APPROXIMATION OF THE DIFFERENTIATION OPERATOR ON THE CLASS OF FUNCTIONS ANALYTIC IN AN ANNULUS[1]

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Abstract: In the class of functions analytic in the annulus $C_r := \{z \in \mathbb{C} : r < |z| < 1\}$ with bounded L^p-norms on the unit circle, we study the problem of the best approximation of the operator taking the nontangential limit boundary values of a function on the circle Γ_r of radius r to values of the derivative of the function on the circle Γ_ρ of radius ρ, $r < \rho < 1$, by bounded linear operators from $L^p(\Gamma_r)$ to $L^p(\Gamma_\rho)$ with norms not exceeding a number N. A solution of the problem has been obtained in the case when N belongs to the union of a sequence of intervals. The related problem of optimal recovery of the derivative of a function from boundary values of the function on Γ_ρ given with an error has been solved.

Key words: Best approximation of operators, Optimal recovery, Analytic functions.

Introduction

The paper is devoted to studying a number of related extremal problems for the differentiation operator on the class of functions analytic in an annulus. Similar problems for the analytic continuation operator and for the differentiation operator on the class of functions analytic in a strip were solved earlier in [1] and [2], respectively. In the present paper, we follow the notation and use some auxiliary statements from [1, 2].

Let $C_r := \{z \in \mathbb{C} : r < |z| < 1\}$ be the annulus centered at the origin of internal radius r and external radius 1. We denote by $A(C_r)$ the set of functions analytic in the annulus C_r. For a function $f \in A(C_r)$ and a number ρ, $r < \rho < 1$, we define the p-average of the function f on the circle $\Gamma_\rho := \{z \in \mathbb{C} : |z| = \rho\}$ by the equality

$$\mathcal{M}^p(f,\rho) := \|f\|_{L^p(\Gamma_\rho)} = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(\rho e^{it})|^p \, dt\right)^{1/p}, & 1 \le p < \infty, \\ \max\left\{|f(\rho e^{it})| : t \in [0,2\pi]\right\}, & p = \infty. \end{cases}$$

Let $H^p = H^p(C_r)$ be the Hardy space of functions $f \in A(C_r)$ such that

$$\sup\{\mathcal{M}^p(f,\rho) : r < \rho < 1\} < +\infty.$$

As is well known, for an arbitrary function $f \in H^p$, nontangential limit boundary values exist almost everywhere on the boundary $\Gamma_r \bigcup \Gamma_1$. We denote these values by $f(re^{it})$ and $f(e^{it})$. These functions belong to $L^p(\Gamma_r)$ and $L^p(\Gamma_1)$, respectively.

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In the Hardy space H^p, we consider the class $Q = Q_r^p$ of functions f whose boundary values on the circle Γ_1 satisfy the inequality $\mathcal{M}^p(f,1) \le 1$.

The problem of the best approximation of an unbounded linear operator by linear bounded operators on a class of elements of a Banach space appeared in 1965 in investigations of Stechkin [4]. In his 1967 paper [4], he gave a statement of the problem, presented the first principal results, and solved the problem for differentiation operators of small orders. Detailed information about studies of Stechkin's problem and related extremal problems can be found in Arestov's review paper [3]. In the present paper, we consider the problem of the best approximation of the (first-order) differentiation operator for a function on the circle Γ_ρ by linear bounded operators on the class Q of functions analytic in the annulus C_r. The precise statement of the problem is as follows.

Problem 1. Let $\mathcal{L}(N)$ be the set of linear bounded operators from $L^p(\Gamma_r)$ to $L^p(\Gamma_\rho)$ with norm $\|T\| = \|T\|_{L^p(\Gamma_r)\mapsto L^p(\Gamma_\rho)}$ not exceeding a number $N > 0$. The quantity

$$U(T) := \sup\left\{\mathcal{M}^p(f' - Tf, \rho) :\ f \in Q\right\}$$

is the deviation of an operator $T \in \mathcal{L}(N)$ from the differentiation operator on the class Q. The quantity

$$E(N) := \inf\left\{U(T) : T \in \mathcal{L}(N)\right\} \tag{0.1}$$

is the best approximation of the differentiation operator by the set of bounded operators $\mathcal{L}(N)$ on the class Q. The problem is to calculate the quantity $E(N)$ and to find an extremal operator at which the infimum in (0.1) is attained.

