ON INTERPOLATION BY ALMOST TRIGONOMETRIC SPLINES[1]

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Abstract: The existence and uniqueness of an interpolating periodic spline defined on an equidistant mesh by the linear differential operator $\mathcal{L}_{2n+2}(D) = D^2(D^2 + 1^2)(D^2 + 2^2)\cdots(D^2 + n^2)$ with $n \in \mathbb{N}$ are reproved under the final restriction on the step of the mesh. Under the same restriction, sharp estimates of the error of approximation by such interpolating periodic splines are obtained.

Key words: Splines, Interpolation, Approximation, Linear differential operator.

Introduction

Let $D = d/dx$, $n \in \mathbb{N}$, and let

$$\mathcal{L}_{2n+2}(D) = D^2(D^2 + 1^2)(D^2 + 2^2)\cdots(D^2 + n^2) \tag{0.1}$$

be the $(2n + 2)$th-order linear differential operator with constant real coefficients. We denote the characteristic polynomial of $\mathcal{L}_{2n+2}(D)$ by p_{2n+2}, and let $T_{2n+2} = \{0, 0, \pm i, \ldots, \pm in\}$ be the set of its zeros, with each zero repeated according to its multiplicity, where i is the imaginary unit. The kernel of the differential operator (0.1) is the linear space spanned by the system of functions $\{1, x, \sin x, \cos x, \ldots, \sin nx, \cos nx\}$.

Denote by $\mathbb{T}$ the circumference considered as the interval $[0, 2\pi]$ with identified ends, and let $\|\cdot\|_{L_p(\mathbb{T})} = \|\cdot\|_p$ $(1 \leq p \leq \infty)$ with the usual modification in the case $p = \infty$.

We associate with the differential operator $\mathcal{L}_{2n+2}(D)$ the standard class of differentiable functions

$$W_\infty(\mathcal{L}_{2n+2}) = \{f \in C^{(2n+1)}(\mathbb{T}): \ f^{(2n+1)} \text{ is abs. cont., } \|\mathcal{L}_{2n+2}(D)f\|_\infty \leq 1\}.$$

Let $N \in \mathbb{N}$ and $h = \pi/N$. Denote by $\Delta_N = \{jh : j = 0, 1, \ldots, 2N - 1\}$ the uniform mesh on $[0, 2\pi)$ which can be extended on $\mathbb{R}$ if required; h is the step of the mesh.

We say that a 2π-periodic function s_{2n+2} is a periodic *almost trigonometric spline* with knots at the points of Δ_N if s_{2n+2} satisfies the following conditions:
1) $s_{2n+2} \in C^{(2n)}(\mathbb{T})$,
2) $\mathcal{L}_{2n+2}(D)s_{2n+2}(x) = 0 \quad \forall x \in (jh, (j+1)h), \quad j \in \mathbb{Z}$.

The set of all almost trigonometric splines is denoted by $S(\mathcal{L}_{2n+2}, \Delta_N)$.

Almost trigonometric splines are a special case of the large family of $\mathcal{L}$-splines defined by linear differential operators (see [2], [3], [8], and others).

The term "almost trigonometric spline" is explained by the fact that such a spline is formed by functions which differ from trigonometric polynomials for only one addend ax, where a is some

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constant. This term is not standard, and we use it only not to specify every time by what differential operator the considered splines are defined.

We interpolate at the knots of the mesh Δ_N by elements from $S(\mathcal{L}_{2n+2}, \Delta_N)$; i.e., for every bounded $2N$-periodic sequence $y = \{y_\nu : \nu \in \mathbb{Z}\}$, $y_\nu = y_{\nu+2N}$, we consider the interpolation problem: to find $s \in S(\mathcal{L}_{2n+2}, \Delta_N)$ such that $s(\nu h) = y_\nu, \nu \in \mathbb{Z}$.

For interpolation by polynomial splines, the existence, uniqueness and estimates of the error of approximation in many classes of functions are well-known (see, for instance, [1, Ch. V], [11], [12], [13], and references therein).

The existence and uniqueness of periodic interpolating L-splines corresponding to an arbitrary linear differential operator with constant real coefficients were established in [10]. As far as almost trigonometric splines are concerned the result in [10] means that if $N > n$, then for every bounded $2N$-periodic interpolated sequence, there exists a unique interpolating almost trigonometric spline.

