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# ASYMPTOTIC BEHAVIOR FOR THE LOTKA–VOLTERRA EQUATION WITH DISPLACEMENTS AND DIFFUSION

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**Abstract:** In this paper, we consider the Lotka–Volterra equation with displacements and diffusion, that is transport-diffusion system describing the evolution of prey and predator populations with their displacements and their diffusion, in a periodic domain in $\mathbb{R}$. It is shown that the solution to this equation and its logarithm are globally bounded, and that, when the solution converges to the stationary solution in mean value, it converges uniformly with respect to the time variable as well as the space variable. These results are obtained by using $L^2$-estimate of the well-known Lyapunov functional, and, in particular, an estimate of the point-wise growth of the solution by means of the application of the fundamental solution of the heat equation.

**Keywords:** Lotka–Volterra equation, Asymptotic behavior, Diffusion, Transport/displacement, Numerical example.

## 1. Introduction

As is well-known, the system of equations called *Lotka–Volterra equation*,

$$\left\{\begin{array}{l} \dfrac{d}{dt}u_1 = \alpha u_1 - \beta u_1 u_2, \\ \dfrac{d}{dt}u_2 = -\gamma u_2 + \delta u_1 u_2, \end{array}\right.$$

$(\alpha, \beta, \gamma, \delta > 0)$ was proposed to model the evolution of prey and predator populations (represented by $u_1$ and $u_2$, respevtively). This system of equations has the particularity that all its (positive) solutions are periodic, as illustrated in [16]. In [16], we also find a detailed analysis of the behavior of the solution and various versions of the equation.

As for the Lotka–Volterra equation with diffusion, Rothe [15] considered the Lotka–Volterra equation with diffusion (with the same diffusion coefficient for both species) in one-dimensional domain $0 < x < 1$ with periodic boundary conditions in $x$ (or homogeneous Neumann conditions) and proved the uniform convergence to the time-periodic solution of the Lotka–Volterra equation (independent of $x$) (see also [14], which had made similar reasoning). On the other hand, Gabbuti and Negro [8] proved the convergence of the solution of the Lotka–Volterra equation with diffusion in a bounded domain of $\mathbb{R}^2$ with the homogeneous Neumann condition to the time-periodic solution of the Lotka–Volterra equation (independent of $x$); in the article [8], the diffusion coefficients are not the same for both species and the convergence is in an integral sense, but sufficiently strong. Successively, the asymptotic behavior of the solution of the Lotka–Volterra equation with diffusion with the Dirichlet condition was studied in [18], while the aspects of spatial propagation of the solution to the Lotka–Volterra equation continue to attract the interest of researchers (see for example [4, 5]).

As far as concerns the Lotka–Volterra equation with diffusion in one spatial dimension, the question concerning the travelling waves has attracted the interest of many researchers. However, the results of [14] and [15] exclude the existence of a travelling wave for the classical Lotka–Volterra equation with simple diffusion. For this reason, several researchers have sought some aspects of travelling wave for slightly modified equations (see for example [2, 3, 6, 10, 17]).

In the context of stochastic equations, the Lotka–Volterra equation with logistic effect and diffusion has been studied in [7] and [9]. In [7] the existence and uniqueness theorem of the solution has been proved, and in [9] the existence of an invariant measure has been shown.

In [13] the author has considered the Lotka–Volterra equation with diffusion and population displacements. The results of this article are essentially numerical. However, the question of population displacement/immigration has attracted the attention of many researchers, as evidenced by several recent publications (see for example [1, 11, 12]).

In this article, we consider the Lotka–Volterra equation for the population density $u_1(t,x)$ and $u_2(t,x)$ with diffusion and population displacements on the periodic domain of $\mathbb{R}$ and prove the uniform boundedness of $u_1(t,x)$, $u_2(t,x)$, $\log u_1(t,x)$, $\log u_2(t,x)$. We also prove that in the case where the solution $(u_1,u_2)$ tends to the stationary solution in mean value, $(u_1,u_2)$ converges uniformly to the stationary solution. In order to obtain this result, we use the function

$$U = -\alpha \log(u_2) - \gamma \log(u_1) + \beta u_2 + \delta u_1,$$

but due to the population displacements we cannot directly deduce a conclusion from the equation for $U$, as the authors of [14] and [15] did. In order to overcome this difficulty, we estimate not only $U$ in $L^2(0,2\pi)$ but also point-wise growth of $u_1(t,x)$ and $u_2(t,x)$.

Our study is motivated not only by the general interest of the effect of displacement/immigration for population dynamics but also by the specific behavior that arises from the numerical calculation of the solution of the Lotka–Volterra equation with population displacement in opposite directions for prey and predator populations. This will be illustrated in the following section.

## 2. Motivation and some numerical examples

As we mentioned in Introduction, the evolution of prey and predator populations is described, in its basic form, by Lotka–Volterra equation

$$\frac{d}{dt}u_1(t) = \alpha u_1(t) - \beta u_1(t)u_2(t), \tag{2.1}$$

$$\frac{d}{dt}u_2(t) = -\gamma u_2(t) + \delta u_1(t)u_2(t), \tag{2.2}$$

where $u_1(t)$ and $u_2(t)$ denote the prey and predator populations, respectively, while the coefficients $\alpha$, $\beta$, $\gamma$ and $\delta$ are assumed to be constants and strictly positive. We consider the system of equations (2.1)–(2.2) with the initial conditions

$$u_1(0) = u_{1,0} > 0, \quad u_2(0) = u_{2,0} > 0. \tag{2.3}$$

We first recall the well-known fundamental properties of the solution of the system of equations (2.1)–(2.2). For this, we define the function $U_0(\cdot,\cdot)$ as

$$U_0(u_1,u_2) = -\alpha \log u_2 - \gamma \log u_1 + \beta u_2 + \delta u_1, \quad u_1 > 0, \quad u_2 > 0.$$

*Remark 1.* For any initial data $u_{1,0} > 0$, $u_{2,0} > 0$, the solution $(u_1(t),u_2(t))$ of the Cauchy problem (2.1)–(2.3) exists for all $t > 0$ and it is periodic in $t$. Furthermore, the function $U_0(u_1(t),u_2(t))$ remains constant, i.e.

$$U_0(u_1(t),u_2(t)) = U_0(u_1(0),u_2(0)) = -\alpha \log(u_2(0)) - \gamma \log(u_1(0)) + \beta u_2(0) + \delta u_1(0)$$

for all $t \geq 0$ and the solution $(u_1(t), u_2(t))$ moves along the closed curve

$$\gamma = \{\, (u_1, u_2) \,|\, u_1 > 0,\ u_2 > 0,\ U_0(u_1, u_2) = U_0(u_1(0), u_2(0)) \,\}$$

with a constant period.

The proof of this fact can be found in [16] (and in many other manuals on population dynamics).

