**I^K-SEQUENTIAL TOPOLOGY**

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**Abstract:** In the literature, I-convergence (or convergence in I) was first introduced in [11]. Later related notions of I-sequential topological space and I*-sequential topological space were introduced and studied. From the definitions it is clear that I*-sequential topological space is larger (finer) than I-sequential topological space. This raises a question: is there any topology (different from discrete topology) on the topological space X which is finer than I*-topological space? In this paper, we tried to find the answer to the question. We define I^K-sequential topology for any ideals I, K and study main properties of it. First of all, some fundamental results about I^K-convergence of a sequence in a topological space (X, T) are derived. After that, I^K-continuity and the subspace of the I^K-sequential topological space are investigated.

**Keywords:** Ideal convergence, I^K-convergence, Sequential topology, I^K-sequential topology.

1. Introduction

The notion of convergence of real or complex valued sequences was generalized using asymptotic density and was called statistical convergence by Fast [7] and Steinhaus [20] in the same year 1951, independently. After some years P. Kostyrko, T. Šalát, W. Wilczyński [11] gave a generalization of statistical convergence and called it as ideal convergence (or converges in ideal). Various fundamental properties (convergence in I and I*) were investigated. Later B.K. Lahiri and P. Das in [12] discussed convergence in I and in I* and investigate some additional results related to mentioned concepts [4, 8–10, 15–17].

The concept of I*-convergence of functions was extended to I^K-convergence by M. Mačaj and M. Sleziak in [13] in 2011. The authors of [2, 3, 5, 6, 14] gave further properties and results about I^K-convergence.

In first part of this paper we introduce I^K-sequential topological (seq.-top.) space, which is a natural generalization of I*-seq.-top. space. Later we discuss the I^K-continuity of the function and in last two section we write about I^K-subspace and I^K-connectedness. We will use further the abbreviation T.S. for a topological space.

2. Definition and preliminaries

In this part, we give some known definitions and necessary results.

**Definition 1** [7, 20]. Let \( A \subset \mathbb{N} \), and for \( m \in \mathbb{N} \) let the set

\[
A_m := \{ x \in A : x < m \}
\]

and \( |A_m| \) stand for the cardinality of \( A_m \). Natural density of \( A \) is defined by

\[
\beta(A) := \lim_{m \to \infty} \frac{|A_m|}{m}
\]
whenever the limit exists. A real sequence \( \tilde{x} = (x_i) \) is said to statistically converges to \( x_0 \) if for any \( \varepsilon > 0 \),

\[
\beta\left(\{ n : |x_i - x_0| > \varepsilon \}\right) = 0
\]

holds.

**Definition 2** [11]. Let \( \mathcal{I} \) be any subfamily of \( \mathcal{P}(\mathbb{N}) \), with \( \mathcal{P}(\mathbb{N}) \) being the family of all subsets of \( \mathbb{N} \). Then, \( \mathcal{I} \) is called an ideal on \( \mathbb{N} \) if the following requirements hold:

(i) finite union of sets in \( \mathcal{I} \) is again in \( \mathcal{I} \);
(ii) any subset of a set in \( \mathcal{I} \) is in \( \mathcal{I} \).

\( \mathcal{I} \) is admissible if all singleton subsets of \( \mathbb{N} \) belong to \( \mathcal{I} \). The ideal \( \mathcal{I} \) is non-trivial if \( \mathcal{I} \neq \emptyset \) and \( \mathcal{I} \neq \mathcal{P}(\mathbb{N}) \). A non-trivial ideal \( \mathcal{I} \) is called proper if \( \mathbb{N} \) is not in \( \mathcal{I} \).

The family of finite subsets of the \( \mathbb{N} \) is an admissible non-trivial ideal denoted by \( F_{\text{in}} \) and the family of the subsets of \( \mathbb{N} \) with natural density zero is also an admissible non-trivial ideal denoted by \( I_\beta \). The set of all non-trivial admissible ideals will be denoted as \( NA \) throughout the study.

**Example 1**. [11] Consider the decomposition of \( \mathbb{N} \) as \( \mathbb{N} = \bigcup_{j=1}^{\infty} \beta_j \) where all \( \beta_j \) are infinite subsets of \( \mathbb{N} \) and are mutually disjoint. Take the family

\[
\mathcal{I} = \{ N \subset \mathbb{N} : N \text{ intersect only finite number of } \beta_j \text{'s} \}.
\]

Then, \( \mathcal{I} \) belongs to \( NA \).

