SHILLA GRAPHS WITH $b = 5$ AND $b = 6$†

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Abstract: A $Q$-polynomial Shilla graph with $b = 5$ has intersection arrays \{105t, 4(21t + 1), 16(t + 1); 1, 4(t + 1), 84t\}, $t \in \{3, 4, 19\}$. The paper proves that distance-regular graphs with these intersection arrays do not exist. Moreover, feasible intersection arrays of $Q$-polynomial Shilla graphs with $b = 6$ are found.

Keywords: Shilla graph, Distance-regular graph, $Q$-polynomial graph.

1. Introduction

We consider undirected graphs without loops or multiple edges. For a vertex $a$ of a graph $\Gamma$, denote by $\Gamma_i(a)$ the $i$th neighborhood of $a$, i.e., the subgraph induced by $\Gamma$ on the set of all vertices at distance $i$ from $a$. Define $[a] = \Gamma_1(a)$ and $a^+= \{a\} \cup [a]$.

Let $\Gamma$ be a graph, and let $a, b \in \Gamma$. Denote by $\mu(a, b)$ (by $\lambda(a, b)$) the number of vertices in $[a] \cap [b]$ if $a$ and $b$ are at distance 2 (are adjacent) in $\Gamma$. Further, the induced $[a] \cap [b]$ subgraph is called $\mu$-subgraph ($\lambda$-subgraph).

If vertices $u$ and $w$ are at distance $i$ in $\Gamma$, then we denote by $b_i(u, w)$ (by $c_i(u, w)$) the number of vertices in the intersection of $\Gamma_{i+1}(u)$ (of $\Gamma_{i-1}(u)$, respectively) with $[w]$. A graph $\Gamma$ of diameter $d$ is called distance-regular with intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if, for each $i = 0, \ldots, d$, the values $b_i(u, w)$ and $c_i(u, w)$ are independent of the choice of vertices $u$ and $w$ at distance $i$ in $\Gamma$. Define $a_i = k - b_i - c_i$. Note that, for a distance regular graph, $b_0$ is the degree of the graph and $a_1$ is the degree of the local subgraph (the neighborhood of the vertex). Further, for vertices $x$ and $y$ at distance $l$ in the graph $\Gamma$, denote by $p_{ij}(x, y)$ the number of vertices in the subgraph $\Gamma_i(x) \cap \Gamma_j(y)$. The numbers $p_{ij}(x, y)$ are called the intersection numbers of $\Gamma$ (see [2]). In a distance-regular graph, they are independent of the choice of $x$ and $y$.

A Shilla graph is a distance-regular graph $\Gamma$ of diameter 3 with second eigenvalue $\theta_1$ equal to $a = a_3$. In this case, $a$ divides $k$ and $b$ is defined by $b = b(\Gamma) = k/a$. Moreover, $a_1 = a - b$ and $\Gamma$ has intersection array $\{ab, (a + 1)(b - 1), b_2, 1, c_2, a(b - 1)\}$. Feasible intersection arrays of Shilla graphs are found in [6] for $b \in \{2, 3\}$.

Feasible intersection arrays of Shilla graphs are found in [1] for $b = 4$ (50 arrays) and for $b = 5$ (82 arrays). At present, a list of feasible intersection arrays of Shilla graphs for $b = 6$ is unknown. Moreover, the existence of $Q$-polynomial Shilla graphs with $b = 5$ also is unknown.

In this paper, we find feasible intersection arrays of $Q$-polynomial Shilla graphs with $b = 6$ and prove that $Q$-polynomial Shilla graphs with $b = 5$ do not exist.

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Theorem 1. A Q-polynomial Shilla graph with $b = 6$ has intersection array

(1) $\{42t, 5(7t + 1), 3(t + 3); 1, 3(t + 3), 35t\}$, where $t \in \{7, 12, 17, 27, 57\}$;

(2) $\{372, 315, 75; 1, 15, 310\}$, $\{744, 625, 125; 1, 25, 620\}$ or $\{930, 780, 150; 1, 30, 775\}$;

(3) $\{312, 265, 48; 1, 24, 260\}$, $\{624, 525, 80; 1, 40, 520\}$, $\{1794, 1500, 200; 1, 100, 1495\}$ or $\{5694, 4750, 600; 1, 300, 4745\}$.