In the present paper, we give another proof of this result and observe such an important feature that the inequality $N > n$ cannot be replaced by a weaker one (Theorem 1). After this, for $N > n$, we obtain a sharp estimate of the error of pointwise approximation by periodic interpolating almost trigonometric splines in the class of functions $W_\infty(\mathcal{L}_{2n+2})$ (Theorem 2).

Theorem 1. *If $N > n$, then, for every bounded $2N$-periodic sequence $\{y_\nu\}_{\nu\in\mathbb{Z}}$, $y_\nu = y_{\nu+2N}$, there exists a unique $s \in S(\mathcal{L}_{2n+2}, \Delta_N)$ such that $s(\nu h) = y_\nu, \nu \in \mathbb{Z}$.*

If $N \leq n$, then periodic interpolating almost trigonometric spline cannot exist.

Let $N > n$. We set

$$A_n(x) = \frac{x(x-h)}{4(n!)^2} + 2\sum_{\nu=1}^{n} \frac{(-1)^\nu \sin\dfrac{\nu x}{2} \ \sin\dfrac{\nu(x-h)}{2}}{\nu^2 \ (n-\nu)! \ (n+\nu)! \ \cos\dfrac{\nu h}{2}} \tag{0.2}$$

for $0 \leq x \leq h$ and extend $A_n(x)$ to the whole real line by the equality $A_n(x+h) = -A_n(x)$ for $x \in \mathbb{R}\backslash[0,h]$.

We show that $A_n \in C^{(2n+1)}(\mathbb{T})$. In the class $W_\infty(\mathcal{L}_{2n+2})$, the deviation from the periodic interpolating almost trigonometric splines is estimated by this function.

Theorem 2. *If $N > n$, then, for every function $f \in W_\infty(\mathcal{L}_{2n+2})$, the inequality*

$$|f(x) - s(f)(x)| \leq 2|A_n(x)| \tag{0.3}$$

holds at any point $x \in \mathbb{R}$. The inequality turns into an equality for $f(x) = 2A_n(x)$.

For interpolation by periodic polynomial splines, inequality (0.3) was proved by Tikhomirov [12]. For $N > 3^{n-1}n$, inequality (0.3) is a particular case of the author's result [3]. For periodic trigonometric splines, the corresponding result was established by Nguen [5], [6, Ch. 2, §6].

1. Auxiliary results

First, we study the properties of the function $A_n(x)$.

Lemma 1. *If $N > n$, then*

$$A_n^{(j)}(x)\mid_{x=h} = -A_n^{(j)}(x)\mid_{x=0}, \quad j = 1, 3, \ldots, 2n+1,$$

and

$$A_n^{(j)}(x)\mid_{x=h} = A_n^{(j)}(x)\mid_{x=0} = 0, \quad j = 0, 2, \ldots, 2n.$$

P r o o f. By easy calculations, we verify that $A_n(h) = A_n(0) = 0$ and $A'_n(x) \mid_{x=h} = -A'_n(x) \mid_{x=0}$. Further,

$$A''_n(x) \mid_{x=h} = A''_n(x) \mid_{x=0} = \frac{1}{2(n!)^2} - \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}}{(n-\nu)! \, (n+\nu)!}.$$

Using the known identity [7, Ch.IV, § 4.2.1, eq. 4], we obtain

$$\sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}}{(n-\nu)! \, (n+\nu)!} = \sum_{m=0}^{n-1} \frac{(-1)^{m+n-1}}{m! \, (2n-m)!} = \frac{(-1)^{n-1}}{(2n)!} \sum_{m=0}^{n-1} (-1)^m \binom{2n}{m} = \frac{1}{2(n!)^2}.$$

From this, it follows that $A''_n(x) \mid_{x=h} = A''_n(x) \mid_{x=0} = 0$.

For $j = 3, 4, \ldots, 2n+1$, we have

$$A_n^{(j)}(x) = \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1} \nu^{j-2} \cos(\nu(x-h/2) + \pi j/2)}{(n-\nu)! \, (n+\nu)! \, \cos(\nu h/2)}.$$

For $j = 2k+1$ $(k = 1, 2, \ldots, n)$, easy calculations yield

$$A_n^{(2k+1)}(x) \mid_{x=h} = -A_n^{(2k+1)}(x) \mid_{x=0} = (-1)^k \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1} \, \nu^{2k-1} \, \tan(\nu h/2)}{(n-\nu)! \, (n+\nu)!}.$$

For $j = 2k$ $(k = 2, 3, \ldots, n)$, we obtain

$$A_n^{(2k)}(x) \mid_{x=h} = A_n^{(2k)}(x) \mid_{x=0} = (-1)^k \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1} \, \nu^{2k-2}}{(n-\nu)! \, (n+\nu)!}$$

$$= \frac{(-1)^{n+k}}{(2n)!} \sum_{m=0}^{n-1} (-1)^m (n-m)^{2k-2} \binom{2n}{m} = 0.$$

Here, we used the identity [7, Ch.IV, § 4.2.2, eq. 34]. The lemma is proved. □

We now extend the function $A_n(x)$ from $[0, h]$ to the whole real line by setting $A_n(x+h) = -A_n(x)$. Lemma 1 gives that A_n belongs to $C^{(2n+1)}(\mathbb{R})$ and is 2π-periodic.

