ON $A^{I_{K}}$–SUMMABILITY

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Abstract: In this paper, we introduce and investigate the concept of $A^{I_{K}}$-summability as an extension of $A^{I^{*}}$-summability which was recently (2021) introduced by O.H.H. Edely, where $A = (a_{nk})_{n,k=1}^{\infty}$ is a non-negative regular matrix and $I$ and $K$ represent two non-trivial admissible ideals in $N$. We study some of its fundamental properties as well as a few inclusion relationships with some other known summability methods. We prove that $A^{K}$-summability always implies $A^{I_{K}}$-summability whereas $A^{I}$-summability not necessarily implies $A^{I_{K}}$-summability. Finally, we give a condition namely $AP(I,K)$ (which is a natural generalization of the condition $AP$) under which $A^{I}$-summability implies $A^{I_{K}}$-summability.

Keywords: Ideal, Filter, $I$-convergence, $I_{K}$-convergence, $A^{I}$-summability, $A^{I_{K}}$-summability.

1. Introduction

In 2000, Kostrkyo and Salat [12] introduced the notion of ideal convergence. They studied several fundamental properties of $I^{*}$ and $I^{*}$-convergence and showed that their idea was the extended version of so many known convergence methods. Based on $I$-convergence several generalizations were made by researchers and several analytical and topological properties have been investigated (see [1, 9, 11, 15–19, 21, 22] where many more references can be found) and this area becomes one of the most focused areas of research.

In 2011, M. Macaj and M. Sleziak [13] generalized the idea of $I^{*}$-convergence to $I_{K}$-convergence by involving two ideals $I$ and $K$. In the case of $I_{K}$-convergence, the convergence along the large set is taken with regard to another ideal $K$ instead of considering ordinary convergence. So from that point of view the concept of $I_{K}$-convergence being an extension of $I^{*}$-convergence shows a strong analogy for further investigation. Recent developments in the direction of $I_{K}$-convergence from topological aspects can be found from the works of Das et al. [4, 5], Banerjee and Paul [2, 3] and many others.

If $x = (x_{k})$ be a real-valued sequence and $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix, then $Ax$ is the sequence having $n^{th}$ term $A_{n}(x) = \sum_{k=1}^{\infty}a_{nk}x_{k}$. A sequence $x = (x_{k})$ is said to be $A$-summable to $L$, if $\lim_{n\to\infty}A_{n}(x) = L$. A matrix $A = (a_{nk})_{n,k=1}^{\infty}$ is said to be regular if it maps a convergent sequence into a convergent sequence keeping the same limit i.e., $A \in (c,c)_{reg}$ if $A \in (c,c)$ and $\lim_{n\to\infty}A_{n}(x) = \lim_{k\to\infty}x_{k}$. Here $c$, $(c,c)$, and $(c,c)_{reg}$ denote the collection of all real-valued convergent sequences, collection of all matrices which maps an element of $c$ to an element of $c$, and the collection of all regular matrices which maps an element of $c$ to an element of $c$, respectively. The necessary and sufficient Silverman–Toeplitz conditions for an infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ to be regular are as follows:

(i) $\sup_{n}\sum_{k=1}^{\infty}|a_{nk}| < \infty;$
(ii) For any \( k \in \mathbb{N} \), \( \lim_{n \to \infty} a_{nk} = 0 \);

(iii) \( \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = 1 \).

In 2008, Edely and Mursaleen [7] generalized the notion of \( A \)-summability to statistical \( A \)-summability by using the concept of natural density. Recently, Edely [6] further extended the notion of statistical \( A \)-summability to \( A^2 \)-summability, where \( I \) represents an ideal in \( \mathbb{N} \). In this paper we intend to introduce the notion of \( A^2 \)-summability which is a natural generalization of \( A^2 \)-summability. For more details regarding summability theory, one may refer to [8, 10, 14, 20].

Throughout the paper, we will use \((y_n)\) to denote the image \((A_n(x))\) of the sequence \(x = (x_k)\) under the transformation of the non-negative regular infinite matrix \(A\).