Problem 1 is closely interconnected with a number of extremal problems. One of them is the following problem of calculating the modulus of continuity of the differentiation operator on a class.

Problem 2. The function

$$\omega(\delta) = \sup\left\{\mathcal{M}^p(f', \rho) :\ f \in Q,\ \mathcal{M}^p(f, r) \le \delta\right\} \tag{0.2}$$

of real variable $\delta \in [0, +\infty)$ is called the modulus of continuity of the differentiation operator on the class Q. The problem is to calculate the quantity $\omega(\delta)$ and to find an extremal function (a sequence of functions) at which the supremum in (0.2) is attained.

Define

$$\Delta(N) := \sup\left\{\omega(\delta) - N\delta :\ \delta \ge 0\right\}, N > 0;$$
$$l(\delta) := \inf\left\{E(N) + N\delta :\ N > 0\right\}, \delta \ge 0.$$

The following statement, which connects (0.1) and (0.2), is a special case of Stechkin's theorem [5].

Theorem A. *The following inequalities hold:*

$$E(N) \ge \Delta(N), \quad N > 0; \tag{0.3}$$

$$\omega(\delta) \le l(\delta), \quad \delta \ge 0. \tag{0.4}$$

Definition (0.2) also implies that the sharp inequality

$$\mathcal{M}^p(f', \rho) \le \mathcal{M}^p(f, 1)\, \omega\left(\frac{\mathcal{M}^p(f, r)}{\mathcal{M}^p(f, 1)}\right)$$

is valid for functions from the space $H^p(C_r)$.

Problems of recovering values of an operator on elements of a class lying in the domain of an operator from some information about the elements of the class given with a known error arise in different areas of mathematics and have been well studied. The recovery is implemented by using some set $\mathcal{R}$ of operators. As a rule, one of the following sets of mappings is taken for $\mathcal{R}$: either the set $\mathcal{O}$ of all single-valued mappings or the set $\mathcal{B}$ of bounded operators or the set $\mathcal{L}$ of linear operators. Monograph [6] is devoted to various problems of optimal recovery, in particular, optimal recovery of derivatives on classes of analytic functions.

Problems 1 and 2 are related to the following optimal recovery problem for the derivative of a function analytic in an annulus from boundary values (on one of the boundary circles) given with an error.

Problem 3. For a number $\delta \geq 0$ and an operator $T \in \mathcal{R}$, define

$$\mathcal{U}(T,\delta) = \sup\left\{\mathcal{M}^p(f' - Tg, \rho) :\ f \in Q,\ g \in L^p(\Gamma_r),\ \mathcal{M}^p(f-g,r) \leq \delta\right\}.$$

Then,

$$\mathcal{E}_{\mathcal{R}}(\delta) = \inf\left\{\mathcal{U}(T,\delta) : T \in \mathcal{R}\right\} \tag{0.5}$$

is the value of the best (optimal) recovery of the differentiation operator (the derivative of an analytic function) by recovery methods $\mathcal{R}$ on functions of the class Q from their boundary values on Γ_r given with an error δ. The problem is to calculate the quantity $\mathcal{E}(\delta)$ and to find an optimal recovery method, i.e., an operator at which the infimum in (0.5) is attained.

The following theorem contains a refinement of inequality (0.4); this theorem is a special case of a more general statement connecting the problem on the modulus of continuity of an operator and Stechkin's problem with optimal recovery problems (see [3]).

Theorem B. *The following inequalities hold:*

$$\omega(\delta) \leq \mathcal{E}_{\mathcal{O}}(\delta) \leq \mathcal{E}_{\mathcal{L}}(\delta) = \mathcal{E}_{\mathcal{B}}(\delta) \leq l(\delta), \quad \delta \geq 0. \tag{0.6}$$

1. Main results

We define a (convolution) operator $T_n^1 = T_n^1[\rho, r]$, $n \in \mathbb{Z}$, from $L^p(\Gamma_r)$ to $L^p(\Gamma_\rho)$ by the formula

$$(T_n^1 f)(\rho e^{ix}) = e^{-ix}\,\frac{1}{2\pi}\int_0^{2\pi} \Lambda_n^1(x-t) f(re^{it})\,dt \tag{1.1}$$

with the kernel

$$\Lambda_n^1(t) = r^{-n}e^{int}\,\lambda_n^1(t), \quad \lambda_n^1(t) = \lambda_{n,0}^1 + 2\sum_{k=1}^{\infty}\lambda_{n,k}^1\cos kt, \tag{1.2}$$

$$\lambda_{n,0}^1 = \frac{\rho^{n-1}}{\ln r}(n\ln\rho + 1), \quad \lambda_{n,k}^1 = \rho^{n-1}\,\frac{(n+k)\rho^k - (n-k)\rho^{-k}}{r^k - r^{-k}}, \quad k \in \mathbb{N}.$$

The following two theorems are the main results of the present paper.

