THE VERTEX DISTANCE COMPLEMENT SPECTRUM OF SUBDIVISION VERTEX JOIN AND SUBDIVISION EDGE JOIN OF TWO REGULAR GRAPHS

Ann Susa Thomas
Department of Mathematics, St Thomas College,
Kozhencherry-689641, Kerala, India
anns11thomas@gmail.com

Sunny Joseph Kalayathankal
Jyothi Engineering College,
Cheruthuruthy, Thrissur-679531, Kerala, India
sjkalayathankal@jecc.ac.in

Joseph Varghese Kureethara
Department of Mathematics, Christ University,
Bangalore-560029, Karnataka, India
frjoseph@christuniversity.in

Abstract: The vertex distance complement (VDC) matrix $C$ of a connected graph $G$ with vertex set consisting of $n$ vertices, is a real symmetric matrix $[c_{ij}]$ that takes the value $n - d_{ij}$ where $d_{ij}$ is the distance between the vertices $v_i$ and $v_j$ of $G$ for $i \neq j$ and 0 otherwise. The vertex distance complement spectrum of the subdivision vertex join, $G_1V G_2$ and the subdivision edge join $G_1\overline{V} G_2$ of regular graphs $G_1$ and $G_2$ in terms of the adjacency spectrum are determined in this paper.

Keywords: Distance matrix, Vertex distance complement spectrum, Subdivision vertex join, Subdivision edge join.

1. Introduction

Spectral graph theory deals with the study of the eigenvalues of various matrices associated with graphs. Initially, the spectrum of the adjacency matrix of a graph was studied. Collatz and Sinogowitz initiated the exploration of this topic in 1957 [2]. Since then spectral theory of graphs is an active research area [1, 3].

In this paper, we consider the matrix derived from a type of distance matrix, viz., vertex distance complement (VDC) matrix. The VDC spectra of some classes of graphs are found in [8, 9]. The VDC matrix $C$ of a graph $G$ [7] is defined as follows

$$C = \begin{cases} 
  n - d_{ij}, & i \neq j, \\
  0, & i = j,
\end{cases}$$

where $d_{ij}$ is the distance between the vertices $v_i$ and $v_j$ of $G$ and $n$ denotes the number of vertices of $G$. 

The subdivision graph $S(G)$ of a graph $G$ is obtained by inserting a new vertex of degree two in every edge of $G$. Let $V(G)$ and $I(G)$ denote respectively the existing vertex set and the set of the newly introduced vertices of the subdivision graph $S(G)$ of a graph $G$. The adjacency spectrum of two joins, $G_1 \lor G_2$ and $G_1 \lor G_2$, based on subdivision graph was determined in [4]. The distance spectrum of the same was calculated in [6].

Throughout this article we consider connected simple graphs of diameter at most two. We determine the VDC spectrum of $G_1 \lor G_2$ and $G_1 \lor G_2$ when $G_1$ and $G_2$ are regular graphs. The eigenvalues of $VDC(G)$ are called the $VDC$-eigenvalues of $G$ and they form the $VDC$ spectrum of $G$, denoted by $spec_{VDC}(G)$. We denote $J$ and $I$ as the all-one matrix and identity matrix, respectively, of appropriate orders.

The definitions of the subdivision graphs are as follows.

**Definition 1** [4]. The subdivision-vertex join $G_1 \lor G_2$ of two vertex disjoint graphs $G_1$ and $G_2$ is the graph obtained from $S(G_1)$ and $G_2$ by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$.

**Definition 2** [4]. The subdivision-edge join $G_1 \lor G_2$ of two vertex disjoint graphs $G_1$ and $G_2$ is the graph obtained from $S(G_1)$ and $G_2$ by joining each vertex of $I(G_1)$ with every vertex of $V(G_2)$.

The following results are very useful for computing the VDC spectrum.

**Lemma 1** [3]. Let $G$ be an $r$-regular graph with adjacency matrix $A$ and incidence matrix $R$. Let $A(L(G))$ denote the adjacency matrix of the line graph $L(G)$ of $G$. Then,

$$RR^T = A + rI, \quad R^TR = A(L(G)) + 2I.$$ 

Also,

$$JR = 2J = R^TJ, \quad JR^T = rJ = RJ.$$ 

**Lemma 2** [3]. Let $G$ be $r$-regular $(n;m)$ graph with $spec(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then

$$spec(L(G)) = \begin{cases} 2r - 2, & i = 2, 3, \ldots, n, \\ \lambda_i + r - 2, & i = 1, \\ -2, & m - n times. \end{cases}$$

Also, $Z$ is an eigenvector corresponding to the eigenvalue $-2$ if and only if $RZ = 0$ where $R$ is the incidence matrix of $G$.

