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$\mathcal{K}$-FUNCTIONALS AND EXACT VALUES OF n-WIDTHS IN THE BERGMAN SPACE

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Abstract: In this paper, we consider the problem of mean-square approximation of complex variables functions which are regular in the unit disk of the complex plane. We obtain sharp estimates of the value of the best approximation by algebraic polynomials in terms of $\mathcal{K}$-functionals. Exact values of some widths of the specified class of functions are calculated.

Key words: Bergman space, Best mean-square approximation, $\mathcal{K}$-functional, n-width.

Introduction and preliminary facts

We consider the problem of mean-square approximation by Fourier sums of complex functions f which are regular in a simply connected domain $\mathcal{D} \subset \mathbb{C}$ and belong to the space $L_2 := L_2(\mathcal{D})$ with the finite norm

$$\|f\| := \|f\|_{L_2(\mathcal{D})} = \left(\frac{1}{\pi}\iint_{(\mathcal{D})} |f(z)|^2 d\sigma\right)^{1/2},$$

where the integral is understood in the Lebesgue sense and $d\sigma$ is an element of area.

The study of the mean-square approximation of functions in the domain $\mathcal{D} \subset \mathbb{C}$ is closely related to the theory of orthogonal functions. A sequence of complex functions $\{\varphi_k(z)\}$ $(k = 0, 1, 2, ...)$ is called an orthogonal system on the domain $\mathcal{D}$ if

$$\frac{1}{\pi}\iint_{(\mathcal{D})} \varphi_k(z)\overline{\varphi_l(z)}d\sigma = 0, \qquad k \neq l.$$

Such a sequence of functions is called orthonormal system if

$$\frac{1}{\pi}\iint_{(\mathcal{D})} \varphi_k(z)\overline{\varphi_l(z)}d\sigma = \delta_{k,l},$$

where $\delta_{k,l} = 0,\ k \neq l$, and $\delta_{k,k} = 1,\ k \in \mathbb{N}$. If $f \in L_2$, then the numbers

$$a_k(f) = \frac{1}{\pi}\iint_{(\mathcal{D})} f(z)\overline{\varphi_k(z)}d\sigma \tag{1}$$

are called the Fourier coefficients of the function f with respect to the orthonormal system $\{\varphi_k(z)\}$ $(k = 0, 1, 2, ...)$. We associate with a given function f its Fourier series with respect to the specified orthogonal system:

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f)\varphi_k(z). \tag{2}$$

Let

$$S_{n-1}(f, z) = \sum_{k=0}^{n-1} a_k(f)\varphi_k(z)$$

be the partial sum of order n of the series (2). We form a linear combination of the first n functions of the system $\{\varphi_k(z)\}$:

$$p_{n-1}(z) = \sum_{k=0}^{n-1} d_k \varphi_k(z),$$

where $d_k \in \mathbb{C}$ are arbitrary complex coefficients. We call this linear combination a generalized polynomial. It is well known (see, for example, [1], p.263) that

$$\begin{aligned} E_{n-1}(f) &= \inf\{\|f - p_{n-1}\| : \ d_k \in \mathbb{C}\} \\ &= \|f - S_{n-1}(f)\| = \left\{\sum_{k=n}^{\infty} |a_k(f)|^2\right\}^{1/2}, \end{aligned} \tag{3}$$

where $a_k(f)$ are the Fourier coefficients of the function f defined by (1).

In the case of the mean approximation of complex functions in a simply connected domain $\mathcal{D} \subset \mathbb{C}$ by Fourier series with respect to an orthogonal system of functions $\{\varphi_k(z)\}_{k=0}^{\infty}$ on $\mathcal{D}$, the problem of finding the exact constant in the Jackson-Stechkin inequality was studied in [2]. Recall that Jackson-Stechkin inequalities are inequalities in which the value of the best approximation of a function by a finite dimensional subspace of a given normed space is estimated by the modulus of smoothness of the function itself or some its derivative. In this paper, we use the same methods as in [2, 3, 5, 15].

