A CHARACTERIZATION OF MEIXNER ORTHOGONAL POLYNOMIALS VIA A CERTAIN TRANSFERT OPERATOR

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Abstract: Here we consider a certain transfert operator $M(c,\omega) = I_P - c \tau_\omega, \omega \neq 0, c \in \mathbb{R} - \{0,1\}$, and we prove the following statement: up to an affine transformation, the only orthogonal sequence that remains orthogonal after application of this transfert operator is the Meixner polynomials of the first kind.

Keywords: Orthogonal polynomials, Regular form, Meixner polynomials, Divided-difference operator, Transfert operator, Hahn property.

1. Introduction and preliminaries

Let $O$ be a linear operator acting on the space of polynomials as a lowering operator (the derivative [4, 18, 19], the $q$-derivative [4, 12, 14, 15], the divided-difference [1], the Dunkl [6, 8, 9, 11, 13], the $q$-Dunkl [5, 7, 13], other [17, 21]), a transfert operator (see [20]) or a raising operator (see [2, 3, 17]). Many researchers in this vast field cited above had the concern to characterize the $O$-classical polynomial sequences that is those which fulfill the so-called Hahn property: the sequences $\{P_n\}_{n \geq 0}$ and $\{OP_n\}_{n \geq 0}$ are orthogonal.

By the way, in [20], the authors characterized the $I(q,\omega)$-classical orthogonal polynomials where $I(q,\omega)$ is a transfert operator acting on the space of polynomials $P$ and defined by [20]

$$I(q,\omega) := I_P + \omega h_q, \quad \omega \in \mathbb{C} \setminus \{0\}, \quad q \in \mathbb{C}_\omega := \{ z \in \mathbb{C}, z \neq 0, z^{n+1} \neq 1, 1 + \omega z^n \neq 0, n \in \mathbb{N}\},$$

with $I_P$ being the identity operator in $P$ and $(h_q f)(x) = f(qx), f \in P$ (homothety). Therefore, our goal is to consider the following transfert operator $M(c,\omega)$ acting on $P$ and defined by

$$M(c,\omega) = I_P - c \tau_\omega, \quad \omega \neq 0, \quad c \in \mathbb{R} - \{0,1\}, \quad (1.1)$$

where

$$(\tau_\omega f)(x) = f(x - \omega), \quad f \in P,$$

(translation) and to characterize all sequences of orthogonal polynomials $\{P_n\}_{n \geq 0}$ having the Hahn property; the resulting up an affine transformation (that is to say up a composition of a homothety and a translation; see (1.4) below), is the Meixner polynomials of the first kind (see Theorem 2
Indeed, in Section 2, firstly we deal with the $M_{(c,\omega)}$-character by presenting some characterizations of it (see Theorem 1), secondly, we establish the system verified by the elements of second-order recurrence relation for the sequences $\{P_n\}_{n \geq 0}$ and $\{M_{(c,\omega)}P_n\}_{n \geq 0}$ and thirdly we solve it to deduce the desired result (Theorem 2). Moreover, the divided-difference equation fulfilled by its canonical form and the second order linear divided-difference equation satisfied by any Meixner polynomial are highlighted.

Let $P$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $P'$ be its dual. We denote by $\langle u, f \rangle$ the action of $u \in P'$ on $f \in P$. In particular, we denote by

$$\langle u \rangle := \langle u, x^n \rangle, \quad n \geq 0$$

the moments of $u$. The form $u$ is called regular if we can associate with it a sequence of monic polynomials $\{P_n\}_{n \geq 0}$ with $\deg P_n = n$, $n \geq 0$ ((MPS) in short) [18] such that

$$\langle u, P_m P_n \rangle = r_n \delta_{n,m}, \quad n, m \geq 0; \quad r_n \neq 0, \quad n \geq 0.$$ 

The sequence $\{P_n\}_{n \geq 0}$ is then called orthogonal with respect to $u$ ((MOPS) in short). In this case, the (MOPS) $\{P_n\}_{n \geq 0}$ fulfills the standard recurrence relation ((TTRR) in short) [10, 18]

$$\begin{cases}
P_0(x) = 1, & P_1(x) = x - \beta_0, \\
P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0,
\end{cases} \quad (1.2)$$

where

$$\beta_n = \frac{\langle u, x P_n^2 \rangle}{r_n}, \quad \gamma_{n+1} = \frac{r_{n+1}}{r_n} \neq 0, \quad n \geq 0.$$ 

Moreover, the regular form $u$ will be supposed normalized that is to say $\langle u \rangle_0 = 1$.

For any form $u$, any polynomial $g$ and $a, \omega \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$, we let $\tau_b u$, $h_a u$, $g u$, $D u = u'$, $D_\omega u$ be the forms defined by duality [18] namely

$$\langle \tau_b u, f \rangle = \langle u, \tau_{-b} f \rangle, \quad \langle h_a u, f \rangle = \langle u, h_a f \rangle, \quad \langle g u, f \rangle = \langle u, g f \rangle, \quad \langle u', f \rangle = -\langle u, f' \rangle, \quad \langle D_\omega u, f \rangle = -\langle u, D_\omega f \rangle$$

where

$$\tau_b f(x) = f(x + b), \quad h_a f(x) = f(ax), \quad (D_\omega f)(x) = \frac{f(x) - f(x - \omega)}{\omega}, \quad f \in P,$$

and due to the well known formulas [1, 18] we have

$$\tau_b (f u) = (\tau_b f) (\tau_b u), \quad h_a (f u) = (h_a^{-1} f) (h_a u), \quad u \in P', \quad f \in P. \quad (1.3)$$

Let $\delta_b$ be the Dirac mass at $b$ defined by

$$\langle \delta_b, f \rangle = f(b), \quad b \in \mathbb{C}, \quad f \in P.$$ 

In addition, let $\{\hat{P}_n\}_{n \geq 0}$ be the (MPS) defined by

$$\hat{P}_n(x) = a^{-n} P_n(ax + b), \quad n \geq 0, \quad a \neq 0, \quad b \in \mathbb{C}.$$ 