2. Definitions and preliminaries

**Definition 1.** A collection \( I \) containing subsets of a nonempty set \( X \) is called an ideal in \( X \) if and only if (i) \( \emptyset \in I \), (ii) \( P, Q \in I \) implies \( P \cup Q \in I \) (Additive), and (iii) \( P \in I, Q \subseteq P \) implies \( Q \in I \) (Hereditary).

If for any \( x \in X \) \( \{x\} \subset I \) then it is said that \( I \) satisfies the admissibility property or simply is called admissible. Also \( I \) is called non-trivial if \( X \notin I \) and \( I \neq \{\emptyset\} \).

Some standard examples of ideal are given below:

(i) The set \( I_f \) consisting of all subsets of \( \mathbb{N} \) having finite cardinality is an admissible ideal in \( \mathbb{N} \).

(ii) The set \( I_d \) of all subsets of natural numbers having natural density 0 is an ideal in \( \mathbb{N} \) which is also admissible.

(iii) The set \( I_c = \{ A \subseteq \mathbb{N} : \sum_{n \in A} a^{-1} < \infty \} \) is an ideal in \( \mathbb{N} \) which also has the so called admissibility property.

(iv) Suppose \( \mathbb{N} = \bigcup_{p=1}^{\infty} D_p \), where \( D_p \subset \mathbb{N} \) for any \( p \in \mathbb{N} \) and for \( i \neq j, D_i \cap D_j = \emptyset \). Then, the set \( I \) of all subsets of \( \mathbb{N} \) which intersects finitely many \( D_p \)'s forms an ideal in \( \mathbb{N} \).

More important examples can be found in [9] and [11].

**Definition 2.** A collection \( F \) containing subsets of a nonempty set \( X \) is called a filter in \( X \) if and only if (i) \( \emptyset \notin F \) (ii) \( M, N \in F \) implies \( M \cap N \in F \) and (iii) \( M \in F, N \supset M \) implies \( N \in F \).

If \( I \) is a proper non-trivial ideal in \( X \), then the collection \( F(I) = \{ M \subset X : \exists P \in I \text{ such that } M = X \setminus P \} \) forms a filter in \( X \). It is known as the filter associated with the ideal \( I \).

**Definition 3** [12]. Let \( I \) be an ideal in \( \mathbb{N} \) which satisfies the admissibility property. A real-valued sequence \( x = (x_k) \) is called \( I \)-convergent to \( l \) if for every \( \varepsilon > 0 \) the set \( \{ k \in \mathbb{N} : |x_k - l| \geq \varepsilon \} \) is contained in \( I \). The number \( l \) is called the \( I \)-limit of the sequence \( x = (x_k) \). Symbolically, \( I - \lim x = l \).

**Definition 4** [12]. Let \( I \) be an ideal in \( \mathbb{N} \) which satisfies the admissibility property. A sequence \( x = (x_k) \) is called \( T^* \)-convergent to \( l \), if there exists a set \( M = \{ m_1 < m_2 < \ldots < m_k < \ldots \} \) in the associated filter \( F(I) \), for which \( \lim_{k} x_{m_k} = l \) holds.
**Definition 5** [13]. Let $\mathcal{I}, \mathcal{K}$ denote two ideals in $\mathbb{N}$. A sequence $x = (x_k)$ is called $\mathcal{I}^\mathcal{K}$-convergent to $l$ if the associated filter $\mathcal{F}(\mathcal{I})$ contains a set $M$ such that the sequence $y = (y_k)$ defined by

$$y_k = \begin{cases} x_k, & k \in M, \\ l, & k \notin M \end{cases}$$

is $\mathcal{K}$-convergent to $l$.

If we consider $\mathcal{K} = \mathcal{I}_f$ then $\mathcal{I}^\mathcal{K}$-convergence concept coincides with $\mathcal{I}^\mathcal{K}$-convergence [12].