**Definition 3** [19]. Assume \( \mathcal{F} \subset \mathcal{P}(\mathbb{N}) \). The collection \( \mathcal{F} \) is a filter on \( \mathbb{N} \) if

(i) a finite intersection of elements of \( \mathcal{F} \) is in \( \mathcal{F} \) and
(ii) if \( C \in \mathcal{F} \land C \subseteq D \), then \( D \in \mathcal{F} \).

If empty set is not in \( \mathcal{F} \) then \( \mathcal{F} \) is proper. If \( \mathcal{I} \in NA \) then the collection

\[
\mathcal{F} = \{ N \subset \mathbb{N} : N^C \in \mathcal{I} \}
\]

is a filter on \( \mathbb{N} \). It is known as the \( \mathcal{I} \)-associated filter.

**Definition 4** [21]. In a T.S. \( (\mathcal{X}, \mathcal{T}) \) a sequence \( \tilde{x} = (x_i) \subset \mathcal{X} \) is called to converging in \( \mathcal{I} \) to a point \( x \in \mathcal{X} \) if

\[
\{ i \in \mathbb{N} : x_i \in v \} \in \mathcal{F}(\mathcal{I})
\]

holds for each neighborhood \( v \) of \( x \). The point \( x \) is referred to as the ideal limit of the sequence \( \tilde{x} = (x_i) \) and it is represented by \( x_i \xrightarrow{\mathcal{I}} x \) (or \( \mathcal{I} - \lim x_i = x \)).

**Remark 1**.

(i) Statistical and \( I_\beta \)-convergence are coincide.
(ii) Classical convergence and \( F_{\text{in}} \)-convergence are coincide.

**Lemma 1** [1]. Assume that \( \mathcal{I}, \mathcal{I}_1 \) and \( \mathcal{I}_2 \) be ideals on the set \( \mathbb{N} \) and consider a T.S. \( (\mathcal{X}, \mathcal{T}) \), then

1. If \( \mathcal{I} \in NA \), then every convergent sequence is \( \mathcal{I} \)-convergent sequence which converges to same point.
2. If \( \mathcal{I}_1 \subseteq \mathcal{I}_2 \) and \( (x_i) \subseteq \mathcal{X} \) is a sequence which \( x_i \xrightarrow{\mathcal{I}_1} x \), then \( x_i \xrightarrow{\mathcal{I}_2} x \).
3. If \( \mathcal{X} \) the Hausdorff space, then the limit of every convergent sequence is unique.
3. $\mathcal{I}^K$-convergence of sequences

In this part we will investigate some results related to $\mathcal{I}^K$-convergence of sequences which is a generalized form of $\mathcal{I}^*$-convergence of sequences. If we consider $\mathcal{F} \text{in}$ instead of $\mathcal{K}$, then we will have $\mathcal{I}^*$-convergence.

**Definition 5** [6]. In a T.S. $(\mathcal{X}, \mathcal{T})$ a sequence $\tilde{x} = (x_i) \subset \mathcal{X}$ is called to be $\mathcal{I}^*$-converging to $x_0 \in \mathcal{X}$ if $\exists M \in \mathcal{F}(\mathcal{I})$ s.t. the sequence $y_i := \begin{cases} x_i, & i \in M, \\ x, & i \notin M \end{cases}$ is $\mathcal{F} \text{in}$ convergent to $x$.

That is, for each neighborhood $\nu$ of $x$,

$$\{i \in \mathbb{N} : y_i \in \nu\} \in \mathcal{F}(\mathcal{F} \text{in}),$$

or

$$\{i \in M : y_i \notin \nu\} \cup \{i \in M^C : y_i \notin \nu\} \in \mathcal{F} \text{in}.$$

So,

$$\{i \in M : x_i \notin \nu\} \cup \{i \in M^C : x \notin \nu\} \in \mathcal{F} \text{in}.$$

This implies that

$$\{i \in M : y_i \notin \nu\} \in \mathcal{F} \text{in}.$$

Therefore,

$$\{i \in M : y_i \notin \nu\} \in \mathcal{F}(\mathcal{F} \text{in}).$$

It is clear that this definition is the same as the definition given in [6]. In the definition of $\mathcal{I}^*$-convergence of sequence if we consider an arbitrary ideal $\mathcal{K}$ instead of the ideal $\mathcal{F} \text{in}$ then it yields the definition of $\mathcal{I}^K$-convergence of a sequence. That is, $\mathcal{I}^K$-convergence is the generalized form of $\mathcal{I}^*$-convergence.