The model (2.1)–(2.2) is constructed for the prey and predators populations homogeneously distributed in a territory. But, if the populations are not homogeneously distributed and if there are population displacements, the relations mentioned in Remark 1 will not be guaranteed. Let us see an example of changing the behavior of the solution.

Consider the equation system

$$\begin{cases} \partial_t u_1(t,x) = -v_1(t)\partial_x u_1(t,x) + \alpha u_1(t,x) - \beta u_1(t,x)u_2(t,x), \\ \partial_t u_2(t,x) = -v_2(t)\partial_x u_2(t,x) - \gamma u_2(t,x) + \delta u_1(t,x)u_2(t,x), \end{cases} \quad t > 0, \quad x \in \mathbb{R}, \tag{2.4}$$

with the initial condition

$$u_1(0,x) = \overline{u}_1(x), \quad u_2(0,x) = \overline{u}_2(x).$$

Let us choose a particular initial data $(\overline{u}_1(x), \overline{u}_2(x))$ defined as follows: consider the equation system (2.1)–(2.2) and write $x$ instead of $t$ in the solution $(\overline{u}_1(\cdot), \overline{u}_2(\cdot))$ to these equations. It is clear that the thus defined functions $\overline{u}_1(x)$ and $\overline{u}_2(x)$ can be defined on $\mathbb{R}$ and are periodic in $x$. Let us further assume that

$$v_1(t) = -v_2(t) \quad \forall t \geq 0$$

and that they are periodic in $t$ with the same period as the solution of the equation system (2.1)–(2.2). Then, for a certain choice of functions $(v_1(t), v_2(t))$ we find the amplification of the oscillation of the solution in certain points $x$ and the contraction in certain points $x$, as illustrated in the graphs obtained by numerical calculation (see Fig. 1–2).

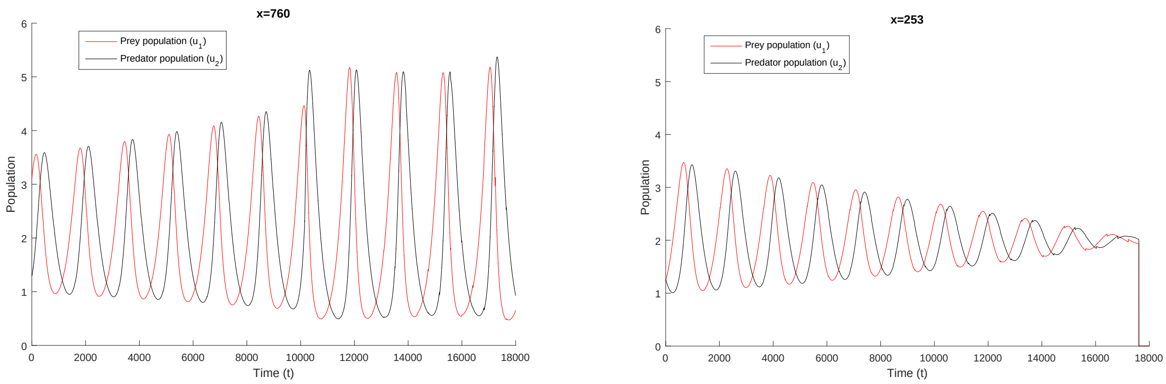


Figure 1. Solution of the equation system (2.4) at a point where amplification occurs and at a point where contraction occurs.

However, even with displacements, the equation system (2.4) in a periodic domain $x \in \mathbb{R}/\mathrm{mod}\,\mathrm{L}$ has a globally similar behavior to what we have seen in Remark 1.

*Remark 2.* Let $L$ be a strictly positive number. Let $u_{1,0}(x)$ and $u_{2,0}(x)$ be two functions with strictly positive values and periodic in $x \in \mathbb{R}$ with period $L$. If the solution $(u_1(t,x), u_2(t,x))$ to the equation system (2.4) with the initial condition

$$u_1(0,x) = u_{1,0}(x), \quad u_2(0,x) = u_{2,0}(x),$$

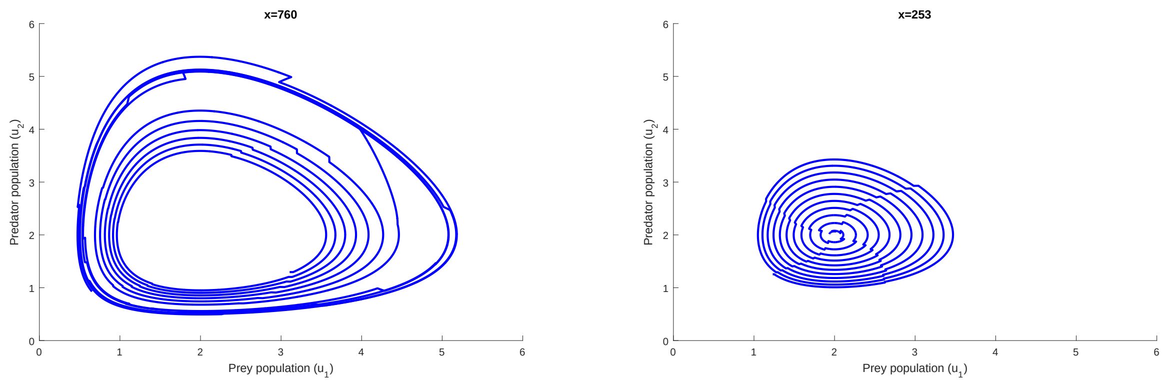


Figure 2. Trajectories of the solution of the equation system (2.4) on the phase plane at a point where amplification occurs and at a point where contraction occurs in the space $(u_1, u_2)$.

exists and is periodic in $x \in \mathbb{R}$ with period $L$, then we have

$$\int_0^L U_0(u_1(t,x), u_2(t,x))dx = \text{Const} = \int_0^L U_0(u_{1,0}(x), u_{2,0}(x))dx. \tag{2.5}$$

Indeed, it follows immediately from (2.4) that

$$\partial_t \log u_1 = -v_1 \partial_x \log u_1 + \alpha - \beta u_2, \tag{2.6}$$
$$\partial_t \log u_2 = -v_2 \partial_x \log u_2 - \gamma + \delta u_1, \tag{2.7}$$

from (2.4), (2.6) and (2.7), by direct calculations, we obtain

$$\partial_t U_0(u_1(t,x), u_2(t,x)) = -v_1 \partial_x(-\gamma \log u_1 + \delta u_1) - v_2 \partial_x(-\alpha \log u_2 + \beta u_2). \tag{2.8}$$

Given the assumption that $u_1(t,x)$ and $u_2(t,x)$ are periodic in $x$ with period $L$, we have

$$\int_0^L \partial_x(-\gamma \log u_1 + \delta u_1)dx = \int_0^L \partial_x(-\alpha \log u_2 + \beta u_2)dx = 0.$$

Thus

$$\frac{d}{dt}\int_0^L U_0(u_1(t,x), u_2(t,x))dx = 0,$$

which implies (2.5). But, we cannot deduce that $\sup_{0\le x\le 2\pi} U_0(u_1(t,x), u_2(t,x))$ is bounded at $t$.