Lemma 2. If $N > n$, then $\mathcal{L}_{2n+2}(D)(2A_n(x)) = \operatorname{sign} \sin Nx$, $x \in \mathbb{R}$.

P r o o f. Let $0 \le x \le h$. Since

$$\sin \frac{\nu x}{2} \, \sin \frac{\nu(x-h)}{2} = A_\nu \cos \nu x + B_\nu \sin \nu x + C_\nu, \quad \nu = 1, 2, \ldots, n,$$

where A_ν, B_ν and C_ν are independent of x, the sum on the right-hand side of (0.2) vanishes by the differential operator $D(D^2 + 1^2)(D^2 + 2^2) \cdots (D^2 + n^2)$. Taking into account that the factors on the right-hand side of (0.1) can be rearranged, we obtain

$$D^2(D^2 + 1^2) \cdots (D^2 + n^2) \left(\frac{x(x-h)}{2(n!)^2} \right) = \frac{1}{2(n!)^2} (D^2 + 1^2) \cdots (D^2 + n^2)(x(x-h))'' = 1.$$

For $x \in \mathbb{R} \backslash [0, h]$, we use the equality $A_n(x+h) = -A_n(x)$. □

The next statement is a special case of a result proved in [9] for an arbitrary linear differential operator with constant real coefficients.

Lemma 3. *If $N > n$, then $x = 0$ is the unique zero of $A_n(x)$ in $[0, h)$ and this zero is simple.*

P r o o f. By Lemma 1, $A_n(0) = 0$. Moreover, the function $A_n(x)$ coincides, up to a nonzero constant, with some function $P_n(x)$ introduced in [9]. It was proved in [9] that if $N > n$, then $P_n(x)$ has a unique zero in $[0, h)$ and this zero is simple. Therefore, $A_n(x)$ has the same property. $\square$

To prove our two theorems, we also need the periodic analog of the Rolle theorem on the relation between the number of zeros of a smooth function $\varphi(x)$ and the number of sign changes of $D(D^2 + 1^2)(D^2 + 2^2) \cdots (D^2 + n^2)\varphi(x)$ on $\mathbb{T}$.

We say that a continuous function f changes sign at some point t_0 if the inequality $f(t_0 - \varepsilon)f(t_0 + \varepsilon) < 0$ holds for all sufficiently small $\varepsilon > 0$. If f has a jump at the point t_0, then, instead of $f(t_0 - \varepsilon)$ and $f(t_0 + \varepsilon)$, we write $\lim_{t \to t_0 - 0} f(t)$ and $\lim_{t \to t_0 + 0} f(t)$, respectively. Denote by $Z(f, \mathbb{T})$ the number of zeros of the function f on $\mathbb{T}$, and by $\nu(f, \mathbb{T})$ the number of sign changes of $f \not\equiv 0$ on $\mathbb{T}$ (the number of sign changes of $f \equiv 0$ is not defined). The number $\nu(f, \mathbb{T})$ is always even. We denote by $G(\mathbb{T})$ the set of 2π-functions of bounded variation with a finite number of jumps on the period and absolutely continuous on all intervals of continuity. We also denote by $G_m(\mathbb{T})$ the set of 2π-periodic functions whose derivatives of order $m - 2$ are absolutely continuous on $\mathbb{T}$ and $f^{(m-1)} \in G(\mathbb{T})$. Let $\mathcal{T}_n$ be the set of trigonometric polynomials of order at most n.

Lemma 4. *For every function $f \in G_{2n+1}(\mathbb{T}) \backslash \mathcal{T}_n$, the following inequality holds:*

$$\nu\big(D(D^2 + 1^2)(D^2 + 2^2) \cdots (D^2 + n^2)f, \mathbb{T}\big) \geq Z(f, \mathbb{T}).$$

This result was established by Nguen [5] (see also [6]) and is the periodic analog of the Rolle theorem for the trigonometric differential operator.