**Definition 6** [13]. Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider a T.S. $(\mathcal{X}, \mathcal{T})$. The sequence $\tilde{x} = (x_i) \subset \mathcal{X}$ is $\mathcal{I}^K$-convergent to a point $x \in \mathcal{X}$ if $\exists M \in \mathcal{F}(\mathcal{I})$ s.t. the sequence $y_i := \begin{cases} x_i, & i \in M, \\ x, & i \notin M \end{cases}$ is $\mathcal{K}$-convergent to $x$. We represent it as $\mathcal{I}^K - \lim(x_i) = x$ or $x_i \overset{\mathcal{K}}{\to} x$.

**Definition 7.** Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T})$ represent a T.S. Consider the sequences $\tilde{x} = (x_i) \subset \mathcal{X}$ and $\tilde{y} = (y_i) \subset \mathcal{X}$. Define a relation $\sim_\mathcal{I}$ as

$$\tilde{x} \sim_\mathcal{I} \tilde{y} \iff \{i : x_i \neq y_i\} \in \mathcal{I}.$$

The relation $\sim_\mathcal{I}$ is an equivalence relation. That is,

1. $\forall \tilde{x} = (x_i) \subset \mathcal{X}, \{i : x_i \neq x_i\} = \emptyset \in \mathcal{I} \Rightarrow \tilde{x} \sim_\mathcal{I} \tilde{x}$.
2. Let $\tilde{x} \sim_\mathcal{I} \tilde{y}$. Since $\{i : y_i \neq x_i\} = \{i : x_i \neq y_i\} \in \mathcal{I}$, then $\tilde{y} \sim_\mathcal{I} \tilde{x}$.
3. Let $\tilde{x} \sim_\mathcal{I} \tilde{y}$ and $\tilde{y} \sim_\mathcal{I} \tilde{z}$. Then, $A := \{i : x_i = y_i\} \in \mathcal{F}(\mathcal{I})$ and $B := \{i : y_i = z_i\} \in \mathcal{F}(\mathcal{I})$. So, $\{i : x_i = z_i\} = A \cap B \in \mathcal{F}(\mathcal{I})$. Hence, $\tilde{x} \sim_\mathcal{I} \tilde{z}$ holds.
Lemma 2. Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X}, \mathcal{T})$ and the sequences $\tilde{x} = (x_i) \subseteq \mathcal{X}$. Assume $x_i \xrightarrow{\mathcal{K}} x$ for any $x \in \mathcal{X}$ and $\tilde{t} = (t_i) \subseteq \mathcal{X}$ is a sequence s.t. $\tilde{x} \sim_{\mathcal{I}} \tilde{t}$. Then, the sequence $t_i \xrightarrow{\mathcal{K}} x$.

Proof. Let $x_i \xrightarrow{\mathcal{K}} x$, then $\exists M \in \mathcal{F}(\mathcal{I})$ s.t. the following sequence
\[
y_i = \begin{cases} x_i, & i \in M, \\ x, & i \notin M \end{cases}
\]
is $\mathcal{K}$-convergent to $x$. Since $(x_i) \sim_{\mathcal{I}} (t_i)$. So $\forall i \in M$, $x_i = t_i$. Therefore, the following sequence
\[
y_i = \begin{cases} t_i, & i \in M, \\ x, & i \notin M \end{cases}
\]
is $\mathcal{K}$-convergent to $x$ which shows that $t_i \xrightarrow{\mathcal{K}} x$ holds.

The Definition 7 gives the possibility that the definition of $\mathcal{I}^{\mathcal{K}}$-convergence of a sequence can be rewritten as follows:

Definition 8. Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X}, \mathcal{T})$. A sequence $\tilde{x} = (x_i) \subseteq \mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to the point $x \in \mathcal{X}$ if there exist a sequence $\tilde{t} = (t_i) \subseteq \mathcal{X}$ s.t. $\tilde{x} \sim_{\mathcal{I}} \tilde{t}$ and $t_i \xrightarrow{\mathcal{K}} x$ holds.