If $\{P_n\}_{n \geq 0}$ is a (MOPS) associated with $u$, then $\{\hat{P}_n\}_{n \geq 0}$ is a (MOPS) associated with

$$\hat{u} = (h_{a-1} \circ \tau_{-b}) u.$$
and fulfilling the (TTRR) in (1.2) \( \beta_n \leftrightarrow \hat{\beta}_n, \gamma_{n+1} \leftrightarrow \hat{\gamma}_{n+1}, n \geq 0 \) with [18]

\[
\hat{\beta}_n = \frac{\beta_n - b}{a}, \quad \hat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0.
\]

Let now \( \{P_n\}_{n \geq 0} \) be a (MPS) and let \( \{u_n\}_{n \geq 0} \) be its dual sequence, \( u_n \in \mathcal{P}' \) defined by

\[
\langle u_n, P_m \rangle = \delta_{n,m}, \quad n, m \geq 0.
\]

Let us recall some results [18].

**Lemma 1** [18]. For any \( u \in \mathcal{P}' \) and any integer \( m \geq 1 \), the following statements are equivalent

\[\begin{align*}(i) \quad & \langle u, P_{m-1} \rangle \neq 0, \quad \langle u, P_n \rangle = 0, \quad n \geq m, \\
(ii) \quad & \exists \lambda_{\nu} \in \mathbb{C}, \quad 0 \leq \nu \leq m-1, \quad \lambda_{m-1} \neq 0,
\end{align*}\]

such that

\[
u = \sum_{\nu=0}^{m-1} \lambda_{\nu} u_{\nu}.
\]

As a consequence,

- the dual sequence \( \{\hat{u}_n\}_{n \geq 0} \) of \( \{\hat{P}_n\}_{n \geq 0} \) is given by

\[
\hat{u}_n = a^n (h_{\alpha-1} \circ \tau_{-b}) u_n, \quad n \geq 0,
\]

- when \( \{P_n\}_{n \geq 0} \) be a (MOPS) then \( u = u_0 \). In this case, we have

\[
u = r^{-1} P_n u_0, \quad n \geq 0
\]

and reciprocally. Lastly, when \( u_0 \) is regular and \( \Phi \) is a polynomial such that \( \Phi u_0 = 0 \), then \( \Phi = 0 \).

The monic Meixner polynomials \( \{M_n(x; \alpha, c)\}_{n \geq 0} \) of the first kind are given by [10, 16]

\[
M_n(x; \alpha, c) = (\alpha + 1)_n \left( \frac{c}{c-1} \right)^n _2F_1 \left( \begin{array}{c} -n, -x \\ \alpha + 1 \end{array} \left| 1 - \frac{1}{c} \right. \right), \quad n \geq 0,
\]

they are orthogonal with respect to the discrete weight

\[
\rho(x) = \frac{c^n x^\alpha}{x!}, \quad x \in \mathbb{N}
\]

for \( \alpha > -1, 0 < c < 1 \). Here, the Pochhammer symbol \((z)_n\) takes the form

\[ (z)_0 = 1, \quad (z)_n = \prod_{k=1}^{n} (z + k - 1), \quad n \geq 1, \]

and \( _2F_1 \) is the hypergeometric function defined by

\[
_2F_1 \left( \begin{array}{c} p, q \\ r \end{array} \left| s \right. \right) = \sum_{k=0}^{\infty} \frac{(p)_k (q)_k}{(r)_k k!} s^k.
\]

By describing exhaustively the \( D_{-1} \)-classical orthogonal polynomials in [1], the authors rediscover the (MOPS) of Meixner \( \{M_n(x; \alpha, c)\}_{n \geq 0} \) orthogonal with respect to the \( D_{-1} \)-classical Meixner form \( \mathcal{M}(\alpha, c) \) for \( \alpha \neq -n - 1, n \geq 0, c \in \mathbb{C} \setminus \{0,1\} \) and the positive definite case occurring for \( \alpha + 1 > 0, c \in (0, \infty) \setminus \{1\} \); they establish successively the (TTRR) elements, the divided-difference
 equation, the modified moments, the discrete representation and the second order linear divided-difference equation (see the following),

\[
\begin{aligned}
\beta_n &= \frac{c}{1-c} (\alpha + 1) + \frac{1+c}{1-c} n, \quad \gamma_{n+1} = \frac{c}{(1-c)^2} (n+1)(n+\alpha+1), \quad n \geq 0, \\
D_{-1}((x + \alpha + 1)M(\alpha, c)) - ((1 - c^{-1})x + \alpha + 1)M(\alpha, c) &= 0, \\
(M(\alpha, c))_{n}^{\phi} &= \left(\frac{c}{1-c}\right)^n \frac{\Gamma(\alpha + 1 + n)}{\Gamma(\alpha + 1)}, \quad n \geq 0, \quad c \in \mathbb{C} - \{0,1\}, \quad \alpha + 1 \in \mathbb{C} - (-\mathbb{N}), \\
M(\alpha, c) &= (1-c)^{\alpha+1} \sum_{k \geq 0} \frac{\Gamma(\alpha + 1 + k)}{\Gamma(\alpha + 1)} \frac{c^{-k}}{k!} \delta_k, \quad 0 < |c| < 1, \quad \alpha \neq -n - 1, \quad n \geq 0, \\
(x + \alpha + 1)(D_{-1} \circ D_{1} M_{n+1})(x;\alpha,c) + ((1 - c^{-1})x + \alpha + 1)(D_{1} M_{n+1})(x;\alpha,c) \\
- (n + 1)(1 - c^{-1})M_{n+1}(x;\alpha,c) &= 0, \quad n \geq 0.
\end{aligned}
\]

(1.5)

2. Main result

2.1. The $M_{(c,\omega)}$-classical character

First of all, let $\omega \neq 0$ and $c \in \mathbb{R} - \{0,1\}$. By virtue of (1.1) we have

\[
(M_{(c,\omega)}f)(x) = f(x) - cf(x - \omega), \quad f \in \mathcal{P}.
\]

(2.1)

Particularly,

\[
(M_{(c,\omega)}1)(x) = 1 - c, \quad (M_{(c,\omega)}x^n)(x) = (1 - c)x^n + \text{lower degree terms}, \quad n \geq 1.
\]

(2.2)

When $c = 1$, $M_{(1,\omega)}$ is not a transfert operator but a lowering one since $M_{(1,\omega)} = \omega D_{-\omega}$.

