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TWO METHODS OF DESCRIBING 2-LOCAL DERIVATIONS AND AUTOMORPHISMS

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Abstract: In the present paper, we investigate 2-local linear operators on vector spaces. Sufficient conditions are obtained for the linearity of a 2-local linear operator on a finite-dimensional vector space. To do this, families of matrices of a certain type are selected and it is proved that every 2-local linear operator generated by these families is a linear operator. Based on these results we prove that each 2-local derivation of a finite-dimensional null-filiform Zinbiel algebra is a derivation. Also, we develop a method of construction of 2-local linear operators which are not linear operators. To this end, we select matrices of a certain type and using these matrices we construct a 2-local linear operator. If these matrices are distinct, then the 2-local linear operator constructed using these matrices is not a linear operator. Applying this method we prove that each finite-dimensional filiform Zinbiel algebra has a 2-local derivation that is not a derivation. We also prove that each finite-dimensional naturally graded quasi-filiform Leibniz algebras of type I has a 2-local automorphism that is not an automorphism.

Keywords: Linear operator, 2-Local linear operator, Leibniz algebra, Zinbiel algebra, Derivation, 2-Local derivations, Automorphism, 2-Local automorphism

1. Introduction

In 1997, P. Šemrl [20] introduced and investigated so-called 2-local derivations and 2-local automorphisms on operator algebras. He described such maps on the algebra $B(H)$ of all bounded linear operators on an infinite-dimensional separable Hilbert space H. Namely, he proved that every 2-local derivation (automorphism) on $B(H)$ is a derivation (respectively an automorphism).

A similar description of 2-local derivations for the finite-dimensional case appeared later in [17]. In the paper [19] 2-local derivations have been described on matrix algebras over finite-dimensional division rings. In [9] Sh. Ayupov and K. Kudaybergenov suggested a new technique and have generalized the above-mentioned results of [20] and [17] for arbitrary Hilbert spaces. Namely, they proved that every 2-local derivation on the algebra $B(H)$ of all linear bounded operators on an arbitrary Hilbert space H is a derivation. They obtained also a similar result for the automorphisms. In [4, 10] the authors extended the above results for 2-local derivations and gave a proof of the theorem for arbitrary von Neumann algebras.

Afterwards, 2-local derivations have been investigated by many authors on different algebras and many results have been obtained. In [15] it was established that every 2-local $*$-homomorphism from a von Neumann algebra into a C*-algebra is a linear $*$-homomorphism. These authors also proved that every 2-local Jordan $*$-homomorphism from a JBW*-algebra into a JB*-algebra is a Jordan $*$-homomorphism. Later, in [14] the authors prove that any 2-local automorphism on an arbitrary AW*-algebra without finite type I direct summands is an automorphism.

In the paper [11] 2-local derivations of finite-dimensional Lie algebras are described. The authors proved that every 2-local derivation on a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero is a derivation. They also showed that each finite-dimensional nilpotent Lie algebra L with $\dim L \geq 2$ admits a 2-local derivation which is not a derivation. At the same time, in [18] X. Lai and Z.X. Chen describe 2-local Lie derivations for the case of finite-dimensional simple Lie algebras.

In the paper [12] the authors proved that every 2-local automorphism on a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero is an automorphism and showed that each finite-dimensional nilpotent Lie algebra with dimension ≥ 2 admits a 2-local automorphism which is not an automorphism. Later, in [13] similar results were obtained in the case of finite-dimensional Leibniz algebras. Many papers were devoted to 2-local derivations and automorphisms on Lie and Leibniz algebras. In particular, in the paper [6]it was proven that every 2-local inner derivation on the Lie ring of skew-symmetric matrices over a commutative ring is an inner derivation. They also proved that every 2-local spatial derivation on various Lie algebras of infinite-dimensional skew-adjoint matrix-valued maps on a set is a spatial derivation. In [8] the previous results were extended of the Lie ring of skew-adjoint matrices over a commutative $*$-ring and various Lie algebras of skew-adjoint operator-valued maps on a set, respectively.

In [5] 2-local inner derivations on the Jordan ring $H_n(\Re)$ of symmetric $n \times n$ matrices over a commutative associative ring $\Re$ were investigated. It was proven that every such 2-local inner derivation is a derivation. In the paper [7], the authors introduced and investigated the notion of 2- local linear maps on vector spaces. A sufficient condition was obtained for the linearity of a 2-local linear map on a finite-dimensional vector space. Based on this result, the authors proved that every 2-local inner derivation on finite-dimensional semi-simple Jordan algebras over an algebraically closed field of characteristics different from 2 and a field of characteristics 0 is a derivation. Also, they showed that every 2-local 1-automorphism (i.e. implemented by single symmetries) of the mentioned Jordan algebra is an automorphism.

The present paper is devoted to 2-local linear operators, 2-local derivations and automorphisms on finite-dimensional vector spaces, Leibniz and Zinbiel algebras. This paper is organized as follows:

In Section 2, we investigate 2-local linear operators on vector spaces. Sufficient conditions are obtained for the linearity of a 2-local linear operator on a finite-dimensional vector space. To do this, families of matrices of a certain type are selected and it is proved that every 2-local linear operator generated by these families is a linear operator.

In Section 3, we develop a method of construction of 2-local linear operators which are not linear operators. For this purpose we select matrices of a certain type and using these matrices we construct a 2-local linear operator. If these matrices are distinct, then the 2-local linear operator constructed using these matrices is not a linear operator.

In Section 4, basing on the results of Section 2 we describe 2-local derivations of finite-dimensional null-filiform Zinbiel algebras. Namely, we prove that each 2-local derivation of a finite-dimensional null-filiform Zinbiel algebra is a derivation. Also, applying the method of Section 3 we prove that n-dimensional filiform Zinbiel algebras, $n \geq 5$, have 2-local derivations that are not derivations.

In Section 5, applying the method of Section 3 we prove that finite-dimensional naturally graded quasi-filiform Leibniz algebras of type I have 2-local automorphisms which are not automorphisms.

2. 2-Local liner operators of finite-dimensional vector spaces which are liner operators

Definition 1. *Let V be a vector space over a field $\mathbb{F}$, $\Delta : V \to V$ be a map such that for each pair v, w of elements in V there exists a linear operator $L_{v,w}$ of V satisfying the following conditions*

$$\Delta(v) = L_{v,w}(v), \quad \Delta(w) = L_{v,w}(w).$$

Then Δ is called a 2-local linear operator.

Definition 2. *Let V be a vector space of dimension n over a field $\mathbb{F}$, and let $\nu = \{e_1, e_2, \ldots e_n\}$ be a basis of the vector space V. Let $\mathcal{M}$ be a set of $n \times n$ matrices. Then a mapping $\Delta : V \to V$ is called a 2-local linear operator generated by matrices in $\mathcal{M}$, if, for each pair v and w of elements in V, there exists a linear operator $L_{v,w}$ generated by a matrix in $\mathcal{M}$ with respect to ν such that*

$$\Delta(v) = L_{v,w}(v), \quad \Delta(w) = L_{v,w}(w).$$

Let n and m be natural numbers such that $m \leq n$. Let, for fixed k, p such that $1 \leq k \leq n$, $1 \leq p \leq m$,

$$f_{ij}(x_1, x_2, \ldots, x_p), \quad i = 1, 2, \ldots, m, \quad j \neq k, \quad j = 1, 2, \ldots, n,$$

be functions with values in a field $\mathbb{F}$ (including the function $f_{ij}(x_1, x_2, \ldots, x_p) \equiv 0$),

$$g_i(x_1, x_2, \ldots, x_p), \quad i = 1, 2, \ldots m,$$

be functions with values in the field $\mathbb{F}$ such that, for any nonzero elements $\{a_1, a_2, \ldots, a_p\} \subset \mathbb{F}$, the following system of equations

$$g_i(x_1, x_2, \ldots, x_p) = g_i(a_1, a_2, \ldots, a_p), \quad i = 1, 2, \ldots m,$$

has a unique solution $x_j = a_j$, $j = 1, 2, \ldots, p$, and let $\mathcal{M}_{m,n}(k, p)$ be a set of $m \times n$ matrices A with components a_{ij} such that, there exist nonzero elements $a_i \in \mathbb{F}$, $i = 1, 2, \ldots, p$, satisfying the following equalities

$$a_{ik} = g_i(a_1, a_2, \ldots, a_p), \quad i = 1, 2, \ldots m,$$
$$a_{ij} = f_{ij}(a_1, a_2, \ldots, a_p), \quad i = 1, 2, \ldots m, \quad j \neq k.$$

Remark 1. Note that, in the definition of the set $\mathcal{M}_{m,n}(k, p)$ components of every matrix A in $\mathcal{M}_{m,n}(k, p)$ are computed using some nonzero elements $a_i \in \mathbb{F}$, $i = 1, 2, \ldots, p$.

Also, note that, by the definition of the set $\mathcal{M}_{m,n}(k, p)$, a matrix of this set may contain a row, all components of which are zeros, since $p \leq m$.