In the following lemma we demonstrate that Definition 6 and Definition 8 are equivalent for any ideals $\mathcal{I}$ and $\mathcal{K}$ and for any T.S. $(\mathcal{X}, \mathcal{T})$.

Lemma 3. Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X}, \mathcal{T})$ and $\tilde{x} = (x_i) \subseteq \mathcal{X}$ be a sequence. Then, $x_i \xrightarrow{\mathcal{K}} x$ iff $\exists \tilde{t} = (t_i) \subseteq \mathcal{X}$ s.t. $\tilde{x} \sim_{\mathcal{I}} \tilde{t}$ and $t_i \xrightarrow{\mathcal{K}} x$ hold.

Proof. Let $x_i \xrightarrow{\mathcal{K}} x$ holds. Then, $\exists M \in \mathcal{F}(\mathcal{I})$ s.t. the following sequence
\[
y_i = \begin{cases} x_i, & i \in M, \\ x, & i \notin M \end{cases}
\]
is $\mathcal{K}$-convergent to $x$. Let us chose $(t_i) = (y_i) \forall i \in \mathbb{N}$. Then, the proof will complete if we show that $\tilde{x} \sim_{\mathcal{I}} \tilde{y}$.

Consider the fact $\{i \in \mathbb{N} : x_i = y_i\} = \{i \in M : x_i = y_i\} \in \mathcal{F}(\mathcal{I})$. Hence, $\tilde{x} \sim_{\mathcal{I}} \tilde{t}$.

Conversely, let $\tilde{x} = (x_i)$ and $\tilde{t} = (t_i)$ be sequences s.t. $\tilde{x} \sim_{\mathcal{I}} \tilde{t}$ and $t_i \xrightarrow{\mathcal{K}} x$ hold. Since $\tilde{x} \sim_{\mathcal{I}} \tilde{t}$, then
\[
M = \{i \in \mathbb{N} : x_i = t_i\} \in \mathcal{F}(\mathcal{I})
\]
holds. Define a sequence
\[
y_i = \begin{cases} x_i, & i \in M, \\ x, & i \notin M \end{cases}
\]
Since $x_i = t_i$ hold $\forall i \in M$, then we can write
\[
t_i = \begin{cases} x_i, & i \in M, \\ x, & i \notin M \end{cases}
\]
Because $\tilde{t} = (t_i)$ is $\mathcal{K}$-convergent to $x$, the sequence $\tilde{y} = (y_i)$ is also $\mathcal{K}$-convergent to $x$. Hence, the sequence $\tilde{x} = (x_i)$ is $\mathcal{I}^{\mathcal{K}}$-convergent to the point $x$ and this completes the proof. \qed
4. \(\mathcal{I}^K\)-seq.-top. space

In this section, we are going to define a new topology on the \(\mathcal{X}\) using the ideal \(\mathcal{I}\) and \(\mathcal{K}\) and investigate some properties of the new T.S. This topology will be an extended version of the \(\mathcal{I}^*\)-seq.-top. space which was discussed in [18]. If we take \(\mathcal{I} = \mathcal{F}\), then \(\mathcal{I}^K\)-seq.-top. space is coincide with \(\mathcal{I}^*\)-T.S.

**Definition 9.** Let \(\mathcal{I}\) and \(\mathcal{K}\) stand for the ideals of \(\mathbb{N}\) and consider the T.S. \((\mathcal{X}, \mathcal{T})\). Then

1. A set \(F \subseteq \mathcal{X}\) is \(\mathcal{I}^K\)-closed, if for each \((x_i) \subseteq F\) with \(x_i \xrightarrow{\mathcal{K}} x\), then \(x \in F\).
2. A set \(V \subset \mathcal{X}\) is \(\mathcal{I}^K\)-open, if its complement \(V^C\) is \(\mathcal{I}^K\)-closed.

**Remark 2.** Consider the T.S. \((\mathcal{X}, \mathcal{T})\). An \(O \subset \mathcal{X}\) is \(\mathcal{I}^K\)-open iff each sequence in \(\mathcal{X} - O\) has \(\mathcal{I}^K\)-limit in \(\mathcal{X} - O\).