**Definition 6** [13]. Let $\mathcal{K}$ be an ideal in $\mathbb{N}$. Then, $P \subset \mathcal{K} Q$ denotes the property $P \setminus Q \in \mathcal{K}$. Also $P \subset \mathcal{K} Q$ and $Q \subset \mathcal{K} P$ together implies $P \sim_\mathcal{K} Q$. Thus $P \sim_\mathcal{K} Q$ if and only if $P \Delta Q \in \mathcal{K}$. A set $P$ is said to be $\mathcal{K}$-pseudointersection of a system $\{P_i : i \in \mathbb{N}\}$ if for every $i \in \mathbb{N}$ $P \subset \mathcal{K} P_i$ holds.

**Definition 7** [13]. Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals on $\mathbb{N}$. Then $\mathcal{I}$ is said to have the additive property with respect to $\mathcal{K}$ or the condition $AP(\mathcal{I}, \mathcal{K})$ holds if every sequence $(F_n)_{n \in \mathbb{N}}$ of sets from $\mathcal{F}(\mathcal{I})$ has $\mathcal{K}$-pseudointersection in $\mathcal{F}(\mathcal{I})$.

**Definition 8** [6]. A real-valued sequence $x = (x_k)$ is said to be $A^\mathcal{I}$-summable to a real number $L$, if the transformed sequence $(A_n(x))$ is $\mathcal{I}$-convergent to $L$. Symbolically, it is written as $A^\mathcal{I} \lim x_k = L$.

**Definition 9** [6]. A real-valued sequence $x = (x_k)$ is said to be $A^{\mathcal{I}}^\mathcal{K}$-summable to a real number $L$, if there exists a set $M = \{m_1 < m_2 < \ldots < m_i < \ldots\} \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_{i \to \infty} \sum_k a_{m_k} x_k = \lim_{i \to \infty} y_{m_i} = L.$$

### 3. Main results

Throughout the section, for a sequence $x = (x_k)$ we will use $y = (y_n)$ to denote the transformed sequence $A_n(x)$ where $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$.

**Definition 10.** Let $A = (a_{nk})_{n,k=1}^{\infty}$ be a non-negative regular matrix and suppose $\mathcal{I}, \mathcal{K}$ be two admissible ideals in $\mathbb{N}$. A real-valued sequence $x = (x_k)$ is said to be $A^{\mathcal{I}}_{\mathcal{K}}$-summable to $L \in \mathbb{R}$, if there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that the sequence $z = (z_k)$ defined by

$$z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases}$$

is $\mathcal{K}$-convergent to $L$, where the sequence $y = (y_n)$ is defined as

$$y_n = A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

In this case we write, $A^{\mathcal{I}}_{\mathcal{K}} \lim x_k = L$.

**Example 1.** Consider the decomposition of $\mathbb{N}$ given by

$$\mathbb{N} = \bigcup_{i=1}^{\infty} D_i, \quad D_i = \left\{2^{s-1}(2s - 1) : s = 1, 2, 3, \ldots \right\}.$$

Let $\mathcal{I}$ denotes the ideal consisting of all subsets of $\mathbb{N}$ which intersects finitely many of $D_i$'s. Consider the sequence $x = (x_k)$ defined by $x_k = 1/i$ if $k \in D_i$ and the infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ as
\[ a_{nk} = \begin{cases} 1, & k = n + 2, \\ 0, & \text{otherwise.} \end{cases} \]

Then, the sequence is \( A^\mathcal{K} \)-summable to 0 for \( \mathcal{K} = \mathcal{I} \).

**Justification:** Clearly,
\[ y_n = \sum_{k=1}^{\infty} a_{nk} x_k = \frac{1}{i}, \quad n + 2 \in D_i. \]

Let \( M = \mathbb{N} \setminus D_1 \). Then, \( M \in \mathcal{F}(\mathcal{I}) \) and it is easy to verify that the sequence \( z = (z_k) \) defined by
\[ z_k = \begin{cases} y_k, & k \in M, \\ 0, & k \notin M \end{cases} \]
is \( \mathcal{I} \)-convergent to 0. Hence, \( A^{\mathcal{I}} - \lim x_k = 0 \).