In view of Theorem 2 from [1], a Q-polynomial Shilla graph with $b = 5$ has intersection array $\{105t, 4(21t + 1), 16(t + 1); 1, 4(t + 1), 84t\}$, $t \in \{3, 4, 19\}$.

Theorem 2. Distance-regular graphs with intersection arrays $\{315, 256, 64; 1, 16, 252\}$ and $\{1995, 1600, 320; 1, 80, 1596\}$ do not exist.

Theorem 3. Distance-regular graphs with intersection array $\{420, 340, 80; 1, 20, 336\}$ do not exist.

2. Proof of Theorem 1

In this section, $\Gamma$ is a Q-polynomial Shilla graph with $b = 6$. Then $(a_2 - 5a_1 - 6)^2 - 4(5b_2 - a_3)$ is the square of an integer. By [6, Lemma 8], we have

$$2a \leq c_2b(b + 1) + b^2 - b - 2;$$

therefore, $a \leq 21c_2 + 14$. It follows from the proof of Theorem 9 in [6] that either $k < b^3 - b = 6 \cdot 35$ or $v < k(2b^3 - b + 1) = 428k$. By [6, Corollary 17 and Theorem 20], the number $b_2 + c_2$ divides $b(b - 1)b_2$ and

$$-34 = -b^2 + 2 \leq \theta_3 \leq -b^2(b + 3)/(3b + 1) \leq -18.$$

Theorem 2 from [7] implies the following lemma.

Lemma 1. If $b_2 = c_2$, then $\Gamma$ has an intersection arrays $\{42t, 5(7t + 1), 3(t + 3); 1, 3(t + 3), 35t\}$ and $t \in \{7, 12, 17, 27, 57\}$.

To the end of this section, assume that $b_2 \neq c_2$ and $k > \theta_1 > \theta_2 > \theta_3$ are eigenvalues of the graph $\Gamma$. Then

$$6(6b_2 + c_2)/(b_2 + c_2) = -\theta_3.$$

On the other hand, according to [6, Lemma 10], the number $c_2$ divides $(a + 6)b_2$, $30a(a + 1)$ and $(a + 6)b_2 \geq (a + 1)c_2$.

Lemma 2. If $-34 \leq \theta_3 \leq -18$, then one of the following statements holds:

(1) $\theta_3 = -31$ and $\Gamma$ has one of the intersection arrays $\{372, 315, 75; 1, 15, 310\}$, $\{744, 625, 125; 1, 25, 620\}$, and $\{930, 780, 150; 1, 30, 775\}$;

(2) $\theta_3 = -26$ and $\Gamma$ has one of the intersection arrays $\{312, 265, 48; 1, 24, 260\}$, $\{624, 525, 80; 1, 40, 520\}$, $\{1794, 1500, 200; 1, 100, 1495\}$, and $\{5694, 4750, 600; 1, 300, 4745\}$;

(3) $\theta_3 = -21$ and $\Gamma$ has one of the intersection arrays $\{42t, 5(7t + 1), 3(t + 3); 1, 3(t + 3), 35t\}$ for $t \in \{7, 12, 17, 27, 57\}$. 
The lemma is proved.

Let \( \theta_3 = -34 \). Then \( 3(6b_2 + c_2) = 17(b_2 + c_2) \) and \( b_2 = 14c_2 \). Further, \( \theta_3 \) is a root of the equation \( x^2 - (a_1 + a_2 - k)x + (b - 1)b_2 - a_2 = 0 \); therefore, \( a = 425/28 \cdot c_2 - 34 \). In this case, the multiplicity of the first nonprincipal eigenvalue is \( m = 6/5 \cdot (2545c_2 - 5544)/c_2 \), a contradiction with the fact that 5 does not divide 6 \cdot 5544.