**Theorem 1 (Perron–Frobenius).** If all entries of an $n \times n$ matrix are positive, then it has a unique maximal eigenvalue. Its eigenvector has positive entries.

2. **The VDC spectrum of $G_1 \lor G_2$**

**Theorem 2.** Let $G_i$ be an $r_i$ regular graph with $n_i$ vertices and $m_i$ edges, for $i = 1, 2$. If $\{\lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{mi}\}$ denotes the adjacency spectrum corresponding to the adjacency matrix $A_i$ of $G_i$, the $spec_{VDC}(G_1 \lor G_2)$ consists of

(i) $2\lambda_{1i} + 2r_i - n + 2$, for $i = 2, 3, \ldots, n_1$;
Let $X$ be an eigenvector corresponding to the eigenvalue $\lambda_{1i} \neq r_1$ of $A_1$. Using Lemma 1, we note that

$$A(L(G_1))R^TX = (\lambda_{1i} + r_1 - 2)R^TX.$$ 

Hence, $\lambda_{1i} + r_1 - 2$ are the eigenvalues of $A(L(G_1))$ with an eigenvector $R^TX$.

By Perron–Frobenius theorem, $X$ and $R^TX$ are orthogonal to the all-one vector $J$.

Let

$$\mathbf{Y} = \begin{pmatrix} X \\ R^TX \\ 0 \end{pmatrix}.$$ 

Then,

$$2\lambda_{1i} + 2r_1 - n + 2, \quad i = 2, 3, \ldots, n_1$$

where $n = n_1 + m_1 + n_2$. 

Proof. Given that $G_1$ and $G_2$ are regular graphs with regularity $r_1$ and $r_2$ respectively. Let $R$ be the incidence matrix of $G_1$ and $A(L(G_1))$ be the adjacency matrix of the line graph of $G_1$. The distance matrix of a graph with diameter at most two and adjacency matrix $G$ of vertices, the VDC matrix of $G$ can be rewritten as $A + 2A$ or $2(J - I) - A$ [5].

The subdivision-vertex join $G_1 \bigvee G_2$ has $n = n_1 + m_1 + n_2$ vertices. With the proper labeling of vertices, the VDC matrix of $G_1 \bigvee G_2$ is a square matrix of order $n$ given by

$$C = \begin{pmatrix} (n - 2)(J - I) & (n - 3)J + 2R & (n - 1)J \\ (n - 3)J + 2R^T & (n - 4)(J - I) + 2A(L(G_1)) & (n - 2)J \\ (n - 1)J & (n - 2)J & (n - 2)(J - I) + A_2 \end{pmatrix}.$$ 

(ii) $-n$, repeated $m_1 - 1$ times;

(iii) $\lambda_{2i} - n + 2, \; \text{for} \; i = 2, 3, \ldots, n_2$;

(iv) the 3 roots of the equation

$$x^3 - (n_1 n - 2n_2 n - 2n_2 + m_1 n^2 - 4m_1 + 4r_1 + r_2 - 3n + 4)x^2$$

$$- (2n_1 n_2 n - 3n_1 m_1 - 2n_1 r_1 n + 2n_1 r_1 - n_1 r_2 n + 2n_1 r_2 + 2n_1 m_1)$$

$$- 6n_1 n + 4n_1 + 2n_2 m_1 n - 4n_2 m_1 - 4n_2 r_1 n + 8n_2 r_1 + 2n_2 n^2 - 6n_2 n + 4n_2 - m_1 r_2 n$$

$$+ 4n_1 r_2 + 2m_1 n^2 - 8m_1 n + 4m_1 - 4n_1 r_2 + 8r_1 n - 8r_1 - 2r_2 - 2r_2 - 3n^2 + 8n - 4)x$$

$$- (2n_1 n_2 m_1 - 4n_1 n_2 r_1 n + 4n_1 n_2 r_1 + 2n_1 n_2 n^2 - 3n_1 n_2 n - n_1 m_1 n_2$$

