for all $\alpha \in \mathbb{N}$. Let $\mathcal{U}(t) = t^r$ with $r \ge 1$, and $\lambda_s = (s)$. Then $(t_\alpha) \in [\mathcal{D}^w_\lambda,\mathcal{U},\Delta^p]_1$ and $(t_\alpha) \in [\mathcal{D}^w_\lambda,\mathcal{U},\Delta^p]_\infty$. Let $(\gamma_\alpha) = ((-1)^\alpha)$. Then $(\gamma_\alpha t_\alpha) \notin [\mathcal{D}^w_\lambda,\mathcal{U},\Delta^p]_1$ and $(\gamma_\alpha t_\alpha) \notin [\mathcal{D}^w_\lambda,\mathcal{U},\Delta^p]_\infty$.

The following example illustrates that $[\mathcal{D}^w_\lambda,\mathcal{U},\Delta^p]_0$ is generally not solid.

Example 3. Let $X = \mathbb{R}$ and consider the function $f(t) = t$ for all $t \in \mathbb{R}$. Let $p = 1$. Consider the sequence (t_α) defined by $t_\alpha = 1$ for all $\alpha \in \mathbb{N}$. Assume $\mathcal{U}(t) = t^r$ with $r = 2$ and $\lambda_s = (s)$. Let $(\gamma_\alpha) = ((-1)^\alpha)$ for all $\alpha \in \mathbb{N}$. Then $(\gamma_\alpha t_\alpha) \notin [\mathcal{D}^w_\lambda,\mathcal{U},\Delta^p]_0$. Thus, the set $[\mathcal{D}^w_\lambda,\mathcal{U},\Delta^p]_0$ is not solid.

The following result is a consequence of Lemma 1 and Theorem 5.

Corollary 3. *The spaces* $[\mathcal{D}_\lambda^w, \mathcal{U}]_0$ *and* $[\mathcal{D}_\lambda^w, \mathcal{U}]_\infty$ *are monotone.*

Remark 1. The space $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$ is not convergence free.

P r o o f. The following example clearly illustrates this point. □

Example 4. Let $p = 1$, $\mathcal{U} = t$ and consider the sequence $\lambda_s = (s)$. Consider the sequence (t_α) defined by

$$t_\alpha = \frac{1}{2}\left(1 - (-1)^\alpha\right).$$

Then $(t_\alpha) \in [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$.

Now consider the sequence (ϖ_α) defined by

$$\varpi_\alpha = \begin{cases} \alpha & \text{if} \quad \alpha \text{ is odd,} \\ \bar{\theta} & \text{if} \quad \alpha \text{ is even.} \end{cases}$$

Then $(\varpi_\alpha) \notin [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$.

Remark 2. The spaces $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\mathcal{K}$, where $\mathcal{K} = 0, 1, \infty$, are generally not symmetric. The following example illustrates this fact.

Example 5. Let $p = 1$, $X = \mathbb{R}$, and consider the function $f(t) = t$ for all $t \in \mathbb{R}$. Let $\mathcal{U}(t) = t$ and $\lambda_s = (s)$. Consider the sequence (t_α) defined by $t_\alpha = \alpha$ for all $\alpha \in \mathbb{N}$. Then $(t_\alpha) \in [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$. Now, rearrange the sequence (t_α) to obtain the sequence (ϖ_α) defined by

$$\varpi_\alpha = (t_1, t_2, t_4, t_3, t_9, \ldots).$$

Then $(\varpi_\alpha) \notin [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\mathcal{K}$, where $\mathcal{K} = 0, 1, \infty$. Hence, the spaces $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\mathcal{K}$, where $\mathcal{K} = 0, 1, \infty$, are generally not symmetric.

3. Conclusion

In this paper, we introduced and analyzed the concept of difference λ-weak convergence for sequences defined by an Orlicz function. Our study provided an in-depth examination of the algebraic and topological properties of these sequences, offering a foundational perspective on their structure and behavior. We also established key inclusion relationships between these newly defined spaces and existing sequence spaces, thereby enhancing the overall framework of sequence space theory. Our results contribute to the broader field of functional analysis, particularly in the context of sequence spaces and Orlicz functions, and open new avenues for future research.

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