Theorem 1. *Let V be a vector space of dimension n over the field $\mathbb{F}$, and let $\nu = \{e_1, e_2, \dots e_n\}$ be a basis of the vector space V. Let Δ be a 2-local linear operator on V generated by matrices in $\mathcal{M}_{n,n}(k,p)$ with respect to the basis ν. Then Δ is a linear operator generated by a matrix in $\mathcal{M}_{n,n}(k,p)$ with respect to the basis ν.*

P r o o f. Without loss of the generality, we suppose that $k = 1$. Indeed, matrices in $\mathcal{M}_{n,n}(k,p)$ depend on the basis $\nu = \{e_1, e_2, \dots e_n\}$. If we swap the vectors e_1 and e_k, then we get the set of matrices $\mathcal{M}_{n,n}(1,p)$, i.e., $k = 1$. By the definition, for every element $x \in V$,

$$x = \sum_{i=1}^{n} x_i e_i,$$

there exists a matrix $A_{x,e_1} = (a_{ij}^{x,e_1})_{i,j=1}^{n}$ in $\mathcal{M}_{n,n}(1,p)$ such that

$$\Delta(x) = \widehat{A_{x,e_1}\bar{x}},$$

where $\bar{x} = (x_1, x_2, \dots, x_n)^T$ is the vector corresponding to x, $\widehat{\bar{x}}$ is an operation on $\bar{x}$ such that $\widehat{\bar{x}} = x$, and

$$\overline{\Delta(e_1)} = A_{x,e_1}\overline{e_1} = (a_{11}^{x,e_1}, a_{21}^{x,e_1}, a_{31}^{x,e_1}, \dots, a_{n1}^{x,e_1})^T.$$

Since $\Delta(e_1) = L_{x,e_1}(e_1) = L_{y,e_1}(e_1)$, we have

$$\overline{\Delta(e_1)} = (a_{11}^{x,e_1}, a_{21}^{x,e_1}, a_{31}^{x,e_1}, \dots, a_{n1}^{x,e_1})^T = (a_{11}^{y,e_1}, a_{21}^{y,e_1}, a_{31}^{y,e_1}, \dots, a_{n1}^{y,e_1})^T$$

for each pair x, y of elements in V. Hence, $a_{q1}^{x,e_1} = a_{q1}^{y,e_1}$, $q = 1, 2, \dots n$. By the condition, there exist nonzero elements a_i^{x,e_1}, $a_i^{y,e_1} \in \mathbb{F}$, $i = 1, 2, \dots, p$ such that

$$a_{q1}^{x,e_1} = g_i(a_1^{x,e_1}, a_2^{x,e_1}, \dots, a_p^{x,e_1}), \quad i = 1, 2, \dots n,$$
$$a_{q1}^{y,e_1} = g_i(a_1^{y,e_1}, a_2^{y,e_1}, \dots, a_p^{y,e_1}), \quad i = 1, 2, \dots n.$$

So, we have

$$g_i(a_1^{x,e_1}, a_2^{x,e_1}, \dots, a_p^{x,e_1}) = g_i(a_1^{y,e_1}, a_2^{y,e_1}, \dots, a_p^{y,e_1}), \quad i = 1, 2, \dots n.$$

By the definition of g_i, $i = 1, 2, \dots n$, we have

$$a_i^{x,e_1} = a_i^{y,e_1}, \quad i = 1, 2, \dots p.$$

By the condition, for every component a_{ij}^{z,e_1}, $j \neq 1$, of A_{z,e_1} we have

$$a_{ij}^{z,e_1} = f_{ij}(a_1^{z,e_1}, a_2^{z,e_1}, \dots, a_p^{z,e_1}), \quad i = 1, 2, \dots n, j \neq 1.$$

where $z \in \{x, y\}$. Therefore $a_{ij}^{x,e_1} = a_{ij}^{y,e_1}$, $i, j = 1, 2, \dots n$, i.e. $A_{x,e_1} = A_{y,e_1}$, and

$$\Delta(x) = \widehat{A_{y,e_1}\bar{x}}$$

for any $x \in V$, and the matrix of $\Delta(x)$ does not depend on x. Hence Δ is a linear operator, and the matrix A_{y,e_1} is the matrix of Δ. The proof is complete. □

Let n be a natural number, and let $\{i_1, i_2, \dots i_p\}$ and $\{j_1, j_2, \dots j_q\}$ be subsets of $\{1, 2, \dots, n\}$ such that

$$p + q = n, \quad \{i_1, i_2, \dots i_p\} \cup \{j_1, j_2, \dots j_q\} = \{1, 2, \dots, n\}.$$

Let, for fixed k, m, l and s such that $1 \leq k, m, l, s \leq n$, $k \neq m$,

$$\mathcal{M}_n(k, m, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$$

be a set of $n \times n$ matrices $A = (a_{ij})_{i,j=1}^n$ such that the $p \times n$ submatrix

$$A_1 : a_{\alpha\beta}, \alpha \in \{i_1, i_2, \dots i_p\}, \quad \beta = 1, 2, \dots, n,$$

belongs to the set $\mathcal{M}_{p,n}(k, l)$ and the $q \times n$ submatrix

$$A_2 : a_{\alpha\beta}, \alpha \in \{j_1, j_2, \dots j_q\}, \quad \beta = 1, 2, \dots, n,$$

belongs to the set $\mathcal{M}_{q,n}(m, s)$. Then the following theorem takes place.

Theorem 2. *Let V be a vector space of dimension n over the field $\mathbb{F}$, and let $\nu = \{e_1, e_2, \dots e_n\}$ be a basis of the vector space V. Let Δ be a 2-local linear operator on V generated by matrices in $\mathcal{M}_n(k, m, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$ with respect to the basis ν. Then Δ is a linear operator generated by a matrix in*

$$\mathcal{M}_n(k, m, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$$

with respect to the basis ν.

P r o o f. Without loss of generality, we suppose that $k = 1$, $m = n$. Indeed, matrices in $\mathcal{M}_n(k, m, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$ depend on the basis $\nu = \{e_1, e_2, \dots e_n\}$. If we swap the vectors e_1 and e_k, e_m and e_n respectively then we get the set of matrices $\mathcal{M}_n(1, n, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$, i.e., $k = 1$, $m = n$. Then, by definition of Δ, for every element $x \in V$,

$$x = \sum_{i=1}^{n} x_i e_i,$$

there exists a matrix

$$A_{x,e_1} = (a_{ij}^{x,e_1})_{i,j=1}^n$$

in $\mathcal{M}_n(1, n, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$ such that

$$\Delta(x) = \widehat{A_{x,e_1}\bar{x}},$$

where $\bar{x} = (x_1, x_2, \dots, x_n)^T$ is the vector corresponding to x, $\widehat{\bar{x}}$ is an operation on $\bar{x}$ such that $\widehat{\bar{x}} = x$, and

$$\overline{\Delta(e_1)} = \overline{L_{x,e_1}(e_1)} = A_{x,e_1}\overline{e_1} = (a_{11}^{x,e_1}, a_{21}^{x,e_1}, a_{31}^{x,e_1}, \dots, a_{n1}^{x,e_1})^T,$$

where L_{x,e_1} is a linear operator, generated by A_{x,e_1}. Since $\Delta(e_1) = L_{x,e_1}(e_1) = L_{y_1,e_1}(e_1)$, we have

$$\overline{\Delta(e_1)} = (a_{11}^{x,e_1}, a_{21}^{x,e_1}, a_{31}^{x,e_1}, \dots, a_{n1}^{x,e_1})^T = (a_{11}^{y_1,e_1}, a_{21}^{y_1,e_1}, a_{31}^{y_1,e_1}, \dots, a_{n1}^{y_1,e_1})^T$$

for each pair, x, y_1 of elements in V. Hence,

$$a_{\alpha 1}^{x,e_1} = a_{\alpha 1}^{y_1,e_1}, \alpha \in \{i_1, i_2, \dots i_p\}. \tag{2.1}$$

By the definition of $\mathcal{M}_n(1, n, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$ the submatrix

$$\{a^{x,e_1}_{\alpha j}\}_{\alpha\in\{i_1,i_2,\dots i_p\},\ j=1,2,\dots,n}$$

belongs to the set of matrices $\mathcal{M}_{p,n}(1, l)$. Hence, by definition of the set $\mathcal{M}_{p,n}(1, l)$ there exist mappings

$$g_i(x_1, x_2, \dots, x_l), \quad i = 1, 2, \dots p,$$

with values in the field $\mathbb{F}$ and nonzero elements $\{a^{x,e_1}_1, a^{x,e_1}_2, \dots, a^{x,e_1}_l\} \subset \mathbb{F}$ depending on x and e_1 such that

$$a^{x,e_1}_{i_\alpha 1} = g_\alpha(a^{x,e_1}_1, a^{x,e_1}_2, \dots, a^{x,e_1}_l), \quad \alpha \in \{1, 2, \dots p\}.$$

Also, there exist nonzero elements $\{a^{x,e_1}_1, a^{x,e_1}_2, \dots, a^{x,e_1}_l\} \subset \mathbb{F}$ depending on x and e_1 such that

$$a^{y_1,e_1}_{\alpha 1} = g_\alpha(a^{y_1,e_1}_1, a^{y_1,e_1}_2, \dots, a^{y_1,e_1}_l), \quad \alpha \in \{i_1, i_2, \dots i_p\}.$$