From (1.1), we have

\[
M_{(c,\omega)} = IP - c \tau_{\omega}.
\]

The transposed $^tM_{(c,\omega)}$ of $M_{(c,\omega)}$ is

\[
^tM_{(c,\omega)} = IP' - c \tau_{-\omega} = M_{(c,-\omega)},
\]

leaving out a light abuse of notation without consequence.

Thus,

\[
\langle M_{(c,-\omega)}u,f \rangle = \langle u,M_{(c,\omega)}f \rangle, \quad u \in \mathcal{P}', \quad f \in \mathcal{P}.
\]

(2.2)

Particularly, by virtue of (2.2) we get

\[
(M_{(c,-\omega)}u)_0 = 1 - c, \quad (M_{(c,-\omega)}u)_n = (1 - c)(u)_n - c \sum_{k=0}^{n-1} \binom{n}{k} (-\omega)^{n-k} (u)_k, \quad n \geq 1.
\]

Lemma 2. The following formulas hold

\[
M_{(c,\omega)}(fg)(x) = f(x)(M_{(1,\omega)}g)(x) + (\tau_{\omega}g)(x)(M_{(c,\omega)}f)(x), \quad f, g \in \mathcal{P},
\]

(2.3)

\[
M_{(c,-\omega)}(fu) = (\tau_{-\omega}f)(M_{(c,-\omega)}u) + (M_{(1,-\omega)}f)u, \quad u \in \mathcal{P}', \quad f \in \mathcal{P},
\]

(2.4)

\[
h_a \circ M_{(c,\omega)} = M_{(c,a^{-1}\omega)} \circ h_a \text{ in } \mathcal{P}, \quad h_a \circ M_{(c,-\omega)} = M_{(c,-a\omega)} \circ h_a \text{ in } \mathcal{P}', \quad a \in \mathbb{C} - \{0\},
\]

(2.5)

\[
f_b \circ M_{(c,\omega)} = M_{(c,\omega)} \circ f_{b} \text{ in } \mathcal{P}, \quad f_b \circ M_{(c,-\omega)} = M_{(c,-\omega)} \circ f_{b} \text{ in } \mathcal{P}', \quad b \in \mathbb{C}.
\]

(2.6)
Proof. The proof is straightforward since definitions and duality.

Now consider a (MPS) \( \{P_n\}_{n \geq 0} \). On account of (2.2), let us define the (MPS) \( \{P_n^{[1]}(\cdot; c, \omega)\}_{n \geq 0} \) by
\[
P_n^{[1]}(x; c, \omega) = \frac{(M(c, \omega)P_n)(x)}{1 - c}, \quad \omega \neq 0, \quad c \in \mathbb{R} - \{0, 1\}, \quad n \geq 0.
\] (2.7)
Denoting by \( \{u_n^{[1]}(c, \omega)\}_{n \geq 0} \) the dual sequence of \( \{P_n^{[1]}(\cdot; c, \omega)\}_{n \geq 0} \), we have the result

Lemma 3. The following formula holds
\[
M_{(c, -\omega)}(u_n^{[1]}(c, \omega)) = (1 - c)u_n, \quad n \geq 0.
\] (2.8)

Proof. Indeed, from the definition it follows
\[
\langle u_n^{[1]}(c, \omega), P_m^{[1]}(\cdot; c, \omega) \rangle = \delta_{n,m}, \quad n, m \geq 0,
\]
so we have
\[
\langle (M_{(c, -\omega)}(u_n^{[1]}(c, \omega)), P_m) = (1 - c)\delta_{n,m}, \quad n, m \geq 0,
\]
therefore,
\[
\langle M_{(c, -\omega)}(u_n^{[1]}(c, \omega)), P_m \rangle = 0, \quad m \geq n + 1, \quad n \geq 0;
\]
\[
\langle M_{(c, -\omega)}(u_n^{[1]}(c, \omega)), P_n \rangle = 1 - c, \quad n \geq 0.
\]
By virtue of Lemma 1, we get
\[
M_{(c, -\omega)}(u_n^{[1]}(c, \omega)) = \sum_{\nu=0}^{n} \lambda_{n,\nu}u_\nu, \quad n \geq 0.
\]
But,
\[
\langle M_{(c, -\omega)}(u_n^{[1]}(c, \omega)), P_\mu \rangle = \lambda_{n,\mu}, \quad 0 \leq \mu \leq n,
\]
with \( \lambda_{n,\mu} = 0, \quad 0 \leq \mu < n \) and \( \lambda_{n,n} = 1 - c \). The formula (2.8) is then established.

Definition 1. The (MPS) \( \{P_n\}_{n \geq 0} \) is called \( M_{(c, \omega)} \)-classical if \( \{P_n\}_{n \geq 0} \) and \( \{P_n^{[1]}(\cdot; c, \omega)\}_{n \geq 0} \) are orthogonal.

Remark 1. When the (MPS) \( \{P_n\}_{n \geq 0} \) is orthogonal, it satisfies the (TTRR) (1.2). When the (MPS) \( \{P_n^{[1]}(\cdot; c, \omega)\}_{n \geq 0} \) is orthogonal, it satisfies the (TTRR) (1.2) with the notations \( (\beta_n \leftrightarrow \beta_n^{[1]}, \quad \gamma_{n+1}^{[1]} \leftrightarrow \gamma_{n+1}^{[1]}, \quad n \geq 0) \).

