ZAGREB INDICES OF A NEW SUM OF GRAPHS

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Abstract: The first and second Zagreb indices, since its inception have been subjected to extensive research
in the physio-chemical analysis of compounds. In [5], Hanyuan Deng et al. computed the first and second
Zagreb indices of four new operations on a graph defined by M. Eliasi, B. Taeri [6]. Motivated by [6], in this
paper we define a new operation on graphs and compute the first and second Zagreb indices of the resultant
graph. We illustrate the results with some examples.

Keywords: First Zagreb index \( M_1(G) \), Second Zagreb index \( M_2(G) \), \( F^* \) sum.

1. Introduction

A graph without loops and also without parallel edges is called a simple graph and if all the
pairs of vertices of the graph are connected by a path then it is said to be connected. Throughout
our discussion, we consider only connected simple graphs. The degree-based structural descriptors
have been a subject of detailed study since their induction from the first degree-based topological
C.F. Wilcox [12] defined another degree based index in connection with studying physical properties
of chemical compounds. At first, both these indices were named as Zagreb group indices [3], but
later I. Gutman named them as first and second Zagreb indices. The first Zagreb index \( M_1(G) \)
is defined as the sum of squares of degrees of all the vertices and the second Zagreb index \( M_2(G) \) is
defined as the sum of product of degrees of end vertices of all the edges. That is,

\[
M_1(G) = \sum_{u \in V(G)} d_G(u)^2, \quad M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v).
\]

Various physical applications of these indices can be found in [8–10, 13, 19, 20]. A more unified and
general approach on degree based indices of graphs were considered by X. Li, H. Zhao in [17, 18]
which lead in defining generalized Zagreb index as

\[
M_\alpha(G) = \sum_{u \in V(G)} d_G(u)^\alpha.
\]
Various particular cases for this generalized Zagreb index were considered separately, one among them is the Forgotten index $F(G)$ (when $\alpha = 3$) defined in 1972 [11] but resurged in 2015 through the works of B. Furtula, I. Gutman [7]. For more works on topological indices, see [2, 15, 17, 18, 21]. The degree based topological indices of graph operations have been a subject of detailed study recently [1, 5]. In [16], M.H. Khalifeh, H. Yousefi–Azari, A.R. Ashrafi computed the first and second Zagreb indices of graph operations such as cartesian product, composition, join, disjunction and symmetric difference of graphs. In [6], M. Eliasi, B. Taeri defined four new operations of graphs related to subdivisions and computed the Wiener index. Motivated by [6], in this paper we define a new sum related to the four subdivision graphs and compute the first and second Zagreb indices of the new sum. We also find the Zagreb indices of some chemical structures and some classes of bridge graphs using the expressions obtained. We refer to this new sum as $F^*$ sums of graphs.

2. $F^*$ sums of graphs

Let $G_1$, $G_2$ be two graphs with vertex set $V_1$, $V_2$ and edge set $E_1$, $E_2$ respectively. The four subdivision graphs $S(G_1)$, $R(G_1)$, $Q(G_1)$, $T(G_1)$ are defined as follows in [4]:

1. $S(G_1)$ is the graph obtained from $G_1$ by replacing each edge $e_i$ of $G_1$ with a vertex and making the new vertex adjacent to the corresponding end vertices of $e_i$ for each $e_i \in E_1$. That is, $S(G_1)$ is a graph with vertex set $V(S(G_1)) = V_1 \cup V_1^*$ where $V_1^*$ is the collection of new vertices and the edge set

$$E(S(G_1)) = \{(v, h), (u, h) : e = vu \in E_1, h \in V_1^*\}.$$ 

2. $R(G_1)$ is the graph obtained from $G_1$ by replacing each edge $e_i$ of $G_1$ with a vertex and making new vertex adjacent to the corresponding end vertices of $e_i$ for each $e_i \in E_1$ also keeping every edge in $G_1$ as well. That is, $R(G_1)$ is a graph with vertex set $V(R(G_1)) = V_1 \cup V_1^*$ where $V_1^*$ is the collection of new vertices and edge set

$$E(R(G_1)) = \{(v, h), (u, h) : e = vu \in E_1, h \in V_1^*\} \cup E_1.$$ 

3. $Q(G_1)$ is the graph obtained from $G_1$ by replacing each edge $e_i$ of $G_1$ with a vertex and making new vertex adjacent to the corresponding end vertices of $e_i$ for each $e_i \in E_1$ along with edges joining vertex in the $i$th copy of $V_1^*$ to the vertex in the $j$th copy of $V_1^*$ whenever $e_i$ adjacent to $e_j$ in $G_1$. That is, $Q(G_1)$ is a graph with vertex set $V(Q(G_1)) = V_1 \cup V_1^*$ where $V_1^*$ is the collection of new vertices and edge set

$$E(Q(G_1)) = \{(v, h), (u, h) : e = vu \in E_1, h \in V_1^*\} \cup E_1^*,$$

$$E_1^* = \{(u_i, u_j) : e_i \text{ adjacent to } e_j \text{ in } E_1, u_i, u_j \in V_1^*\},$$

where $u_i, u_j$ are the vertices corresponding to the edges $e_i, e_j \in E_1$.