Theorem 1. *Assume that the parameter N has the representation*

$$N = \frac{\rho^{n-1}\,|n\ln\rho + 1|}{r^n|\ln r|},$$

in which $n \in \mathbb{Z}$ is such that

$$|n| \geq \frac{\pi}{\ln^2 r}\,\sin^{-1}\left(\frac{\ln\rho}{\ln r}\pi\right). \tag{1.3}$$

Then, quantity (0.1) *satisfies the equality*

$$E(N) = \frac{\rho^{n-1}}{|\ln r|}\left|n \ln\frac{r}{\rho} - 1\right|.$$

In this case, the operator T_n^1 *defined by* (1.1) *and* (1.2) *is extremal in problem* (0.1).

Theorem 2. *Let* $\delta_n = r^n$, *where* $n \in \mathbb{Z}$ *satisfies condition* (1.3). *Then, quantities* (0.2) *and* (0.5) *satisfy the relations*

$$\omega(\delta_n) = \mathcal{E}_{\mathcal{O}}(\delta_n) = \mathcal{E}_{\mathcal{L}}(\delta_n) = \mathcal{E}_{\mathcal{B}}(\delta_n) = n\rho^{n-1}. \tag{1.4}$$

In this case, the linear bounded operator T_n^1 *defined by* (1.1) *and* (1.2) *is an optimal recovery method in problem* (0.5). *The functions* $f_n(z) = cz^n$, $|c| = 1$, *are extremal in problem* (0.2).

2. Auxiliary statements

In addition, we introduce a (convolution) operator $V_n^1 = V_n^1[\rho, r]$, $n \in \mathbb{Z}$, from $L^p(\Gamma_1)$ to $L^p(\Gamma_\rho)$ by the formula

$$(V_n^1 f)(\rho e^{ix}) = e^{-ix}\,\frac{1}{2\pi}\int_0^{2\pi} \mathcal{V}_n^1(x-t) f(re^{it})\,dt \tag{2.1}$$

with the kernel

$$\mathcal{V}_n^1(t) = e^{int}\,\mu_n^1(t), \quad \mu_n^1(t) = \mu_{n,0}^1 + 2\sum_{k=1}^{\infty} \mu_{n,k}^1 \cos kt, \tag{2.2}$$

$$\mu_{n,0}^1 = \frac{\rho^{n-1}}{\ln r}\left(n\ln\frac{r}{\rho} - 1\right), \quad \mu_{n,k}^1 = \rho^{n-1}\,\frac{(n+k)(\rho/r)^k - (n-k)(\rho/r)^{-k}}{r^{-k} - r^k}, \quad k \in \mathbb{N}.$$

Lemma 1. *For an arbitrary function* f *from the class* Q *and* $n \in \mathbb{Z}$, *we have the equality*

$$f'(\rho e^{ix}) = (T_n^1 f)(\rho e^{ix}) + (V_n^1 f)(\rho e^{ix}), \quad x \in [0, 2\pi]. \tag{2.3}$$

P r o o f. The function f in the annulus C_r is representable as the sum of the Laurent series

$$f(z) = \sum_{k=-\infty}^{+\infty} \varphi_k\, z^k, \quad z \in C_r.$$

Then, from the definitions of operators (1.1)–(1.2) and (2.1)–(2.2), we obtain the relations

$$(T_n^1 f)(\rho e^{ix}) + (V_n^1 f)(\rho e^{ix}) = \sum_{k=-\infty}^{+\infty} (\lambda_{n,k}^1 r^k + \mu_{n,k}^1)\,\varphi_{n+k}\, e^{i(n+k-1)x}.$$

Now, from the equality

$$\lambda_{n,k}^1 r^k + \mu_{n,k}^1 = (n+k)\rho^{n+k-1},$$

the assertion of Lemma 1 follows. □

Lemma 2. *Let a number* $n \in \mathbb{Z}$ *satisfy condition* (1.3). *Then the functions* λ_n^1 *and* μ_n^1 *defined by* (1.2) *and* (2.2) *are of the same sign, which remains unchanged on the period, i.e.,* $\lambda_n^1(x)\mu_n^1(x) > 0$, $x \in [0, 2\pi]$.