We study in more detail the case where $\mathcal{D}$ is the unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$. In this case, it is clear that the system of functions $\varphi_k(z) = z^k (k = 0, 1, 2, ...)$ is orthogonal in the disk U:

$$\frac{1}{\pi}\iint_{(U)} \varphi_k(z)\overline{\varphi_l(z)}d\sigma = \frac{1}{\pi}\int_0^1\int_0^{2\pi} r^{k+l+1}e^{i(k-l)t}drdt = 0, \quad k \neq l.$$

However, this system is not orthonormal, since

$$\frac{1}{\pi}\iint_{(U)} |\varphi_k(z)|^2 d\sigma = \frac{1}{\pi}\int_0^1\int_0^{2\pi} r^{2k+1}drdt = \frac{1}{k+1}.$$

Therefore, the system of functions $\varphi_k^*(z) = \sqrt{k+1}z^k$ $(k = 0, 1, 2, ...)$ is orthonormal. We denote by $\mathcal{A}(U)$ the set of all functions f analytic in U. The Maclaurin series of such a function has the form

$$f(z) = \sum_{k=0}^{\infty} c_k(f)z^k, \tag{4}$$

where $c_k(f)$ are the Maclaurin coefficients of f. We note that

$$\|f\|^2 = \sum_{k=0}^{\infty} \frac{|c_k(f)|^2}{k+1}, \quad E_{n-1}^2(f) = \sum_{k=n}^{\infty} \frac{|c_k(f)|^2}{k+1}. \tag{5}$$

It was proved in the monograph [1] that the Fourier series of a function f with respect to the orthonormal system $\varphi_k^*(z) = \sqrt{k+1}z^k$, $k = 0, 1, 2, ...$, coincides with the series (4) for $f \in \mathcal{A}(U)$; i.e.,

$$f(z) = \sum_{k=0}^{\infty} a_k(f)\varphi_k^*(z) = \sum_{k=0}^{\infty} c_k(f)z^k. \tag{6}$$

Therefore, the series (6) can be differentiated term by term any number of times and, according to the Weierstrass theorem [6, p.107], for any $r \in \mathbb{N}$, we get

$$f^{(r)}(z) = \sum_{k=r}^{\infty} c_k(f)k(k-1)\cdots(k-r+1)z^{k-r} := \sum_{k=r}^{\infty} \alpha_{k,r}c_k(f)z^{k-r}, \tag{7}$$

where

$$\alpha_{k,r} := k(k-1)\cdots(k-r+1), \quad k \in \mathbb{N}, \quad r \in \mathbb{Z}_+, \quad k \geq r.$$

We denote by $L_2^{(r)} := L_2^{(r)}(U)$ $(L_2^{(0)} := L_2(U))$ the class of all functions $f \in L_2$ such that $f^{(r)} \in L_2$ $(r \in \mathbb{Z}_+,\ f^{(0)} \equiv f)$.

1. Sharp estimates of the value of the best approximation by means of $\mathcal{K}$-functionals

In this section, we prove some sharp inequalities relating the value $E_{n-1}(f)$ of the best approximation of functions in the class $L_2^{(r)}$ and Peetre $\mathcal{K}$-functionals. The definition and some properties of Peetre $\mathcal{K}$-functionals are given in [7]. The direct and inverse theorems of the theory of approximation by means of $\mathcal{K}$-functionals were proved in [8, 9]. We define the $\mathcal{K}$-functional constructed by the spaces L_2 and $L_2^{(m)}$ as follows:

$$\mathcal{K}_m(f,t^m)_2 := \mathcal{K}\left(f,t^m;L_2;L_2^{(m)}\right) = \inf\left\{\|f-g\|_2 + t^m\cdot\|g^{(m)}\|_2 : g \in L_2^{(m)}\right\}, \tag{8}$$

where $m \in \mathbb{N}$ and $0 < t \leq 1$. We note that a weak equivalence of the $\mathcal{K}$-functional defined by (8) and a special generalized modulus of continuity of order m was established in [8].