Theorem 1. For any (MOPS) \( \{P_n\}_{n \geq 0} \), the following assertions are equivalent.

a) The sequence \( \{P_n\}_{n \geq 0} \) is \( M_{(c, \omega)} \)-classical.

b) There exist a polynomial \( \phi \) monic, \( \deg \phi \leq 1 \) and a constant \( K \neq 0 \) such that
\[
M_{(c, -\omega)}(\phi u_0) - K^{-1}(1 - c)u_0 = 0, \quad (2.9)
\]
\[
1 - c - K\phi'(0) \omega n \neq 0, \quad n \geq 0. \quad (2.10)
\]
c) There exist a polynomial \( \phi \) monic, \( \deg \phi \leq 1 \), a constant \( K \neq 0 \) and a sequence of complex numbers \( \{\lambda_n\}_{n \geq 0} \), \( \lambda_n \neq 0 \), \( n \geq 0 \), such that
\[
(K\phi(x) - 1 + c)(M_{(c, -\omega)} \circ M_{(c, \omega)} P_n)(x)
+ (c - 1)(K\phi(x) - 1)(M_{(c, \omega)} P_n)(x) = \lambda_n P_n(x), \quad n \geq 0.
\] (2.11)

Proof. a) \( \Rightarrow \) b), a) \( \Rightarrow \) c).

From (2.8) and the regularity of \( u_0 \) and \( u_0^{[1]}(c, \omega) \), we have
\[
M_{(c, -\omega)}(P_n^{[1]}(\cdot; c, \omega)u_0^{[1]}(c, \omega)) = \zeta_n P_n u_0, \quad n \geq 0,
\]
with
\[
\zeta_n = (1 - c) \frac{\langle u_0^{[1]}(c, \omega), (P_n^{[1]}(\cdot; c, \omega))^2 \rangle}{\langle u_0, P_n^2 \rangle}, \quad n \geq 0.
\]
By (2.4), we get
\[
(\tau_{-\omega} P_n^{[1]}(\cdot; c, \omega))M_{(c, -\omega)}(u_0^{[1]}(c, \omega)) + (M_{(c, -\omega)} P_n^{[1]}(\cdot; c, \omega))u_0^{[1]}(c, \omega) = \zeta_n P_n u_0, \quad n \geq 0.
\]
In accordance with the definition of \( M_{(c, -\omega)} \), one may write
\[
M_{(c, -\omega)}(u_0^{[1]}(c, \omega)) = u_0^{[1]}(c, \omega) - c(\tau_{-\omega} u_0^{[1]}(c, \omega)),
\]
which yields
\[
P_n^{[1]}(\cdot; c, \omega)u_0^{[1]}(c, \omega) - c(\tau_{-\omega} P_n^{[1]}(\cdot; c, \omega))(\tau_{-\omega} u_0^{[1]}(c, \omega)) = \zeta_n P_n u_0, \quad n \geq 0.
\] (2.12)

Taking \( n = 0 \) in (2.12) leads to
\[
\begin{align*}
  u_0^{[1]}(c, \omega) - c(\tau_{-\omega} u_0^{[1]}(c, \omega)) &= (1 - c) u_0.
\end{align*}
\] (2.13)

Injecting (2.13) in (2.12) gives
\[
\{ P_n^{[1]}(\cdot; c, \omega) - (\tau_{-\omega} P_n^{[1]}(\cdot; c, \omega)) \} u_0^{[1]}(c, \omega) = \{ \zeta_n P_n - (1 - c)(\tau_{-\omega} P_n^{[1]}(\cdot; c, \omega)) \} u_0, \quad n \geq 0.
\] (2.14)

Now, taking \( n = 1 \) in (2.14), we obtain
\[
\begin{align*}
  u_0^{[1]}(c, \omega) &= K\phi(x) u_0, \quad n \geq 0.
\end{align*}
\] (2.15)

where \( K \) be a normalization constant since \( \phi \) monic and
\[
K\phi(x) = \frac{1 - c}{\omega} \left\{ (1 - \frac{\gamma_1^{[1]}(c, \omega)}{\gamma_1})(x + \omega - \frac{\gamma_1^{[1]}(c, \omega)}{\gamma_1} \beta_0 - \beta_0^{[1]}(c, \omega)).
\]

Applying the operator \( \tau_{-\omega} \) to (2.15), we get
\[
(\tau_{-\omega} u_0^{[1]}(c, \omega)) = K(\tau_{-\omega} \phi)(x)(\tau_{-\omega} u_0).
\] (2.16)

Replacing (2.16) and (2.15) in (2.13) leads to the desired result (2.9). By virtue of (2.15), the formula in (2.14) becomes
\[
\begin{align*}
  \left\{ K\phi\left(P_n^{[1]}(\cdot; c, \omega) - (\tau_{-\omega} P_n^{[1]}(\cdot; c, \omega)) \right) + (1 - c)(\tau_{-\omega} P_n^{[1]}(\cdot; c, \omega)) - \zeta_n P_n \right\} u_0 &= 0, \quad n \geq 0.
\end{align*}
\]
Hence, and by comparing the degrees we obtain

\[
K \phi \left( F_n^1(\cdot;c,\omega) - (\tau_\omega P_n^1(\cdot;c,\omega)) \right) + (1 - c)(\tau_\omega P_n^1(\cdot;c,\omega)) - \zeta_n P_n = 0, \quad n \geq 0,
\]

thanks to the regularity of \( u_0 \). Moreover, from (2.1) with the change \( \omega \leftarrow -\omega \), we may write

\[
(\tau_\omega P_n^1(\cdot;c,\omega)) = c^{-1}\left( F_n^1(\cdot;c,\omega) - (M_{(c,\omega)} P_n^1(\cdot;c,\omega)) \right), \quad n \geq 0.
\]