4. $T(G_1)$ is the graph obtained from $G_1$ by replacing each edge $e_i$ of $G_1$ with a vertex and making new vertex adjacent to the corresponding end vertices of $e_i$ for each $e_i \in E_1$ along with edges joining vertex in the $i$th copy of $V_1^*$ to the vertex in the $j$th copy of $V_1^*$ whenever $e_i$ adjacent to $e_j$ in $G_1$ and keeping every edge of $G_1$ as well. That is, $T(G_1)$ is a graph with vertex set $V(T(G_1)) = V_1 \cup V_1^*$ where $V_1^*$ is the collection of new vertices and edge set

$$E(T(G_1)) = \{(v, h), (u, h) : e = vu \in E_1, h \in V_1^*\} \cup E_1^*,$$

$$E_1^* = \{(u_i, u_j) : e_i \text{ adjacent to } e_j \text{ in } E_1, u_i, u_j \in V_1^*\} \cup E_1,$$

where $u_i, u_j$ are the vertices corresponding to the edges $e_i, e_j \in E_1$. 
In each of these new subdivision graphs the vertices \( V_1 \) can be termed as black vertices and the vertices \( V_1^* \) can be termed as white vertices. In [6], M. Eliasi, B. Taeri defined four new sums called \( F \) sums with the operation cartesian product on black vertices on copies of subdivision graphs. Motivated by this we define a sum on copies of white vertices related to the cartesian product. Let \( F \) be any one of the symbols \( S, R, Q, T \), then the \( F^* \) sum of two graphs \( G_1 \) and \( G_2 \) is denoted by \( G_1 *_F G_2 \), is a graph with the vertex set \( V(G_1 *_F G_2) = V(F(G_1)) \times V_2 \) and the edge set \( E(G_1 *_F G_2) = \{ (a, b)(c, d) : a = c \in V_1^* \text{ and } bd \in E_2 \text{ or } ac \in E(F(G_1)) \text{ and } b = d \in V_2 \} \).

Fig. 1 is an example with \( G_1 = P_4, G_2 = P_6 \).

\[ \begin{align*}
&\text{Fig. 1. (a) } P_4 *_S P_6, \quad (b) \ P_4 *_R P_6, \quad (c) \ P_4 *_Q P_6, \quad (d) \ P_4 *_T P_6.
\end{align*} \]

\[ 3. \text{ Zagreb index of } F^* \text{ sum} \]

In this section we compute the first and second Zagreb indices of \( F^* \) sums of graphs.

**Theorem 1.** Let \( G_1 \) and \( G_2 \) be two connected graphs, then

\[ \begin{align*}
(a) \quad M_1(G_1 *_S G_2) &= |V_2| M_1(G_1) + |E_1| M_1(G_2) + 4|M_1(G_2)| (2|E_2| + |V_2|), \\
(b) \quad M_2(G_1 *_S G_2) &= 2|E_1| M_1(G_2) + |E_1| M_2(G_2) + 4|E_1| (2|E_2| + |V_2|). \\
\end{align*} \]

**Proof.** From the definition of first Zagreb index, we have

\[ \begin{align*}
M_1(G_1 *_S G_2) &= \sum_{(a,b) \in V(G_1 *_S G_2)} (d_{G_1 *_S G_2}(a,b))^2 \\
&= \sum_{(u,v)(x,y) \in E(G_1 *_S G_2)} (d_{G_1 *_S G_2}(u,v) + d_{G_1 *_S G_2}(x,y)) \\
&= \sum_{u \in V_1^*} \sum_{v \in E_2} (d_{G_1 *_S G_2}(u,v) + d_{G_1 *_S G_2}(u,y)) \\
&+ \sum_{v \in V_2} \sum_{u \in E(S(G_1))} (d_{G_1 *_S G_2}(u,v) + d_{G_1 *_S G_2}(x,v)).
\end{align*} \]
Now we separately find the values of the each parts in the sum. Firstly we consider the sum in which $u \in V_1^*$ and $vy \in E_2$

$$\sum_{u \in V_1^*} \sum_{vy \in E_2} \left( d_{(G_1 \ast S G_2)}(u, v) + d_{(G_1 \ast S G_2)}(u, y) \right)$$

$$= \sum_{u \in V_1^*} \sum_{vy \in E_2} [d_{S(G_1)}(u) + d_{G_2}(v) + (d_{S(G_1)}(u) + d_{G_2}(y))]$$

$$= \sum_{u \in V_1^*} \sum_{vy \in E_2} [2d_{S(G_1)}(u) + d_{G_2}(v) + d_{G_2}(y)]$$

$$= \sum_{u \in V_1^*} \left[ 4|E_2| + M_1(G_2) \right] = 4|E_1||E_2| + |E_1|M_1(G_2).$$

Now for each edge $ux \in E(S(G_1))$, $v \in V_2$

$$\sum_{v \in V_2} \sum_{ux \in E(S(G_1))} \left( d_{(G_1 \ast S G_2)}(u, v) + d_{(G_1 \ast S G_2)}(x, v) \right)$$

$$= \sum_{v \in V_2} \sum_{ux \in E(S(G_1))} \left[ d_{S(G_1)}(u) + (d_{G_2}(v) + d_{S(G_1)}(x)) \right]$$

$$= \sum_{v \in V_2} \left( 2|E_1|d_{G_2}(v) + M_1(G_1) + 4|E_1| \right) = 4|E_1||E_2| + |V_2|M_1(G_1) + 4|E_1||V_2|.$$

From the expressions we obtain

$$M_1(G_1 \ast S G_2) = |V_2|M_1(G_1) + |E_1|M_1(G_2) + 4|E_1|(2|E_2| + |V_2|).$$

Next consider

$$M_2(G_1 \ast S G_2) = \sum_{(u,v) \in E(G_1 \ast S G_2)} (d_{G_1 \ast S G_2}(u, v))$$

$$= \sum_{u \in V_1^*} \sum_{vy \in E_2} (d_{G_1 \ast S G_2}(u, v)d_{G_1 \ast S G_2}(u, y)) + \sum_{v \in V_2} \sum_{ux \in E(S(G_1))} (d_{G_1 \ast S G_2}(u, v)d_{G_1 \ast S G_2}(x, v))$$

$$= \sum_{u \in V_1^*} \sum_{vy \in E_2} [d_{S(G_1)}(u) + d_{G_2}(v)] [d_{S(G_1)}(u) + d_{G_2}(y)]$$

$$+ \sum_{v \in V_2} \sum_{ux \in E(S(G_1))} \left[ d_{S(G_1)}(u) (d_{G_2}(v) + d_{S(G_1)}(x)) \right]$$

$$= \sum_{u \in V_1^*} \sum_{vy \in E_2} \left[ 4 + 2(d_{G_2}(v) + d_{G_2}(y)) + d_{G_2}(v)d_{G_2}(y) \right] + \sum_{v \in V_2} \left( 2(|E_1|)d_{G_2}(v) + 4|E_1| \right)$$