Let \( \theta_3 = -33 \). Then \( 2(6b_2 + c_2) = 11(b_2 + c_2) \) and \( b_2 = 9c_2 \). Further, \( a = 275/27 \cdot c_2 - 33 \) and the multiplicity of the first nonprincipal eigenvalue is equal to \( m_1 = 6/5 \cdot (1645c_2 - 5184)/c_2 \), a contradiction as above.

Let \( \theta_3 = -32 \). Then \( 3(6b_2 + c_2) = 16(b_2 + c_2) \) and \( 2b_2 = 13c_2 \). Further, \( a = 100/13 \cdot c_2 - 32 \) and the multiplicity of the first nonprincipal eigenvalue is \( m_1 = 6/5 \cdot (1195c_2 - 4836)/c_2 \), a contradiction as above.

Let \( \theta_3 = -31 \). Then \( 6(6b_2 + c_2) = 31(b_2 + c_2) \) and \( b_2 = 5c_2 \). Further, \( a = 31/5 \cdot c_2 - 31 \) and the multiplicity of the first nonprincipal eigenvalue is \( m_1 = 30(37c_2 - 180)/c_2 = 1110 - 5400/c_2 \). The number of vertices in the graph is \( 31/5 \cdot (222c_2^2 - 2005c_2 + 4500)/c_2 \); hence, \( c_2 \) divides 900 and is a multiple of 5. By computer enumeration, we find that, only for \( c_2 = 15, 25 \) and 30, we have admissible intersection arrays \( \{372, 315, 75; 1, 15, 310\} \), \( \{744, 625, 125; 1, 25, 620\} \) and \( \{930, 780, 150; 1, 30, 775\} \).

Let \( \theta_3 = -30 \). Then \( 6(b_2 + c_2) = 5(b_2 + c_2) \) and \( b_2 = 4c_2 \). Further, \( a = 125/24 \cdot c_2 - 30 \) and the multiplicity of the first nonprincipal eigenvalue is \( m_1 = 6/5 \cdot (745c_2 - 4176)/c_2 \), a contradiction as above.

Let \( \theta_3 = -27 \). Then \( 2(b_2 + c_2) = 9(b_2 + c_2) \) and \( 3b_2 = 7c_2 \). Further, \( a = 25/7 \cdot c_2 - 25 \) and the multiplicity of the first nonprincipal eigenvalue is \( m_1 = 6/5 \cdot (445c_2 - 3276)/c_2 \), a contradiction as above.

Let \( \theta_3 = -26 \). Then \( 3(6b_2 + c_2) = 13(b_2 + c_2) \) and \( b_2 = 12c_2 \). Further, \( a = 13/4 \cdot c_2 - 26 \) and the multiplicity of the first nonprincipal eigenvalue is \( m_1 = 6/5 \cdot (77c_2 - 600)/c_2 = 462 - 3600/c_2 \). The number of vertices in the graph is \( 13/8 \cdot (231c_2^2 - 3340c_2 + 12000)/c_2 \); hence, \( c_2 \) divides 1200 and is a multiple of 4. By computer enumeration, we find that only for \( c_2 = 24, 40, 100, \) and 300 we have admissible intersection arrays \( \{312, 265, 48; 1, 24, 260\} \), \( \{624, 525, 80; 1, 40, 520\} \), \( \{1794, 1500, 200; 1, 100, 1495\} \), and \( \{5694, 4750, 600; 1, 300, 4745\} \).

Let \( \theta_3 = -21 \). Then \( 2(b_2 + c_2) = 7(b_2 + c_2) \) and \( b_2 = c_2 \). Further, \( a = 7/3 \cdot c_2 - 21 \) and the multiplicity of the first nonprincipal eigenvalue is \( m_1 = 6/5 \cdot (41c_2 - 360)/c_2 = 246 - 2160/c_2 \). The number of vertices in the graph is \( 7/3 \cdot (82c_2^2 - 1335c_2 + 5400)/c_2 \); hence, \( c_2 \) divides 1080 and is a multiple of 3. By computer enumeration, we find that, only for \( c_2 = 18, 30, 45, 60, 90, \) and 180, we have admissible intersection arrays \( \{42t, 5(7t+1), 3(t+3); 1, 3(t+3), 35t\} \) for \( t \in \{3, 7, 12, 17, 27, 57\} \).