Consequently, the last equation becomes

\[
(K \phi(x) - 1 + c)(M_{(c,\omega)} \circ M_{(c,\omega)} P_n)(x) + (c - 1)(K \phi(x) - 1)(M_{(c,\omega)} P_n)(x) = c(1 - c)\zeta_n P_n(x), \quad n \geq 0.
\]

Writing into (2.17)

\[
\begin{align*}
\phi(x) &= \phi'(0)x + \phi(0), \\
(M_{(c,\omega)} P_n)(x) &= P_n(x) - c P_n(x - \omega), \\
(M_{(c,\omega)} \circ M_{(c,\omega)} P_n)(x) &= (1 + c^2)P_n(x) - c (P_n(x - \omega) + P_n(x + \omega)), \\
P_n(x) &= \sum_{k=0}^{n} a_{n,k} x^k, \quad a_{n,n} = 1, \quad n \geq 0,
\end{align*}
\]

and by comparing the degrees we obtain

\[
1 - c - K \phi'(0) \omega n = \zeta_n \neq 0, \quad n \geq 0.
\]

Hence (2.10) and a) ⇒ b).

Finally, (2.17) is (2.11) with \( \lambda_n = c(1 - c)\zeta_n \neq 0, \quad n \geq 0 \). We have also proved that a) ⇒ c).

b) ⇒ a) Let us suppose that there exist a polynomial \( \phi \) monic, \( \deg \phi \leq 1 \) and a constant \( K \neq 0 \) such that (2.9)–(2.10) are valid. From (2.9), we have

\[
0 = \langle M_{(c,\omega)}(\phi u_0) - K^{-1}(1 - c)u_0, 1 \rangle = (1 - c)(\langle u_0, \phi \rangle - K^{-1}).
\]

Thus,

\[
K^{-1} = \langle u_0, \phi \rangle = \phi'(0)\beta_0 + \phi(0) = \phi(\beta_0).
\]

Necessarily, \( \phi(\beta_0) \neq 0 \). Let \( v = K\phi u_0 \). We are going to prove that the (MPS) \( \{ P_n^1(\cdot;c,\omega) \}_{n \geq 0} \) is orthogonal with respect to \( v \). We have successively

\[
\langle v, P_0^1(\cdot;c,\omega) \rangle = K \langle u_0, \phi \rangle = 1,
\]

for all \( n \geq 1, \)

\[
\langle v, P_n^1(\cdot;c,\omega) \rangle = \frac{K}{1 - c} \langle \phi u_0, M_{(c,\omega)} P_n \rangle = \frac{K}{1 - c} \langle M_{(c,\omega)}(\phi u_0), P_n \rangle
\]

\[
= \frac{K}{(2.9) \frac{1}{1 - c}} (K^{-1}(1 - c)u_0, P_n) = 0,
\]
and for $m \geq 1$, $n \geq 0$,

\[
\langle v, x^m P_n^1(\cdot; c, \omega) \rangle = \frac{K}{1 - c} \langle \phi u_0, x^m (P_n(x) - cP_n(x - \omega)) \rangle
\]

\[
= \frac{K}{1 - c} \langle \phi u_0, x^m P_n(x) \rangle - \frac{Kc}{1 - c} \langle \phi u_0, \tau_\omega ((\xi + \omega)^m P_n(\xi))(x) \rangle
\]

\[
= \frac{K}{1 - c} \langle \phi u_0, x^m P_n(x) \rangle - \frac{K}{1 - c} (c\tau_\omega(\phi u_0), (x + \omega)^m P_n(x))
\]

\[
\equiv c\tau_\omega(\phi u_0) = (\phi - K^{-1}(1 - c)u_0) + \langle u_0, (x + \omega)^m P_n(x) \rangle,
\]

or equivalently, for $m \geq 1$, $n \geq 0$,

\[
\langle v, x^m P_n^1(\cdot; c, \omega) \rangle = -\frac{K\phi'(0)}{1 - c} \sum_{k=1}^{m} \binom{m}{k-1} \omega^{m-k} \langle u_0, x^k P_n(x) \rangle
\]

\[
-\frac{K\phi(0)}{1 - c} \sum_{k=0}^{m-1} \binom{m}{k} \omega^{m-k} \langle u_0, x^k P_n(x) \rangle + \sum_{k=0}^{m} \binom{m}{k} \omega^{m-k} \langle u_0, x^k P_n(x) \rangle
\]

from which thanks to the orthogonality of $\{P_n\}_{n \geq 0}$ and (2.10) we get

\[
\begin{aligned}
\left\{ \begin{array}{l}
\langle v, x^m P_n^1(\cdot; c, \omega) \rangle = 0, & 1 \leq m \leq n - 1, & n \geq 2,
\langle v, x^m P_n^1(\cdot; c, \omega) \rangle = (1 - \frac{K\phi'(0)}{1 - c} n \omega) \langle u_0, P_n^2 \rangle \neq 0, & n \geq 1.
\end{array} \right.
\end{aligned}
\]

(2.19)

By the identities in (2.18)–(2.19), we see that $\{P_n^1(\cdot; c, \omega)\}_{n \geq 0}$ is orthogonal with respect to $v$. We then obtain the desired result.

c) $\Rightarrow$ b) Comparing the degrees in (2.11), we can deduce (2.10). Making $n = 0$ into (2.11), we obtain

\[
\lambda_0 = c(1 - c)^2.
\]

(2.20)