**Theorem 1.** Let \( A^{\mathcal{I}} - \lim x_k = L \) then \( A^{\mathcal{K}} - \lim x_k = L \).

**Proof.** Let \( A^{\mathcal{I}} - \lim x_k = L \). Then, there exists a set
\[ M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(\mathcal{I}) \]
such that \( \lim_{i} y_{m_i} = L \). This implies that the sequence \( z = (z_k) \) defined as
\[ z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases} \]
is usual convergent to \( L \). Now by Theorem 2.1 of [11], we can say that for any ideal \( \mathcal{K} \), the sequence \( z = (z_k) \) is \( \mathcal{K} \)-convergent to \( L \). Hence, \( A^{\mathcal{K}} - \lim x_k = L \). \( \square \)

**Theorem 2.** Let \( A^\mathcal{K} - \lim x_k = L \) then \( A^{\mathcal{I}}^\mathcal{K} - \lim x_k = L \).

**Proof.** Since \( A^\mathcal{K} - \lim x = L \), so for every \( \varepsilon > 0 \),
\[ \{k \in \mathbb{N} : |y_k - L| \geq \varepsilon\} \in \mathcal{K}. \] (3.1)
Choose \( M = \mathbb{N} \) from \( \mathcal{F}(\mathcal{I}) \). Consider the sequence \( z = (z_k) \) defined by \( z_k = y_k, \ k \in M \). Then, using (3.1), we get for every \( \varepsilon > 0 \),
\[ \{k \in \mathbb{N} : |z_k - L| \geq \varepsilon\} \in \mathcal{K} \]
i.e. \( z = (z_k) \) is \( \mathcal{K} \)-convergent to \( L \). Hence \( A^{\mathcal{K}} - \lim x_k = L \). \( \square \)

**Remark 1.** Converse of the above theorem is not necessarily true.
Example 2. Consider the ideals
\[ I_c = \{ B \subseteq \mathbb{N} : \sum_{b \in B} b^{-1} < \infty \}, \quad I_d = \{ B \subseteq \mathbb{N} : d(B) = 0 \} \]
and the infinite matrix \( A = (a_{nk})_{n,k=1}^{\infty} \) defined by
\[ a_{nk} = \begin{cases} 1, & k = n, \\ 0, & \text{otherwise}. \end{cases} \]
Let \( x = (x_k) \) be the sequence defined as
\[ x_k = \begin{cases} 1, & k \text{ is prime}, \\ 0, & k \text{ is not prime}. \end{cases} \]
Then, there exists set \( M \) of all non prime numbers \( \in \mathcal{F}(I_d) \) such that the sequence \( z = (z_k) \) defined as
\[ z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases} \]
is \( I_d \)-convergent to 0. Hence, \( A^{I_c} - \lim x_k = 0 \). But we claim that \( A^{I_c} - \lim x_k \neq 0 \). Because if \( A^{I_c} - \lim x_k = 0 \), then for any particular \( \varepsilon \) with \( 0 < \varepsilon < 1 \), we have the set
\[ \{ k \in \mathbb{N} : |y_k - 0| \geq \varepsilon \} = \text{set of all prime numbers } \in I_c, \]
it is a contradiction.

The next theorem gives the condition under which \( A^{I_c} \)-summability implies \( A^{K} \)-summability.

**Theorem 3.** Let \( I \) and \( K \) be two admissible ideals in \( \mathbb{N} \). If \( I \subseteq K \) then \( A^{I_c} - \lim x_k = L \) implies \( A^{K} - \lim x_k = L \).