$$= 4|E_1||E_2| + 2|E_1|M_1(G_2) + |E_1|M_2(G_2) + 4|E_1|(|E_2| + |V_2|).$$

Thus,

$$M_2(G_1 \ast S G_2) = 2|E_1|M_1(G_2) + |E_1|M_2(G_2) + 4|E_1|(2|E_2| + |V_2|).$$

□
Theorem 2. Let $G_1$ and $G_2$ be two connected graphs, then

(a) $M_1(G_1 * R G_2) = 4|V_2|M_1(G_1) + |E_1|M_1(G_2) + 4|E_1||E_2| + |V_2|$, 
(b) $M_2(G_1 * R G_2) = 4M_1(G_1)(1 + |E_2|) + M_2(G_2)(4|V_2| + |E_1|) + 2|E_1|M_1(G_2) + 4|E_1||E_2|$.

Proof. We have

$$M_1(G_1 * R G_2) = \sum_{(a,b)\in V(G_1 * R G_2)} (d(G_1 * R G_2)(a, b))^2$$

$$= \sum_{(u,v)(x,y)\in E(G_1 * R G_2)} (d(G_1 * R G_2)(u, v) + d(G_1 * R G_2)(x, y))$$

$$= \sum_{u\in V_1^*} \sum_{v\in E_2} (d(G_1 * R G_2)(u, v) + d(G_1 * R G_2)(u, y)) + \sum_{v\in V_2} \sum_{u,x\in E(R(G_1))} (d(G_1 * R G_2)(u, v) + d(G_1 * R G_2)(x, v)).$$

Now we separately find the values of each part in the sum. First we consider the sum in which $u \in V_1^*$ and $vy \in E_2$

$$\sum_{u\in V_1^*} \sum_{v\in E_2} (d(G_1 * R G_2)(u, v) + d(G_1 * R G_2)(u, y))$$

$$= \sum_{u\in V_1^*} \sum_{v\in E_2} [(d(R(G_1))(u) + d(G_2)(v)) + (d(R(G_1))(u) + d(G_2)(y))]$$

$$= \sum_{u\in V_1^*} \sum_{v\in E_2} [4 + d(G_2)(v) + d(G_2)(y)] = \sum_{u\in V_1^*} [4|E_2| + M_1(G_2)] = 4|E_1||E_2| + |E_1|M_1(G_2).$$

Now for each edge $ux \in E(R(G_1))$, $v \in V_2$

$$\sum_{v\in V_2} \sum_{ux\in E(R(G_1))} (d(G_1 * S G_2)(u, v) + d(G_1 * S G_2)(x, v))$$

$$= \sum_{v\in V_2} \sum_{ux\in E(R(G_1)), u,x\in V_1^*} (d(G_1 * S G_2)(u, v) + d(G_1 * S G_2)(x, v))$$

$$+ \sum_{v\in V_2} \sum_{ux\in E(R(G_1)), u,x\in V_1^*} (d(G_1 * S G_2)(u, v) + d(G_1 * S G_2)(x, v)).$$

Now we calculate the each sum separately

$$\sum_{v\in V_2} \sum_{ux\in E(R(G_1)), u,x\in V_1^*} (d(G_1 * R G_2)(u, v) + d(G_1 * R G_2)(x, v))$$

$$= \sum_{v\in V_2} \sum_{ux\in E(R(G_1)), u,x\in V_1^*} (d(R(G_1))(u) + d(R(G_1))(x))$$

$$= \sum_{v\in V_2} \sum_{ux\in E(R(G_1)), u,x\in V_1^*} 2(d(R(G_1))(u) + d(G_1)(x)) = 2|V_2|M_1(G_1).$$
By considering the case where $ux \in E(R(G_1))$, $u \in V_1$, $x \in V_1^*$

$$
\sum_{v \in V_2} \sum_{ux \in E(R(G_1)), \ u \in V_1^*, \ x \in V_1^*} (d_{G_1 \ast R(G_2)}(u,v) + d_{G_1 \ast R(G_2)}(x,v)) = \sum_{v \in V_2} \sum_{ux \in E(R(G_1)), \ u \in V_1, \ x \in V_1^*} d_{R(G_1)}(u) + (d_{G_2}(v) + 2)
$$

$$
= \sum_{v \in V_2} \sum_{ux \in E(R(G_1)), \ u \in V_1, \ x \in V_1^*} 2d_{G_1}(u) + d_{G_2}(v) + 2 = 2|V_2|M_1(G_1) + 4|E_1|(|E_2| + |V_2|).
$$

Thus we obtain

$$
M_1(G_1 \ast R G_2) = 4|V_2|M_1(G_1) + |E_1|M_1(G_2) + 4|E_1|(|E_2| + |V_2|).
$$

Similarly,

$$
M_2(G_1 \ast R G_2) = \sum_{(u,v)(x,y) \in E(G_1 \ast R G_2)} (d_{G_1 \ast R G_2}(u,v)d_{G_1 \ast R G_2}(x,y))
$$

$$
= \sum_{u \in V_1^*} \sum_{v \in E_2} (d_{G_1 \ast R G_2}(u,v)d_{G_1 \ast R G_2}(u,y)) + \sum_{v \in V_2} \sum_{ux \in E(R(G_1))} (d_{G_1 \ast R G_2}(u,v)d_{G_1 \ast R G_2}(x,v)).
$$