Moreover, from definitions, (2.11) may be written as

\[
\phi((M_{c,\omega} P_n) - (\tau_\omega \circ M_{c,\omega} P_n)) + K^{-1}(1 - c)(\tau_\omega \circ M_{c,\omega} P_n) = c^{-1} K^{-1} \lambda_n P_n, \quad n \geq 0,
\]

then,

\[
\langle u_0, \phi((M_{c,\omega} P_n) - (\tau_\omega \circ M_{c,\omega} P_n)) + K^{-1}(1 - c)(\tau_\omega \circ M_{c,\omega} P_n) \rangle = c^{-1} K^{-1} \lambda_n \langle u_0, P_n \rangle, \quad n \geq 0.
\]

Equally,

\[
\langle M_{c,\omega} \phi u_0 - (M_{c,\omega} \circ \tau_\omega)(\phi u_0) + K^{-1}(1 - c)(M_{c,\omega} \circ \tau_\omega u_0)(P_n) \rangle = c^{-1} K^{-1} \lambda_n \langle u_0, P_n \rangle, \quad n \geq 0.
\]

By virtue of Lemma 1 and (2.20), we get

\[
M_{c,\omega}(\phi u_0) - (M_{c,\omega} \circ \tau_\omega)(\phi u_0) + K^{-1}(1 - c)(M_{c,\omega} \circ \tau_\omega u_0) - K^{-1}(1 - c)^2 u_0 = 0.
\]

A similar expression is

\[
M_{c,\omega}(\phi u_0) - K^{-1}(1 - c)u_0 = (M_{c,\omega} \circ \tau_\omega)(\phi u_0)
\]

\[
- K^{-1}(1 - c)(M_{c,\omega} \circ \tau_\omega u_0) - K^{-1}(1 - c)cu_0.
\]

(2.21)
Therefore, \((2.6)\) and definition of the operator \((M_{(c, -\omega)}\), we have for the right side of \((2.21)\),

\[
(M_{(c, -\omega)} \circ \tau_\omega)(\phi u_0) - K^{-1}(1 - c)(M_{(c, -\omega)} \circ \tau_\omega u_0) - K^{-1}(1 - c)cu_0
\]

\[
= \tau_\omega(M_{(c, -\omega)}(\phi u_0)) - K^{-1}(1 - c)\tau_\omega(M_{(c, -\omega)}u_0 + c\tau_{-\omega}u_0)
\]

\[
= \tau_\omega(M_{(c, -\omega)}(\phi u_0) - K^{-1}(1 - c)u_0).
\]

Therefore, \((2.21)\) becomes

\[
M_{(1, \omega)}(M_{(c, -\omega)}(\phi u_0) - K^{-1}(1 - c)u_0) = 0.
\]

From the fact that the operator \(M_{(1, \omega)}\) is injective in \(P'\) we get \((2.9)\).

\[\square\]

**Lemma 4.** If \(u_0\) satisfies \((2.9)\), then \(\tau_\omega = (h_a - \circ \tau_{-b})u_0\) fulfills the equality

\[
M_{(c, -\omega)a^{-1}}(a^{-\deg \phi}(ax + b)\tau_{-b}u_0) - a^{-\deg \phi}K^{-1}(1 - c)\tau_{-b}u_0 = 0.
\]

**Proof.** We need the following formulas which are easy to prove from \((1.3)\)

\[
g(\tau_{-b}u) = \tau_h((\tau_{-b}\phi)u); \quad g(h_a) = (h_a)(\phi u), \quad g \in P, \quad u \in P'.
\]

Now, with \(u_0 = (\tau_h \circ (h_a))\tau_{-b}u_0\), we have

\[
-K^{-1}(1 - c)u_0 = (\tau_h \circ (h_a)(\phi u)) - K^{-1}(1 - c)\tau_{-b}u_0).
\]

Further,

\[
M_{(c, -\omega)}(\phi u_0) = M_{(c, -\omega)}(\phi (\tau_h(h_a)\tau_{-b}u_0)) = M_{(c, -\omega)}(\tau_h((\tau_{-b}\phi)(h_a)\tau_{-b}u_0))
\]

\[
= (\tau_h \circ M_{(c, -\omega)})(\tau_{-b}\phi(h_a)\tau_{-b}u_0)) = (\tau_h \circ M_{(c, -\omega)})(h_a((h_a \circ \tau_{-b}\phi)\tau_{-b}u_0))
\]

\[
= (\tau_h \circ h_a \circ M_{(c, -\omega)a^{-1}})((h_a \circ \tau_{-b}\phi)\tau_{-b}u_0).
\]

Consequently, equation \((2.9)\) becomes

\[
\tau_h \circ h_a(M_{(c, -\omega)a^{-1}}(\phi(ax + b))\tau_{-b}u_0) - K^{-1}(1 - c)\tau_{-b}u_0 = 0.
\]

This leads to the desired equality. \[\square\]

### 2.2. Determination of all \(M_{(c, \omega)}\)-classical (MOPSs)

**Lemma 5.** Let \(\{P_n\}_{n \geq 0}\) be a \(M_{(c, \omega)}\)-classical (MOPS). The following equality holds

\[
\frac{c}{1 - c} \omega P_{n+1}(x - \omega) = (\beta_{n+1} - \beta_{n+1}^h)P_{n+1}^h(x; c, \omega) + (\gamma_{n+1} - \gamma_{n+1}^h)P_n^h(x; c, \omega), \quad n \geq 0.
\]

**Proof.** From the (TTRR) \((1.2)\) we have

\[
P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0.
\]

Applying the transfert operator to \((2.24)\), using \((2.3)\) and \((2.7)\) we obtain

\[
(1 - c)P_{n+2}^h(x; c, \omega) = (1 - c)(x - \beta_{n+1})P_{n+1}^h(x; c, \omega) + c\omega P_{n+1}(x - \omega) - \gamma_{n+1}(1 - c)P_n^h(x; c, \omega), \quad n \geq 0.
\]

\[\square\]
But from the (TTRR) of \( \{P_n^{[1]}(c; \omega)\}_{n \geq 0} \), one may write
\[
x_n P_n^{[1]}(c; \omega) = P_{n+2}^{[1]}(c; \omega) + \beta_n^{[1]} P_{n+1}^{[1]}(c; \omega) + \gamma_n^{[1]} P_n^{[1]}(c; \omega), \quad n \geq 0.
\] (2.26)

Now, injecting (2.26) in (2.25) leads to the desired result (2.23).