By the equalities (2.1), we have

$$g_\alpha(a^{x,e_1}_1, a^{x,e_1}_2, \dots, a^{x,e_1}_l) = g_\alpha(a^{y_1,e_1}_1, a^{y_1,e_1}_2, \dots, a^{y_1,e_1}_l), \quad \alpha \in \{1, 2, \dots p\}.$$

By the definition of the functions g_v, $v = 1, 2, \dots p$ in the definition of the set $\mathcal{M}_{p,n}(1, l)$, we have

$$a^{x,e_1}_i = a^{y_1,e_1}_i, \quad i = 1, 2, \dots l. \tag{2.2}$$

By the definition of the set $\mathcal{M}_{p,n}(1, l)$, there exist functions

$$f_{\alpha j}(x_1, x_2, \dots, x_p), \quad \alpha \in \{i_1, i_2, \dots i_p\}, \quad j = 2, \dots, n,$$

with values in the field $\mathbb{F}$ such that, for every component $a^{z,e_1}_{\alpha j}$, $\alpha \in \{i_1, i_2, \dots i_p\}$, $j = 2, 3, \dots, n$, of A_{z,e_1} we have

$$a^{z,e_1}_{\alpha j} = f_{\alpha,j}(a^{z,e_1}_1, a^{z,e_1}_2, \dots, a^{z,e_1}_p), \quad \alpha \in \{i_1, i_2, \dots i_p\}, \quad j = 2, 3, \dots, n.$$

where $z \in \{x, y_1\}$. Therefore, by (2.2), $a^{x,e_1}_{\alpha j} = a^{y_1,e_1}_{\alpha j}$, $\alpha \in \{i_1, i_2, \dots i_p\}$, $j = 1, 2, \dots n$. Hence, for the elements $v \in V_1$, where V_1 is the vector subspace, generated by the vectors $\{e_{i_1}, e_{i_2}, \dots, e_{i_p}\}$, i.e.,

$$V_1 = \langle e_{i_1}, e_{i_2}, \dots, e_{i_p}\rangle$$

and $w \in V_2$, where V_2 is the vector subspace, generated by the vectors $\{e_{j_1}, e_{j_2}, \dots, e_{j_p}\}$, i.e.,

$$V_2 = \langle e_{j_1}, e_{j_2}, \dots, e_{j_p}\rangle$$

such that

$$\widehat{A_{x,e_1}}\bar{x} = v + w,$$

the elements $t \in V_1$ and $r \in V_2$ such that

$$\widehat{A_{y_1,e_1}}\bar{x} = t + r$$

we have

$$v = t.$$

Similarly, from $L_{x,e_n}(e_n) = L_{y_2,e_n}(e_n)$ it follows that

$$a_{\alpha n}^{x,e_n} = a_{\alpha n}^{y_2,e_n}, \quad \alpha \in \{j_1, j_2, \dots j_q\}$$

and

$$a_{\alpha j}^{x,e_n} = a_{\alpha j}^{y_2,e_n}, \quad \alpha \in \{j_1, j_2, \dots j_q\}, \quad j = 1, 2, \dots n.$$

Hence, for the elements $a \in V_1$ and $b \in V_2$ such that

$$\widehat{A_{x,e_n}}\bar{x} = a + b,$$

the elements $c \in V_1$ and $d \in V_2$ such that

$$\widehat{A_{y_2,e_n}}\bar{x} = c + d$$

we have

$$b = d.$$

Therefore, if we take $y_1 = e_n$, $y_2 = e_1$, then, for the elements $f \in V_1$ and $g \in V_2$ such that

$$\widehat{A_{e_1,e_n}}\bar{x} = f + g,$$

we have

$$\widehat{A_{x,e_1}}\bar{x} = v + w = f + w = f + b = f + g = \widehat{A_{e_1,e_n}}\bar{x}$$

since $v = f$, $A_{x,e_1}\bar{x} = A_{x,e_n}\bar{x}$ and $b = g$. So,

$$L_{x,e_1}(x) = L_{x,e_n}(x) = L_{e_1,e_n}(x).$$

for any $x \in V$, and the matrix of $\Delta(x)$ does not depend on x. Hence Δ is a linear operator and the matrix A_{e_1,e_n} is the matrix of Δ. This ends the proof. □

Example 1. Let $\mathcal{J}_{56}$ be the Jordan algebra with a basis $\{e_1, n_1, n_2, n_3\}$ such that

$$n_1^2 = n_2, \quad e_1 n_3 = \frac{1}{2} n_3, \quad e_1 n_i = n_i, \quad i = 1, 2$$

(see Table 3 in [16]). Then the matrix of its arbitrary derivation has the following form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & \beta & 2\alpha & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}.$$

If we take $k = 2$, $m = 4$, $i_1 = 2$, $i_2 = 3$, $j_1 = 4$, $l = 2$, $s = 1$, then the set of such matrices we can take as the set $\mathcal{M}_4(k, m, i_1, i_2, j_1, l, s)$.

Therefore, by Theorem 2, each 2-local automorphism of the Jordan algebra $\mathcal{J}_{56}$ is an automorphism. In this case, $\mathcal{M}_4(k, m, i_1, i_2, j_1, l, s)$ is a set of 4×4 matrices such that the 3×4 submatrix

$$A_1 : a_{\alpha,\beta}, \quad \alpha \in \{1, 2, 3\}, \quad \beta = 1, 2, 3, 4,$$

belongs to the set $\mathcal{M}_{3,4}(2, 2)$, and, the 1×4 submatrix

$$A_2 : a_{\alpha,\beta}, \quad \alpha = 4, \quad \beta = 1, 2, 3, 4,$$

belongs to the set $\mathcal{M}_{1,4}(4, 1)$.

3. 2-Local liner operators on finite-dimensional vector spaces which are not linear operators

Let n be a natural number, V be a vector space of dimension n over a field $\mathbb{F}$ with a basis $\{e_1, e_2, \dots, e_n\}$. Let, for fixed k, m, α, β, γ, η such that

$$1 \le k, m, \alpha, \beta \le n, \quad 2 \le \eta \le n, \quad k \ne m, \quad \alpha \le \beta, \quad 0 \le \gamma \le (n-\beta)n + \beta(n-\eta)$$

and, for fixed subsets $\{i_1, i_2, \dots, i_\beta\}$ and $\{j_1, j_2, \dots, j_\eta\}$ of natural numbers from $\{1, 2, \dots, n\}$ such that $k, m \in \{j_1, j_2, \dots, j_\eta\}$,

$$\begin{aligned} &f_{ij}(x_1, x_2, \dots, x_\alpha), \quad i \in \{i_1, i_2, \dots, i_\beta\}, \quad j \in \{j_1, j_2, \dots, j_\eta\}, \quad j \ne k, \quad j \ne m, \\ &f_{ij}(x_1, x_2, \dots, x_\gamma), \quad i \in \{1, 2, \dots, n\} \setminus \{i_1, i_2, \dots, i_\beta\}, \quad j \in \{1, 2, \dots, n\} \quad \text{if} \quad \beta \ne n, \\ &f_{ij}(x_1, x_2, \dots, x_\gamma), \quad i \in \{1, 2, \dots, n\}, \quad j \in \{1, 2, \dots, n\} \setminus \{j_1, j_2, \dots, j_\eta\} \quad \text{if} \quad \eta \ne n \end{aligned}$$

be functions with values in the field $\mathbb{F}$ (including the function $f_{ij} \equiv 0$) and, for fixed nonzero elements a_1, a_2, $\dots$, a_α, b_1, b_2, $\dots$, b_β, z_1, z_2, $\dots$, z_γ in $\mathbb{F}$,

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, \dots, a_\alpha, b_1, b_2, \dots, b_\beta, z_1, z_2, \dots, z_\gamma)$$

be a $n \times n$ matrix with components a_{ij}, $i, j = 1, 2, \dots, n$, such that

1) for $i \in \{i_1, i_2, \dots, i_\beta\}$, $a_{ik} \in \{a_1, a_2, \dots a_\alpha\}$ or $a_{ik} = 0$ and for any $a \in \{a_1, a_2, \dots a_\alpha\}$ there exists $l \in \{i_1, i_2, \dots, i_\beta\}$ such that $a_{lk} = a$;
2) for every component a_{ij}, $i \in \{i_1, i_2, \dots, i_\beta\}$, $j \in \{j_1, j_2, \dots, j_\eta\}$, $j \ne k$, $j \ne m$, of $\mathcal{M}_n^{k,m,\eta}(a_1, a_2, \dots, a_\alpha, b_1, b_2, \dots, b_\beta, z_1, z_2, \dots, z_\gamma)$,
$$a_{ij} = f_{ij}(a_1, a_2, \dots, a_\alpha);$$
3) $a_{i_s m} = b_s$, $s = 1, 2, \dots, \beta$;
4) every component a_{ij} of the submatrices
$$\begin{aligned} &B : a_{ij}, i \in \{1, 2, \dots, n\} \setminus \{i_1, i_2, \dots, i_\beta\}, \quad j \in \{1, 2, \dots, n\}, \\ &C : a_{ij}, i \in \{1, 2, \dots, n\}, \quad j \in \{1, 2, \dots, n\} \setminus \{j_1, j_2, \dots, j_\eta\} \end{aligned}$$
is equal to $f_{ij}(z_1, z_2, \dots, z_\gamma)$;
5) if $\beta = n$ and $\eta = n$, then $\gamma = 0$ and we use the designation
$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, \dots, a_\alpha, b_1, b_2, \dots, b_n)$$
instead of $\mathcal{M}_n^{k,m,\eta}(a_1, a_2, \dots, a_\alpha, b_1, b_2, \dots, b_\beta, z_1, z_2, \dots, z_\gamma)$.