Now we find the sums separately

$$
\sum_{u \in V_1^*} \sum_{v \in E_2} (d_{G_1 \ast R G_2}(u,v)d_{G_1 \ast R G_2}(u,y)) = \sum_{u \in V_1^*} \sum_{v \in E_2} [(d_{R(G_1)}(u) + d_{G_2}(v)) (d_{R(G_1)}(u) + d_{G_2}(y))]
$$

$$
= \sum_{u \in V_1^*} \sum_{v \in E_2} [d_{R(G_1)}(u)^2 + d_{R(G_1)}(u) (d_{G_2}(v) + d_{G_2}(y)) + d_{G_2}(v)d_{G_2}(y)]
$$

$$
= \sum_{u \in V_1^*} \sum_{v \in E_2} [4 + 2(d_{G_2}(v) + d_{G_2}(y)) + d_{G_2}(v)d_{G_2}(y)] = 4|E_1||E_2| + 2|E_1|M_1(G_2) + |E_1|M_2(G_2).
$$

Also,

$$
\sum_{v \in V_2} \sum_{ux \in E(R(G_1))} (d_{G_1 \ast R G_2}(u,v)d_{G_1 \ast R G_2}(x,v)) = \sum_{v \in V_2} \sum_{ux \in E(R(G_1)), \ u \in V_1} (d_{G_1 \ast R G_2}(u,v)d_{G_1 \ast R G_2}(x,v))
$$

$$
+ \sum_{v \in V_2} \sum_{ux \in E(R(G_1)), \ u \in V_1, \ x \in V_1^*} (d_{G_1 \ast R G_2}(u,v)d_{G_1 \ast R G_2}(x,v)).
$$

Finding the sums separately, we get

$$
\sum_{v \in V_2} \sum_{ux \in E(R(G_1)), \ u \in V_1} (d_{G_1 \ast R G_2}(u,v)d_{G_1 \ast R G_2}(x,v)) = \sum_{v \in V_2} \sum_{ux \in E(R(G_1)), \ u \in V_1} (d_{R(G_1)}(u)d_{R(G_1)}(x))
$$

$$
= \sum_{v \in V_2} \sum_{ux \in E(R(G_1)), \ u \in V_1} 4d_{G_1}(u)d_{G_1}(x) = 4|V_2|M_2(G_2).
$$

Now,

$$
\sum_{v \in V_2} \sum_{ux \in E(R(G_1)), \ u \in V_1, \ x \in V_1^*} (d_{G_1 \ast R G_2}(u,v)d_{G_1 \ast R G_2}(x,v)) = \sum_{v \in V_2} \sum_{ux \in E(R(G_1)), \ u \in V_1, \ x \in V_1^*} (d_{R(G_1)}(u)(d_{G_2}(v) + d_{R(G_1)}(x))
$$

$$
= \sum_{v \in V_2} \sum_{ux \in E(R(G_1)), \ u \in V_1, \ x \in V_1^*} 4d_{G_1}(u) + 2d_{G_1}(u)d_{G_2}(v) = 4M_1(G_1) + 4|E_2|M_1(G_1).
Now collecting all the previous terms, we get
\[ M_2(G_1 \ast_R G_2) = 4M_1(G_1)(1 + |E_2|) + M_2(G_2)(4|V_2| + |E_1|) + 2|E_1|M_1(G_2) + 4|E_1||E_2|. \]

\[ \square \]

**Theorem 3.** Let \( G_1 \) and \( G_2 \) be two connected graphs, then

(a) \[ M_1(G_1 \ast_Q G_2) = (|V_2| + 2|E_2|)M_1(G_1) + |E_1|M_1(G_2) + 2|V_2|M_2(G_1) + |V_2|F(G_1) + 2|E_2| \]

(b) \[ M_2(G_1 \ast_Q G_2) = |E_2|M_2(G_1) + |E_1|M_2(G_2) + M_2(G_1)M_2(G_2) + 2|E_2|M_1(G_1) \]

where \( r_{ij} \) denotes the number of neighbouring common vertices adjacent to both \( u_i \) and \( u_j \).

\[ \text{Proof.} \] We have
\[ M_1(G_1 \ast_Q G_2) = \sum_{(a,b) \in V(G_1 \ast_Q G_2)} (d_{G_1 \ast_Q G_2}(a,b))^2 \]
\[ = \sum_{(u,v),(x,y) \in E(G_1 \ast_Q G_2)} (d_{G_1 \ast_Q G_2}(u,v) + d_{G_1 \ast_Q G_2}(x,y)) \]
\[ = \sum_{u \in V_1} \sum_{v \in V_2} (d_{G_1 \ast_Q G_2}(u,v) + d_{G_1 \ast_Q G_2}(u,y)) \]
\[ + \sum_{v \in V_2} \sum_{u \in E(G_1 \ast_Q G_2)} (d_{G_1 \ast_Q G_2}(u,v) + d_{G_1 \ast_Q G_2}(x,v)). \]

First we consider the sum in which \( u \in V_1^* \) and \( v \in E_2 \)
\[ \sum_{u \in V_1^*} \sum_{v \in E_2} (d_{G_1 \ast_Q G_2}(u,v) + d_{G_1 \ast_Q G_2}(u,y)) \]
\[ = \sum_{u \in V_1^*} \sum_{v \in E_2} [(d_{Q(G_1)}(u) + d_{G_2}(v)) + (d_{Q(G_1)}(u) + d_{G_2}(y))] \]
\[ = \sum_{u \in V_1^*} \sum_{v \in E_2} [2d_{Q(G_1)}(u) + d_{G_2}(v) + d_{G_2}(y)] \]
\[ = \sum_{e \in E_2} (d_{G_1}(e) + d_{G_2}(e)) + \sum_{u \in V_1^*} M_1(G_2) = 2|E_2|M_1(G_1) + |E_1|M_1(G_2). \]