\section*{Proposition 1}

The coefficients \( \beta_n, \gamma_{n+1}, \beta_n^{[1]}, \gamma_{n+1}^{[1]} \) satisfy the following system
\[
\beta_n - \beta_n^{[1]} = \omega \frac{c}{1 - c}, \quad n \geq 0, \tag{2.27}
\]
\[
\gamma_{n+1} - \gamma_{n+1}^{[1]} = -\omega^2 \frac{c}{(1 - c)^2} (n + 1), \quad n \geq 0, \tag{2.28}
\]
\[
\beta_{n+1} - \beta_n = \omega \frac{1 + c}{1 - c}, \quad n \geq 0, \tag{2.29}
\]
\[
\gamma_n^{[1]} = \frac{n}{n + 1} \gamma_{n+1}, \quad n \geq 1. \tag{2.30}
\]

\textbf{Proof.} Firstly, the higher degree test in (2.23) yields
\[
\beta_{n+1} - \beta_n^{[1]} = \omega \frac{c}{1 - c}, \quad n \geq 0. \tag{2.31}
\]

Secondly, \( n = 0 \) in (2.23) gives
\[
\gamma_1 - \gamma_1^{[1]} = -\omega^2 \frac{c}{(1 - c)^2}. \tag{2.32}
\]

Thirdly, applying the transfert operator \( M_{(c; \omega)} \) to
\[
P_1(x) = x - \beta_0
\]
and by virtue of (2.7) and (2.31)–(2.32) we get (2.27) and
\[
\gamma_1 - \gamma_1^{[1]} = -\omega^2 \frac{c}{(1 - c)^2}. \tag{2.33}
\]

Thanks to (2.27), the formula in (2.23) becomes
\[
c \omega P_{n+1}(x) - \omega = c \omega P_{n+1}^{[1]}(c; \omega) + (1 - c)(\gamma_{n+1} - \gamma_{n+1}^{[1]}) P_n^{[1]}(c; \omega), \quad n \geq 0. \tag{2.34}
\]

Moreover, multiplication of (2.24) by \( c \omega \) with the change \( x \leftarrow x - \omega \) yields
\[
c \omega P_{n+2}(x) - \omega = (x - \omega - \beta_{n+1}) c \omega P_{n+1}(x - \omega) - \gamma_{n+1} c \omega P_n(x - \omega), \quad n \geq 0. \tag{2.35}
\]

Replacing (2.34) for the index \( n, n+1, n+2 \) in (2.35), using (2.26) for the index \( n, n+1 \), the formula in (2.27) and the fact that \( \{P_n^{[1]}(c; \omega)\}_{n \geq 0} \) is a basis, we obtain successively
\[
(\gamma_{n+2}^{[1]} - \gamma_{n+2}) - (\gamma_{n+1}^{[1]} - \gamma_{n+1}) = \omega^2 \frac{c}{(1 - c)^2}, \quad n \geq 0, \tag{2.36}
\]
\[
(\gamma_{n+1}^{[1]} - \gamma_{n+1}) \left\{ (1 - c)(\beta_n - \beta_{n+1}) + (1 + c) \omega \right\} = 0, \tag{2.37}
\]
\[
(\gamma_{n+1}^{[1]} - \gamma_{n+1}) \gamma_n^{[1]} = (\gamma_n^{[1]} - \gamma_n) \gamma_{n+1}, \quad n \geq 1. \tag{2.38}
\]

Summing on (2.36) and taking into account (2.33) lead to (2.28) and (2.37) yields (2.29).

Lastly, (2.30) is a direct consequence of (2.38) and (2.28).
Now, we are able to solve the system (2.27)–(2.30).

Summing on (2.29) leads to
\[
\beta_n = \beta_0 + \omega \frac{1 + c}{1 - c} n, \quad n \geq 0.
\] (2.39)

Injecting (2.39) in (2.27) yields
\[
\beta_n^{[1]} = \beta_0 - \omega \frac{c}{1 - c} + \omega \frac{1 + c}{1 - c} n, \quad n \geq 0.
\] (2.40)

Also, injecting (2.30) in (2.28) gives
\[
\frac{\gamma_{n+2}}{n+2} - \frac{\gamma_{n+1}}{n+1} = \omega^2 \frac{c}{(1-c)^2} n, \quad n \geq 0.
\]

Summing the previous equality leads to
\[
\gamma_{n+1} = (n+1) \left( \gamma_1 + \omega^2 \frac{c}{(1-c)^2} n \right), \quad n \geq 0.
\] (2.41)

After replacing (2.41) in (2.30) we deduce the following
\[
\gamma_n^{[1]} = (n+1) \left( \gamma_1 + \omega^2 \frac{c}{(1-c)^2} (n+1) \right), \quad n \geq 0.
\] (2.42)

\[\square\]

**Corollary 1.** Let \( \{P_n\}_{n \geq 0} \) be a \( M(c,\omega) \)-classical (MOPS). The following statements hold.

1) The recurrence elements of \( \{P_n\}_{n \geq 0} \) are
\[
\begin{aligned}
\beta_n &= \omega \left( \frac{\beta_0}{\omega} + \frac{1 + c}{1 - c} n \right), \quad n \geq 0, \\
\gamma_{n+1} &= \omega^2 \frac{c}{(1-c)^2} (n+1) \left( n + \frac{(1-c)^2}{c} \frac{\gamma_1}{\omega^2} \right), \quad n \geq 0.
\end{aligned}
\] (2.43)

2) The recurrence elements of \( \{P_n^{[1]}(c; c, \omega)\}_{n \geq 0} \) are
\[
\begin{aligned}
\beta_n^{[1]} &= \omega \left( \frac{\beta_0}{\omega} - \frac{c}{1 - c} + \frac{1 + c}{1 - c} n \right), \quad n \geq 0, \\
\gamma_{n+1}^{[1]} &= \omega^2 \frac{c}{(1-c)^2} (n+1) \left( n + 1 + \frac{(1-c)^2}{c} \frac{\gamma_1}{\omega^2} \right), \quad n \geq 0.
\end{aligned}
\] (2.44)

**Proof.** The formula (2.43) is a consequence of (2.39) and (2.41). Also, (2.44) is a direct result from (2.40) and (2.42).