Let V_1, V_2 be vector subspaces generated by the sets of vectors

$$\{e_j : j \ne m, \ j \in \{j_1, j_2, \dots, j_\eta\}\}$$

and $\{e_m\}$ respectively, i.e.,

$$V_1 = \langle \{e_j : j \ne m, \ j \in \{j_1, j_2, \dots, j_\eta\}\} \rangle, \quad V_2 = \langle e_m \rangle.$$

If $\eta \ne n$, then let V_3 be a vector subspace generated by the set of vectors

$$\{e_j : j \in \{1, 2, \dots, n\} \setminus \{j_1, j_2, \dots, j_\eta\}\},$$

i.e.,

$$V_3 = \langle \{e_j : j \in \{1, 2, \dots, n\} \setminus \{j_1, j_2, \dots, j_\eta\}\} \rangle.$$

Lemma 1. *If $\eta \neq n$, then, for any $v \in V_3$ and $x_1, x_2, \dots x_\alpha,\ y_1, y_2, \dots, y_\beta \in \mathbb{F}$,*

$$\mathcal{M}_n^{k,m,\eta}(x_1, x_2, \dots x_\alpha, y_1, y_2, \dots, y_\beta, z_1, z_2, \dots, z_\gamma)\bar{v}$$
$$= \mathcal{M}_n^{k,m,\eta}(a_1, a_2, \dots, a_\alpha, b_1, b_2, \dots, b_\beta, z_1, z_2, \dots, z_\gamma)\bar{v}.$$

P r o o f. We have

$$\mathcal{M}_n^{k,m,\eta}(x_1, x_2, \dots x_\alpha, y_1, y_2, \dots, y_\beta, z_1, z_2, \dots, z_\gamma)\bar{v} = \overline{\sum_{i=1}^{n} \sum_{j\in\{1,2,\dots,n\}\setminus\{j_1,j_2,\dots,j_\eta\}} a_{ij} v^j e_i} = C\bar{v},$$

where

$$v = \sum_{j\in\{1,2,\dots,n\}\setminus\{j_1,j_2,\dots,j_\eta\}} v^j e_j,$$

C is a matrix from item 4) of the definition of $\mathcal{M}_n^{k,m,\eta}(a_1, a_2, \dots, a_\alpha, b_1, b_2, \dots, b_\beta, z_1, z_2, \dots, z_\gamma)$ above. Since $x_1, x_2, \dots x_\alpha, y_1, y_2, \dots, y_\beta$ in $\mathbb{F}$ are chosen arbitrarily we have the statement of the lemma. □

Theorem 3. *Let V be a vector space of dimension n over a field $\mathbb{F}$ with a basis $\{e_1, e_2, \dots, e_n\}$. Then, for any nonzero elements $c_1, c_2, \dots, c_\alpha$ from the field $\mathbb{F}$, a mapping Δ on V defined as follows*

(I) *in the case $\eta \neq n$,*

1) *if $v = v_1 + v_3$ or $v = v_3,\ v_1 \in V_1,\ v_1 \neq 0,\ v_3 \in V_3$ then*

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, \dots a_\alpha, b_1, b_2, \dots b_\beta, z_1, z_2, \dots, z_\gamma)\bar{v},$$

2) *if $v = v_1 + v_2 + v_3,\ v_1 \in V_1,\ v_2 \in V_2,\ v_2 \neq 0,\ v_3 \in V_3$, then*

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(c_1, c_2, \dots, c_\alpha, b_1, b_2, \dots b_\beta, z_1, z_2, \dots, z_\gamma)\bar{v},$$

(II) *in the case $\eta = n$,*

1) *if $v = v_1,\ v_1 \in V_1,\ v_1 \neq 0$, then*

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, \dots a_\alpha, b_1, b_2, \dots b_\beta, z_1, z_2, \dots, z_\gamma)\bar{v},$$

2) *if $v = v_1 + v_2,\ v_1 \in V_1,\ v_2 \in V_2,\ v_2 \neq 0$, then*

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(c_1, c_2, \dots, c_\alpha, b_1, b_2, \dots b_\beta, z_1, z_2, \dots, z_\gamma)\bar{v}$$

is a 2-local linear operator, and Δ is a linear operator if and only if

$$a_i = c_i, \quad i = 1, 2, \dots, \alpha.$$

P r o o f. We will prove the theorem in the case (I). In the case (II), the theorem is proved similarly. We prove that the mapping Δ, defined in the theorem, is a 2-local linear operator on V. Take the subspace $V_1 \oplus V_3$ and arbitrary two elements v, w from $V_1 \oplus V_3$. Then, by the definition of Δ, item 1) of the theorem and by Lemma 1, for the linear operator $L_{v,w}$ with the matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma),$$

we have $\Delta(v) = L_{v,w}(v),\ \Delta(w) = L_{v,w}(w)$.

Take the subspace $V_2 \oplus V_3$ and two elements v, w from $V_2 \oplus V_3$ such that

$$v = v_2 + v_3, \quad v_2 \in V_2, \quad v_2 \neq 0, \quad v_3 \in V_3, \quad w = w_2 + w_3, \quad w_2 \in V_2, \quad w_2 \neq 0, \quad w_3 \in V_3.$$

Then, by item 2) of the theorem, for the linear operator $L_{v,w}$ with the matrix

$$\mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma),$$

we have $\Delta(v) = L_{v,w}(v)$, $\Delta(w) = L_{v,w}(w)$.

Now, if we take elements $v \in V_1 \oplus V_3$ such that

$$v = v_1 + v_3, \quad v_1 \in V_1, \quad v_1 \neq 0, \quad v_3 \in V_3, \quad w \in V_2 \oplus V_3$$

such that

$$w = w_2 + w_3, \quad w_2 \in V_2, \quad w_2 \neq 0, \quad w_3 \in V_3,$$

then, by items 1) and 2) of the theorem

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v},$$

and

$$\overline{\Delta(w)} = \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{w}$$

respectively. In this case, by Lemma 1, for the linear operator $T_{v,w}$ with the matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma),$$

we have

$$\Delta(v) = T_{v,w}(v), \quad \Delta(w) = T_{v,w}(w).$$

Now, if $v \in V_1 \oplus V_2 \oplus V_3$ such that

$$v = v_1 + v_2 + v_3, \quad v_1 \in V_1, \quad v_2 \in V_2, \quad v_2 \neq 0, \quad v_3 \in V_3, \quad w \in V_1 \oplus V_3$$

such that

$$w = w_1 + w_3, \quad w_1 \in V_1, \quad w_1 \neq 0, \quad w_3 \in V_3,$$

then, by items 2) and 1) of the theorem,

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}$$

and

$$\overline{\Delta(w)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{w}$$

respectively. In this case, there exist elements $\lambda_1, \lambda_2, ..., \lambda_\beta$ in the field $\mathbb{F}$ such that for the linear operator $L_{v,w}$ with the matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, \lambda_1, \lambda_2, ..., \lambda_\beta, z_1, z_2, ..., z_\gamma),$$

we have

$$\Delta(v) = L_{v,w}(v), \quad \Delta(w) = L_{v,w}(w).$$

Indeed, the equality $\Delta(w) = L_{v,w}(w)$ is obviously true for any λ_1, λ_2, ... λ_β in $\mathbb{F}$ by Lemma 1. As for the equality $\Delta(v) = L_{v,w}(v)$, we rewrite it in the following form

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, \lambda_1, \lambda_2, ..., \lambda_\beta, z_1, z_2, ..., z_\gamma)\bar{v}$$

$$= \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}.$$

The last equality is a system of linear equations with respect to the variables λ_1, λ_2, ... λ_β. By Lemma 1, this system can be written in the following way

$$h_i + v_2^m \lambda_i = g_i + v_2^m b_i, \quad i \in \{i_1, i_2, ..., i_\beta\}, \quad h_j = h_j, \quad j \in \{1, 2, ..., n\} \setminus \{i_1, i_2, ..., i_\beta\},$$

for some elements h_i, $i = 1, 2, ..., n$ and g_j, $j \in \{i_1, i_2, ..., i_\beta\}$, from $\mathbb{F}$, where $v_2 = v_2^m e_m$. Since, $v_2^m \neq 0$, this system of linear equations has the solution

$$\lambda_i = \frac{1}{v_2^m}(g_i + v_2^m b_i - h_i), \quad i \in \{i_1, i_2, ..., i_\beta\}.$$

Hence,

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, \lambda_1, \lambda_2, ..., \lambda_\beta, z_1, z_2, ..., z_\gamma)$$

is a desired matrix.

The case

$$v = v_1 + v_2 + v_3, \quad v_1 \in V_1, \quad v_2 \in V_2, \quad v_2 \neq 0, \quad v_3 \in V_3,$$
$$w = w_1 + w_2 + w_3, \quad w_1 \in V_1, \quad w_2 \in V_2, \quad w_2 \neq 0, \quad w_3 \in V_3$$

is also trivial, i.e., by item 2) of the theorem, for the linear operator $L_{v,w}$ with the matrix

$$\mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma),$$

we have $\Delta(v) = L_{v,w}(v)$, $\Delta(w) = L_{v,w}(w)$.