Note that the periodic analog of the Rolle theorem in the form of Lemma 4 exists not for any linear differential operator. More detailed information on some results and open problems in this area can be found in [4] and references therein.

2. Proofs of Theorems

We now pass directly to the proofs of Theorems 1 and 2.

P r o o f of Theorem 1. Let $N > n$. We prove that if $s \in S(\mathcal{L}_{2n+2}, \Delta_N)$ and $s(jh) = 0$ $\forall j \in \mathbb{Z}$, then $s \equiv 0$. After this, the existence and uniqueness of the interpolating periodic almost trigonometric spline for every interpolated periodic sequence is a simple consequence of the Kramer theorem for the corresponding system of linear algebraic equations.

Suppose that there exist $s_1, s_2 \in S(\mathcal{L}_{2n+2}, \Delta_N)$ such that $s_k(jh) = 0$ $\forall j \in \mathbb{Z}$ $(k = 1, 2)$ and $s_1 \not\equiv s_2$. This means that there is a point $x_* \notin \Delta_N$ such that $s_1(x_*) \neq s_2(x_*)$. Let $s_1(x_*) \neq 0$ and $C = s_2(x_*)/s_1(x_*)$. Then the function $\varphi(x) = Cs_1(x) - s_2(x)$ has the following properties:

1) $\varphi \in S(\mathcal{L}_{2n+2}, \Delta_N)$;
2) $\varphi(jh) = 0, \quad j = 0, 1, \ldots, 2N - 1$;
3) $\varphi(x_*) = 0$.

Thus, $\varphi(x)$ has at least $2N + 1$ zeros on the period. From Lemma 4, we have

$$\nu(D(D^2 + 1^2)(D^2 + 2^2) \cdots (D^2 + n^2)\varphi, \mathbb{T}) \geq 2N + 1.$$

But $D(D^2 + 1^2)(D^2 + 2^2) \cdots (D^2 + n^2)\varphi(x)$ is a piecewise constant function with possible jumps at the points of the mesh Δ_N. Therefore, this function cannot change sign more than $2N$ times on $\mathbb{T}$. We have a contradiction from which it easily follows that $s_1 \equiv s_2 \equiv 0$.

The inequality $N > n$ cannot be replaced by a weaker one. Indeed, if $N = n$, then the function $\sin nx$ interpolates the sequence $y \equiv 0$ at the points of Δ_N. This function lies in the kernel of the linear differential operator (0.1) and can be interpreted as an element of the space $S(\mathcal{L}_{2n+2}, \Delta_N)$. Theorem 1 is proved. $\qquad\square$

P r o o f of Theorem 2 is based on the ideas of [12]. Let $N > n$. Suppose that (0.3) fails; i.e., there exist a point $x_* \in [0, 2\pi)$ and a function $f \in W_\infty(\mathcal{L}_{2n+2})$ such that the inequality

$$|f(x_*) - s(f)(x_*)| > 2|A_n(x_*)|$$

holds. Define $\delta(x) = f(x) - s(f)(x)$. This function is zero at the points of the mesh Δ_N. According to Lemma 3, the function $A_n(x)$ vanishes at the same points. From these facts, we have $x_* \notin \Delta_N$. Therefore, there is a number λ, $0 < |\lambda| < 1$, such that $\lambda\delta(x_*) = 2A_n(x_*)$.

We now introduce the function $\Delta(x) = \lambda\delta(x) - 2A_n(x)$. It is zero at all points of the set $\Delta_N \cup \{x_*\}$ and possibly also at some other points. Therefore $Z(\Delta(x), \mathbb{T}) \geq 2N + 1$. It is clear that $\Delta \in G_{2n+1}(\mathbb{T}) \backslash \mathcal{T}_n$. We denote $\mathcal{L}_{2n+1}(D) = D(D^2 + 1^2)(D^2 + 2^2) \cdots (D^2 + n^2)$, apply Lemma 4, and obtain

$$\nu\left(\mathcal{L}_{2n+1}(D)\Delta(x), \mathbb{T}\right) \geq 2N + 1. \tag{2.1}$$