**Theorem 2.** Up to an affine transformation, the only \( M(c,1) \)-classical (MOPS) is the Meixner’s one of the first kind.

**Proof.** The classification of the canonical situations depends on the fact that \( \beta_0 \neq 0 \) or \( \beta_0 = 0 \).

\( \beta_0 \neq 0. \) For (2.43)–(2.44), put
\[
\omega \beta_0 = (1-c) \gamma_1
\]
and
\[
\frac{(1-c)^2}{c} \frac{\gamma_1}{\omega^2} = \alpha + 1.
\]
Then,
\[
\frac{\beta_0}{\omega} = \frac{c}{1-c} (\alpha + 1).
\]

Now, for (2.43), choosing \( a = \omega, \ b = 0 \) in (1.4) and thanks to (2.5)–(2.6) this yields
\[
\begin{align*}
\hat{\beta}_n &= \frac{c}{1-c} (\alpha + 1) + \frac{1+c}{1-c} n, \quad n \geq 0, \\
\hat{\gamma}_{n+1} &= \frac{c}{(1-c)^2} (n+1)(n+\alpha +1), \quad n \geq 0.
\end{align*}
\]

Therefore (see (1.5)),
\[
\hat{P}_n = M_n(:, \alpha, c), \quad n \geq 0,
\]
with \( \alpha \neq -n - 1, \ n \geq 0 \). Next, for (2.44), choosing
\[
a = \omega, \quad b = -\frac{2\omega c}{1-c}
\]
in (1.4) and thanks to (2.5)–(2.6) this yields
\[
\begin{align*}
\beta_0 &= 0. \text{ In this case, (2.43)–(2.44) become successively,}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
\beta_n &= \omega \frac{1+c}{1-c} n, \quad n \geq 0, \\
\gamma_{n+1} &= \omega^2 \frac{c}{(1-c)^2} (n+1) \left( n + \frac{(1-c)^2}{c} \frac{\gamma_1}{\omega^2} \right), \quad n \geq 0,
\end{cases} \quad (2.45)
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
\beta_0^{[1]} &= \omega \left( -\frac{c}{1-c} + \frac{1+c}{1-c} n \right), \quad n \geq 0, \\
\gamma_{n+1}^{[1]} &= \omega^2 \frac{c}{(1-c)^2} (n+1) \left( n + 1 + \frac{(1-c)^2}{c} \frac{\gamma_1}{\omega^2} \right), \quad n \geq 0.
\end{cases} \quad (2.46)
\end{align*}
\]

For (2.45), putting
\[
\frac{(1-c)^2}{c} \frac{\gamma_1}{\omega^2} = \alpha + 1,
\]
and choosing in (1.4)
\[
a = \omega, \quad b = -\frac{\omega c}{1-c} (\alpha + 1),
\]
we obtain
\[
\begin{align*}
\begin{cases}
\hat{\beta}_n &= \frac{c}{1-c} (\alpha + 1) + \frac{1+c}{1-c} n, \quad n \geq 0, \\
\hat{\gamma}_{n+1} &= \frac{c}{(1-c)^2} (n+1)(n+\alpha +1), \quad n \geq 0.
\end{cases}
\end{align*}
\]

Consequently,
\[
\hat{P}_n = M_n(:, \alpha, c), \quad n \geq 0,
\]
with $\alpha \neq -n - 1$, $n \geq 0$. For (2.46), putting
\[
\frac{(1 - c)^2}{c} \frac{\gamma_1}{\omega^2} = \alpha + 1
\]
and choosing in (1.4)
\[
a = \omega, \quad b = -\frac{\omega c}{1 - c} (\alpha + 3),
\]
we get
\[
\begin{align*}
\hat{\gamma}[1]_{n+1} &= \frac{c}{1 - c} (n + 1)(n + \alpha + 2), \quad n \geq 0, \\
\hat{\beta}[1]_n &= \frac{c}{1 - c} (\alpha + 2) + \frac{1 + c}{1 - c} n, \quad n \geq 0,
\end{align*}
\]
Equivalently,
\[
\hat{P}[1]_n = M_n(\cdot; \alpha + 1, c), \quad n \geq 0,
\]
with $\alpha \neq -n - 2$, $n \geq 0$.

The theorem is then proved. \hfill \Box

**Remark 2.** On account of Theorem 1, Theorem 2 and after some easy calculations we get for the divided-difference equation (2.9) fulfilled by the Meixner form $M(\alpha, c)$,
\[
M_{(c,-1)}\left(\left(1 - \frac{1 + c}{1 - c} (\alpha + 1)\right)M(\alpha, c)\right) + (\alpha + 1)M(\alpha, c) = 0,
\]
and also for the second order linear divided-difference equation (2.11) satisfied by any Meixner polynomial $M_n(\cdot; \alpha, c)$, for all $n \geq 0$,
\[
\left(\frac{1 - c}{\alpha + 1} x + 2c\right) (M_{(c,-1)} \circ M_{(c,1)} M_n)(x; \alpha, c) + (1 - c)\left(\frac{1 - c}{\alpha + 1} x - c\right) (M_{(c,1)} M_n)(x; \alpha, c) = c(1 - c)^2 \frac{n + \alpha + 1}{\alpha + 1} M_n(x; \alpha, c).
\]

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