A graph with the array obtained for \( t = 3 \) does not exist by [5].

Let \( \theta_3 = -18 \). Then \( 6(b_2 + c_2) = 19(b_2 + c_2) \), so \( 3b_2 = c_2 \). Further, \( a = 2512 \cdot c_2 - 18 \) and the multiplicity of the first nonprincipal eigenvalue is \( m_1 = 6/5 \cdot (145c_2 - 1224)/c_2 \), a contradiction. The lemma is proved.

Theorem 1 follows from Lemmas 1–2.

3. Triple intersection numbers

In the proof of Theorem 3, the triple intersection numbers [3] are used.
Let $\Gamma$ be a distance-regular graph of diameter $d$. If $u_1, u_2, u_3$ are vertices of the graph $\Gamma$, then $r_1, r_2, r_3$ are non-negative integers not greater than $d$. Denote by $\{u_1 u_2 u_3\over r_1 r_2 r_3\}$ the set of vertices $w \in \Gamma$ such that $d(w, u_i) = r_i$ and by $\{u_1 u_2 u_3\over r_1 r_2 r_3\}$ the number of vertices in $\{u_1 u_2 u_3\over r_1 r_2 r_3\}$. The numbers $\{u_1 u_2 u_3\over r_1 r_2 r_3\}$ are called the triple intersection numbers. For a fixed triple of vertices $u_1, u_2, u_3$, instead of $\{u_1 u_2 u_3\over r_1 r_2 r_3\}$, we will write $[r_1 r_2 r_3]$. Unfortunately, there are no general formulas for the numbers $[r_1 r_2 r_3]$. However, [3] outlines a method for calculating some numbers $[r_1 r_2 r_3]$.

Let $u, v, w$ be vertices of the graph $\Gamma$, $W = d(u, v), U = d(v, w)$, and let $V = d(u, w)$. Since there is exactly one vertex $x = u$ such that $d(x, u) = 0$, then the number $[0 jh]$ is 0 or 1. Hence $[0 jh] = \delta_{W} \delta_{U}$. Similarly, $[i0h] = \delta_{V} \delta_{U}$ and $[ij0] = \delta_{U} \delta_{V}$.

Another set of equations can be obtained by fixing the distance between two vertices from $\{u, v, w\}$ and counting the number of vertices located at all possible distances from the third:

\[
\begin{align*}
\sum_{l=1}^{d} [l jh] &= p_{jh}^{U} - [0 jh] \\
\sum_{l=1}^{d} [i lh] &= p_{jh}^{V} - [i0h] \\
\sum_{l=1}^{d} [i jl] &= p_{ijh}^{W} - [ij0]
\end{align*}
\]

(3.1)

However, some triplets disappear. For $|i - j| > W$ or $i + j < W$, we have $p_{ijh}^{W} = 0$; therefore, $[ijh] = 0$ for all $h \in \{0, ..., d\}$.

We set

\[S_{i j h}(u, v, w) = \sum_{r, s, t = 0}^{d} Q_{r i} Q_{s j} Q_{t h} \left[u w v\right]_{r s t}.\]

If the Krein parameter $q_{ij}^{h} = 0$, then $S_{i j h}(u, v, w) = 0$.

We fix vertices $u, v, w$ of a distance-regular graph $\Gamma$ of diameter 3 and set

\[
\begin{align*}
\{ijh\} &= \\{uvw\over i j h\}, \quad [ijh] = \left[u w v\right]_{i j h}, \quad [ijh]' = \left[u w v\right]_{i h j}, \quad [ijh]^* = \left[v w u\right]_{j i h}, \quad [ijh]^~ = \left[w v u\right]_{h j i}.
\end{align*}
\]

Calculating the numbers

\[
\begin{align*}
[ijh]' &= \left[u w v\right]_{i h j}, \quad [ijh]^* = \left[v w u\right]_{j i h}, \quad [ijh]^~ = \left[w v u\right]_{h j i}
\end{align*}
\]

(symmetrization of the triple intersection numbers) can give new relations that make it possible to prove the nonexistence of a graph.