For each edge \( ux \in E(Q(G_1)) \) and the vertex \( v \in V_2 \)
\[ \sum_{v \in V_2} \sum_{ux \in E(Q(G_1))} (d_{G_1 \ast_Q G_2}(u,v) + d_{G_1 \ast_Q G_2}(x,v)) \]
\[ = \sum_{v \in V_2} \sum_{ux \in E(Q(G_1)), u \in V_1, x \in V_1^*} (d_{G_1 \ast_Q G_2}(u,v) + d_{G_1 \ast_Q G_2}(x,v)) \]
\[ + \sum_{v \in V_2} \sum_{ux \in E(Q(G_1)), u \in V_1, x \in V_1^*} (d_{G_1 \ast_Q G_2}(u,v) + d_{G_1 \ast_Q G_2}(x,v)). \]
Now we separately find both the sums. First,

\[
\sum_{v \in V_2} \sum_{ux \in E(Q(G_1))} (d_{(G_1 \ast Q G_2)}(u, v) + d_{(G_1 \ast Q G_2)}(x, v)) = \sum_{v \in V_2} \sum_{ux \in E(Q(G_1))) \cap u \in V_1, x \in V_1^*} (d_{Q(G_1)}(u) + (d_{G_2}(v) + d_{Q(G_1)}(x)) = \sum_{v \in V_2} \sum_{ux \in E(Q(G_1))) \cap u \in V_1, x \in V_1^*} d_{G_1}(u) + d_{G_2}(v) + d_{Q(G_1)}(x) = \sum_{v \in V_2} M_1(G_1) + 2|E_1|d_{G_2}(v) + 2 \sum_{v \in V_2} \sum_{ux \in E(Q(G_1))) \cap u \in V_1, x \in V_1^*} (d_{G_1}(u_i) + d_{G_1}(v_i)) = |V_2| |M_1(G_1)| + 4|E_1||E_2| + 2|V_2| |M_1(G_1)|.
\]

The second part of the sum is the following

\[
\sum_{v \in V_2} \sum_{ux \in E(Q(G_1))) \cap u \in V_1, x \in V_1^*} (d_{(G_1 \ast Q G_2)}(u, v) + d_{(G_1 \ast Q G_2)}(x, v)) = \sum_{v \in V_2} \sum_{ux \in E(Q(G_1))) \cap u \in V_1, x \in V_1^*} (d_{Q(G_1)}(u) + (d_{G_2}(v) + d_{Q(G_1)}(x) + d_{G_2}(v)) = \sum_{v \in V_2} \left( \sum_{ux \in E(Q(G_1))) \cap u \in V_1, x \in V_1^*} 2d_{G_2}(v) \right) + \sum_{v \in V_2} \left( \sum_{ux \in E(Q(G_1))) \cap u \in V_1, x \in V_1^*} (d_{Q(G_1)}(u) + d_{Q(G_1)}(x)) \right)
= \sum_{v \in V_2} \left( \sum_{ux \in E(Q(G_1))) \cap u \in V_1, x \in V_1^*} 2d_{G_2}(v) \right) + \sum_{v \in V_2} \left( \sum_{ux \in E(Q(G_1))) \cap u \in V_1, x \in V_1^*} (d_{G_1}(u_i) + d_{G_1}(u_j) + d_{G_1}(u_j) + d_{G_1}(u_k)) \right)
= 4(|E(Q(G_1))| - 2|E_1||E_2| + |V_2| \left( 2 \sum_{u_j \in V_1} C_{d_{G_1}(u_j)}^2 d_{G_1}(u_j) + \sum_{u_j \in V_1} (d_{G_1}(u_j) - 1) \sum_{u_i \in V_1, u_i, u_j \in E_1} d_{G_1}(u_i) \right)
= 4(|E(Q(G_1))| - 2|E_1||E_2| + |V_2| \left( \sum_{u_j \in V_1} (d_{G_1}(u_j)^3 - d_{G_1}(u_j))^2 + \sum_{u_j \in V_1} (d_{G_1}(u_j) - 1) \sum_{u_i \in V_1, u_i, u_j \in E_1} d_{G_1}(u_i) \right)
= 4(|E(Q(G_1))| - 2|E_1||E_2| + |V_2| (F(G_1) + 2M_2(G_1) - 2M_1(G_1)).
\]

Here \(u_iu_j\) is the edge corresponding to the vertex \(u\) and \(u_ju_k\) is the edge corresponding to the vertex \(x\).

Thus we obtain

\[
M_1(G_1 \ast Q G_2) = |V_2| + 2|E_2| |M_1(G_1)| + |E_1| |M_1(G_2)| + 2|V_2| |M_2(G_1) + |V_2| F(G_1) + 2|E_2| (|E(Q(G_1))| - 2|E_1|) + 3|E_1|).
\]

Similarly,

\[
M_2(G_1 \ast Q G_2) = \sum_{(u,v)(x,y) \in E(G_1 \ast Q G_2)} (d_{(G_1 \ast Q G_2)}(u, v)d_{(G_1 \ast Q G_2)}(x, y)) = \sum_{ux \in E_1} \sum_{vy \in E_2} (d_{(G_1 \ast Q G_2)}(u, v)d_{(G_1 \ast Q G_2)}(u, y)) + \sum_{uv \in E(Q(G_1))} \sum_{vx \in E_1} (d_{(G_1 \ast Q G_2)}(u, v)d_{(G_1 \ast Q G_2)}(x, v))
\]
Now we separately find the values of each part in the sum

\[ \sum_{u \in V_1^*} \sum_{v \in E_2} (d_{G_1 \ast G_2}(u, v) d_{G_1 \ast G_2}(u, y)) = \sum_{u \in V_1^*} \sum_{v \in E_2} \left[ (d_{Q(G_1)}(u) + d_{G_2}(v)) (d_{Q(G_1)}(u) + d_{G_2}(y)) \right] \]

\[ = \sum_{u \in V_1^*} \sum_{v \in E_2} \left[ d_{Q(G_1)}(u)^2 + d_{Q(G_1)}(u) (d_{G_2}(v) + d_{G_2}(y)) + d_{G_2}(v) d_{G_2}(y) \right] \]

\[ = \sum_{v \in E_2} \sum_{u \in E_1} (d_{G_1}(u_i) + d_{G_1}(u_j))^2 + \sum_{u \in E_1} \sum_{v \in E_2} (d_{G_1}(u_i) + d_{G_1}(u_j)) (d_{G_2}(v) + d_{G_2}(y)) \]

\[ + \sum_{u \in E_1 \cup E_2} \sum_{v \in E_2} d_{G_2}(v) d_{G_2}(y) \]

\[ = \sum_{v \in E_2} \sum_{u \in E_1} (d_{G_1}(u_i) + d_{G_1}(u_j))^2 + 2d_{G_1}(u_i) d_{G_1}(u_j) + M_2(G_1) M_2(G_2) + \left| E_1 \right| \left| M_2(G_2) \right| \]

\[ = |E_2| F(G_1) + 2 |E_2| M_2(G_1) + M_2(G_1) M_2(G_2) + |E_1| M_2(G_2). \]