The case $v \in V_3$ and $w \in V_1 \oplus V_2 \oplus V_3$ such that

$$w = w_1 + w_2 + w_3, \quad w_1 \in V_1, \quad w_1 \neq 0, \quad w_2 \in V_2, \quad w_2 \neq 0, \quad w_3 \in V_3$$

follows by Lemma 1. Indeed, we have

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}$$

by item 1 of the theorem, and,

$$\overline{\Delta(w)} = \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{w}$$

by item 2 of the theorem. At the same time,

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}$$

by Lemma 1. Hence,

$$\Delta(v) = L_{v,w}(v), \quad \Delta(w) = L_{v,w}(w)$$

for the linear operator $L_{v,w}$, generated by the matrix $\mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)$.

Thus, in all cases, for any pair v and w of elements from V, there exists a linear operator $L_{v,w}$ on V such that $\Delta(v) = L_{v,w}(v)$, $\Delta(w) = L_{v,w}(w)$, i.e., Δ is a 2-local linear operator.

Now, if $a_i = c_i$, $i = 1, 2, ..., \alpha$, then, by items 1) and 2) of the theorem, for any $v \in V$,

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}.$$

So Δ is linear.

Suppose that $(a_1, a_2, ..., a_\alpha) \neq (c_1, c_2, ..., c_\alpha)$. Then there exists a vector $v \in V_1$, $v \neq 0$, such that

$$\mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v} \neq \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}.$$

Then, for any $w \in V_2$, $w \neq 0$, we have

$$\begin{aligned}\overline{\Delta(v+w)} &= \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\overline{(v+w)},\\ \overline{\Delta(v)} &= \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v},\\ \overline{\Delta(w)} &= \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{w}.\end{aligned}$$

So,

$$\begin{aligned}\overline{\Delta(v+w) - (\Delta(v) + \Delta(w))} &= \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}\\ &-\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v} \neq 0,\end{aligned}$$

i.e., Δ is not additive. This ends the proof. □

4. 2-Local derivations of complex null-filiform and filiform Zinbiel algebras

An algebra $\mathcal{A}$ over a field $\mathbb{F}$ is called Zinbiel algebra if, for any $x, y, z \in \mathcal{A}$, the identity

$$(xy)z = x(yz) + x(zy)$$

holds. For a given Zinbiel algebra $\mathcal{A}$, we define the following sequence:

$$\mathcal{A}^1 = \mathcal{A}, \quad \mathcal{A}^{i+1} = \sum_{k=1}^{i} \mathcal{A}^k \mathcal{A}^{i+1-k}, \quad i \geq 1.$$

A Zinbiel algebra $\mathcal{A}$ is said to be nilpotent if $\mathcal{A}^i = 0$ for some $i \in \mathbb{N}$. The minimal number i satisfying $\mathcal{A}^i = 0$ is called index of nilpotency or nilindex of the algebra $\mathcal{A}$.

It is clear that the index of nilpotency of an arbitrary n-dimensional nilpotent Zinbiel algebra does not exceed the number $n+1$.

Definition 3. *An n-dimensional Zinbiel algebra $\mathcal{A}$ is said to be null-filiform if*

$$\dim \mathcal{A}^i = (n+1) - i,$$

where $\dim \mathcal{A}^i$ is the dimension of $\mathcal{A}^i$, $1 \leq i \leq n+1$.

It is evident that the last definition is equivalent to the fact that the Zinbiel algebra $\mathcal{A}$ has maximal index of nilpotency.

Theorem 4 [2]. *An arbitrary n-dimensional null-filiform Zinbiel algebra over the field $\mathbb{C}$ of complex numbers is isomorphic to the algebra*

$$F_n^0: \; e_i e_j = C_{i+j-1}^{j} e_{i+j}, \quad 2 \leq i+j \leq n,$$

where omitted products $e_k e_l$ are equal to zero and $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra, the symbols C_s^t are binomial coefficients defined as

$$C_s^t = \frac{s!}{t!(s-t!)}.$$

Definition 4. *An n-dimensional Zinbiel algebra $\mathcal{A}$ is said to be filiform if*

$$\dim \mathcal{A}^i = n - i, \quad 2 \leq i \leq n.$$

Theorem 5 [2]. *An arbitrary n-dimensional, $n \geq 5$, filiform Zinbiel algebra over the field $\mathbb{C}$ of complex numbers is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$F_n^1: \ e_i e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq n-1,$$
$$F_n^2: \ e_i e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq n-1, \quad e_n e_1 = e_{n-1},$$
$$F_n^3: \ e_i e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq n-1, \quad e_n e_n = e_{n-1},$$

where omitted products $e_k e_l$ are equal to zero and $\{e_1, e_2, \dots, e_n\}$ is a basis of the appropriate algebra.

Theorem 6 [21]. *A linear map $\triangle : F_n^0 \to F_n^0$ is a derivation if and only if $\triangle$ is of the following form:*

$$\triangle(e_i) = \sum_{j=i}^{n} C_j^{i-1} \alpha_{j-i+1} e_j, \quad 1 \leq i \leq n,$$

where $\alpha_i \in \mathbb{C}$, $1 \leq i \leq n$.

Theorem 7 [21]. *A linear map $\triangle : F_n^1 \to F_n^1$ is a derivation if and only if $\triangle$ is of the following form*:

$$\triangle(e_1) = \sum_{j=1}^{n} \alpha_j e_j, \quad \triangle(e_i) = \sum_{j=i}^{n} C_j^{i-1} \alpha_{j-i+1} e_j, \quad 2 \leq i \leq n-1, \quad \triangle(e_n) = b_{n-1} e_{n-1} + b_n e_n,$$

where $\alpha_i \in \mathbb{C}$, $1 \leq i \leq n$.

Theorem 8 [21]. *A linear map $\triangle : F_n^2 \to F_n^2$ is a derivation if and only if $\triangle$ is of the following form:*

$$\triangle(e_1) = \sum_{j=1}^{n} \alpha_j e_j, \quad \triangle(e_2) = \sum_{j=2}^{n-1} C_j^1 \alpha_{j-1} e_j + \alpha_n e_{n-1},$$

$$\triangle(e_i) = \sum_{j=i}^{n-1} C_j^{i-1} \alpha_{j-i+1} e_j, \quad 3 \leq i \leq n-1, \quad \triangle(e_n) = b_{n-1} e_{n-1} + (n-2)\alpha_1 e_n,$$

where $\alpha_i \in \mathbb{C}$, $1 \leq i \leq n$.

Theorem 9 [21]. *A linear map $\triangle : F_n^3 \to F_n^3$ is a derivation if and only if $\triangle$ is of the following form:*

$$\triangle(e_1) = \sum_{j=1}^{n} \alpha_j e_j, \quad \triangle(e_i) = \sum_{j=i}^{n-1} C_j^{i-1} \alpha_{j-i+1} e_j, \quad 2 \leq i \leq n-1,$$
$$\triangle(e_n) = -\alpha_n e_{n-2} + b_{n-1} e_{n-1} + \frac{n-1}{2} \alpha_1 e_n,$$

where $\alpha_i \in \mathbb{C}$, $1 \leq i \leq n$.

The following theorems are the main theorems of the present section.

Theorem 10. *Each 2-local derivation on F_n^0 is a derivation.*

P r o o f. Let Δ be an arbitrary 2-local derivation on F_n^0. By the definition, for any x, $y \in F_n^0$ there exists a derivation $D_{x,y}$ on F_n^0 such that

$$\Delta(x) = D_{x,y}(x), \quad , \Delta(x) = D_{x,y}(x).$$

By Theorem 6, the matrix of the derivation $D_{x,y}$ has the following matrix form:

$$D_{x,y} = \begin{pmatrix} \alpha_1^{x,y} & 0 & 0 & \dots & 0 & 0 \\ \alpha_2^{x,y} & C_2^1\alpha_1^{x,y} & 0 & \dots & 0 & 0 \\ \alpha_3^{x,y} & C_3^1\alpha_2^{x,y} & C_3^2\alpha_1^{x,y} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-1}^{x,y} & C_{n-1}^1\alpha_{n-2}^{x,y} & C_{n-1}^2\alpha_{n-3}^{x,y} & \dots & C_{n-1}^{n-2}\alpha_1^{x,y} & 0 \\ \alpha_n^{x,y} & C_n^1\alpha_{n-1}^{x,y} & C_n^2\alpha_{n-2}^{x,y} & \dots & C_n^{n-2}\alpha_2^{x,y} & C_n^{n-1}\alpha_1^{x,y} \end{pmatrix}.$$

Clearly, the set of all $n \times n$ matrices of the form above we can set as a set $\mathcal{M}_{m,n}(k,p)$ defined in Section 2, where $m = n$, $k = 1$, $p = n$, i.e., $\mathcal{M}_{m,n}(k,p) = \mathcal{M}_{n,n}(1,n)$

Each 2-local derivation on F_n^0 is a 2-local linear operator on F_n^0 generated by matrices in $\mathcal{M}_{n,n}(1,n)$ with respect to the basis $\{e_1, e_2, ..., e_n\}$. Conversely, every 2-local linear operator on F_n^0 generated by matrices in $\mathcal{M}_{n,n}(1,n)$ is a 2-local derivation on F_n^0 by Theorem 6.