**Proof.** Let \( I \subseteq K \). Then, \( A^{I_c} - \lim x_k = L \) gives the assurance of the existence of a set \( M \in \mathcal{F}(I) \) such that the sequence \( z = (z_k) \) defined as
\[ z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases} \]
is \( K \)-convergent to \( L \) and subsequently, we have
\[ \forall \varepsilon > 0, \quad \{ k \in M : |y_k - L| \geq \varepsilon \} \in K. \tag{3.2} \]
Now as the inclusion
\[ \{ k \in \mathbb{N} : |y_k - L| \geq \varepsilon \} \subseteq \{ k \in M : |y_k - L| \geq \varepsilon \} \cup (\mathbb{N} \setminus M) \]
holds and by our assumption, \( \mathbb{N} \setminus M \in I \subseteq K \), from (3.2) we have
\[ \{ k \in \mathbb{N} : |y_k - L| \geq \varepsilon \} \in K. \]
Hence, \( A^{K} - \lim x_k = L \). \( \square \)
Theorem 4. If every subsequence of \( x = (x_k) \) is \( A^{\mathcal{K}} \)-summable to \( L \), then \( x \) is \( A^{\mathcal{I}} \)-summable to \( L \).

Proof. If possible let us assume the contrary. Then, for every \( M \in \mathcal{F}(\mathcal{I}) \), the sequence \( z = (z_k) \) defined as

\[
z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases}
\]

is not \( K \)-convergent to \( L \). In other words, for any \( M \in \mathcal{F}(\mathcal{I}) \), there exists an \( \varepsilon_M > 0 \) such that

\[
B = M \cap \{ k \in \mathbb{N} : |y_k - L| \geq \varepsilon_M \} \notin \mathcal{K}.
\]

Since \( K \) is admissible, so \( B \) is infinite. Let \( B = \{ b_1 < b_2 < \ldots < b_k < \ldots \} \). Construct a subsequence \( w = (w_k) \) defined as \( w_k = y_{b_k} \) for \( k \in \mathbb{N} \). Then, \( A^{\mathcal{K}} - \lim w_k \neq L \), we get a contradiction to the hypothesis. \( \square \)

Theorem 5. Let \( x = (x_k) \) be a sequence such that \( A^{\mathcal{K}} - \lim x_k = L \). Then, every subsequence of \( x \) is \( A^{\mathcal{K}} \)-summable to \( L \) if and only if both \( \mathcal{I} \) and \( \mathcal{K} \) does not contain infinite sets.

Proof. There are two possible cases.

Case I. Let \( \mathcal{K} \) contain an infinite set. Suppose \( C \) be an infinite set and \( C \in \mathcal{K} \). Then, \( \mathbb{N} \setminus C \in \mathcal{F}(\mathcal{K}) \) and \( \mathbb{N} \setminus C \) is also infinite. Let \( \varepsilon > 0 \) be arbitrary. Choose \( L_1 \in \mathbb{R} \) such that \( L_1 \neq L \). Consider the infinite matrix \( A = (a_{nk})_{n,k=1}^{\infty} \), defined as

\[
a_{nk} = \begin{cases} 1, & k = n, \\ 0, & otherwise, \end{cases}
\]

and the sequence \( x = (x_k) \) such that

\[
x_k = \begin{cases} L_1, & k \in C, \\ L, & k \in \mathbb{N} \setminus C. \end{cases}
\]

Then,

\[
\{ k \in \mathbb{N} : |y_k - L| \geq \varepsilon \} \subseteq C \in \mathcal{K}.
\]

This means that \( x \) is \( A^{\mathcal{K}} \)-summable to \( L \). Therefore by Theorem 2, \( x \) is \( A^{\mathcal{I}} \)-summable to \( L \). But clearly the subsequence \( (x_k)_{k\in C} \) of \( x \) is \( A^{\mathcal{I}} \)-summable to \( L_1 \) and not to \( L \).

Case II. Let \( \mathcal{K} \) does not contain an infinite set. Then \( \mathcal{K} = \mathcal{I}_f \) and \( A^{\mathcal{K}} \)-summability concept coincides with \( A^{\mathcal{I}} \)-summability.