From (0.1) and the definition of almost trigonometric splines, we have the equalities $\mathcal{L}_{2n+1}(D)\delta(x) = \mathcal{L}_{2n+1}(D)f(x) - c_j$ on every interval $[jh, (j+1)h)$, $j = 0, 1, \ldots, 2N - 1$, where c_j are constants. Using the Lagrange finite increments formula and the inequality $|\lambda| < 1$, we obtain

$$|\mathcal{L}_{2n+1}(D)(\lambda\delta(t')) - \mathcal{L}_{2n+1}(D)(\lambda\delta(t''))| < |\mathcal{L}_{2n+1}(D)f(t') - \mathcal{L}_{2n+1}(D)f(t'')|$$

$$= |\mathcal{L}_{2n+2}(D)f(\xi)| \cdot |t' - t''| \ \leq |t' - t''|$$

on an arbitrary subinterval $[t', t''] \subset [jh, (j+1)h)$ for every interpolated function of our class. From (0.2), it follows that $\mathcal{L}_{2n+1}(D)(2A_n(x)) = x - h/2 \ \forall x \in [0, h)$. Hence, $|t' - t''| = |\mathcal{L}_{2n+1}(D)(2A_n(t')) - \mathcal{L}_{2n+1}(D)(2A_n(t''))|$. Thus,

$$|\mathcal{L}_{2n+1}(D)(\lambda\delta(t')) - \mathcal{L}_{2n+1}(D)(\lambda\delta(t''))| < |\mathcal{L}_{2n+1}(D)(2A_n(t')) - \mathcal{L}_{2n+1}(D)(2A_n(t''))|.$$

It is easy to see that if $|a| < |b|$, then $\text{sign}(b - a) = \text{sign}\, b$. Applying this fact, we come to the conclusion that the function $\mathcal{L}_{2n+1}(D)\Delta(x)$ changes sign no more than once in every interval $[jh, (j+1)h)$. If $\mathcal{L}_{2n+1}(D)\Delta(x)$ changes sign at the point jh (this is possible if the function is discontinuous at jh), then $\mathcal{L}_{2n+1}(D)\Delta(x)$ preserves sign in one of two adjacent intervals $((j-1)h, jh)$ or $(jh, (j+1)h)$. Thus, we arrive at the inequality

$$\nu\left(\mathcal{L}_{2n+1}(D)\Delta(x), \mathbb{T}\right) \leq 2N.$$

The obtained inequality contradicts to (2.1). The simple observation that inequality (0.3) turns into an equality for $f = 2A_n$ completes the proof. $\qquad\square$

Corollary 1. *If $N > n$, then*

$$\sup_{f \in W_\infty(\mathcal{L}_{2n+2})} \|f - s(f)\|_p = 2\|A_n\|_p, \quad 1 \leq p < \infty,$$

and

$$\sup_{f \in W_\infty(\mathcal{L}_{2n+2})} \|f - s(f)\|_\infty = \left| \frac{h^2}{8(n!)^2} + 4 \sum_{\nu=1}^{n} \frac{(-1)^\nu \sin^2 \dfrac{\nu h}{4}}{\nu^2 \, (n - \nu)! \, (n + \nu)! \, \cos \dfrac{\nu h}{2}} \right|.$$

We now consider separately the case $n = 1$, i.e., the case of $\mathcal{L}$-splines corresponding to the differential operator $\mathcal{L}_4(D) = D^2(D^2 + 1)$. They belong piecewise to the space $span\{1, t, \sin t, \cos t\}$. These splines generalize the well-known cubic splines and have many applications in numerical analysis for the shape preserving approximation, the description of curves and their parametrization, and other problems (see, for instance, [14], [15], [16], and references therein). In particular (see [15]) these splines are attractive from a geometrical point of view, because they are able to provide parameterizations of conic sections with respect to the arc length so that equally spaced points in the parameter domain correspond to equally spaced points on the described curve.

The restriction on the grid step is the least strong here: $h \leq \pi/2$, and the "minimal" equidistant grid on the period is $\{0, \pi/2, \pi, 3\pi/2\}$. Theorem 1 gives the existence and uniqueness of spline interpolants for $N \geq 2$. According to Corollary 1, the error of approximation in the class $W_\infty(\mathcal{L}_4)$ is

$$
\sup_{f \in W_\infty(\mathcal{L}_4)} \|f - s(f)\|_\infty = \left| 1 + \frac{\pi^2}{8N^2} - \frac{1}{\cos\dfrac{\pi}{2N}} \right|.
$$

3. Conclusion

We established that, for 2π-periodic $\mathcal{L}$-splines corresponding to the differential operator (0.1) on the equidistant mesh with the step $h = \pi/N$, the restriction $N > n$ provides the existence and uniqueness of the $\mathcal{L}$-spline interpolant as well as the exact estimates of the error of approximation. This restriction is final, i.e., cannot be replaced by a weaker one.

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