Given these circumstances, we are interested in the asymptotic behavior of the solution $(u_1(t,x), u_2(t,x))$ of the Lotka–Volterra equation with displacements and diffusion (see (3.1)–(3.2) in the next section).

## 3. Position of problem and preliminary result

We consider in this article the following equation system

$$\partial_t u_1(t,x) = -v_1(t)\partial_x u_1(t,x) + \kappa \partial_x^2 u_1(t,x) + \alpha u_1(t,x) - \beta u_1(t,x) u_2(t,x), \tag{3.1}$$
$$\partial_t u_2(t,x) = -v_2(t)\partial_x u_2(t,x) + \kappa \partial_x^2 u_2(t,x) - \gamma u_2(t,x) + \delta u_1(t,x) u_2(t,x), \tag{3.2}$$

for $t \geq 0$ and $x \in \mathbb{R}$, where $\alpha, \beta, \gamma, \delta$ and $\kappa$ are strictly positive constants and $v_1(t)$ and $v_2(t)$ are functions of $t$. The system (3.1)–(3.2) will be considered with the initial condition

$$u_i(t,x) = u_{i,0}(x), \quad i = 1,2. \tag{3.3}$$

For the functions $u_{1,0}(x)$ and $u_{2,0}(x)$, it is assumed that

$$u_{i,0}(x) > 0, \quad u_{i,0}(x) = u_{i,0}(x+2\pi) \quad \forall x \in \mathbb{R}, \quad u_{i,0}(\cdot) \in L^\infty(\mathbb{R}), \quad i = 1,2. \tag{3.4}$$

Since the equations (3.1)–(3.2) are parabolic equations subject to the conditions (3.3)–(3.4), the existence and uniqueness of the local solution follow from the classical theory of parabolic equations. Furthermore, considering the equations (3.1)–(3.2) on $\mathbb{R}_+ \times \mathbb{T}$ with the torus $\mathbb{T} = \mathbb{R}/\mathrm{mod}\, 2\pi$, the periodicity in $x$ of the solution $(u_1(t,x), u_2(t,x))$ follows automatically. As far as concerns the global solution, we will first prove the inequality (4.3) on the interval of the existence of the solution $(u_1(t,x), u_2(t,x))$ and then deduce from the inequality (4.3) and the theorem of the existence and the uniqueness of the local solution the existence and the uniqueness of the global solution.

We now define the functions $U_1(u_1)$, $U_2(u_2)$ and $U(u_1,u_2)$:

$$U_1(u_1) = -\gamma\Big(\log u_1 - \log\frac{\gamma}{\delta}\Big) + \delta\left(u_1 - \frac{\gamma}{\delta}\right), \tag{3.5}$$

$$U_2(u_2) = -\alpha\Big(\log u_2 - \log\frac{\alpha}{\beta}\Big) + \beta\Big(u_2 - \frac{\alpha}{\beta}\Big), \tag{3.6}$$

$$U(u_1,u_2) = U_1(u_1) + U_2(u_2). \tag{3.7}$$

Since

$$\min_{u_1>0}(-\gamma\log u_1 + \delta u_1) = -\gamma\log\left(\frac{\gamma}{\delta}\right) + \gamma, \tag{3.8}$$

$$\min_{u_2>0}(-\alpha\log u_2 + \beta u_2) = -\alpha\log\left(\frac{\alpha}{\beta}\right) + \alpha, \tag{3.9}$$

it follows that $U_1(u_1) \geq 0$, $U_2(u_2) \geq 0$ and $U(u_1,u_2) \geq 0$ for any $u_1 > 0$ and $u_2 > 0$. Thus

$$\min_{u_1>0} U_1(u_1) = \min_{u_2>0} U_2(u_2) = \min_{u_1>0,\ u_2>0} U(u_1,u_2) = 0, \tag{3.10}$$

$$U(u_1,u_2) = 0 \iff u_1 = \frac{\gamma}{\delta} \quad \text{and} \quad u_2 = \frac{\alpha}{\beta}. \tag{3.11}$$

Let us set

$$\tilde{U}(t) = \frac{1}{2\pi}\int_0^{2\pi} U(u_1(t,x), u_2(t,x))dx. \tag{3.12}$$

Let us first note the following fact, which can be proved in a manner similar to the raisoning presented in [14] and [15].

**Proposition 1.** *Assume that*

$$\sup_{0\leq x\leq 2\pi} U(u_{1,0}(x), u_{2,0}(x)) < \infty$$

*and that the problem* (3.1)–(3.3) *with* (3.4) *admits the unique solution* $(u_1(t,x), u_2(t,x))$ *in the time interval* $[0,\tau[$ $(0 < \tau \leq \infty)$. *Then, the function* $\tilde{U}(t)$ *is decreasing on the interval* $[0,\tau[$.

P r o o f. In a manner similar to deriving (2.8), but adding the terms that result from the diffusion termes, we obtain

$$\partial_t U = \kappa\partial_x^2 U - \kappa\sigma - v_1\partial_x U_1 - v_2\partial_x U_2, \tag{3.13}$$

where

$$\sigma = \sigma(t,x) = \gamma\left(\frac{\partial_x u_1(t,x)}{u_1(t,x)}\right)^2 + \alpha\left(\frac{\partial_x u_2(t,x)}{u_2(t,x)}\right)^2.$$

By integrating both sides of the equality (3.13) with respect to $x$ from 0 to $2\pi$, we obtain

$$\int_0^{2\pi} \partial_t U dx = \int_0^{2\pi} \left(\kappa\partial_x^2 U - \kappa\sigma - v_1\partial_x U_1 - v_2\partial_x U_2\right) dx.$$

Since the functions $U(u_1(t,x), u_2(t,x))$, $U_1(u_1(t,x))$ and $U_2(u_2(t,x))$ are $2\pi$-periodic in $x$, we have

$$\frac{d}{dt}\int_0^{2\pi} U(u_1(t,x), u_2(t,x))dx = -\kappa\int_0^{2\pi} \sigma(t,x)dx.$$

This, together with the relation $\sigma \geq 0$, implies that the function $\tilde{U}(t)$ is decreasing. □

**Corollary 1.** *If the solution $(u_1(t,x), u_2(t,x))$ of the problem (3.1)–(3.3) (with (3.4)) exists for all $t > 0$, then the function $\tilde{U}(t)$ converges to a value $\tilde{U}_\infty$ for $t \to \infty$.*

P r o o f. It immediately follows from Proposition 1 and the relation (3.10). □

## 4. Main result

Our main result is the following.