Theorem 1. *Let $n, m \in \mathbb{N}$ and $r \in \mathbb{Z}_+$ be arbitrary numbers such that $n \geq r+m$. Then the following equality holds*:

$$\sup_{\substack{f\in L_2^{(r)}\\ f\notin\mathcal{P}_r}} \frac{\sqrt{(n+1)/(n-r+1)}\cdot\alpha_{n,r}E_{n-1}(f)}{\mathcal{K}_m\left(f^{(r)},\sqrt{\dfrac{n-r-m+1}{n-r+1}}\cdot\dfrac{1}{\alpha_{n-r,m}}\right)} = 1. \tag{9}$$

P r o o f. Using (7), we easily find that

$$E^2_{n-r-1}(f^{(r)}) = \sum_{k=n}^{\infty}\alpha^2_{k,r}\frac{|c_k(f)|^2}{k-r+1}, \quad r \in \mathbb{Z}_+. \tag{10}$$

Taking into account equality (10), we obtain

$$\begin{aligned} E^2_{n-1}(f) &= \sum_{k=n}^{\infty}\frac{|c_k(f)|^2}{k+1} = \sum_{k=n}^{\infty}\frac{k-r+1}{(k+1)\alpha^2_{k,r}}\cdot\alpha^2_{k,r}\cdot\frac{|c_k(f)|^2}{k-r+1} \\ &\leq \max_{\substack{k\in\mathbb{N}\\ k\geq n}}\left\{\frac{k-r+1}{(k+1)\alpha^2_{k,r}}\right\}\cdot\sum_{k=n}^{\infty}\alpha^2_{k,r}\frac{|c_k(f)|^2}{k-r+1} \\ &= \frac{n-r+1}{n+1}\cdot\frac{1}{\alpha^2_{n,r}}\cdot E^2_{n-r-1}\left(f^{(r)}\right). \end{aligned} \tag{11}$$

Now, for an arbitrary function $f \in L_2^{(r)}$, we write

$$E_{n-1}(f) \leq \sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}E_{n-r-1}\left(f^{(r)}\right) \leq \sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}\|f^{(r)} - S_{n-r-1}(g)\|, \tag{12}$$

where $S_{n-r-1}(g)$ is the partial sum of order $n-r$ of the Fourier series of an arbitrary function $g \in L_2^{(m)}$. In view of (2) and (11), we get

$$\begin{aligned} \|g - S_{n-r-1}(g)\| = E_{n-r-1}(g) &\leq \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} E_{n-r-m-1}\left(g^{(m)}\right) \\ &\leq \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} \left\|g^{(m)}\right\|. \end{aligned} \tag{13}$$

It follows from inequalities (12) and (13) that

$$\begin{aligned} E_{n-1}(f) &\leq \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \left\{\|f^{(r)} - g\| + \|g - S_{n-r-1}(g)\|\right\} \\ &\leq \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \left\{\|f^{(r)} - g\| + \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} \left\|g^{(m)}\right\|\right\}. \end{aligned} \tag{14}$$

Now, we note that the left-hand side of inequality (14) does not depend on $g \in L_2^{(m)}$. Therefore, passing to the infimum over all functions $g \in L_2^{(m)}$ on the right-hand side of (14) and using the definition (8) of $\mathcal{K}$, we get

$$E_{n-1}(f) \leq \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \mathcal{K}_m\left(f^{(r)}, \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right).$$

This implies the following upper bound:

$$\sup_{\substack{f \in L_2^{(r)} \\ f \notin \mathcal{P}_r}} \frac{\sqrt{(n+1)/(n-r+1)} \cdot \alpha_{n,r} E_{n-1}(f)}{\mathcal{K}_m\left(f^{(r)}, \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right)} \leq 1, \tag{15}$$

where $\mathcal{P}_r$ is the subspace of complex algebraic polynomials of degree at most r.