**Proof.** The proof is evident from Definition 9. Therefore, it is omitted here. \(\Box\)

**Definition 10.** Let \(\mathcal{I}\) and \(\mathcal{K}\) stand for the ideals of \(\mathbb{N}\) and consider the T.S. \((\mathcal{X}, \mathcal{T})\). For any subset \(A \subseteq \mathcal{X}\) define a set \(\overline{A}^{\mathcal{I}K}\) (it is called \(\mathcal{I}^K\)-closure of \(A\)) by

\[
\overline{A}^{\mathcal{I}K} := \{x \in \mathcal{X} : \exists (x_i) \subseteq A, \ x_i \xrightarrow{\mathcal{I}K} x\}.
\]

It is clear that \(\overline{\emptyset}^{\mathcal{I}K} = \emptyset\), \(\overline{\mathcal{X}}^{\mathcal{I}K} = \mathcal{X}\), and \(A \subseteq \overline{A}^{\mathcal{I}K}\) holds \(\forall A \subseteq \mathcal{X}\).

**Remark 3.** A subset \(C\) of the T.S. \(\mathcal{X}\) is \(\mathcal{I}^K\) closed set iff \(C^{\mathcal{I}K} = C\).

**Proof.** Proof is obvious from the Definition 10. So, it is omitted here. \(\Box\)

**Lemma 4.** Let \(\mathcal{I}\) and \(\mathcal{K}\) stand for the ideals of \(\mathbb{N}\) and let \((\mathcal{X}, \mathcal{T})\) represent a T.S. For any subset \(A \subseteq \mathcal{X}\), \(\mathcal{I}^K\)-closure of \(A\) is \(\mathcal{I}^K\)-closed.

**Proof.** We must show that \(\overline{(\overline{A}^{\mathcal{I}K})^{\mathcal{I}K}} = \overline{A}^{\mathcal{I}K}\).

It is clear that \(\overline{A}^{\mathcal{I}K} \subset \overline{(\overline{A}^{\mathcal{I}K})^{\mathcal{I}K}}\).

Let \(x \in \overline{(\overline{A}^{\mathcal{I}K})^{\mathcal{I}K}}\). Then, there exist a sequence \((x_i) \subset \overline{A}^{\mathcal{I}K}\) s.t. \(x_i \xrightarrow{\mathcal{I}K} x\) holds. Since \((x_i) \subset \overline{A}^{\mathcal{I}K}\), then there exist sequences \((x^n_i) \subset A\) s.t. \(x^n_i \xrightarrow{\mathcal{I}K} x_i\). Therefore there exist the sets \(M_n \in \mathcal{F}(\mathcal{I})\) s.t.

\[
\{i \in M_n : x^n_i \notin v^n\} \in \mathcal{K}
\]

for each neighborhood \(v^n\) of \(x_i\). Choose \(m_1\) the \(i\) where \(x^1_i\) is belonging to neighborhood \(v^1\) of \(x_1\), similarly \(m_2\) the \(i\) where \(x^2_i\) is belonging to neighborhood \(v^2\) of \(x_2\). If we continue this process and take \(m_p\) the \(i\) where \(x^p_i\) is belonging to neighborhood \(v^p\) of \(x_p\). The obtained sequence \((x^n_{m_p})\) belongs to \(A\). The theorem will be proved if we show that \(x_{m_p} \xrightarrow{\mathcal{I}K} x\). Since \(x_i \xrightarrow{\mathcal{I}K} x\), so \(\exists M \in \mathcal{F}(\mathcal{I})\) s.t. the sequence

\[
y_i = \begin{cases} x_i, & i \in M, \\ x, & i \notin M, \end{cases} \quad y_i \xrightarrow{\mathcal{K}} x.
\]
So,
\[ \{ i \in M : x_i \notin v \} \in \mathcal{K} \]
for each neighborhood \( v \) of \( x \). Now,
\[ \{ i \in M : v^n \subset v \} \subseteq \{ i \in M : x_i \notin v \} \in \mathcal{K}. \]
Therefore,
\[ \{ i \in M : v^n \subset v \} \in \mathcal{K} \]
and
\[ \{ i \in M : x_m \notin v \} \subseteq \{ i \in M : v^n \subset U \} \in \mathcal{K} \]
hold. So, \( x_m \xrightarrow{\mathcal{I}^\mathcal{K}} x \) and \( x \in \overline{A}^{\mathcal{I}^\mathcal{K}}. \)

\[ \square \]

**Definition 11.** Let \( \mathcal{I} \) and \( \mathcal{K} \) stand for the ideals of \( \mathbb{N} \) and \((\mathcal{X}, \mathcal{T})\) represent a T.S. Then, for \( A \subset \mathcal{X} \), \( \mathcal{I}^\mathcal{K} \)-interior of \( A \) is defined as
\[
A^{\mathcal{I}^\mathcal{K}} := A - (\overline{\mathcal{X} - A}^{\mathcal{I}^\mathcal{K}}).
\]

**Proposition 1.** Let \( \mathcal{V} \) be a subset of T.S. \( \mathcal{X} \), then \( \mathcal{V} \) is \( \mathcal{I}^\mathcal{K} \)-open iff \( \mathcal{V}^{\mathcal{I}^\mathcal{K}} = \mathcal{V} \).