4. **Graphs with intersection arrays** $\{315, 256, 64; 1, 16, 252\}$ and $\{1995, 1600, 320; 1, 80, 1596\}$

Let $\Gamma$ be a distance-regular graph with intersection array $\{315, 256, 64; 1, 16, 252\}$. By [2, Theorem 4.4.3], the eigenvalues of the local subgraph of the graph $\Gamma$ are contained in the interval $[-5, 59/5]$. Since the Terwilliger polynomial (see [4]) is $-4(5x - 59)(x + 5)(x + 1)(x - 43)$, then these eigenvalues lie in $[-5, -1] \cup (59/5, 43]$. Hence, all nonprincipal eigenvalues are negative and the
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Local subgraph is a union of isolated \((a_1 + 1)\)-cliques, a contradiction with the fact that \( a_1 + 1 = 49 \) does not divide \( k = 315 \).

Thus, a distance-regular graph with intersection array \( \{315, 256, 64; 1, 16, 252\} \) does not exist.

Let \( \Gamma \) be a distance-regular graph with intersection array \( \{1995, 1600, 320; 1, 80, 1596\} \). Then \( \Gamma \) has \( 1 + 1995 + 39900 + 8000 = 49896 \) vertices, spectrum \( 1995^1, 399^{495}, 15^{23275}, -21^{26125} \), and the dual matrix of eigenvalues

\[
Q = \begin{pmatrix}
1 & 495 & 23275 & 26125 \\
1 & 99 & 175 & -275 \\
1 & 0 & -56 & 55 \\
1 & -99/4 & 931/4 & -209 \\
\end{pmatrix}
\]

The Terwilliger polynomial of the graph \( \Gamma \) is \(-20(x+5)(x+1)(x-79)(x-299)\); hence, the eigenvalues of the local subgraph are contained in \([-5, -1] \cup \{79\} \cup \{394\}\).

Note that the multiplicity \( m_1 = 495 \) of the eigenvalue \( \theta_1 = 399 \) is less than \( k \). By the corollary to Theorem 4.4.4 from [2] for \( b = b_1/(\theta_1 + 1) = 4 \), the graph \( \Sigma = [u] \) has an eigenvalue \(-1 - b = -5\) of multiplicity at least \( k - m_1 = 1500 \).

Let the number of eigenvalues \( 79 \) of the graph \( \Sigma \) be equal to \( y \). Then the sum of eigenvalues of the graph \( \Sigma \) is at most \(-7500 - (494 - y) + 79y + 394\); therefore, \( y \geq 95 \). Now twice the number of edges in \( \Sigma \) is equal to \( 786030 = 1995 \cdot 394 = \sum_i m_i \theta_i^2 \) but not less than \( 25 \cdot 1500 + 399 \cdot 79^2 + 394^2 = 786030 \).

Hence, \( \Sigma \) has spectrum \( 394^1, 79^{95}, -1^{399}, -5^{1500} \).

Now the number \( t = k_\Sigma \lambda_\Sigma/2 \) of triangles in \( \Sigma \) containing this vertex is equal to \( \sum_i m_i \theta_i^3/(2v) \).

Therefore,

\[
t = \sum_i m_i \theta_i^3/(2v) = (394^3 + 79^3 \cdot 95 - 399 - 125 \cdot 1500)/3990 = 27021
\]

and \( \lambda_\Sigma = 54042/394 \) is approximately equal to 137.16, a contradiction.

Thus, a distance-regular graph with intersection array \( \{1995, 1600, 320; 1, 80, 1596\} \) does not exist.

Theorem 2 is proved.