To obtain a lower bound of the extremal characteristic on the left-hand side of (15), in (8), we put $f(z) := p_n(z)$, where $p_n(z)$ is an arbitrary complex algebraic polynomial in $\mathcal{P}_n$. Since the function $g(z) \equiv 0$ belongs to the class $L_2^{(m)}$, we obtain from (8) the upper bound

$$\mathcal{K}_m(p_n; t^m)_2 \leq \|p_n\|.$$

Since the function $g(z) := p_n(z)$ also belongs to the class $L_2^{(m)}$, we find from (8) that

$$\mathcal{K}_m(p_n; t^m)_2 \leq t^m \|p_n^{(m)}\|.$$

Thus, the last two relations imply that, for any element $p_n(z) \in \mathcal{P}_n$,

$$\mathcal{K}_m(p_n; t^m)_2 \leq \min\left\{\|p_n\|; t^m \|p_n^{(m)}\|\right\}. \tag{16}$$

We consider the function $f_0(z) = z^n$. Since

$$f_0^{(r+m)} = n(n-1)\cdots(n-r+1)\cdots(n-r-m+1)z^{n-r-m} = \alpha_{n,r} \cdot \alpha_{n-r,m} z^{n-r-m},$$

according to (16), we have

$$\begin{aligned} \mathcal{K}\left(f_0^{(r)}; \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right) &\leq \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} \|f_0^{(r+m)}\| \\ = \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} \cdot \frac{\alpha_{n,r} \cdot \alpha_{n-r,m}}{\sqrt{n-r-m+1}} &= \frac{\alpha_{n,r}}{\sqrt{n-r+1}}. \end{aligned}$$

Using the obtained inequality and the second equality in (5), we establish that

$$\begin{aligned}&\sup_{\substack{f\in L_2^{(r)}\\ f\in\mathcal{P}_r}}\frac{\sqrt{(n+1)/(n-r+1)}\cdot\alpha_{n,r}E_{n-1}(f)}{\mathcal{K}_m\left(f^{(r)},\sqrt{\dfrac{n-r-m+1}{n-r+1}}\cdot\dfrac{1}{\alpha_{n-r,m}}\right)}\\ &\geq\frac{\sqrt{(n+1)/(n-r+1)}\cdot\alpha_{n,r}E_{n-1}(f_0)}{\mathcal{K}_m\left(f_0^{(r)},\sqrt{\dfrac{n-r-m+1}{n-r+1}}\cdot\dfrac{1}{\alpha_{n-r,m}}\right)}\geq 1.\end{aligned}\tag{17}$$

We obtain equality (9) by comparing the upper bound (15) with the lower bound (17). The theorem is proved.

2. Exact values of n-widths of a class of functions

We assume that S is the unit ball in the space L_2, $\Lambda_n\subset L_2$ is an n-dimensional subspace, and $\Lambda^n\subset L_2$ is a subspace of codimension n. Let $\mathcal{L}:L_2\to\Lambda_n$ be a continuous linear operator, let $\mathcal{L}^\perp:L_2\to\Lambda_n$ be a continuous linear projection operator, and let $\mathfrak{M}$ be a convex centrally symmetric subset of L_2. The quantities

$$b_n(\mathfrak{M},L_2)=\sup\left\{\sup\{\varepsilon>0;\ \varepsilon S\cap\Lambda_{n+1}\subset\mathfrak{M}\}:\Lambda_{n+1}\subset L_2\right\},$$

$$d_n(\mathfrak{M},L_2)=\inf\left\{\sup\{\inf\{\|f-g\|:g\in\Lambda_n\}:f\in\mathfrak{M}\}:\Lambda_n\subset L_2\right\},$$

$$\delta_n(\mathfrak{M},L_2)=\inf\left\{\inf\{\sup\{\|f-\mathcal{L}f\|:f\in\mathfrak{M}\}:\mathcal{L}L_2\subset\Lambda_n\}:\Lambda_n\subset L_2\right\},$$

$$d^n(\mathfrak{M},L_2)=\inf\left\{\sup\{\|f\|_{2,\gamma}:f\in\mathfrak{M}\cap\Lambda^n\}:\Lambda^n\subset L_2\right\},$$

$$\Pi_n(\mathfrak{M},L_2)=\inf\left\{\inf\left\{\sup\left\{\|f-\mathcal{L}^\perp f\|:f\in\mathfrak{M}\right\}:\mathcal{L}^\perp L_2\subset\Lambda_n\right\}:\Lambda_n\subset L_2\right\}$$

are called, respectively, the *Bernstein*, *Kolmogorov*, *linear*, *Gelfand*, and *projection* n-widths of the subset $\mathfrak{M}$ in the space L_2. These widths are monotone with respect to n, and the following relation holds (see, for example, [10, 11]):

$$b_n(\mathfrak{M},L_2)\leq d^n(\mathfrak{M},L_2)\leq d_n(\mathfrak{M},L_2)=\delta_n(\mathfrak{M},L_2)=\Pi_n(\mathfrak{M},L_2).\tag{18}$$