**Theorem 1.** *Assume that*

$$\sup_{t\geq 0} |v_1(t) - v_2(t)| \equiv C_v < \infty, \tag{4.1}$$

$$\sup_{0\leq x\leq 2\pi} U(u_{1,0}(x), u_{2,0}(x)) < \infty. \tag{4.2}$$

*Then, the problem (3.1)–(3.3) with (3.4) admits a unique solution $(u_1(t,x), u_2(t,x))$ for all $t > 0$ and we have*

$$\sup_{t\geq 0,\ 0\leq x\leq 2\pi} U(u_1(t,x), u_2(t,x)) < \infty. \tag{4.3}$$

*More precisely,*

i) *there exists a continuous and increasing function $\Lambda_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\limsup_{t\to\infty} \sup_{0\leq x\leq 2\pi} U(u_1(t,x), u_2(t,x)) \leq \Lambda_1(\tilde{U}_\infty),$$

ii) *if $\tilde{U}_\infty = 0$, then we have*

$$\lim_{t\to\infty} \sup_{0\leq x\leq 2\pi} U(u_1(t,x), u_2(t,x)) = 0,$$

*where $\tilde{U}_\infty = \lim_{t\to\infty} \tilde{U}(t)$ with $\tilde{U}(t)$ defined in (3.12).*

For the proof of Theorem 1 we use the proposition.

**Proposition 2.** *Assume that the conditions (4.1)–(4.2) and (3.4) are satisfied and that the problem (3.1)–(3.3) admits a unique solution $(u_1(t,x), u_2(t,x))$ for all $t > 0$. Then, there exists an increasing and continuous function $\Lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\limsup_{t\to\infty} \|U(u_1(t,\cdot), u_2(t,\cdot))\|^2_{L^2(0,2\pi)} \le \Lambda_2(\tilde{U}_\infty), \tag{4.4}$$
$$\Lambda_2(0) = 0.$$

The function $\Lambda_2(\cdot)$ can be given for example by the formula (5.13).

In the following section, we will prove Proposition 2. Theorem 1 will be proved in the successive section.

## 5. Proof of Proposition 2

In order to prove Proposition 2, we begin with the following lemma.

**Lemma 1.** *Let $U = U(x)$ be a positive and $2\pi$-periodic function such that*

$$\left\| \frac{d}{dx} U \right\|_{L^2(0,2\pi)} < \infty.$$

*If*

$$\|U\|_{L^2(0,2\pi)} > \sqrt{8\pi}\,\overline{U}, \tag{5.1}$$

*then we have*

$$\left\| \frac{d}{dx} U \right\|^2_{L^2(0,2\pi)} \ge \frac{1}{256\pi^3\overline{U}^2}\left(1 - \frac{4\sqrt{2\pi}\,\overline{U}}{3\|U\|_{L^2(0,2\pi)}}\right)\|U\|^4_{L^2(0,2\pi)}, \tag{5.2}$$

*where*

$$\overline{U} = \frac{1}{2\pi}\int_0^{2\pi} U(x)dx.$$

P r o o f. Set

$$\mu = \frac{\|U\|_{L^2(0,2\pi)}}{2\sqrt{2\pi}}, \quad D_\mu = \left\{ x \in [0,2\pi] \,|\, U(x) > \mu \right\}, \tag{5.3}$$

and denote by $|D_\mu|$ the measure of the set $D_\mu$. Since $U(x) > \mu$ on $D_\mu$, it follows from the definition of $\overline{U}$ and $\mu$ that

$$\mu|D_\mu| \le 2\pi\overline{U}. \tag{5.4}$$

Since

$$U(x)^2 = (U(x)-\mu)^2 + 2\mu(U(x)-\mu) + \mu^2,$$

it follows that

$$\int_{D_\mu} |U(x)|^2 dx = \int_{D_\mu} (U(x)-\mu)^2 dx + 2\int_{D_\mu} \mu(U(x)-\mu)dx + \int_{D_\mu} \mu^2 dx.$$

Hence

$$\int_{D_\mu} (U(x)-\mu)^2 dx = \int_{D_\mu} |U(x)|^2 dx - 2\int_{D_\mu} \mu(U(x)-\mu)dx - |D_\mu|\mu^2$$
$$\ge \int_{D_\mu} |U(x)|^2 dx - 3|D_\mu|\mu^2 - \frac{1}{2}\int_{D_\mu} (U(x)-\mu)^2 dx.$$

Thus, taking into account (5.3), we have

$$\int_{D_\mu} (U(x)-\mu)^2 dx \geq \frac{2}{3}\int_{D_\mu} |U(x)|^2 dx - 2|D_\mu|\mu^2 = \frac{2}{3}\int_{D_\mu} |U(x)|^2 dx - \frac{|D_\mu|\|U\|^2_{L^2(0,2\pi)}}{4\pi}. \tag{5.5}$$

On the other hand, we have

$$\int_{D_\mu^c} |U(x)|^2 dx \leq (2\pi - |D_\mu|)\mu^2.$$

Hence, taking into account (5.3), we have

$$\int_{D_\mu} |U(x)|^2 dx \geq \|U\|^2_{L^2(0,2\pi)} - (2\pi-|D_\mu|)\mu^2 = \Big(\frac{3}{4}+\frac{|D_\mu|}{8\pi}\Big)\|U\|^2_{L^2(0,2\pi)}. \tag{5.6}$$

From (5.5) and (5.6) we obtain

$$\int_{D_\mu} (U(x)-\mu)^2 dx \geq \left(\frac{1}{2}-\frac{|D_\mu|}{6\pi}\right)\|U\|^2_{L^2(0,2\pi)}. \tag{5.7}$$

Recall that under the condition (5.1) the relation (5.4) implies that $|D_\mu| < 2\pi$, and thus there exists at least one $\bar{x} \in \mathbb{R}$ such that $U(\bar{x}) \leq \mu$. Since $U(x)$ is $2\pi$-periodic, it is not restrictive to assume that $\bar{x} = 0$ (and thus $U(\bar{x}+2\pi) \leq \mu$).

We first consider the case

$$D_\mu = ]x_0, x_0+|D_\mu|[\,.$$

In this case, since we have

$$\int_{D_\mu} (U(x)-\mu)^2 dx = \int_{D_\mu} 2\int_{x_0}^{x} (U(x')-\mu)\frac{d}{dx'}U(x')dx'dx,$$

and thus

$$\int_{D_\mu} (U(x)-\mu)^2 dx \leq 2|D_\mu|\Big(\int_{D_\mu} (U(x)-\mu)^2 dx\Big)^{1/2}\Big(\int_{D_\mu}\Big(\frac{d}{dx}U(x)\Big)^2 dx\Big)^{1/2},$$

we obtain

$$\int_{D_\mu} (U(x)-\mu)^2 dx \leq 4|D_\mu|^2 \int_{D_\mu}\left(\frac{d}{dx}U(x)\right)^2 dx. \tag{5.8}$$

Even in the general case with

$$D_\mu = \bigcup_{k=0}^{N} ]x_k, x'_k[, \quad |D_\mu| = \sum_{k=1}^{N}(x'_k - x_k), \quad N\in\mathbb{N}, \quad N\geq 2 \quad \text{or} \quad N=+\infty,$$

on every interval $]x_k, x'_k[$ we have

$$\int_{x_k}^{x'_k} (U(x)-\mu)^2 dx \leq 4|D_\mu|^2 \int_{x_k}^{x'_k}\left(\frac{d}{dx}U(x)\right)^2 dx.$$

By summing these inequalities, we obtain (5.8).