Therefore, by Theorem 1, each 2-local derivation on F_n^0 is a linear operator generated by a matrix from $\mathcal{M}_{n,n}(1,n)$. Hence, each 2-local derivation on F_n^0 is a derivation by Theorem 6. This ends the proof. □

Theorem 11. *The algebras F_n^1, F_n^2 and F_n^3 have 2-local derivations which are not derivations.*

P r o o f. Let D be an arbitrary derivation on F_n^1. By Theorem 7, the matrix of the derivation D has the following form:

$$\begin{pmatrix} \alpha_1 & 0 & 0 & \dots & 0 & 0 \\ \alpha_2 & C_2^1\alpha_1 & 0 & \dots & 0 & 0 \\ \alpha_3 & C_3^1\alpha_2 & C_3^2\alpha_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-1} & C_{n-1}^1\alpha_{n-2} & C_{n-1}^2\alpha_{n-3} & \dots & C_{n-1}^{n-2}\alpha_1 & \beta_{n-1} \\ \alpha_n & 0 & 0 & \dots & 0 & \beta_n \end{pmatrix}.$$

Let $a_1 = \alpha_{n-1}$, $a_2 = \alpha_n$, $b_1 = \beta_{n-1}$, $b_2 = \beta_n$ and

$$z_1 = \alpha_1, \quad z_2 = \alpha_2, \quad ..., \quad z_{n-2} = \alpha_{n-2}.$$

Then, if this matrix we denote by $\mathcal{M}_n^{1,n,n}(a_1, a_2, b_1, b_2, z_1, z_2, ..., z_{n-2})$, then $\mathcal{M}_n^{1,n,n}(a_1, a_2, b_1, b_2, z_1, z_2, ..., z_{n-2})$ satisfies the all conditions of the definition in Section 3 of a matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)$$

in the case of $k = 1$, $m = n$, $\eta = n$, $\alpha = 2$, $\beta = 2$ and $\gamma = n - 2$.

Therefore, by Theorem 3, we can find a 2-local derivation on F_n^1 which is not linear.

Now we take the algebra F_n^2 and a derivation D on F_n^2. By Theorem 8, the matrix of the derivation D has the following form:

$$\begin{pmatrix} \alpha_1 & 0 & 0 & \dots & 0 & 0 \\ \alpha_2 & C_2^1\alpha_1 & 0 & \dots & 0 & 0 \\ \alpha_3 & C_3^1\alpha_2 & C_3^2\alpha_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-1} & C_{n-1}^1\alpha_{n-2}+\alpha_n & C_{n-1}^2\alpha_{n-3} & \dots & C_{n-1}^{n-2}\alpha_1 & \beta_{n-1} \\ \alpha_n & 0 & 0 & \dots & 0 & (n-2)\alpha_1 \end{pmatrix}.$$

Similar to the previous case, we take $a_1 = \alpha_{n-1}$, $b_1 = \beta_{n-1}$ and

$$z_1 = \alpha_1, \quad z_2 = \alpha_2, \quad ..., \quad z_{n-2} = \alpha_{n-2}, \quad z_{n-1} = \alpha_n.$$

Then, if this matrix we denote by $\mathcal{M}_n^{1,n,n}(a_1, b_1, z_1, z_2, ..., z_{n-1})$, then $\mathcal{M}_n^{1,n,n}(a_1, b_1, z_1, z_2, ..., z_{n-1})$ satisfies the all conditions of the definition in Section 3 of a matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)$$

in the case of $k = 1, \ m = n, \ \eta = n, \ \alpha = 1, \ \beta = 1$ and $\gamma = n - 1$.

Therefore, by Theorem 3, we can find a 2-local derivation on F_n^1 which is not linear.

Similarly we prove that F_n^3 has 2-local derivations which are not derivations. This ends the proof. □

5. 2-Local automorphisms of naturally graded quasi-filiform Leibniz algebras of type I

A vector space with a bilinear bracket $(\mathcal{L}, [\cdot, \cdot])$ is called a Leibniz algebra if, for any $x, \ y, \ z \in L$, the so-called Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds. For a given Leibniz algebra $(\mathcal{L}, [\cdot, \cdot])$, the sequence of two-sided ideals is defined recursively as follows:

$$\mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}], \quad k \geq 1.$$

This sequence is said to be the lower central series of $\mathcal{L}$.

A Leibniz algebra $\mathcal{L}$ is said to be nilpotent, if there exists $n \in \mathbb{N}$ such that $\mathcal{L}^n = \{0\}$.

It is easy to see that the sum of two nilpotent ideals of a Leibniz algebra is also nilpotent. Therefore, the maximal nilpotent ideal of a finite-dimensional Leibniz algebra always exists. The maximal nilpotent ideal of a Leibniz algebra is said to be the nilradical of the algebra.

Now we give the definitions of automorphisms and 2-local automorphisms.

Let $\mathcal{A}$ be an algebra. A linear bijective map $\varphi : \mathcal{A} \to \mathcal{A}$ is called an automorphism if it satisfies

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] \quad \text{for all} \quad x, y \in \mathcal{A}.$$

Let $\mathcal{A}$ be an algebra. A (not necessarily linear) map $\Delta : \mathcal{A} \to \mathcal{A}$ is called a 2-local automorphism if, for any elements $x, \ y \in \mathcal{A}$, there exists an automorphism $\varphi_{x,y} : \mathcal{A} \to \mathcal{A}$ such that

$$\Delta(x) = \varphi_{x,y}(x), \quad \Delta(y) = \varphi_{x,y}(y).$$

Below we define the notion of a quasi-filiform Leibniz algebra.

An n-dimensional Leibniz algebra $\mathcal{L}$ is called quasi-filiform if $\mathcal{L}^{n-2} \neq \{0\}$ and $\mathcal{L}^{n-1} = \{0\}$.

Given an n-dimensional nilpotent Leibniz algebra $\mathcal{L}$ such that $\mathcal{L}^{s-1} \neq \{0\}$ and $\mathcal{L}^s = \{0\}$, put

$$\mathcal{L}_i = \mathcal{L}^i/\mathcal{L}^{i+1}, \quad 1 \leq i \leq s-1,$$

and

$$\mathrm{gr}(\mathcal{L}) = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_{s-1}.$$

Due to $[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}$ we obtain the graded algebra $\mathrm{gr}(\mathcal{L})$. If $\mathrm{gr}(\mathcal{L})$ and $\mathcal{L}$ are isomorphic, i.e., if $\mathrm{gr}(\mathcal{L}) \cong \mathcal{L}$, then we say that $\mathcal{L}$ is naturally graded.

Let x be a nilpotent element of the set $\mathcal{L}\backslash\mathcal{L}^2$. For the nilpotent operator of right multiplication $\mathcal{R}_x$ we define a decreasing sequence $C(x) = (n_1, n_2, \ldots, n_k)$, where $n = n_1 + n_2 + \cdots + n_k$, which consists of the dimensions of Jordan blocks of the operator $\mathcal{R}_x$. On the set of such sequences we consider the lexicographic order, that is,

$$C(x) = (n_1, n_2, \ldots, n_k) \leq C(y) = (m_1, m_2, \ldots, m_t)$$

iff there exists $i \in \mathbb{N}$ such that $n_j = m_j$ for any $j < i$ and $n_i < m_i$.

The sequence

$$C(\mathcal{L}) = \max_{x \in \mathcal{L}\backslash\mathcal{L}^2} C(x)$$

is called the characteristic sequence of the algebra $\mathcal{L}$.

A quasi-filiform non Lie Leibniz algebra $\mathcal{L}$ is called an algebra of the type I (respectively, type II) if there exists an element $x \in \mathcal{L}\backslash\mathcal{L}^2$ such that the operator $\mathcal{R}_x$ has the form

$$\begin{pmatrix} J_{n-2} & 0 \\ 0 & J_2 \end{pmatrix}, \quad (\text{respectively}, \quad \begin{pmatrix} J_2 & 0 \\ 0 & J_{n-2} \end{pmatrix}).$$

The following theorem obtained in [1] gives the classification of naturally graded quasifiliform Leibniz algebras of type I.

Theorem 12. *An arbitrary n-dimensional naturally graded quasi-filiform Leibniz algebra of type I is isomorphic to one of the pairwise non-isomorphic algebras of the following families*:

$$\mathcal{L}_n^{1,\lambda}: \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_{n-1}, e_1] = e_n, \\ [e_1, e_{n-1}] = \lambda e_n, & \lambda \in \mathbb{C}, \end{cases} \qquad \mathcal{L}_n^{2,\lambda}: \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_{n-1}, e_1] = e_n, \\ [e_1, e_{n-1}] = \lambda e_n, & \lambda \in \{0, 1\}, \\ [e_{n-1}, e_{n-1}] = e_n, \end{cases}$$

$$\mathcal{L}_n^{3,\lambda}: \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_{n-1}, e_1] = e_n + e_2, \\ [e_1, e_{n-1}] = \lambda e_n, & \lambda \in \{-1, 0, 1\}, \end{cases} \qquad \mathcal{L}_n^{4,\mu}: \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_{n-1}, e_1] = e_n + e_2, \\ [e_{n-1}, e_{n-1}] = \mu e_n, & \mu \neq 0, \end{cases}$$

$$\mathcal{L}_n^{5,\lambda,\mu}: \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_{n-1}, e_1] = e_n + e_2, \\ [e_1, e_{n-1}] = \lambda e_n, & (\lambda, \mu) = (1, 1) \ \textit{or} \ (2, 4), \\ [e_{n-1}, e_{n-1}] = \mu e_n, \end{cases}$$

where $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra.