We recall (see, for example, [12, p. 25]) that a nondecreasing function Ψ on $\mathbb{R}_+$ is called a k-majorant if the function $t^{-k}\Psi(t)$ is nonincreasing in $\mathbb{R}_+$, $\Psi(0)=0$, and $\Psi(t)\to 0$ as $t\to 0$. For $k=1$, the function Ψ is simply called a majorant.

Let $W_2^{(r)}(\mathcal{K}_m,\Psi)$, $r\in\mathbb{Z}_+,m\in\mathbb{N}$, be the class of all functions $f\in L_2^{(r)}$ whose derivatives $f^{(r)}$ satisfy the condition

$$\mathcal{K}_m(f^{(r)},t^m)\leq\Psi(t^m),\quad 0<t<1.$$

In this definition, Ψ is a majorant, $L_2^{(0)}\equiv L_2$, and $W_2^{(0)}(\mathcal{K}_m,\Psi)=W_2(\mathcal{K}_m,\Psi)$. For any subset $\mathfrak{M}\subset L_2$, we define

$$E_{n-1}(\mathfrak{M})_{L_2}:=\sup\left\{E_{n-1}(f):\ f\in\mathfrak{M}\right\}.$$

We note that, in the Bergman space, values of widths of some classes of analytic functions in a disk were calculated, for example, in [13–19].

Theorem 2. *Let Ψ be the majorant defining the class $W_2^{(r)}(\mathcal{K}_m, \Psi)$, $m \in \mathbb{N}$, and $r \in \mathbb{R}_+$. Then, for any natural number $n \geq m+r$, we have*

$$\begin{aligned} &\lambda_n\left(W_2^{(r)}(\mathcal{K}_m, \Psi), L_2\right) = E_{n-1}\left(W_2^{(r)}(\mathcal{K}_m, \Psi)\right) \\ &= \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right), \end{aligned} \tag{19}$$

where $\lambda_n(\cdot)$ is any of the n-widths $b_n(\cdot)$, $d_n(\cdot)$, $d^n(\cdot)$, $\delta_n(\cdot)$, and $\Pi_n(\cdot)$.

P r o o f. Let n be a natural number such that $n \geq m+r$. In view of the definition of the class $W_2^{(r)}(\mathcal{K}_m, \Psi)$, relations (15) and (18) imply that

$$\begin{aligned} &\lambda_n\left(W_2^{(r)}(\mathcal{K}_m, \Psi), L_2\right) \leq d_n\left(W_2^{(r)}(\mathcal{K}_m, \Psi), L_2\right) \\ &\leq E_{n-1}\left(W_2^{(r)}(\mathcal{K}_m, \Psi)\right) \leq \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right). \end{aligned} \tag{20}$$

To find the corresponding lower bound, in view of (18), it suffices to estimate the Bernstein n-width of the class $W_2^{(r)}(\mathcal{K}_m, \Psi)$. On the set $\mathcal{P}_n \cap L_2$, we define the ball

$$\mathcal{M}_{n+1} := \left\{p_n \in \mathcal{P}_n : \|p_n\| \leq \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right)\right\}.$$

Now, we note that, in view of formula (7) and the identity $\alpha_{k,r+m} = \alpha_{k,r}\,\alpha_{k-r,m}$, for an arbitrary $p_n(z) = \sum_{k=0}^n a_k(p_n) z^k \in \mathcal{P}_n$, the following equality holds:

$$p_n^{(r+m)}(z) = \sum_{k=r+m}^{n} a_k(p_n)\alpha_{k,r+m} z^{k-r-m} := \sum_{k=r+m}^{n} a_k(p_n)\alpha_{k,r} \cdot \alpha_{k-r,m} z^{k-r-m}.$$