From (5.7) and (5.8) we have

$$\int_{D_\mu}\left(\frac{d}{dx}U(x)\right)^2 dx \geq \frac{1}{4|D_\mu|^2}\Big(\frac{1}{2}-\frac{|D_\mu|}{6\pi}\Big)\|U\|^2_{L^2(0,2\pi)}. \tag{5.9}$$

Since, according to (5.4), we have

$$|D_\mu| \leq \frac{4\pi\sqrt{2\pi}\,\overline{U}}{\|U\|_{L^2(0,2\pi)}},$$

from (5.9) we obtain

$$\int_{D_\mu} \left(\frac{d}{dx}U(x)\right)^2 dx \geq \frac{1}{256\pi^3\overline{U}^2}\left(1 - \frac{4\sqrt{2\pi}\,\overline{U}}{3\|U\|_{L^2(0,2\pi)}}\right)\|U\|^4_{L^2(0,2\pi)}.$$

Since

$$\int_0^{2\pi} \left(\frac{d}{dx}U(x)\right)^2 dx \geq \int_{D_\mu} \left(\frac{d}{dx}U(x)\right)^2 dx,$$

we deduce the inequality (5.2). This completes the proof of the lemma. □

Lemma 1 leads to the following property.

**Lemma 2.** *Assume that the conditions (4.1)–(4.2) and (3.4) are satisfied and that the problem (3.1)–(3.3) admits a unique solution $(u_1(t,x), u_2(t,x))$ for all $t > 0$. Let $U(\cdot,\cdot)$ and $\tilde{U}(t)$ be the functions defined in (3.7) and (3.12), respectively. If*

$$\|U(u_1(t,\cdot), u_2(t,\cdot))\|_{L^2(0,2\pi)} > \sqrt{8\pi}\tilde{U}(t),$$

*then we have*

$$\frac{d}{dt}\|U\|^2_{L^2} \leq \left(\frac{C_v^2}{\kappa} - \frac{\kappa}{256\pi^3\tilde{U}^2}\left(1 - \frac{4\sqrt{2\pi}}{3\|U\|_{L^2}}\tilde{U}\right)\|U\|^2_{L^2}\right)\|U\|^2_{L^2}, \tag{5.10}$$

*where $\tilde{U} = \tilde{U}(t)$ and*

$$\|U\|_{L^2} = \|U(u_1(t,\cdot), u_2(t,\cdot))\|_{L^2(0,2\pi)}.$$

P r o o f. By writing $v_1(t) - v_2(t) + v_2(t)$ instead of $v_1(t)$ in (3.13), we have

$$\partial_t U = \kappa\partial_x^2 U - \kappa\sigma(t,x) - v_2(t)\partial_x U - (v_1(t) - v_2(t))\partial_x U_1. \tag{5.11}$$

If we multiply both sides of (5.11) by $U$ and integrate them on $[0, 2\pi]$, then, using integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} |U|^2 dx = -\kappa\int_0^{2\pi} |\partial_x U|^2 dx - \kappa\int_0^{2\pi} \sigma U dx + (v_1(t) - v_2(t))\int_0^{2\pi} U_1\partial_x U dx.$$

Note that due to relations $U = U_1 + U_2$, $U_1 \geq 0$, $U_2 \geq 0$ (see (3.5)–(3.9)), we have

$$\int_0^{2\pi} U_1\partial_x U dx \leq \frac{1}{2\kappa}\int_0^{2\pi} |U_1|^2 dx + \frac{\kappa}{2}\int_0^{2\pi} |\partial_x U|^2 dx \leq \frac{1}{2\kappa}\int_0^{2\pi} |U|^2 dx + \frac{\kappa}{2}\int_0^{2\pi} |\partial_x U|^2 dx.$$

Thus, taking into account the relation $\sigma U \geq 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} |U|^2 dx \leq -\frac{\kappa}{2}\int_0^{2\pi} |\partial_x U|^2 dx + \frac{|v_1(t) - v_2(t)|^2}{2\kappa}\int_0^{2\pi} |U|^2 dx. \tag{5.12}$$

Applying the inequality (5.2) to the first term on the right-hand side of the inequality (5.12) and taking into account the condition (4.1), we obtain (5.10). This completes the proof of the lemma. □

P r o o f (of Proposition 2). Note that if $\|U\|_{L^2} > \sqrt{8\pi}\tilde{U}$, then we have

$$1 - \frac{4\sqrt{2\pi}}{3\|U\|_{L^2}}\tilde{U} \geq \frac{1}{3}.$$

Thus, in this case, the right-hand side of the inequality (5.10) is bounded from above by

$$\left(\frac{C_v^2}{\kappa} - \frac{\kappa}{256\pi^3\tilde{U}^2}\frac{\|U\|_{L^2}^2}{3}\right)\|U\|_{L^2}^2.$$

Therefore, from Lemma 2 it follows that

$$\limsup_{t\to\infty}\int_0^{2\pi}|U(u_1(t,x),u_2(t,x))|^2dx \leq \Lambda_2(\tilde{U}_\infty),$$

where $\Lambda_2(\cdot)$ is defined by

$$\Lambda_2(a) = \max\left(8\pi, \frac{768\pi^3C_v^2}{\kappa^2}\right)a^2, \tag{5.13}$$

which completes the proof of Proposition 2. □

## 6. Proof of Theorem 1

In order to prove Theorem 1, we begin with an estimate of the $\|\partial_x U(u_1(t,\cdot),u_2(t,\cdot))\|_{L^2(0,2\pi)}$. We have the following lemma (in Lemmas 3–9, we assume that the hypothesis of Proposition 2 is satisfied).