P r o o f. We introduce the notation

$$g_{\pm}(x,y) := \frac{e^{ny}\sin\left(\dfrac{\ln\rho}{\ln r}\pi\right)}{\cosh\dfrac{x\pi}{\ln r} \pm \cos\left(\dfrac{\ln\rho}{\ln r}\pi\right)}, \quad y = \ln\rho/r.$$

For the functions $g_{\pm}$, the following assertion is true [2, Lemma 3]. Condition (1.3) is necessary and sufficient for the functions $\dfrac{\partial g_{\pm}}{\partial y}$ to maintain sign for arbitrary $x \in \mathbb{R}$ and $0 < y < \ln 1/r$. Moreover, for the functions

$$\Lambda_{\pm}(x) := -\pi\ln^{-1} r e^{-ny} \sum_{k=-\infty}^{+\infty} g_{\pm}(x + 2\pi k, y), \quad y = \ln\rho/r,$$

the following equalities hold [1, Lemma 1]:

$$\Lambda_{\pm}(x) = \lambda_0^{\pm} + 2\sum_{k=1}^{\infty} \lambda_k^{\pm}\cos kx,$$

$$\lambda_0^+ = \frac{\ln\rho}{\ln r}, \quad \lambda_k^+ = \frac{\rho^k - \rho^{-k}}{r^k - r^{-k}}, \quad \lambda_0^- = \frac{\ln r/\rho}{\ln r}, \quad \lambda_k^- = \frac{(\rho/r)^k - (\rho/r)^{-k}}{r^{-k} - r^k}.$$

Hence, for the functions λ_n^1 и μ_n^1 defined by equalities (1.2) и (2.2), we have

$$\lambda_n^1(x) = \frac{\partial}{\partial\rho}\left(\rho^n\Lambda_+(x)\right) = -\frac{\pi r^n}{\rho\ln r}\sum_{k=-\infty}^{+\infty}\frac{\partial}{\partial y} g_+(x + 2\pi k, y),$$

$$\mu_n^1(x) = \frac{\partial}{\partial\rho}\left(\rho^n\Lambda_-(x)\right) = -\frac{\pi r^n}{\rho\ln r}\sum_{k=-\infty}^{+\infty}\frac{\partial}{\partial y} g_-(x + 2\pi k, y).$$

If $n \in \mathbb{Z}$ satisfies condition (1.3), then the right-hand sides of these equalities have the same sign, which remains unchanged on the period. Lemma 2 is proved. □

Corollary 1. *Let $n \in \mathbb{Z}$ satisfy condition* (1.3). *Then the equality $|\lambda_{n,0}^1| + |\mu_{n,0}^1| = n\rho^{n-1}$ holds.*

P r o o f. The proof follows from Lemma 2 and the chain of relations

$$|\lambda_{n,0}^1| + |\mu_{n,0}^1| = \left|\frac{1}{2\pi}\int_0^{2\pi}\lambda_n^1(t)\,dt\right| + \left|\frac{1}{2\pi}\int_0^{2\pi}\mu_n^1(t)\,dt\right| =$$

$$= \left|\frac{1}{2\pi}\int_0^{2\pi}\lambda_n^1(t)\,dt + \frac{1}{2\pi}\int_0^{2\pi}\mu_n^1(t)\,dt\right| = |\lambda_{n,0}^1 + \mu_{n,0}^1| = n\rho^{n-1}.$$