$$+ n_1 m_1 n - 2n_1 m_1 + 2n_1 r_1 r_2 n - 2n_1 r_1 r_2 - 2n_1 r_1 n^2 + 6n_1 r_1 n - 4n_1 r_1 - n_1 r_2 n^2$$

$$+ 2n_1 r_2 n + n_1 n^3 - 4n_1 n^2 + 4n_1 n + 2n_2 m_1 n^2 - 8n_2 m_1 - 4n_2 r_1 n^2 + 8n_2 r_1 n + n_2 n^3$$

$$- 4n_2 n^2 + 4n_2 n - m_1 r_2 n^2 + 2m_1 r_2 n + 4m_1 r_2 + m_1 n^3 - 4m_1 n^2 + 8m_1 - 4r_1 r_2 n$$

$$+ 4r_1 n^2 - 8r_1 n + r_2 n^2 - 2r_2 n - n^3 + 4n^2 - 4n) = 0,$$
is an eigenvalue of the VDC matrix of $G_1 \bigvee G_2$ corresponding to the eigenvector $\Psi$. This is because

$$
\begin{pmatrix}
(n-2)(J-I) & (n-3)J+2R \\
(n-3)J+2R^T & (n-4)(J-I)+2A(L(G)) \\
(n-1)J & (n-2)J
\end{pmatrix}
\begin{pmatrix}
\Psi
\end{pmatrix}
= (2\lambda_{1i} + 2r_1 - n + 2) \begin{pmatrix} \Psi \end{pmatrix}.
$$

By a similar reasoning, if $Y$ is an eigenvector of $A(L(G))$ corresponding to the eigenvalue $\lambda_{1i} + r_1 - 2$, for $i = 2, 3, \ldots, n_1$,

$$
\Phi = \begin{pmatrix} RY \\ -Y \\ 0 \end{pmatrix}
$$

is an eigenvector of VDC matrix of $G_1 \bigvee G_2$ corresponding to the eigenvalue $-n$. (Note that the line graph of a regular graph is also regular).

Hence, $-n$ is an eigenvalue of $G_1 \bigvee G_2$ repeated $n_1 - 1$ times.

Now, $-2$ is an eigenvalue of $A(L(G))$ with multiplicity $m_1 - n_1$. Let $Z$ be an eigenvector of $A(L(G))$ corresponding to the eigenvalue $-2$. Then, by Lemma 2, $RZ = 0$ and by Perron–Frobenius theorem, $JZ = 0$.

Let

$$
\Omega = \begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix}.
$$

Then $-n$ is an eigenvalue of the VDC matrix of $G_1 \bigvee G_2$ repeated $m_1 - n_1$ times with an eigenvector $\Omega$. This is because

$$
\begin{pmatrix}
(n-2)(J-I) & (n-3)J+2R \\
(n-3)J+2R^T & (n-4)(J-I)+2A(L(G)) \\
(n-1)J & (n-2)J
\end{pmatrix}
\begin{pmatrix}
\Omega
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ -n \end{pmatrix}.
$$

In total, $-n$ is an eigenvalue of $G_1 \bigvee G_2$ repeated $m_1 - 1$ times.

Now, let $\lambda_{2i} \neq r_2$ be an eigenvalue of $G_2$ with an eigenvector $W$. Since $G_2$ is regular, $JW = 0$.

Hence

$$
\Psi = \begin{pmatrix} 0 \\ 0 \\ W \end{pmatrix}
$$

is an eigenvector of the VDC matrix of $G_1 \bigvee G_2$ corresponding to the eigenvalue $\lambda_{2i} - n + 2$, for $i = 2, 3, \ldots, n_2$. Thus, we have obtained $n_1 + m_1 + n_2 - 3$ eigenvalues.

The remaining three eigenvalues are to be determined. We note that all the eigenvectors constructed so far, are orthogonal to

$$
\begin{pmatrix} J \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ J \end{pmatrix}.
$$
The remaining three eigenvectors are spanned by these three vectors and is of the form

$$\Theta = \begin{pmatrix} \alpha J \\ \beta J \\ \gamma J \end{pmatrix}.$$  

for some $\alpha, \beta, \gamma \neq (0, 0, 0)$.