**Proof.** Let \( \mathcal{V} \) be an \( \mathcal{I}^\mathcal{K} \)-open set. Then, \( \mathcal{X} - \mathcal{V} \) is \( \mathcal{I}^\mathcal{K} \)-closed set and
\[
\overline{\mathcal{X} - V}^{\mathcal{I}^\mathcal{K}} = \mathcal{X} - \mathcal{V}
\]
holds. So, we have
\[
\mathcal{V}^{\mathcal{I}^\mathcal{K}} = \mathcal{V} - (\mathcal{X} - \mathcal{V}) = \mathcal{V}.
\]
Conversely assume that
\[
\mathcal{V}^{\mathcal{I}^\mathcal{K}} = \mathcal{V}
\]
holds. From the definition of \( \mathcal{I}^\mathcal{K} \)-interior of \( \mathcal{V} \) we have
\[
\mathcal{V} = \mathcal{V} - (\overline{\mathcal{X} - \mathcal{V}}^{\mathcal{I}^\mathcal{K}}).
\]
Hence,
\[
\mathcal{V} \cap \overline{\mathcal{X} - \mathcal{V}}^{\mathcal{I}^\mathcal{K}} = \emptyset.
\]
Consequently
\[
\overline{\mathcal{X} - \mathcal{V}}^{\mathcal{I}^\mathcal{K}} \subset \mathcal{X} - \mathcal{V}.
\]
Thus,
\[
\overline{\mathcal{X} - \mathcal{V}}^{\mathcal{I}^\mathcal{K}} = \mathcal{X} - \mathcal{V}
\]
is satisfied. Therefore, \( \mathcal{X} - \mathcal{V} \) is \( \mathcal{I}^\mathcal{K} \)-closed and \( \mathcal{V} \) is \( \mathcal{I}^\mathcal{K} \)-open. \( \square \)

**Definition 12** [21]. A sequence \((x_i)\) in a T.S. \( \mathcal{X} \) is \( \mathcal{I} \)-eventually in a subset \( A \) of \( \mathcal{X} \) if
\[
\{ i \in \mathbb{N} : x_i \notin A \} \in \mathcal{F}(\mathcal{I}).
\]
Definition 13. Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X}, \mathcal{T})$. A sequence $\tilde{x} = (x_i) \subseteq \mathcal{X}$ is $\mathcal{I}^K$-eventually in a subset $\mathcal{V}$ of $\mathcal{X}$. If there exist a sequence $\tilde{y} = (y_i) \subseteq \mathcal{X}$ s.t. $\tilde{y} \sim_{\mathcal{I}} \tilde{x}$ and $\tilde{y}$ is $\mathcal{K}$-eventually in $\mathcal{V}$.

In the next theorem, we will provide a sequence characterization of $\mathcal{I}^K$-open set.

Theorem 1. Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X}, \mathcal{T})$. A subset $\mathcal{v}$ of $\mathcal{X}$ is $\mathcal{I}^K$-open iff each $\mathcal{I}^K$-convergent sequence to $x_0 \in \mathcal{v}$ is $\mathcal{I}^K$-eventually in $\mathcal{v}$.