In this section we use the following theorem from [3] concerning automorphisms of naturally graded quasi-filiform Leibniz algebras of type I.

Theorem 13. *A linear map $\varphi : \mathcal{L} \to \mathcal{L}$ is an automorphism if and only if φ has the following form*:

$$\varphi\left(\mathcal{L}_n^{1,\lambda}\right) : \begin{cases} \varphi(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 \Big(\sum_{i=2}^{n-2} \alpha_{i-1} e_i + \alpha_{n-1}(1+\lambda) e_n \Big), \\ \varphi(e_j) = \alpha_1^{j-1} \sum_{i=j}^{n-2} \alpha_{i-j+1} e_i, \quad 3 \le j \le n-2, \\ \varphi(e_{n-1}) = \sum_{i=n-3}^{n} b_i e_i, \\ \varphi(e_n) = \alpha_1 (b_{n-3} e_{n-2} + b_{n-1} e_n), \end{cases}$$

where $\alpha_i \in \mathbb{C}$, $1 \le i \le n$, $\alpha_1 b_{n-1} \ne 0$;

$$\varphi\left(\mathcal{L}_n^{2,0}\right) : \begin{cases} \varphi(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 \sum_{i=2}^{n-2} \alpha_{i-1} e_i + \alpha_{n-1} (\alpha_1 + \alpha_{n-1}) e_n, \\ \varphi(e_j) = \alpha_1^{j-1} \sum_{i=j}^{n-2} \alpha_{i-j+1} e_i, \quad 3 \le j \le n-2, \\ \varphi(e_{n-1}) = b_{n-2} e_{n-2} + b_{n-1} e_{n-1} + b_n e_n, \\ \varphi(e_n) = (\alpha_1 + \alpha_{n-1}) b_{n-1} e_n, \end{cases}$$

where $\alpha_i \in \mathbb{C}$, $1 \le i \le n$, $\alpha_1 b_{n-1} \ne 0$, $b_{n-1} = \alpha_1 + \alpha_{n-1}$;

$$\varphi\left(\mathcal{L}_n^{2,1}\right) : \begin{cases} \varphi(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 \sum_{i=2}^{n-2} \alpha_{i-1} e_i + \alpha_{n-1} (2\alpha_1 + \alpha_{n-1}) e_n, \\ \varphi(e_j) = \alpha_1^{j-1} \sum_{i=j}^{n-2} \alpha_{i-j+1} e_i, \quad 3 \le j \le n-2, \\ \varphi(e_{n-1}) = b_{n-2} e_{n-2} + b_{n-1} e_{n-1} + b_n e_n, \\ \varphi(e_n) = (\alpha_1 + \alpha_{n-1}) b_{n-1} e_n, \end{cases}$$

where $\alpha_i \in \mathbb{C}$, $1 \le i \le n$, $\alpha_1 b_{n-1} \ne 0$, $b_{n-1} = \alpha_1 + \alpha_{n-1}$;

$$\varphi\left(\mathcal{L}_n^{3,-1}\right) : \begin{cases} \varphi(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \\ \varphi(e_j) = \alpha_1^{j-1} (\alpha_1 + \alpha_{n-1}) e_j + \alpha_1^{n-1} \sum_{i=j+1}^{n-2} \alpha_{i-j+1} e_i, \quad 2 \le j \le n-2, \\ \varphi(e_{n-1}) = \sum_{i=2}^{n-3} \alpha_i e_i + b_{n-2} e_{n-2} + (\alpha_1 + \alpha_{n-1}) e_{n-1} + b_n e_n, \\ \varphi(e_n) = \alpha_1 (\alpha_1 + \alpha_{n-1}) e_n, \end{cases}$$

where $\alpha_i \in \mathbb{C}$, $1 \le i \le n$, $\alpha_1 (\alpha_1 + \alpha_{n-1}) \ne 0$;

$$\varphi\left(\mathcal{L}_n^{3,0}\right) : \begin{cases} \varphi(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 (\alpha_1 + \alpha_{n-1}) e_2 + \alpha_1 \sum_{i=3}^{n-2} \alpha_{i-1} e_i + \alpha_1 \alpha_{n-1} e_n, \\ \varphi(e_j) = \alpha_1^{j-1} (\alpha_1 + \alpha_{n-1}) e_j + \alpha_1^{j-1} \sum_{i=j+1}^{n-2} \alpha_{i-j+1} e_i, \quad 2 \le j \le n-2, \\ \varphi(e_{n-1}) = \sum_{i=2}^{n-4} \alpha_i e_i + b_{n-3} e_{n-3} + b_{n-2} e_{n-2} + (\alpha_1 + \alpha_{n-1}) e_{n-1} + b_n e_n, \\ \varphi(e_n) = (b_{n-3} - \alpha_{n-3}) \alpha_1 e_{n-2} + \alpha_1^2 e_n, \end{cases}$$

where $\alpha_i \in \mathbb{C}$, $1 \le i \le n$, $\alpha_1 (\alpha_1 + \alpha_{n-1}) \ne 0$; *for the algebras* $\mathcal{L}_n^{3,1}, \mathcal{L}_n^{4,\mu}, \mathcal{L}_n^{5,\lambda,\mu}$

$$\begin{cases} \varphi(e_1) = \sum_{i=1}^{n-2} \alpha_i e_i + \alpha_n e_n, \\ \varphi(e_j) = \alpha_1^{i-1} \sum_{i=j}^{n-2} \alpha_{i-j+1} e_i, \quad 2 \le j \le n-2, \\ \varphi(e_{n-1}) = b_{n-2} e_{n-2} + \alpha_1 e_{n-1} + b_n e_n, \\ \varphi(e_n) = 2\alpha_1^2 e_n, \end{cases}$$

where $\alpha_i \in \mathbb{C}$, $1 \le i \le n-2$, $\alpha_n \in \mathbb{C}$, $\alpha_1 \ne 0$.

The following theorem is one of the main results of the present paper concerning 2-local automorphisms.

Theorem 14. *The algebras $\mathcal{L}_n^{1,\lambda}$, $\mathcal{L}_n^{2,\lambda}$, where $\lambda \in \{0, 1\}$, $\mathcal{L}_n^{3,\lambda}$, where $\lambda \in \{-1, 0, 1\}$, $\mathcal{L}_n^{4,\mu}$ and $\mathcal{L}_n^{5,\lambda,\mu}$, where $(\lambda, \mu) = (1, 1)$ or $(2, 4)$, have 2-local automorphisms which are not automorphisms.*

P r o o f. Let φ be an arbitrary automorphism on $\mathcal{L}_n^{1,\lambda}$. By Theorem 13, the matrix of the automorphism φ has the following form:

$$\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\alpha_2 & \alpha_1^2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\alpha_3 & \alpha_1\alpha_2 & \alpha_1^3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{n-4} & \alpha_1\alpha_{n-5} & \alpha_1^2\alpha_{n-6} & \dots & \alpha_1^{n-6}\alpha_2 & \alpha_1^{n-4} & 0 & 0 & 0 & 0 \\
\alpha_{n-3} & \alpha_1\alpha_{n-4} & \alpha_1^2\alpha_{n-5} & \dots & a_1^{n-6}a_3 & \alpha_1^{n-5}\alpha_2 & \alpha_1^{n-3} & 0 & \beta_{n-3} & 0 \\
\alpha_{n-2} & \alpha_1\alpha_{n-3} & \alpha_1^2\alpha_{n-4} & \alpha_1^3\alpha_{n-5} & \dots & \alpha_1^{n-5}\alpha_3 & \alpha_1^{n-4}\alpha_2 & \alpha_1^{n-2} & \beta_{n-2} & \alpha_1\beta_{n-3} \\
\alpha_{n-1} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \beta_{n-1} & 0 \\
\alpha_n & \alpha_{n-1}(1+\lambda) & 0 & 0 & 0 & \dots & 0 & 0 & \beta_n & \alpha_1\beta_{n-1}
\end{pmatrix}.$$

Let $a_1 = \alpha_n, \ \alpha_{n-1} = 0, \ b_1 = \beta_n$ and

$$z_1 = \alpha_1, \quad z_2 = \alpha_2, \quad ..., \quad z_{n-2} = \alpha_{n-2}, \quad z_{n-1} = \beta_{n-1}, \quad z_n = \beta_{n-2}, \quad z_{n+1} = \beta_{n-3}.$$

Then, denoting this matrix by $\mathcal{M}_n^{1,n,n}(a_1, b_1, z_1, z_2, ..., z_{n+1})$, we see that $\mathcal{M}_n^{1,n,n}(a_1, b_1, z_1, z_2, ..., z_{n+1})$ satisfies all conditions of the definition in Section 3 of a matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)$$

in the case of $k = 1, \ m = n-1, \ \eta = n-1, \ \alpha = 1, \ \beta = 1$ and $\gamma = n+1$.