**Lemma 3.** *For all* $t_2 > t_1 \geq 0$, *we have*

$$\begin{aligned}&\int_{t_1}^{t_2}\|\partial_x U(u_1(t,\cdot),u_2(t,\cdot))\|_{L^2(0,2\pi)}^2dt\\ &\leq \frac{C_v}{\kappa^2}\int_{t_1}^{t_2}\|U(u_1(t,\cdot),u_2(t,\cdot))\|_{L^2_{(0,2\pi)}}^2dt + \frac{1}{\kappa}\|U(u_1(t_1,\cdot),u_2(t_1,\cdot))\|_{L^2(0,2\pi)}^2.\end{aligned}\tag{6.1}$$

P r o o f. From (5.12) we deduce that

$$\int_0^{2\pi}|\partial_x U(t,x)|^2dx \leq \frac{|v_1(t)-v_2(t)|^2}{\kappa^2}\int_0^{2\pi}|U(t,x)|^2dx - \frac{1}{\kappa}\frac{d}{dt}\int_0^{2\pi}|U(t,x)|^2dx,$$

where $U(t,x) = U(u_1(t,x),u_2(t,x))$. Integrating both sides of this inequality with respect to $t$ from $t_1$ to $t_2$, we obtain

$$\int_{t_1}^{t_2}\|\partial_x U(t,\cdot)\|_{L^2(0,2\pi)}^2dt \leq \frac{C_v}{\kappa^2}\int_{t_1}^{t_2}\|U(t,\cdot)\|_{L^2(0,2\pi)}^2dt - \frac{1}{\kappa}\left(\|U(t_2,\cdot)\|_{L^2(0,2\pi)}^2 - \|U(t_1,\cdot)\|_{L^2(0,2\pi)}^2\right). \tag{6.2}$$

Eliminating the negative terme of the right-hand side of the inequality (6.2), we obtain (6.1). □

We deduce from Lemma 3 the following relation.

**Lemma 4.** *We have*

$$\begin{aligned}&\int_t^{t+1}\sup_{0\leq x\leq 2\pi}U(u_1(t',x),u_2(t',x))dt'\\ &\leq \tilde{U}(t) + \sqrt{2\pi}\Big(\frac{C_v}{\kappa^2}\int_t^{t+1}\|U(u_1(t',\cdot),u_2(t',\cdot))\|_{L(0,2\pi)^2}^2dt' + \frac{1}{\kappa}\|U(u_1(t,\cdot),u_2(t,\cdot))\|_{L^2(0,2\pi)}^2\Big)^{1/2},\end{aligned}\tag{6.3}$$

*where* $\tilde{U}(t)$ *is the notation introduced in* (3.12).

P r o o f. We use the notation $U(t,x) = U(u_1(t,x), u_2(t,x))$ as in the proof of Lemma 3. Since

$$\|\varphi\|_{L^1(0,2\pi)} \leq \sqrt{2\pi}\|\varphi\|_{L^2(0,2\pi)}$$

for all $\varphi \in L^2(0,2\pi)$, from the relation

$$\sup_{0\leq x\leq 2\pi} U(t,x) \leq \tilde{U}(t) + \|\partial_x U(t,\cdot)\|_{L^1(0,2\pi)},$$

we obtain

$$\sup_{0\leq x\leq 2\pi} U(t,x) \leq \tilde{U}(t) + \sqrt{2\pi}\|\partial_x U(t,\cdot)\|_{L^2(0,2\pi)}. \tag{6.4}$$

Taking into account the decreasing of $\tilde{U}(t)$, the inequality (6.3) follows immediatly from (6.1) and (6.4). □

We will now estimate the growth of

$$\sup_{0\leq x\leq 2\pi} u_1(t,x), \quad \sup_{0\leq x\leq 2\pi} u_2(t,x), \quad \sup_{0\leq x\leq 2\pi} (-\log u_1(t,x)), \quad \sup_{0\leq x\leq 2\pi} (-\log u_2(t,x)).$$

To this end, we return to the equations (3.1) and (3.2). Note that, if we introduce the function

$$\xi_1(t,x) = x + \int_0^t v_1(t')dt',$$

and if we consider the variables $(t,\xi_1)$ instead of $(t,x)$, then the equation (3.1) is rewritten as

$$\partial_t u_1(t,\xi_1) = \kappa\partial_{\xi_1}^2 u_1(t,\xi_1) + \alpha u_1(t,\xi_1) - \beta u_1(t,\xi_1)u_2(t,\xi_1). \tag{6.5}$$

Analogously, if we introduce the function

$$\xi_2(t,x) = x + \int_0^t v_2(t')dt',$$

and if we consider the variables $(t,\xi_2)$ instead of $(t,x)$, then the equation (3.2) is rewritten as

$$\partial_t u_2(t,\xi_2) = \kappa\partial_{\xi_2}^2 u_2(t,\xi_2) - \gamma u_2(t,\xi_2) + \delta u_1(t,\xi_2)u_2(t,\xi_2). \tag{6.6}$$

Using (6.5) and (6.6), we will prove the four following lemmas.

**Lemma 5.** *Set*

$$u_1^+(t) = \sup_{0\leq x\leq 2\pi} u_1(t,x) = \sup_{\xi_1\in\mathbb{R}} u_1(t,\xi_1). \tag{6.7}$$

*Then, for* $0 \leq t_0 \leq t$, *we have*

$$u_1^+(t) \leq u_1^+(t_0)e^{\alpha(t-t_0)} \equiv \Phi_1(u_1^+(t_0), t-t_0). \tag{6.8}$$

P r o o f. By formally applying the fundamental solution of the heat equation to (6.5), we have

$$u_1(t,\xi_1) = \int_{-\infty}^{+\infty} \Theta(t-t_0,\xi'-\xi_1)u_1(t_0,\xi')d\xi'$$
$$+\int_{t_0}^{t}\int_{-\infty}^{+\infty} \Theta(t-t',\xi'-\xi_1)\left(\alpha u_1(t',\xi') - \beta u_1(t',\xi')u_2(t',\xi')\right)d\xi'dt',$$

where

$$\Theta(\tau,\eta) = \frac{1}{\sqrt{(4\pi\tau\kappa)}} \exp(-\frac{|\eta|^2}{4\tau\kappa}).$$

Since

$$\int_{-\infty}^{+\infty} \Theta(\tau,\eta)d\eta = 1$$

for all $\tau > 0$, we have

$$u_1^+(t) \leq u_1^+(t_0) + \alpha \int_{t_0}^{t} u_1^+(t')dt',$$

so that we obtain (6.8). □

**Lemma 6.** *Set*

$$w_2^+(t) = \sup_{0\leq x\leq 2\pi} (-\log u_2(t,x)) = \sup_{\xi_2\in\mathbb{R}} (-\log u_2(t,\xi_2)).$$

*Then, for* $0 \leq t_0 \leq t$, *we have*

$$w_2^+(t) \leq w_2^+(t_0) + \gamma(t-t_0) \equiv \Psi_2(w_2^+(t_0), t-t_0). \tag{6.9}$$

P r o o f. If we divide both sides of (6.6) by $-u_2(t,\xi_2)$, we have

$$\partial_t(-\log(u_2(t,\xi_2))) = \kappa\partial_{\xi_2}^2(-\log(u_2(t,\xi_2))) - (\partial_{\xi_2}\log(u_2(t,\xi_2)))^2 + \gamma - \delta u_1(t,\xi_2). \tag{6.10}$$

By formally applying the fundamental solution of the heat equation to (6.10), we have

$$-\log(u_2(t,\xi_2)) \leq \int_{-\infty}^{+\infty} \Theta(t-t_0,\xi'-\xi_2)(-\log(u_2(t_0,\xi')))d\xi' + \gamma(t-t_0),$$

and this inequality implies (6.9). □

**Lemma 7.** *Set*

$$u_2^+(t) = \sup_{0\leq x\leq 2\pi} u_2(t,x) = \sup_{\xi_2\in\mathbb{R}} u_2(t,\xi_2).$$