Proof. Let $\mathcal{v}$ be $\mathcal{I}^K$-open. Then, $\mathcal{X} - \mathcal{v}$ is $\mathcal{I}^K$-closed and $\overline{\mathcal{X} - \mathcal{v}}^{\mathcal{I}^K} = \mathcal{X} - \mathcal{v}$ holds. Let $\tilde{x} = (x_i) \subset \mathcal{X}$ be a sequence s.t. $x_i \xrightarrow{\mathcal{I}^K} x$ and $x \in \mathcal{v}$. Then, $\exists \mathcal{M} \in \mathcal{F}(\mathcal{I})$ s.t. the sequence

$$t_i = \begin{cases} x_i, & i \in \mathcal{M}, \\ x, & i \notin \mathcal{M} \end{cases}$$

is $\mathcal{K}$-convergent to $x$. Since $\mathcal{v}$ is a neighborhood of $x$, then we have

$$\mathcal{H} = \{i \in \mathbb{N} : x_i \notin \mathcal{v}\} \subset \mathcal{K}.$$ 

If we choose $y_i = t_i$, then

$$\{i \in \mathbb{N} : y_i = x_i\} = \{i \in \mathbb{N} : t_i = x_i\} = \mathcal{M} \in \mathcal{F}(\mathcal{I})$$

holds. So, $(y_i) \sim_{\mathcal{I}} (x_i)$ holds and $(y_i)$ is eventually in $\mathcal{v}$.

Conversely, let $\tilde{x} = (x_i) \subset \mathcal{X}$ be a sequence which is $\mathcal{I}^K$-convergent sequence to a point $x \in \mathcal{v}$ and it is $\mathcal{I}^K$-eventually in $\mathcal{v}$. Assume that $\mathcal{v}$ is not $\mathcal{I}^K$-open subset of $\mathcal{X}$. So there exists $x_0 \in \overline{\mathcal{X} - \mathcal{v}}^{\mathcal{I}^K}$ which $x_0 \notin \mathcal{X} - \mathcal{v}$. This means that there exists a sequence $(x_i) \subset \mathcal{X} - \mathcal{v}$ which is $\mathcal{I}^K$-convergence to $x_0 \in \mathcal{v}$. So, $(x_i)$ is $\mathcal{I}^K$-eventually in $\mathcal{v}$.

Therefore, $\exists \tilde{y} = (y_i) \subset \mathcal{X}$ which $\tilde{x} \sim_{\mathcal{I}} \tilde{y}$ and $\tilde{y}$ is $\mathcal{K}$-eventually in $\mathcal{v}$. This implies that $\tilde{y}$ is $\mathcal{K}$-eventually in $\mathcal{v}$ which is not in case. \qed

Theorem 2. Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X}, \mathcal{T})$. A subset $\mathcal{C} \subset \mathcal{X}$ is $\mathcal{I}^K$-closed iff

$$\mathcal{C} = \cap \{\mathcal{A} : \mathcal{A} \text{ is } \mathcal{I}^K-\text{closed and } \mathcal{C} \subset \mathcal{A}\}.$$ 

Proof. Let

$$\mathcal{C} = \cap \{\mathcal{A} : \mathcal{A} \text{ is } \mathcal{I}^K-\text{closed and } \mathcal{C} \subset \mathcal{A}\}.$$ 

Let $x$ be any element of $\mathcal{I}^K$-closure of $\mathcal{C}$. Then there exists $(x_i) \subset \mathcal{C}$ s.t. $x_i \xrightarrow{\mathcal{I}^K} x$. Let $x \notin \mathcal{C}$ so

$$x \notin \cap \{\mathcal{A} : \mathcal{A} \text{ is } \mathcal{I}^K-\text{closed and } \mathcal{C} \subset \mathcal{A}\}.$$ 

This implies that $\exists \mathcal{I}^K$-closed subset $\mathcal{F}$ of $\mathcal{X}$ s.t. $x \notin \mathcal{A}$, but $\mathcal{C}$ is $\mathcal{I}^K$-closed and it is a subset of $\mathcal{A}$, which is a contradiction.

The converse is obvious. \qed

Theorem 3. Let $\mathcal{I}$ and $\mathcal{K}$ be ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T})$ be a T.S. A function $\mathcal{cl}_{\mathcal{I}^K} : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ defined as $\mathcal{cl}_{\mathcal{I}^K}(\mathcal{A}) = \overline{\mathcal{A}}^{\mathcal{I}^K}$ is satisfying Kuratowski closure axioms

$(K1)$ $\mathcal{cl}_{\mathcal{I}^K}(\emptyset) = \emptyset$ and $\mathcal{cl}_{\mathcal{I}^K}(\mathcal{X}) = \mathcal{X},$
(K2) \( A \subseteq \text{cl}_{\mathcal{K}}(A) \quad \forall A \subseteq \mathcal{X} \),
(K3) \( \text{cl}_{\mathcal{K}}(A) = \text{cl}_{\mathcal{K}}(\text{cl}_{\mathcal{K}}(A)) \quad \forall A \subseteq \mathcal{X} \),
(K4) \( \text{cl}_{\mathcal{K}}(A \cup B) = \text{cl}_{\mathcal{K}}(A) \cup \text{cl}_{\mathcal{K}}(B) \quad \forall A, B \subseteq \mathcal{X} \).