Therefore, by Theorem 3, we can find a 2-local automorphism on $\mathcal{L}_n^{1,\lambda}$ which is not linear.

Now we take the algebra $\mathcal{L}_n^{2,0}$ and an automorphism φ on $\mathcal{L}_n^{2,0}$. By Theorem 13, the matrix of the automorphism φ has the following form:

$$\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\alpha_2 & \alpha_1^2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\alpha_3 & \alpha_1\alpha_2 & \alpha_1^3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{n-4} & \alpha_1\alpha_{n-5} & \alpha_1^2\alpha_{n-6} & \dots & \alpha_1^{n-6}\alpha_2 & \alpha_1^{n-4} & 0 & 0 & 0 & 0 \\
\alpha_{n-3} & \alpha_1\alpha_{n-4} & \alpha_1^2\alpha_{n-5} & \dots & a_1^{n-6}a_3 & \alpha_1^{n-5}\alpha_2 & \alpha_1^{n-3} & 0 & 0 & 0 \\
\alpha_{n-2} & \alpha_1\alpha_{n-3} & \alpha_1^2\alpha_{n-4} & \alpha_1^3\alpha_{n-5} & \dots & \alpha_1^{n-5}\alpha_3 & \alpha_1^{n-4}\alpha_2 & \alpha_1^{n-2} & \beta_{n-2} & 0 \\
\alpha_{n-1} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \alpha_1+\alpha_{n-1} & 0 \\
\alpha_n & \alpha_{n-1}(\alpha_1+\alpha_{n-1}) & 0 & 0 & 0 & \dots & 0 & 0 & \beta_n & (\alpha_1+\alpha_{n-1})^2
\end{pmatrix}.$$

Similar to the previous case, we take $a_1 = \alpha_n, \ \alpha_{n-1} = 0, \ b_1 = \beta_n$ and

$$z_1 = \alpha_1, \quad z_2 = \alpha_2, \quad ..., \quad z_{n-2} = \alpha_{n-2}, \quad z_{n-1} = \beta_{n-2}.$$

Then, if this matrix we denote by $\mathcal{M}_n^{1,n,n}(a_1, b_1, z_1, z_2, ..., z_{n-1})$, then $\mathcal{M}_n^{1,n,n}(a_1, b_1, z_1, z_2, ..., z_{n-1})$ satisfies all conditions of definition in Section 3 of a matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)$$

in the case of $k = 1$, $m = n-1, \ \eta = n-1, \ \alpha = 1, \ \beta = 1$ and $\gamma = n-1$. Therefore, by Theorem 3, we can find a 2-local automorphism on $\mathcal{L}_n^{2,\lambda}$ which is not linear.

Similarly we prove that $\mathcal{L}_n^{2,1}$ has 2-local automorphisms which are not automorphisms.

Now, we take $\mathcal{L}_n^{3,-1}$, $\mathcal{L}_n^{3,0}$, $\mathcal{L}_n^{3,1}$, $\mathcal{L}_n^{4,\mu}$ and $\mathcal{L}_n^{5,\lambda,\mu}$. By Theorem 13, the matrix of automorphisms of $\mathcal{L}_n^{3,-1}$ and $\mathcal{L}_n^{3,0}$ has the following forms respectively:

$$\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
\alpha_2 & \lambda_2 & 0 & 0 & 0 & 0 & \dots & 0 & \alpha_2 & 0\\
\alpha_3 & \alpha_1^{n-1}\alpha_2 & \lambda_3 & 0 & 0 & 0 & \dots & 0 & \alpha_3 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\alpha_{n-4} & \alpha_1^{n-1}\alpha_{n-5} & \alpha_1^{n-1}\alpha_{n-6} & \dots & \alpha_1^{n-1}\alpha_2 & \lambda_{n-4} & 0 & 0 & \alpha_{n-4} & 0\\
\alpha_{n-3} & \alpha_1^{n-1}\alpha_{n-4} & \alpha_1^{n-1}\alpha_{n-5} & \dots & \alpha_1^{n-1}\alpha_3 & \alpha_1^{n-1}\alpha_2 & \lambda_{n-3} & 0 & \alpha_{n-3} & 0\\
\alpha_{n-2} & \alpha_1^{n-1}\alpha_{n-3} & \alpha_1^{n-1}\alpha_{n-4} & \alpha_1^{n-1}\alpha_{n-5} & \dots & \alpha_1^{n-1}\alpha_3 & \alpha_1^{n-1}\alpha_2 & \lambda_{n-2} & \beta_{n-2} & \alpha_1\beta_{n-3}\\
\alpha_{n-1} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \alpha_1+\alpha_{n-1} & 0\\
\alpha_n & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \beta_n & \alpha_1(\alpha_1+\alpha_{n-1})
\end{pmatrix}$$

and

$$\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
\alpha_2 & \lambda_2 & 0 & 0 & \dots & 0 & \alpha_2 & 0\\
\alpha_3 & \alpha_1\alpha_2 & \lambda_3 & 0 & \dots & 0 & \alpha_3 & 0\\
\alpha_4 & \alpha_1\alpha_3 & \alpha_1^2\alpha_2 & \lambda_4 & \dots & 0 & \alpha_4 & 0\\
\alpha_5 & \alpha_1\alpha_4 & \alpha_1^2\alpha_3 & \alpha_1^3\alpha_2 & \dots & 0 & \alpha_5 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\alpha_{n-4} & \alpha_1\alpha_{n-5} & \alpha_1^2\alpha_{n-6} & \alpha_1^3\alpha_{n-7} & \dots & 0 & \alpha_{n-4} & 0\\
\alpha_{n-3} & \alpha_1\alpha_{n-4} & \alpha_1^2\alpha_{n-5} & \alpha_1^3\alpha_{n-6} & \dots & 0 & \beta_{n-3} & 0\\
\alpha_{n-2} & \alpha_1\alpha_{n-3} & \alpha_1^2\alpha_{n-4} & \alpha_1^3\alpha_{n-5} & \dots & \lambda_{n-2} & \beta_{n-2} & (\beta_{n-3}-\alpha_{n-3})\alpha_1\\
\alpha_{n-1} & 0 & 0 & 0 & \dots & 0 & \alpha_1+\alpha_{n-1} & 0\\
\alpha_n & \alpha_1\alpha_{n-1} & 0 & 0 & \dots & 0 & \beta_n & \alpha_1^2
\end{pmatrix},$$

where $\lambda_i = \alpha_1^{i-1}(\alpha_1+\alpha_{n-1})$, $i = 2, 3, \dots, n-2$.

For the algebras $\mathcal{L}_n^{3,1}$, $\mathcal{L}_n^{4,\mu}$ and $\mathcal{L}_n^{5,\lambda,\mu}$ the matrix of their automorphisms has the following form

$$\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
\alpha_2 & \alpha_1^2 & 0 & 0 & \dots & 0 & 0 & 0\\
\alpha_3 & \alpha_1^2\alpha_2 & \alpha_1^3 & 0 & \dots & 0 & 0 & 0\\
\alpha_4 & \alpha_1^3\alpha_3 & \alpha_1^3\alpha_2 & \alpha_1^4 & \dots & 0 & 0 & 0\\
\alpha_5 & \alpha_1^4\alpha_4 & \alpha_1^4\alpha_3 & \alpha_1^4\alpha_2 & \dots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\alpha_{n-2} & \alpha_1^{n-3}\alpha_{n-3} & \alpha_1^{n-3}\alpha_{n-4} & \alpha_1^{n-3}\alpha_{n-5} & \dots & \alpha_1^{n-2} & \beta_{n-2} & 0\\
0 & 0 & 0 & 0 & \dots & 0 & \alpha_1 & 0\\
\alpha_n & 0 & 0 & 0 & \dots & 0 & \beta_n & 2\alpha_1^2
\end{pmatrix}$$

By these forms and Theorem 3, similar to the cases of $\mathcal{L}_n^{1,\lambda}$ and $\mathcal{L}_n^{2,0}$ we can prove that the algebras $\mathcal{L}_n^{3,-1}$, $\mathcal{L}_n^{3,0}$, $\mathcal{L}_n^{3,1}$, $\mathcal{L}_n^{4,\mu}$ and $\mathcal{L}_n^{5,\lambda,\mu}$ also have 2-local automorphisms which are not automorphisms. This ends the proof. □

Conclusion

In conclusion, it can be said that the article generalizes the methods of studying 2-local derivations and automorphisms of algebras. The method proposed in the second section allows one to make a direct conclusion about whether all 2-local derivations (respectively, automorphisms) are derivations (respectively, automorphisms) based on the general matrix form of the matrix of a derivation (respectively, an automorphism) of an algebra. This method is useful since often the derivation (automorphism) of an algebra has the matrix form in the method under consideration. In the third section, a method is developed that allows one to obtain an entire subspace (an entire subgroup) of 2-local derivations (respectively, 2-local automorphisms) that are not derivations (respectively, automorphisms). As is known, the set of all 2-local derivations (2-local automorphisms) of an algebra forms a vector space (respectively, a group) and the description of this vector space (this group) is an open problem. We think that the method developed in the third section allows to solve this problem.

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