□

Lemma 3. *Let $n \in \mathbb{Z}$ satisfy condition* (1.3). *Then, for the norm and the deviations of the operator T_n^1 given by relations* (1.1), *the following equalities hold:*

$$\|T_n^1\| = \frac{\rho^{n-1}\,|n\ln\rho + 1|}{r^n|\ln r|}, \tag{2.4}$$

$$U(T_n^1) = \frac{\rho^{n-1}}{|\ln r|}\left|n\ln\frac{r}{\rho} - 1\right|, \tag{2.5}$$

$$\mathcal{U}(T_n^1, r^n) = n\rho^{n-1}. \tag{2.6}$$

P r o o f. Using the definition of the operator T_n^1 and Lemma 2, we obtain the following upper bound for the norm:

$$\|T_n^1\| \le \|\Lambda_n^1\|_{L^1(0,2\pi)} = \left|\frac{1}{2\pi r^n}\int_0^{2\pi} \lambda_n^1(t)\,dt\right| = r^{-n}|\lambda_{n,0}^1| = \frac{\rho^{n-1}\,|n\ln\rho+1|}{r^n|\ln r|}.$$

Now equality (2.4) follows from the lower bound due to the functions $r^{-n}f_n(z) = cr^{-n}z^n$, $|c| = 1$.

From equality (2.3) of Lemma 1, we obtain the representation

$$f'(\rho e^{ix}) - (T_n^1 f)(\rho e^{ix}) = (V_n^1 f)(\rho e^{ix}), \quad x \in [0, 2\pi].$$

Then, from the definition of the deviation and taking into account that the inequality $\|f\|_{L^p(\Gamma_1)} \le 1$ holds for functions f from the class Q , we obtain the estimate $U(T_n^1) \le \|V_n^1\|$. Arguing as in the first part of the proof, we can obtain the equality $\|V_n^1\| = |\mu_{n,0}^1|$. To complete the proof of equality(2.5), we note that the deviation $U(T_n^1)$ and the norm of the operator V_n^1 are attained at the functions $f_n(z) = cz^n$, $|c| = 1$.

Finally, using the following standard reasoning, we show that equality (2.6) is true. For arbitrary functions $f \in Q$ and $g \in L^p(\Gamma_r)$, we have

$$\mathcal{M}^p(f' - T_n^1 g, \rho) \le \mathcal{M}^p(f' - T_n^1 f, \rho) + \mathcal{M}^p(T_n^1(f-g), \rho) \le U(T_n^1) + \|T_n^1\|\,\mathcal{M}^p(f-g, r).$$

Then the equalities (2.4) and (2.5) and Corollary 1 imply the upper estimate

$$\mathcal{U}(T_n^1, r^n) \le U(T_n^1) + \|T_n^1\|\,r^n = |\mu_{n,0}^1| + |\lambda_{n,0}^1| = n\rho^{n-1}.$$

To obtain a lower bound, it is sufficient to consider $f(z) = f_n(z) = cz^n$ and $g \equiv 0$. The lemma is proved. □

Lemma 4. *For an arbitrary $n \in \mathbb{Z}$, the following inequalities hold:*

$$\omega(r^n) \ge n\rho^{n-1}, \tag{2.7}$$

$$\Delta\left(\frac{\rho^{n-1}\,|n\ln\rho+1|}{r^n|\ln r|}\right) \ge \frac{\rho^{n-1}}{|\ln r|}\left|n\ln\frac{r}{\rho} - 1\right|. \tag{2.8}$$

P r o o f. The function $f_n(z) = z^n$ belongs to the class Q. Then the following inequality holds:

$$\omega(r^n) \ge \mathcal{M}^p(f_n', \rho) = n\rho^{n-1}.$$

We have

$$\Delta(N) = \sup\{\omega(\delta) - N\delta \,:\, \delta \ge 0\} \ge \omega(r^n) - Nr^n \ge n\rho^{n-1} - Nr^n.$$

Substituting

$$N = \frac{\rho^{n-1}\,|n\ln\rho+1|}{r^n|\ln r|}$$

into the latter inequality and using Corollary 1, we obtain inequality (2.8). Lemma 4 is proved. □

3. Proof of the main results

P r o o f of Theorem 1. Assume that the parameter N has the representation

$$N = \frac{\rho^{n-1}\,|n\ln\rho+1|}{r^n|\ln r|},$$

in which $n\in\mathbb{Z}$ satisfies (1.3). Combining inequalities (0.3) from Theorem A, (2.8) from Lemma 4, and equality (2.5) from Lemma 3, we obtain the chain of relations

$$\frac{\rho^{n-1}}{|\ln r|}\left|n\ln\frac{r}{\rho}-1\right| \le \Delta(N) \le E(N) \le U(T^1_n) = \frac{\rho^{n-1}}{|\ln r|}\left|n\ln\frac{r}{\rho}-1\right|.$$