5. Graph with array \( \{420, 340, 80; 1, 20, 336\} \)

Let \( \Gamma \) be a distance-regular graph with intersection array \( \{420, 340, 80; 1, 20, 336\} \). Then \( \Gamma \) is a formally self-dual graph having \( 1 + 420 + 7140 + 1700 = 9261 \) vertices, spectrum \( 420^1, 84^{420}, 0^{140}, -21^{1700} \), and the dual matrix of eigenvalues

\[
Q = \begin{pmatrix}
1 & 420 & 7140 & 1700 \\
1 & 84 & 0 & -85 \\
1 & 0 & -21 & 20 \\
1 & -21 & 84 & -64 \\
\end{pmatrix}
\]

The Terwilliger polynomial of the graph \( \Gamma \) is \(-20(x+5)(x+1)(x-16)(x-59)\) and the eigenvalues of the local subgraph are contained in \([-5, -1] \cup \{16\} \cup \{79\}\). If the nonprincipal eigenvalues of a local subgraph are negative, then this subgraph is a union of isolated \((a_1 + 1)\)-cliques, a contradiction with the fact that \( a_1 + 1 = 80 \) does not divide \( k = 420 \). Hence, the local subgraph has eigenvalue 6.
Lemma 3. Intersection numbers of a graph $\Gamma$ satisfy the equalities

1. $p_{11}^1 = 79, p_{21}^1 = 340, p_{32}^1 = 1360, p_{22}^1 = 5440, p_{33}^1 = 340,$
2. $p_{11}^2 = 20, p_{12}^2 = 320, p_{23}^2 = 80, p_{22}^2 = 5519, p_{23}^2 = 1300, p_{33}^2 = 320;$
3. $p_{12}^3 = 336, p_{13}^3 = 84, p_{22}^3 = 5460, p_{23}^3 = 1344, p_{33}^3 = 271.$

Proof. Direct calculations. \(\square\)

Let $u, v,$ and $w$ be vertices of a graph $\Gamma,$ $[rst] = [uvw]$, $\Omega = \Gamma_3(u)$, and let $\Sigma = \Omega_2$. Then $\Sigma$ is a regular graph of degree 1344 on 1700 vertices.

Lemma 4. Let $d(u,v) = d(u,w) = 3$ and $d(v,w) = 1$. Then the following equalities hold:

1. $[122] = 2r_6/5 - 136, [123] = [132] = -2r_6/5 + 472, [133] = 2r_6/5 - 388;$
2. $[211] = r_6/10 - 38, [212] = [221] = -r_6/10 + 374, [222] = -14r_6/10 + 5576,$
   $[223] = [232] = 3r_6/2 - 490, [233] = -3r_6/2 + 1834;$
   $[333] = 11r_6/10 - 1107,$

where $r_6 \in \{1010, 1020, \ldots, 1170\}.$

Proof. A simplification of formulas (3.1) taking into account the equalities $S_{113}(u,v,w) = S_{131}(u,v,w) = S_{311}(u,v,w) = 0.$ \(\square\)

By Lemma 4, we have $1010 \leq [322] = r_6 \leq 1170.$

Lemma 5. Let $d(u,v) = d(u,w) = d(v,w) = 3$. Then the following equalities hold:

1. $[122] = -r_{17} + 336, [123] = [132] = r_{17}, [133] = -r_{17} + 84;$
2. $[213] = [231] = r_{17}, [212] = [221] = -r_{17} + 336, [222] = 39r_{17}/4 + 3444,$
   $[223] = [232] = -35r_{17}/4 + 1680, [233] = 31r_{17}/4 - 336;$
3. $[313] = [331] = -r_{17} + 84, [312] = [321] = r_{17}, [322] = -35r_{17}/4 + 1680,$
   $[323] = [332] = 31r_{17}/4 - 336, [333] = -27r_{17}/4 + 522,$

where $r_{17} \in \{44, 48, \ldots, 76\}$.