*Then, for* $0 \leq t_0 \leq t$, *we have*

$$\begin{aligned} u_2^+(t) &\leq u_2^+(t_0)\left(1 + \delta u_1^+(t_0)\int_{t_0}^{t} e^{\alpha(t'-t_0)} e^{\delta/\alpha\cdot u_1^+(t_0)(e^{\alpha(t-t_0)}-e^{\alpha(t'-t_0)})}dt'\right) \\ &\equiv \Phi_2(u_1^+(t_0), u_2^+(t_0), t-t_0). \end{aligned} \tag{6.11}$$

P r o o f. We formally apply the fundamental solution of the heat equation to (6.6), so that we have

$$u_2(t,\xi_2) \leq \int_{-\infty}^{+\infty} \Theta(t-t_0,\xi'-\xi_2)u_2(t_0,\xi')d\xi' + \delta\int_{t_0}^{t}\int_{-\infty}^{+\infty} \Theta(t-t',\xi'-\xi_2)u_1(t',\xi')u_2(t',\xi')d\xi'dt'.$$

Hence, using the inequality (6.8), we have

$$u_2^+(t) \leq u_2^+(t_0) + \delta u_1^+(t_0)\int_{t_0}^{t} e^{\alpha(t'-t_0)} u_2^+(t')dt',$$

or

$$Y'(t) \le e^{\alpha(t-t_0)} u_2^+(t_0) + \delta u_1^+(t_0) e^{\alpha(t-t_0)} Y(t), \quad Y(t) = \int_{t_0}^{t} e^{\alpha(t'-t_0)} u_2^+(t')dt'.$$

From this inequality follows (6.11). □

**Lemma 8.** *Set*

$$w_1^+(t) = \sup_{0\le x\le 2\pi} (-\log u_1(t,x)) = \sup_{\xi_1\in\mathbb{R}} (-\log u_1(t,\xi_1)).$$

*Then, for* $0 \le t_0 \le t$, *we have*

$$w_1^+(t) \le w_1^+(t_0) + \beta \int_{t_0}^{t} \Phi_2(t_0, u_2^+(t_0), t')dt' \equiv \Psi_1(u_1^+(t_0), u_2^+(t_0), w_1^+(t_0), t-t_0). \tag{6.12}$$

P r o o f. From the equation

$$\partial_t(-\log(u_1(t,\xi_1))) = \kappa\partial_{\xi_1}^2(-\log(u_1(t,\xi_1))) - \kappa(\partial_{\xi_1}\log(u_1(t,\xi_1)))^2 - \alpha + \beta u_2(t,\xi_1),$$

we deduce (in a similar way to the previous case)

$$-\log(u_1(t,\xi_1)) \le w_1^+(t_0) + \beta \int_{t_0}^{t} u_2^+(t')dt'.$$

Hence, using (6.11) we obtain (6.12). □

Let us define $w_1^+(U)$, $u_1^+(U)$, $w_2^+(U)$ and $u_2^+(U)$, for all $U \ge 0$, as follows:

$$\begin{aligned} w_1^+(U) &= -\log(\bar{u}_1), \quad U_1(\bar{u}_1) = U, \quad 0 < \bar{u}_1 \le \frac{\gamma}{\delta}, \\ u_1^+(U) &= \bar{\bar{u}}_1, \quad U_1(\bar{\bar{u}}_1) = U, \quad \bar{\bar{u}}_1 \ge \frac{\gamma}{\delta}, \\ w_2^+(U) &= -\log(\bar{u}_2), \quad U_2(\bar{u}_2) = U, \quad 0 < \bar{u}_2 \le \frac{\alpha}{\beta}, \\ u_2^+(U) &= \bar{\bar{u}}_2, \quad U_2(\bar{\bar{u}}_2) = U, \quad \bar{\bar{u}}_2 \ge \frac{\alpha}{\beta}. \end{aligned}$$

It is clear that

$$U = U_1(e^{-w_1^+(U)}) = U_1(u_1^+(U)) = U_2(e^{-w_2^+(U)}) = U_2(u_2^+(U)).$$

These definitions are justified due to the definition (3.5)–(3.6) of $U_1(u_1)$ and $U_2(u_2)$.

**Lemma 9.** *If we set*

$$U^+(t) = \sup_{0\le x\le 2\pi} U(u_1(t,x), u_2(t,x)),$$

*we have*

$$U^+(t) \le \tilde{M}(U^+(t_0), t-t_0), \quad t \ge t_0,$$

*where*

$$\tilde{M}(U^+(t_0), t-t_0) = U_1^{\max}(U^+(t_0), t-t_0) + U_2^{\max}(U^+(t_0), t-t_0), \tag{6.13}$$

$$U_1^{\max}(U^+(t_0), t-t_0)$$
$$= \max(U_1(\Phi_1(u_1^+(U^+(t_0)), t-t_0)), U_1(e^{-\Psi_1(u_1^+(U^+(t_0)), u_2^+(U^+(t_0)), w_1^+(U^+(t_0)), t-t_0)})),$$

$$U_2^{\max}(U^+(t_0), t-t_0) = \max(U_2(\Phi_2(u_1^+(U^+(t_0)), u_2^+(U^+(t_0)), t-t_0)), U_2(e^{-\Psi_2(w_2^+(U^+(t_0)), t-t_0)})).$$

P r o o f. The lemma follows immediatly from the definition of $\tilde{M}(U^+(t_0), t-t_0)$ and Lemmas 5–8. □

*Remark 3.* The function $\tilde{M}(a, b)$ can be defined for any values $a \geq 0$ and $b \geq 0$ (independently of the solution $(u_1(t,x), u_2(t,x))$ of the problem (3.1)–(3.3)). Furthermore, the function $\tilde{M}(a, b)$ is continuous and increasing with respect to either $a \geq 0$ or $b \geq 0$.

Indeed, this remark follows immediately from the definition (6.13).

We are now able to prove the main result.