Now,

\[ \sum_{v \in V_2} \sum_{u \in E(Q(G_1))} (d_{G_1 \ast G_2}(u, v) d_{G_1 \ast G_2}(x, v)) = \sum_{v \in V_2} \sum_{u \in E(Q(G_1)), \forall \in V_1, x \in V_1^*} (d_{G_1 \ast G_2}(u, v) d_{G_1 \ast G_2}(x, v)) \]

\[ + \sum_{v \in V_2} \sum_{u \in E(Q(G_1)), \forall \in V_1, x \in V_1^*} (d_{G_1 \ast G_2}(u, v) d_{G_1 \ast G_2}(x, v)). \]

Now we find each sum separately

\[ \sum_{v \in V_2} \sum_{u \in E(Q(G_1)), \forall \in V_1, x \in V_1^*} (d_{G_1 \ast G_2}(u, v) d_{G_1 \ast G_2}(x, v)) = \sum_{v \in V_2} \sum_{u \in E(Q(G_1)), \forall \in V_1, x \in V_1^*} d_{Q(G_1)}(u) (d_{Q(G_1)}(x) + d_{G_2}(v)) \]

\[ = \sum_{v \in V_2} \sum_{u \in E(Q(G_1)), \forall \in V_1, x \in V_1^*} d_{Q(G_1)}(u) d_{Q(G_1)}(x) + d_{G_2}(v) d_{Q(G_1)}(u) \]

\[ = \sum_{v \in V_2} \sum_{u \in E(Q(G_1)), \forall \in V_1, x \in V_1^*} d_{G_1}(u) d_{Q(G_1)}(x) + \sum_{v \in V_2} \sum_{u \in E(Q(G_1)), \forall \in V_1, x \in V_1^*} d_{G_1}(u) d_{G_2}(v) \]

\[ = |V_2| (F(G_1) + 2 M_2(G_1)) + 2 |E_2| M_1(G_1). \]

The second part is

\[ \sum_{v \in V_2} \left( \sum_{u \in E(Q(G_1)), \forall \in V_1, x \in V_1^*} (d_{Q(G_1)}(u) + d_{G_2}(v)) (d_{Q(G_1)}(x) + d_{G_2}(v)) \right) \]

\[ = \sum_{v \in V_2} \sum_{u \in E(Q(G_1)), \forall \in V_1, x \in V_1^*} (d_{Q(G_1)}(u) d_{Q(G_1)}(x) + d_{G_2}(v) (d_{Q(G_1)}(u) + d_{Q(G_1)}(x)) + d_{G_2}(v)^2) \]

\[ = \sum_{v \in V_2} \left( \sum_{u \in E_1, u \in E_1} (d_{G_1}(u_i) + d_{G_1}(u_j)) (d_{G_1}(u_j) + d_{G_1}(u_k)) \right) \]

\[ + \sum_{v \in V_2} d_{G_2}(v) \left( \sum_{u \in E_1, u \in E_1} (d_{G_1}(u_i) + d_{G_1}(u_j) + d_{G_1}(u_j) + d_{G_1}(u_k)) \right) + |E(Q(G_1))| - 2 |E_1| M_1(G_2). \]
\[
\begin{align*}
&= |V_2| \left( \sum_{u_j \in V_1} C_{d_G(u_j)}^2 d_G(u_j)^2 + \sum_{u_i, u_j \in V_1} r_{ij} d_G(u_i) d_G(u_j) + \sum_{u_i \in V_1} (d_G(u_j) - 1) d_G(u_j) \sum_{u_i \in V_1} d_G(u_i) \right) \\
&\quad + 2|E_2| \left( 2 \sum_{u_j \in V_1} C_{d_G(u_j)}^2 d_G(u_j) + \sum_{u_i \in V_1} (d_G(u_j) - 1) \sum_{u_i \in V_1} d_G(u_i) + (|E(Q(G_1))| - 2|E_1|) M_1(G_2) \right) \\
&\quad + |V_2| \left( \frac{1}{2} \sum_{u_j \in V_1} (d_G(u_j)^4 - d_G(u_j)^2) + \sum_{u_i, u_j \in V_1} r_{ij} d_G(u_i) d_G(u_j) \right) \\
&\quad + |V_2| \left( \sum_{u_i \in V_1} d_G(u_j)^2 \sum_{u_i, u_j \in E_1} d_G(u_i) - 2M_2(G_1) \right) \\
&+ 2|E_2| \left( \sum_{u_j \in V_1} (d_G(u_j)^3 - d_G(u_j)^2) + \sum_{u_i \in V_1} (d_G(u_j) - 1) \sum_{u_i \in V_1} d_G(u_i) + (|E(Q(G_1))| - 2|E_1|) M_1(G_2) \right) \\
&= |V_2| \left( \frac{1}{2} M_4(G_1) - \frac{1}{2} F(G_1) + \sum_{u_i, u_j \in V_1} r_{ij} d_G(u_i) d_G(u_j) + \sum_{u_j \in V_1} d_G(u_j)^2 \sum_{u_i \in V_1} d_G(u_i) - 2M_2(G_1) \right) \\
&\quad + 2|E_2| \left( F(G_1) + 2M_2(G_2) - 2M_1(G_1) + (|E(Q(G_1))| - 2|E_1|) M_1(G_2) \right).
\end{align*}
\]