Thus, if $\rho$ is an eigenvalue of the VDC matrix with an eigenvector $\Theta$, then from $C \Theta = \rho \Theta$, we can see that the remaining three eigenvalues are obtained from the matrix

$$\begin{pmatrix} (n-2)(n_1-1) & (n-3)m_1 + 2r_1 & (n-1)n_2 \\ (n-3)n_1 + 4 & n(m_1-1) - 4(m_1 - r_1) & (n-2)n_2 \\ (n-1)n_1 & (n-2)m_1 & (n-2)(n_2-1) + r_2 \end{pmatrix}.$$  

Thus we determine the VDC spectrum of $G_1 \upharpoonright G_2$.  

3. The VDC spectrum of $G_1 \upharpoonright G_2$

In this section we present the VDC spectrum of $G_1 \upharpoonright G_2$.

**Theorem 3.** Let $G_i$ be $r_i$ regular graph with $n_i$ vertices and $m_i$ edges, for $i = 1, 2$. If $\{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{im_i}\}$ denotes the adjacency spectrum corresponding to the adjacency matrix $A_i$ of $G_i$, then the spec$_{VDC}(G_1 \upharpoonright G_2)$ consists of

(i) $\lambda_{i1} + 3 \pm \sqrt{(\lambda_{i1} + 1)^2 + 4(\lambda_{i1} + r_1) - n_i}$, for $i = 2, 3, \ldots, n_1$;

(ii) $-n + 2$, repeated $m_1 - n_1$ times;

(iii) $\lambda_{i2} - n + 2$, for $i = 2, 3, \ldots, n_2$;

(iv) the 3 roots of the equation

$$x^3 - (n_1n - 4n_1 + 2n_2 - 2n_1m_1 - 2n_1r_1 + 2r_1 = -3n + 8)x^2$$

$$- (2n_1n_2 - 4n_1n_2 + n_1m_1 + 2n_1r_1 - 6n_1n_1 - 4n_1r_2 + 2n_1n_2^2 - 12n_1n + 16n_1$$

$$+ 2n_2m_1n - 3n_2m_1 - 2n_2r_1 + 4n_2r_1 + 2n_2n^2 - 10n_2 + 12n_2 - 2m_1r_1 + 4m_1r_1 + 2m_1n^2$$

$$- 6m_4 - m_1r_2 - 2r_1 + 4r_1 + 2n_1r_2 + 2r_2 - 6r_2 - 3n^2 + 16n - 20)x$$

$$-(2n_1n_2 + 4n_1n_2 - 8n_1n_2 + 8n_1n_2^2 - 8n_1r_2 - 16n_1$$

$$- 4n_2m_1n + 6n_2m_1r - 3n_2m_1 - 16n_2 + 2m_1r_2 - 2m_1r_2 - 2m_1n^2$$

$$+ 8n_1r_2 - 8n_1r_1 - 4r_1 + 2n_1n_2^2 - 8n_1n_2 - n_1m_1 + n_1m_1 - 2n_1m_1$$

$$- 2n_1r_2 + 6n_1r_2 + 2n_1r_2 + 10n_1r_1 + 12n_1r_1 - n_1r_2n^2 + 6n_1r_2n$$

$$+ n_1n^3 - 8n_1n^2 + 20n_1n + 2n_2m_1n^2 - 4n_2m_1 - 2n_2r_1n^2 + 8n_2r_1 + n_2n^3$$

$$- 8n_2n^2 + 20n_2n - m_1r_2n^2 + 2m_1n_2 + 4m_1r_2 + m_1n_1 - 4m_1n^2 + 8m_1 - 2r_1r_2n$$

$$+ 2r_1n^2 - 8r_1 + r_2n^2 - 6r_2n - n^3 + 8n^2 - 20n + 8r_1 + 16) = 0.$$  

where $n = n_1 + m_1 + n_2$.  

Proof. Given that $G_1$ and $G_2$ are regular graphs with regularity $r_1$ and $r_2$ respectively. Let $R$ be the incidence matrix of $G_1$. $G_1 \bigvee G_2$ has $n = n_1 + m_1 + n_2$ vertices. With the proper labeling of vertices, the VDC matrix of $G_1 \bigvee G_2$ of order $n$ is given by

$$C = \begin{pmatrix}
(n-4)(J-I) + 2A_1 & (n-3)J + 2R & (n-2)J \\
(n-3)J + 2R & (n-2)(J-I) & (n-1)J \\
(n-2)J & (n-1)J & (n-2)(J-I) + A_2
\end{pmatrix}.$$  

Let $\lambda_{i1} \neq r_1$ be an eigenvalue of $A_1$ with an eigenvector $X$. By Perron–Frobenius theorem, $X$ is orthogonal to the all-one vector $J$.