P r o o f (of Theorem 1). In this proof we use the notations $\tilde{U}(t)$ introduced in (3.12) and $U(t,x) = U(u_1(t,x), u_1(t,x))$. Lemma 2 (see also (5.13)) implies that, if

$$\|U(t,\cdot)\|^2_{L^2(0,2\pi)} > \Lambda_2(\tilde{U}(t)),$$

then $\|U(t,\cdot)\|^2_{L^2(0,2\pi)}$ decreases. Taking into account that $\tilde{U}(t)$ is decreasing, we have

$$\|U(t,\cdot)\|^2_{L^2(0,2\pi)} \leq \max\Big(\|U(0,\cdot)\|^2_{L^2(0,2\pi)}, \Lambda_2(\tilde{U}(0))\Big) \equiv B_U, \quad \forall t \geq 0.$$

This inequality, together with (6.3) and Proposition 1, allows us to conclude the existence of $\tau$ in each interval $[t, t+1]$ such that

$$\sup_{0\leq x\leq 2\pi} U(\tau, x) \leq \tilde{U}(0) + \sqrt{2\pi}\Big(\frac{C_v}{\kappa^2} + \frac{1}{\kappa}\Big)^{1/2}\sqrt{B_U} \equiv A_U.$$

On the other hand, it follows from Lemma 9 (see also Remark 3) that

$$\sup_{0\leq x\leq 2\pi} U(t,x) \leq \tilde{M}(A_U, t-\tau),$$

for $t \geq \tau$. Thus, from these relations it follows that, for all $t \geq 0$, we have

$$\sup_{0\leq x\leq 2\pi} U(t',x) \leq \tilde{M}(A_U, 1), \quad \forall t' \in [t, t+1],$$

in other words, we have

$$\sup_{0\leq x\leq 2\pi} U(t,x) \leq \tilde{M}(A_U, 1), \quad \forall t \geq 0,$$

with $\tilde{M}(A_U, 1) < \infty$ (see (6.13)), which completes the proof of (4.3).

We now set

$$\Lambda_1(\tilde{U}_\infty) = \tilde{M}(A^*_U(\tilde{U}_\infty), 1),$$

where

$$A^*_U(\tilde{U}_\infty) = \tilde{U}_\infty + \sqrt{2\pi}\Big(\frac{C_v}{\kappa^2} + \frac{1}{\kappa}\Big)^{1/2}\sqrt{\Lambda_2(\tilde{U}_\infty)}. \tag{6.14}$$

We note that the right-hand side of (6.14) does not depend on $t$ and we can deduce from the definition of $\tilde{M}$ that the function $\Lambda_1(\tilde{U}_\infty)$ is continuous and increasing. From the reasoning of the proof of (4.3), taking into account (4.4), we deduce that

$$\limsup_{t\to\infty} \sup_{0\leq x\leq 2\pi} U(t,x) \leq \Lambda_1(\tilde{U}_\infty),$$

which completes the proof of the statement *i*) of Theorem 1.

We now assume that $\tilde{U}_\infty = 0$. Then, according to Lemma 4, we have

$$\int_{t-1}^{t} \sup_{0\le x\le 2\pi} U(\tau,x)d\tau \le \tilde{U}(t-1) + \sqrt{2\pi}\Big(\frac{C_v}{\kappa^2}\int_{t-1}^{t} \|U(\tau,\cdot)\|^2_{L(0,2\pi)^2}d\tau + \frac{1}{\kappa}\|U(t-1,\cdot)\|^2_{L^2(0,2\pi)}\Big)^{1/2}.$$

According to Proposition 2 the upper limit of the right-hand side of this inequality is $A_U^*(\tilde{U}_\infty)$, as given in (6.14). Thus, we have

$$\lim_{t\to\infty}\int_{t-1}^{t} \sup_{0\le x\le 2\pi} U(\tau,x)d\tau = 0. \tag{6.15}$$

In order to prove the statement ii) of Theorem 1, we argue by contradiction by assuming that

$$\lim_{t\to\infty} \sup_{0\le x\le 2\pi} U(t,x) \ne 0,$$

in other words, suppose that there exists $\varepsilon > 0$ such that, for each $t > 0$, there exists $t' \ge t$ such that

$$\sup_{0\le x\le 2\pi} U(t',x) \ge \varepsilon. \tag{6.16}$$

Let us define the function $U^{(\varepsilon)}(s)$, for each $s > 0$, as

$$\tilde{M}(U^{(\varepsilon)}(s), s) = \varepsilon. \tag{6.17}$$

Then, from the definition of $\tilde{M}$ it follows that, for $t'$ satisfaying (6.16), we have for $\tau < t'$

$$U^{(\varepsilon)}(t'-\tau) \le \sup_{0\le x\le 2\pi} U(\tau,x).$$

Thus

$$\int_{t'-1}^{t'} U^{(\varepsilon)}(t'-\tau)d\tau \le \int_{t'-1}^{t'} \sup_{0\le x\le 2\pi} U(\tau,x)d\tau. \tag{6.18}$$

Recall that the definition of $\tilde{M}$ (and also of $U_1^{\max}$ and $U_2^{\max}$; see (6.13)) implies that for any $t_0 > 0$, we have

$$\lim_{t\to t_0^+} U_1^{\max}(U^+(t_0), t-t_0) = \max(U_1(u_1^+(U^+(t_0))), U_1(e^{-w_1^+(U^+(t_0))})) = U^+(t_0),$$
$$\lim_{t\to t_0^+} U_2^{\max}(U^+(t_0), t-t_0) = \max(U_2(u_2^+(U^+(t_0))), U_2(e^{-w_2^+(U^+(t_0))})) = U^+(t_0),$$

and thus

$$\lim_{t\to t_0^+} \tilde{M}(U^+(t_0), t-t_0) = 2U^+(t_0).$$

This relation also implies that

$$\lim_{\tau\to t'^-} U^{(\varepsilon)}(t'-\tau) = \frac{1}{2}\varepsilon > 0. \tag{6.19}$$

From the continuity of $\tilde{M}(a,b)$ we can deduce that $U^{(\varepsilon)}(s)$ is continuous (see (6.17)). Thus, from (6.19) it follows that there exists some $s_\varepsilon > 0$ such that $U^{(\varepsilon)}(s) > 0$ for $0 < s < s_\varepsilon$, and we have

$$\int_{t'-s_\varepsilon}^{t'} U^{(\varepsilon)}(t'-\tau)d\tau = \int_0^{s_\varepsilon} U^{(\varepsilon)}(s)ds \equiv c_\varepsilon > 0.$$

Thus, it follows from (6.18) that

$$\int_{t'-1}^{t'} \sup_{0\le x\le 2\pi} U(\tau,x)d\tau \ge c_\varepsilon > 0,$$

where $c_\varepsilon$ is independant of $t'$. This result contradicts (6.15). Therefore we have

$$\lim_{t\to\infty} \sup_{0\le x\le 2\pi} U(t,x) = 0.$$

This completes the proof of the theorem. □

## 7. Conclusion

In this article, we have analyzed the asymptotic behavior of the solution to the Lotka–Volterra equation with diffusion and population displacements in a periodic domain of $\mathbb{R}$. From this analysis we have obtained the global boundedness of the solution and its logarithm and also its uniform convergence to the stationary solution in the case in which the solution converges in mean-value to the stationary solution. This result guarantees that, even if there can be the growth of oscillation of the solution in certain points as we have seen in the example of numerical calculation in the Section 2, these phenomena cannot develop infinitely, and the growth of oscillation is limited.

Moreover we have developed some particular techniques of estimate of the solution. Even if the conditions we have set for the equation are relatively simple, the techniques we have developed here can, with possible adaptation, be used also for analogous problem with more complex conditions.

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