Subcase I: if \( \mathcal{I} \) contains an infinite set. Let \( B \) be any infinite set such that \( B \in \mathcal{I} \). Then, \( \mathbb{N} \setminus B \in \mathcal{F}(\mathcal{I}) \) and \( \mathbb{N} \setminus B \) is also infinite. Define a sequence \( x = (x_k) \) as

\[
x_k = \begin{cases} \xi, & k \in B, \\ L, & k \in \mathbb{N} \setminus B, \end{cases}
\]

where \( \xi (\neq L) \in \mathbb{R} \). Clearly \( x \) is \( A^{\mathcal{I}} \)-summable to \( L \) for the infinite matrix considered in Case I. But clearly the subsequence \( (x_k)_{k\in B} \) of \( x \) is not \( A^{\mathcal{I}} \)-summable to \( L \).

Subcase II: if \( \mathcal{I} \) does not contain an infinite set. In this subcase, we have \( \mathcal{I} = \mathcal{K} = \mathcal{I}_f \) and therefore \( A^{\mathcal{K}} \)-summability concept coincides with ordinary summability ([10]) so any subsequence of \( x \) is ordinary summable to \( L \). \( \square \)
Remark 2. If a sequence is $A^{I,K}$-summable then it may not be $A^I$-summable.

Example 3. Let us consider the ideal $I$ which is defined in Example 1 and the ideal $I_c = \{ A \subseteq \mathbb{N} : \sum_{a \in A} a^{-1} < \infty \}.$

Let $M = \{ k \in \mathbb{N} : k = 2^p \text{ for some non-negative integer } p \}.$ Then, for the regular matrix $A = (a_{nk})_{n,k=1}^{\infty}$ defined as

$$a_{nk} = \begin{cases} 1, & k = n, \\ 0, & \text{otherwise}, \end{cases}$$

the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & k \in M, \\ 0, & k \not\in M \end{cases}$$

is $A^{I_c}$-summable to 0 but $x$ is not $A^I$-summable to 0.

Theorem 6. Let $I$ and $K$ be two ideals in $\mathbb{N}.$ Let $x = (x_k)$ be any real-valued sequence. Then, $A^{I,K} - \lim x_k = L$ implies $A^I - \lim x_k = L$ if and only if $K \subseteq I.$

Proof. Let $K \subseteq I$ and suppose $A^{I,K} - \lim x_k = L.$ Then, the result follows directly from the following inclusion

$$\{ k \in \mathbb{N} : |y_k - L| \geq \varepsilon \} \subseteq \{ k \in M : |y_k - L| \geq \varepsilon \} \cup (\mathbb{N} \setminus M).$$

For the converse part, we assume the contrary. Then, there exists a set say $C \in K \setminus I.$ Let $L_1$ and $L_2$ be two real numbers such that $L_1 \neq L_2.$ Define a sequence $x = (x_k)$ as

$$x_k = \begin{cases} L_1, & k \in C, \\ L_2, & k \in \mathbb{N} \setminus C \end{cases}$$

and the regular matrix $A = (a_{nk})_{n,k=1}^{\infty}$ as

$$a_{nk} = \begin{cases} 1, & k = n, \\ 0, & \text{otherwise}. \end{cases}$$

Then, for any $\varepsilon > 0$ we have,

$$\{ k \in \mathbb{N} : |y_k - L_2| \geq \varepsilon \} \subseteq C \subseteq I$$

which means that $x$ is $A^K$-summable to $L_2$. Therefore by Theorem 2, $x$ is $A^{I,K}$-summable to $L_2.$ By hypothesis $x$ is $A^I$-summable to $L_2.$ Therefore for $\varepsilon = |L_1 - L_2|,$

$$\{ k \in \mathbb{N} : |y_k - L_2| \geq |L_1 - L_2| \} = C \subseteq I,$$

it is a contradiction. Hence we must have $K \subseteq I.$ \hfill \Box

Remark 3. If a sequence is $A^I$-summable then it may not be $A^{I,K}$-summable. Consider the ideal $I$ and the sequence $x = (x_k)$ defined in Example 1. Then, proceeding as Example 1 of [6], we can prove that $A^{I,I} - \lim x_k \neq 0$ although $A^I - \lim x_k = 0.$
Theorem 7. Let $I$ and $K$ be two admissible ideals of $\mathbb{N}$ such that the condition $AP(I, K)$ holds. Then, for a sequence $x = (x_k)$, $A^I$-summability implies $A^K$-summability to the same limit.