Hence, using the Parseval equality and the inequality $\alpha_{k,r} \leq \alpha_{n,r}, k \leq n$, we obtain the Bernstein type inequality

$$\|p_n^{(r+m)}\| = \left\{\sum_{k=r+m}^{n} |a_k(p_n)|^2 \alpha_{k,r}^2 \cdot \alpha_{k-r,m}^2\right\}^{1/2} \leq \alpha_{n,r} \cdot \alpha_{n-r,m} \|p_n\|. \tag{21}$$

By definition, for the majorant Ψ and for any $0 < \tau_1 \leq \tau_2 \leq 1$, we have the inequality $\tau_1 \Psi(\tau_2) \leq \tau_2 \Psi(\tau_1)$. Therefore, for any $0 < t_1 \leq t_2 \leq 1$, setting $\tau_1 = t_1^m$ and $\tau_2 = t_2^m$, we obtain

$$t_1^{-m} \Psi(t_1^m) \geq t_2^{-m} \Psi(t_2^m). \tag{22}$$

We now show that $\mathcal{M}_{n+1} \subset W_2^{(r)}(\mathcal{K}_m, \Psi)$. Thus, we need to prove that, for any polynomial $p_n \subset \mathcal{M}_{n+1}$,

$$\mathcal{K}_m(p_n^{(r)}, t^m) \leq \Psi(t^m),\ 0 < t \leq 1.$$

Since, by assumption, $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, and $n \geq m+r$, we consider two cases:

$$0 < t \leq \left(\sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right)^{1/m}$$

and
$$\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)^{1/m}\le t\le 1.$$
First, assume that
$$0<t\le\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)^{1/m}.$$
In this case, using inequality (22) with
$$t_1=t,\quad t_2=\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)^{1/m}$$
and applying (12) and (21), for any $p_n\in\mathcal{M}_{n+1}$, we obtain
$$\begin{aligned}\mathcal{K}_m(p_n^{(r)},t^m)_2&\le t^m\cdot\|p_n^{(r+m)}\|\le t^m\cdot\alpha_{n,r}\cdot\alpha_{n-r,m}\|p_n\|\\&\le t^m\cdot\alpha_{n,r}\cdot\alpha_{n-r,m}\cdot\sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\\&\le t^m\cdot\alpha_{n-r,m}\cdot\sqrt{\frac{n-r+1}{n-r-m+1}}\cdot\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\le\Psi(t^m).\end{aligned}\tag{23}$$

Now, let
$$\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)^{1/m}\le t\le 1.$$
Then using (16) and the Bernstein type inequality
$$\|p_n^{(r)}\|\le\alpha_{n,r}\cdot\|p_n\|$$
and taking into account that the majorant Ψ is nondecreasing, we find that
$$\begin{aligned}\mathcal{K}_m(p_n^{(r)},t^m)_2&\le\|p_n^{(r)}\|_2\le\alpha_{n,r}\|p_n\|_2\\&\le\alpha_{n,r}\sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\\&\le\sqrt{\frac{n-r+1}{n+1}}\cdot\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\\&\le\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\le\Psi(t^m).\end{aligned}\tag{24}$$
The definition of the class $W_2^{(r)}(\mathcal{K}_m,\Psi)$ along with (23) and (24) implies that $\mathcal{M}_{n+1}\subset W_2^{(r)}(\mathcal{K}_m,\Psi)$. Then, taking into account the definition of the Bernstein n-width and (18), we obtain
$$\begin{aligned}\lambda_n\left(W_2^{(r)}(\mathcal{K}_m,\Psi),L_2\right)&\ge b_n\left(W_2^{(r)}(\mathcal{K}_m,\Psi),L_2\right)\\&\ge b_n(\mathcal{M}_{n+1};L_2)\ge\sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right).\end{aligned}\tag{25}$$
Comparing the upper bound (20) and the lower bound (25), we get the required equality (19). The theorem is proved.

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