Hence,

$$E(N) = \frac{\rho^{n-1}}{|\ln r|}\left|n\ln\frac{r}{\rho}-1\right|.$$

This means that the operator T^1_n is extremal in Problem 1. Theorem 1 is proved. □

P r o o f of Theorem 2. Let $\delta_n = r^n$, where $n\in\mathbb{Z}$ satisfies condition (1.3). Combining inequalities (0.6) from Theorem B, (2.7) from Lemma 4, and equality (2.6) from Lemma 3, we obtain the chain of relations

$$n\rho^{n-1} \le \omega(\delta_n) \le \mathcal{E}_{\mathcal{O}}(\delta_n) \le \mathcal{E}_{\mathcal{L}}(\delta_n) = \mathcal{E}_{\mathcal{B}}(\delta_n) \le \mathcal{U}(T^1_n,\delta_n) = n\rho^{n-1}.$$

Hence,

$$\omega(\delta_n) = \mathcal{E}_{\mathcal{O}}(\delta_n) \le \mathcal{E}_{\mathcal{L}}(\delta_n) = \mathcal{E}_{\mathcal{B}}(\delta_n) = n\rho^{n-1}.$$

This means that the (bounded linear) operator T^1_n is extremal in Problem 3. Theorem 2 is proved. □

4. Generalization of the extremal operator and Theorem 1

It is proved in Lemma 2 that, if $n\in\mathbb{Z}$ satisfies condition (1.3), then the continuous 2π-periodic functions λ^1_n and μ^1_n do not vanish on $[0,2\pi]$, more precisely, $\lambda^1_n(t)\mu^1_n(t)>0$, $t\in[0,2\pi]$. This means that there exists an interval I_n (of positive length) defined by the equality

$$I_n = \left\{\gamma\in\mathbb{R} \,:\, (\lambda^1_n(t)+\gamma)(\mu^1_n(t)-\gamma)>0,\ t\in[0,2\pi]\right\}.$$

The interval $I_n=(\gamma^-_n,\gamma^+_n)$ has the boundary points

$$\gamma^-_n = \max_{t\in[0,2\pi]}\min\{-\lambda^1_n(t),\mu^1_n(t)\},\quad \gamma^+_n = \min_{t\in[0,2\pi]}\max\{-\lambda^1_n(t),\mu^1_n(t)\}$$

related by the inequality $\gamma^-_n<0<\gamma^+_n$. Let S_n be the interval $[\gamma^-_n,\gamma^+_n]$.

We define a (convolution) operator $T^1_{n,\gamma}=T^1_{n,\gamma}[\rho,r]$, $n\in\mathbb{Z}$, from $L^p(\Gamma_r)$ to $L^p(\Gamma_\rho)$ by the formula

$$(T^1_{n,\gamma}f)(\rho e^{ix}) = e^{-ix}\,\frac{1}{2\pi}\int_0^{2\pi}\Lambda^1_{n,\gamma}(x-t)f(re^{it})\,dt \tag{4.1}$$

with the kernel

$$\Lambda^1_{n,\gamma}(t) = r^{-n}e^{int}\left(\lambda^1_n(t)+\gamma\right). \tag{4.2}$$

The following statement is a generalization of Theorem 1.

Theorem 3. *Assume that the parameter N has the representation*

$$N = \frac{1}{r^n}\left|\frac{\rho^{n-1}(n\ln\rho + 1)}{\ln r} + \gamma\right|,$$

in which $n \in \mathbb{Z}$ satisfies (1.3) *and $\gamma \in S_n$. Then, quantity* (0.1) *satisfies the equality*

$$E(N) = \left|\frac{\rho^{n-1}(n\ln(r/\rho) - 1)}{\ln r} - \gamma\right|.$$

In this case, the operator $T^1_{n,\gamma}$ defined by (4.1) *and* (4.2) *is extremal in problem* (0.1).

P r o o f. The theorem can be proved by the scheme of the proof of Theorem 1. □

Remark 1. In the case when $n \in \mathbb{Z}$ satisfies (1.3) and $\gamma \in S_n$, the operators $T^1_{n,\gamma}$ defined by (4.1) and (4.2) are also extremal in Problem 3. However, these operators do not give solutions of this problem in new cases. More precisely, the equality $\mathcal{U}(T^1_{n,\gamma}, r^n) = n\rho^{n-1}$ holds.

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