Let us test the condition under which

$$\forall \Theta = \begin{pmatrix} tX \\ R^T X \\ 0 \end{pmatrix}$$

is an eigenvector of the given VDC matrix.

If $\Theta$ is an eigenvector of the VDC matrix of $G_1 \bigvee G_2$ corresponding to the eigenvalue $\eta$, then $C\Theta = \eta \Theta$ implies

$$\left(\begin{array}{ccc}
(n-4)(J-I) + 2A_1 & (n-3)J + 2R & (n-2)J \\
(n-3)J + 2R & (n-2)(J-I) & (n-1)J \\
(n-2)J & (n-1)J & (n-2)(J-I) + A_2
\end{array}\right)\begin{pmatrix} tX \\ R^T X \\ 0 \end{pmatrix} = \eta \begin{pmatrix} tX \\ R^T X \\ 0 \end{pmatrix}$$

i. e.,

$$-(n-4)t + 2t\lambda_{i1} + 2\lambda_{i1} + 2r_1 = \eta t$$ (3.1)

and

$$2t - (n-2) = \eta.$$ (3.2)

Substituting the value of $\eta$ from equation (3.2) in equation (3.1), we get a quadratic equation in $t$ as

$$t^2 - (1 + \lambda_{i1})t - (\lambda_{i1} + r_1) = 0$$

Hence

$$t = \frac{(1 + \lambda_{i1}) \pm \sqrt{(1 + \lambda_{i1})^2 + 4(\lambda_{i1} + r_1)}}{2}.$$  

Thus corresponding to each eigenvalue $\lambda_{i1} \neq r_1$ of $A_1$, we get two VDC eigenvalues $\eta = 2t + 2 - n$ of $G_1 \bigvee G_2$ and hence a total of $2(n_1 - 1)$ VDC eigenvalues are obtained.

Now, $-2$ is an eigenvalue of $A(L(G_1))$ with multiplicity $m_1 - n_1$. Let $Z$ be an eigenvector of $A(L(G_1))$ with eigenvalue $-2$. Then, by Lemma 2, $RZ = 0$.

However,

$$\Omega = \begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix}$$

is an eigenvector of the VDC matrix of $G_1 \bigvee G_2$ corresponding to the eigenvalue $-n + 2$.

Let $\lambda_{2i} \neq r_2$ be an eigenvalue of $G_2$ with an eigenvector $W$. Then,

$$\Psi = \begin{pmatrix} 0 \\ 0 \\ W \end{pmatrix}$$

is an eigenvector of the VDC matrix of $G_1 \bigvee G_2$ corresponding to the eigenvalue $\lambda_{2i} - n + 2$, for $i = 2, 3, \ldots, n_2$. 


Thus, we have obtained $n_1 + m_1 + n_2 - 3$ eigenvalues. Next, we will determine the remaining three eigenvalues. We note that all the eigenvectors constructed are orthogonal to

\[
\begin{pmatrix} J \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ J \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ J \end{pmatrix}.
\]

The remaining three eigenvectors are spanned by these three vectors and is of the form

\[
\Theta = \begin{pmatrix} \alpha J \\ \beta J \\ \gamma J \end{pmatrix}
\]

for some $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Thus, if $\rho$ is an eigenvalue of $C$ with an eigenvector $\Theta$ then from $C\Theta = \rho\Theta$, we can see that the remaining three eigenvalues are obtained from the matrix

\[
\begin{pmatrix}
(n - 4)(n_1 - 1) + 2r_1 & (n - 3)m_1 + 2r_1 & (n - 2)n_2 \\
(n - 1)n_1 + 4 & (n - 2)(m_1 - 1) & (n - 1)n_2 \\
(n - 2)n_1 & (n - 1)m_1 & (n - 2)(n_2 - 1) + r_2
\end{pmatrix}.
\]

\[\square\]

4. Conclusion

In this paper we have computed the Vertex Distance Complement Spectrum of Subdivision Vertex Join, $G_1 \vee G_2$, and Subdivision Edge Join, $G_1 \overleftarrow{\vee} G_2$ of regular graphs $G_1$ and $G_2$. The work can be extended to graphs with diameter greater than two, graphs that are not regular etc. It is worth exploring the nature of the spectrum of graphs with arbitrary subdivisions.

REFERENCES