Proof. Let $A^I - \lim x_k = L$. Choose a sequence of rationales $(\varepsilon_i)_{i \in \mathbb{N}}$. Then, for every $i$, $M_i = \{k \in \mathbb{N} : |y_k - L| < \varepsilon_i\} \in \mathcal{F}(I)$.

Thus by Definition 7, there exists a set $M \in \mathcal{F}(I)$ such that for any $i \in \mathbb{N}$, $M \setminus M_i \in K$. Consider the sequence $z = (z_k)_{k \in \mathbb{N}}$ defined by

$$ z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M. \end{cases} $$

To complete the proof, it is sufficient to show that the sequence $z = (z_k)$ is $K$-convergent to $L$. Now,

$$ \{k \in \mathbb{N} : |z_k - L| < \varepsilon_i\} = \{k \in M : |z_k - L| < \varepsilon_i\} \cup \{k \in \mathbb{N} \setminus M : |z_k - L| < \varepsilon_i\} $$

$$ = (\mathbb{N} \setminus M) \cup \{k \in M : |z_k - L| < \varepsilon_i\} $$

$$ = (\mathbb{N} \setminus M) \cup (M_i \cap M) $$

$$ = \mathbb{N} \setminus (M \setminus M_i). $$

Now as $M \setminus M_i \in K$, so $\mathbb{N} \setminus (M \setminus M_i) \in \mathcal{F}(K)$ and consequently we have

$$ \{k \in \mathbb{N} : |z_k - L| < \varepsilon_i\} \in \mathcal{F}(K) $$

i.e. $K - \lim z_k = L$. Hence, $A^K - \lim x_k = L$. This completes the proof. $\square$

Theorem 8. Let $I, I_1, I_2, K, K_1, K_2$ be admissible ideals in $\mathbb{N}$ satisfying $I_1 \subseteq I_2$ and $K_1 \subseteq K_2$. Then,

(i) $A^{I_1} - \lim x_k = L$ implies $A^{I_2} - \lim x_k = L$;

(ii) $A^{K_1} - \lim x_k = L$ implies $A^{K_2} - \lim x_k = L$.

Proof. (i) Suppose $A^{I_1} - \lim x_k = L$. Then, by Definition 10, there exists $M \in \mathcal{F}(I_1)$ such that the sequence $z = (z_k)$ defined as

$$ z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases} $$

is $K$-convergent to $L$. Now since $M \in \mathcal{F}(I_1)$, we have $\mathbb{N} \setminus M \in I_2$ and therefore by hypothesis $\mathbb{N} \setminus M \in I_2$, which again implies $M \in \mathcal{F}(I_2)$. Hence we must have that $A^{I_2} - \lim x_k = L$.

(ii) Suppose $A^{K_1} - \lim x_k = L$. Then, by Definition 10, there exists $M \in \mathcal{F}(I_1)$ such that the sequence $z = (z_k)$ defined as,

$$ z_k = \begin{cases} y_k, & k \in M, \\ L, & k \notin M \end{cases} $$

satisfies the following property $\forall \varepsilon > 0$,

$$ \{k \in \mathbb{N} : |z_k - l| \geq \varepsilon\} \in K_1. $$

Now by hypothesis the inclusion $K_1 \subseteq K_2$ holds, so we must have for $\forall \varepsilon > 0$,

$$ \{k \in \mathbb{N} : |z_k - l| \geq \varepsilon\} \in K_2. $$

Hence $A^{K_2} - \lim x_k = L$. $\square$
4. Conclusion

Summability plays an important role in mathematics, particularly in mathematical analysis. In this paper, we introduce and investigate a few properties of $A^I_*$-summability. We generate a few examples and counterexamples in order to study some inclusion relationships with some known methods of summability. But the main focus was to link $A^I$ and $A^I_*$-summability with $A^I_*$-summability. We prove that the condition $AP(I, K)$ plays a crucial role in this regard. In the future, this idea can be utilized by the researchers to develop some other forms of summability.

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