Proof. A simplification of formulas (3.1) taking into account the equalities $S_{113}(u,v,w) = S_{131}(u,v,w) = S_{311}(u,v,w) = 0.$ \(\square\)

By Lemma 5, we have $1015 \leq [322] = -35r_{17}/4 + 1680 \leq 1295.$

The number $d$ of edges between $\Sigma(w)$ and $\Sigma - (\{w\} \cup \Lambda(w))$ satisfies the inequalities

$$359905 = 84 \cdot 1010 + 271 \cdot 1015 \leq d \leq 84 \cdot 1170 + 271 \cdot 1295 = 449225,$$
$$267.786 \leq 1343 - \lambda \leq 334.245,$$
$$1008.755 \leq \lambda \leq 1075.214,$$

where $\lambda$ is the mean value of the parameter $\lambda(\Sigma).$
Lemma 6. Let \( d(u, v) = d(u, w) = 3 \) and \( d(v, w) = 2 \). Then the following equalities hold:

1. \( [122] = (-64r_{15} + 4r_{16} + 7364)/27 \), \( [123] = [132] = (64r_{15} - 4r_{16} + 1708)/27 \), \( [133] = (-64r_{15} + 4r_{16} + 560)/27 \);

2. \( [211] = -r_{15} + 20, [212] = [221] = (71r_{15} + 4r_{16} + 6392)/27, [222] = (-17r_{15} - 13r_{16} + 38311)/9 \), \( [223] = [232] = (-20r_{15} + 35r_{16} + 26095)/27, [233] = (64r_{15} - 31r_{16} + 8053)/27 \);

3. \( [311] = r_{15}, [312] = [321] = (71r_{15} - 4r_{16} + 2248)/27, [313] = (44r_{15} + 4r_{16} + 20)/27 \), \( [322] = (115r_{15} + 35r_{16} + 26716)/27, [323] = [332] = (44r_{15} - 31r_{16} + 7297)/27, [333] = r_{16}, \)

where \(-10r_{15} + 4r_{16} + 20\) is a multiple of 27, \( r_{15} \in \{0, 1, \ldots, 20\} \), and \( r_{16} \in \{0, 1, \ldots, 235\} \).

Proof. A simplification of formulas (3.1) taking into account the equalities \( S_{113}(u, v, w) = S_{131}(u, v, w) = S_{311}(u, v, w) = 0 \).

By Lemma 6, we have

\[ 998 \leq [322] = (115r_{15} + 35r_{16} + 26716)/27 \leq 1294. \]

Let us count the number \( h \) of pairs of vertices \( y \) and \( z \) at distance 3 in the graph \( \Omega \), where

\[ y \in \left\{ \begin{array}{c} uv \\ 31 \end{array} \right\}, \quad z \in \left\{ \begin{array}{c} uv \\ 32 \end{array} \right\}. \]

On the one hand, by Lemma 4, we have \( [323] = -11r_{6}/10 + 1378 \), where \( r_{6} \in \{1010, 1020, \ldots, 1170\} \), therefore

\[ 7644 = 8491 \leq h \leq 84267 = 22428. \]

On the other hand, by Lemma 6, we have \( [313] = (44r_{15} + 4r_{16} + 20)/27 \), where \( r_{15} \in \{0, 1, \ldots, 20\} \), \( r_{16} \in \{0, 1, \ldots, 235\} \), therefore

\[ 7644 \leq \sum_{i} (44r_{15}^{i} + 4r_{16}^{i}) + 995.55 \leq 22428, \]

\[ 6648.44 \leq \sum_{i} (44r_{15}^{i} + 4r_{16}^{i}) \leq 21432.45, \]

\[ 4.96 \leq \sum_{i} (11r_{15}^{i} + r_{16}^{i})/1344 \leq 15.947. \]

If \( r_{15} = 0 \), then \( r_{16} + 5 \) is a multiple of 27 and \( r_{16} = 22.49, \ldots \).
If \( r_{15} = 1 \), then \( 2r_{16} + 5 \) is a multiple of 27 and \( r_{16} = 11.38, \ldots \).

In any case,

\[ \sum_{i} (11r_{15}^{i} + r_{16}^{i})/1344 \geq 22, \]

a contradiction.

Theorem 3 is proved. \( \square \)

Conclusion

The following are the main steps in creating a theory of Shilla graphs:

1. finding a list of feasible intersection arrays of Shilla graphs with \( b = 6 \);
2. classification of \( Q \)-polynomial Shilla graphs with \( b_{2} = c_{2} \).
REFERENCES