Here \( u, u_j \) is the edge corresponding to the vertex \( u \) and \( u_j, u_k \) is the edge corresponding to the vertex \( x \), \( r_{ij} \) denotes the number of common vertices adjacent to both \( u_i \) and \( u_j \). Thus we obtain

\[
M_2(G_1 * Q G_2) = |E_2| |M_2(G_1) + |E_1| |M_2(G_2) + M_2(G_1) M_2(G_2) + 2|E_2| |M_1(G_1) \\
+ \frac{1}{2} |V_2| |M_4(G_1) + (2|E_2| + |V_2| F(G_1))| \\
+ |V_2| \left( \sum_{u_i, u_j \in V_1} r_{ij} d_G(u_i) d_G(u_j) + \sum_{u_j \in V_1} d_G(u_j)^2 \sum_{u_i \in V_1} d_G(u_i) \right).
\]

\[ \tag*{\Box} \]

**Theorem 4.** Let \( G_1 \) and \( G_2 \) be two connected graphs, then

\( a \) \quad \( M_1(G_1 * T G_2) = 2|E_2| |M_1(G_1) + |E_1| |M_1(G_2) + 2|V_2| |M_2(G_1) \\
+ |V_2| |F(G_1) + 4(|E(T(G_1))| - 3|E_1|) |E_2|. \)

\( b \) \quad \( M_2(G_1 * T G_2) = 5|E_2| |M_2(G_1) + (4|V_2| + |E_1|) |M_2(G_2) + M_2(G_1) M_2(G_2) - 2|E_2| |M_1(G_1) \\
+ (|E(T(G_1))| - 3|E_1|) M_1(G_2) + \frac{1}{2} |V_2| |M_4(G_1) + (2|E_2| + |V_2|) F(G_1)| \\
+ |V_2| \left( \sum_{u_i, u_j \in V_1} r_{ij} d_G(u_i) d_G(u_j) + \sum_{u_j \in V_1} d_G(u_j)^2 \sum_{u_i \in V_1} d_G(u_i) \right). \)

where \( r_{ij} \) denotes the number of common vertices adjacent to both \( u_i, u_j \).

**Proof.** We prove this theorem using Theorem 2 and Theorem 3. When \( u \in V_1^* \) and \( vy \in E_2 \)

\[
\sum_{u \in V_1^*} \sum_{vy \in E_2} (d_{G_1 * T G_2}(u, v) + d_{G_1 * T G_2}(u, y)) = \sum_{u \in V_1^*} \sum_{vy \in E_2} (d_{G_1 * Q G_2}(u, v) + d_{G_1 * Q G_2}(u, y)).
\]
From Theorem 3
\[ \sum_{u \in V_1^*} \sum_{v \in E_2} \left( d_{(G_1 \ast_T G_2)}(u, v) + d_{(G_1 \ast_T G_2)}(u, y) \right) = 2|E_2|M_1(G_1) + |E_1|M_1(G_2). \]

Also
\[ \sum_{v \in V_2} \sum_{u \in E(T(G_1))} \left( d_{(G_1 \ast_T G_2)}(u, v) + d_{(G_1 \ast_T G_2)}(x, v) \right) \]
\[ = \sum_{u \in V_1, x \in V_1^*} \sum_{u \in V_2, x \in E(T(G_1))} \left( d_{(G_1 \ast_T G_2)}(u, v) + d_{(G_1 \ast_T G_2)}(x, v) \right) \]
\[ + \sum_{u \in V_1, x \in V_1^*} \sum_{v \in V_2} \sum_{u \in E(T(G_1))} \left( d_{(G_1 \ast_T G_2)}(u, v) + d_{(G_1 \ast_T G_2)}(x, v) \right) \]
\[ + \sum_{u \in V_1, x \in V_1^*} \sum_{v \in V_2} \sum_{u \in E(T(G_1)}, \, u \in V_1} d_{(G_1 \ast_T G_2)}(u, v) + d_{(G_1 \ast_T G_2)}(x, v). \]

Also from Theorem 2 and Theorem 3
\[ \sum_{v \in V_2} \sum_{u \in E(T(G_1))} \left( d_{(G_1 \ast_T G_2)}(u, v) + d_{(G_1 \ast_T G_2)}(x, v) \right) \]
\[ = 4\left( |E(T(G_1))| - 3|E_1| \right) |E_2| + |V_2| \left( F(G_1) + 2M_2(G_1) - 2M_1(G_1) \right) + 2|V_2|M_1(G_1). \]

Thus,
\[ M_1(G_1 \ast_T G_2) = 2|E_2|M_1(G_1) + |E_1|M_1(G_2) + 2|V_2|M_2(G_1) \]
\[ + |V_2|F(G_1) + 4\left( |E(T(G_1))| - 3|E_1| \right) |E_2|. \]

Similarly for \( M_2 \), from Theorem 3
\[ \sum_{u \in V_1^*} \sum_{v \in E_2} \left( d_{(G_1 \ast_T G_2)}(u, v) + d_{(G_1 \ast_T G_2)}(u, y) \right) \]
\[ = |E_2|F(G_1) + |E_2|M_2(G_1) + M_2(G_1)M_2(G_2) + |E_1|M_2(G_2). \]

The second part of the sum is
\[ \sum_{u \in V_1} \sum_{v \in E(T(G_1))} \left( d_{(G_1 \ast_T G_2)}(u, v) + d_{(G_1 \ast_T G_2)}(x, v) \right) \]
\[ = \sum_{u \in V_1} \sum_{v \in E(T(G_1))} \left( d_{(G_1 \ast_T G_2)}(u, v) + d_{(G_1 \ast_T G_2)}(x, v) \right) \]
\[ + \sum_{u \in V_1} \sum_{v \in E(T(G_1))} \left( d_{(G_1 \ast_T G_2)}(u, v) + d_{(G_1 \ast_T G_2)}(x, v) \right) \]
\[ + \sum_{u \in V_1} \sum_{v \in E(T(G_1))} \left( d_{(G_1 \ast_T G_2)}(u, v) + d_{(G_1 \ast_T G_2)}(x, v) \right). \]

From Theorem 2 and Theorem 3 we get
\[ \sum_{v \in V_2} \sum_{u \in E(T(G_1))} \left( d_{(G_1 \ast_T G_2)}(u, v) + d_{(G_1 \ast_T G_2)}(x, v) \right) \]
\[ = |V_2| \left( F(G_1) + 2M_2(G_1) \right) + 2|E_2|M_1(G_1) \]
\[ + |V_2| \left( \frac{1}{2}M_4(G_1) - \frac{1}{2}F(G_1) + \sum_{u_j \in V_1} r_{ij} d_{G_1}(u_i) + \sum_{u_j \in V_1} d_{G_1}(u_j) + \sum_{u_j \in V_1} d_{G_1}(u_j)^2 \right) \]
\[ + 2|E_2| \left( F(G_1) + 2M_2(G_1) - 2M_1(G_1) \right) + \left( |E(Q(G_1))| - 2|E_1| \right) M_1(G_2) + 4|V_2|M_2(G_2), \]
here $r_{ij}$ denotes the number of common vertices adjacent to both $u_i, u_j$. Thus we obtain
\[
M_2(G_1 \ast_T G_2) = 5|E_2|M_2(G_1) + (4|V_2| + |E_1|)M_2(G_2) + M_2(G_1)M_2(G_2) - 2|E_2|M_1(G_1) \\
+ \frac{1}{2}(|V_1|M_4(G_1) + (2|E_2| + |V_2|)F'(G_1)) \\
+ |V_2|\left(\sum_{u_i, u_j \in V_1} r_{ij}d_{G_1}(u_i)d_{G_1}(u_j) + \sum_{u_j \in V_1} d_{G_1}(u_j)^2 \sum_{u_i, u_j \in E_1} d_{G_1}(u_i)\right).
\]

\[
\square
\]

4. Applications with illustration

The above computational procedure can be used to find the respective indices for many classes of graphs very easily. As an illustration we provide the following.

**Example 1.** When $G_1 = P_n$, $G_2 = P_m$, $n, m > 3$, using the theorem, we easily obtain the following results

1. $M_1(P_n \ast_S P_m) = 20mn - 22m - 14n + 14$,
   $M_2(P_n \ast_S P_m) = 32mn - 40m - 24n + 38$;
2. $M_1(P_n \ast_R P_m) = 32mn - 40m - 14n + 14$,
   $M_2(P_n \ast_R P_m) = 64mn - 48m + 24n - 80$;
3. $M_1(P_n \ast_Q P_m) = 40mn - 64m - 22n + 30$,
   $M_2(P_n \ast_Q P_m) = 96mn - 184m + 18n + 134$;
4. $M_1(P_n \ast_T P_m) = 48mn - 82m - 22n + 30$,
   $M_2(P_n \ast_T P_m) = 136mn - 258m - 86n + 146$.

Let $T_{n,m}$ denote the torus grid graph obtained from the cycle $C_n$ and $C_m$. Using $F^*$ sums, we can compute the Zagreb indices of torus grid graph $T_{2n,m}$ since $T_{2n,m} = C_n \ast_S C_m$.

**Example 2.** When $G_1 = C_n$, $G_2 = C_m$, $n, m > 3$, using the theorem, we easily obtain the following results

1. $M_1(C_n \ast_S C_m) = 20mn$,
   $M_2(C_n \ast_S C_m) = 32mn$;
2. $M_1(C_n \ast_R C_m) = 32mn$,
   $M_2(C_n \ast_R C_m) = 48mn$;
3. $M_1(C_n \ast_Q C_m) = 40mn$,
   $M_2(C_n \ast_Q C_m) = 96mn$;
4. $M_1(C_n \ast_T C_m) = 52mn$,
   $M_2(C_n \ast_T C_m) = 136mn$.

We can also find the Zagreb indices of some chemical structures using the expressions of $F^*$ sums.

**Example 3.** Let $n \geq 3$ be an integer, then Zagreb indices of the the zigzag polyhex nanotube $TUHC6[2n, 2]$
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\[ M_1(TUHC6[2n, 2]) = 26n, \]
\[ M_2(TUHC6[2n, 2]) = 33n. \]

Since \( TUHC6[2n, 2] = C_n \ast_P 2 \), then by Theorem 1.

Using \( F^* \) sums, we can also find the Zagreb indices of some classes of bridge graphs. Let \( v_1, v_2, \ldots, v_n \) be vertices of graphs \( G_1, G_2, \ldots, G_n \) respectively. The bridge graph using \( v_1, v_2, \ldots, v_n \) is secured by joining the vertices \( v_i \) of \( G_i \) to \( v_i+1 \) of \( G_{i+1} \) for \( i = 1, 2, \ldots, n-1 \) and it is denoted by \( B(G_1, G_2, \ldots, G_n; v_1, v_2, \ldots, v_n) \). If \( G_i \cong G_i+1 \cong G \) and \( v_i = v_{i+1} = v \) for all \( i = 1, 2, \ldots, n \), then \( B(G, G, \ldots, G; v, v, \ldots, v) = G_n(G, v) \). Let \( B_n = G_n(P_3, v) \) where the degree \( d(v) = 2 \) and \( T_{n,3} = G_n(C_3, v) \) be two class of bridge graphs.

**Example 4.** Let \( n \geq 2 \) be an integer, then
\[ M_1(B_n) = 18n - 14, \]
\[ M_2(B_n) = 24n - 28; \]
\[ M_1(T_{n,3}) = 24n - 14, \]
\[ M_2(T_{n,3}) = 36n - 32. \]

Since \( B_n = P_2 \ast_S P_n \) and \( T_{n,3} = P_2 \ast_R P_n \) and by Theorem 1 and Theorem 2.

5. **Summary and Conclusion**

The \( F \) sum of graphs was a new sum defined by M. Eliasi, B. Taeri in [6], a lot of research has been done on this to compute various topological indices of this \( F \) sum. In this paper we have defined a similar new operation and computed the first and second Zagreb index of this sum. Computing other topological indices on these sums is an area which researchers may find helpful.

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**REFERENCES**