Proof. (K1) and (K2) are clear from the definition of \( \mathcal{K} \)-closure function. By Lemma 4, \( \text{cl}_{\mathcal{K}}(A) \) is closed. So, \( \text{cl}_{\mathcal{K}}(\text{cl}_{\mathcal{K}}(A)) = \text{cl}_{\mathcal{K}}(A) \). Therefore, (K3) holds.

To prove (K4), let \( x \in \text{cl}_{\mathcal{K}}(A) \cup \text{cl}_{\mathcal{K}}(B) \). Then, \( x \in \text{cl}_{\mathcal{K}}(A) \) or \( x \in \text{cl}_{\mathcal{K}}(B) \). Without loss of generality assume that \( x \in \text{cl}_{\mathcal{K}}(A) \). So, \( \exists (x_i) \subset A \) s.t. \( x_i \overset{\mathcal{K}}{\to} x \). Therefore, \( \exists (x_i) \subset A \cup B \) s.t. \( x_i \overset{\mathcal{K}}{\to} x \). So, \( x \in \text{cl}_{\mathcal{K}}(A) \cup \text{cl}_{\mathcal{K}}(B) \).

Conversely, let \( x \in \text{cl}_{\mathcal{K}}(A \cup B) \). Then, there exist a sequence \( (x_i) \subset (A \cup B) \) s.t. \( x_i \overset{\mathcal{K}}{\to} x \). Assume that \( x \notin \text{cl}_{\mathcal{K}}(A) \) and \( x \notin \text{cl}_{\mathcal{K}}(B) \). So, neither set \( A \) nor set \( B \) contains a sequence s.t. \( \mathcal{K} \)-converges to the point \( x \). Consequently, there is not any sequence in the \( A \cup B \) which is convergent to \( x \). But \( x \in \text{cl}_{\mathcal{K}}(A \cup B) \) which is a contradiction. Hence,
\[
\text{cl}_{\mathcal{K}}(A \cup B) = \text{cl}_{\mathcal{K}}(A) \cup \text{cl}_{\mathcal{K}}(B)
\]
holds. \( \square \)

Corollary 1. A subset \( A \) of \( \mathcal{X} \) is \( \mathcal{K} \)-closed iff \( \text{cl}_{\mathcal{K}}(A) = A \) and a subset \( O \subset \mathcal{X} \) is \( \mathcal{K} \)-open iff \( \mathcal{X} \setminus O \) is \( \mathcal{K} \)-closed.

Theorem 4. Let \( \mathcal{I} \) and \( \mathcal{K} \) stand for the ideals of \( \mathbb{N} \) and consider the T.S. \( (\mathcal{X}, \mathcal{T}) \). Then,
\[
\mathcal{T}_{\mathcal{K}} := \{ A \subset \mathcal{X} : \text{cl}_{\mathcal{K}}(\mathcal{X} \setminus A) = \mathcal{X} \setminus A \}
\]
is a topology over the set \( \mathcal{X} \).

Proof. By (K1), it is clear that \( \mathcal{X} \in \mathcal{T}_{\mathcal{K}} \) and \( \emptyset \in \mathcal{T}_{\mathcal{K}} \) hold. Let \( A, B \in \mathcal{T}_{\mathcal{K}} \) be arbitrary sets. To prove \( A \cup B \in \mathcal{T}_{\mathcal{K}} \) we must to prove that
\[
\mathcal{X} \setminus A \cup B = \text{cl}_{\mathcal{K}}(\mathcal{X} \setminus A \cup B)
\]
holds. By (K2), we have
\[
\mathcal{X} \setminus A \cup B \subseteq \text{cl}_{\mathcal{K}}(\mathcal{X} \setminus A \cup B).
\]

Now, let \( x \in \text{cl}_{\mathcal{K}}(\mathcal{X} \setminus A \cup B) \) be an arbitrarily element. Then, \( \exists (x_i) \subset \mathcal{X} \setminus (A \cup B) \) s.t. it is \( \mathcal{K} \)-convergent to \( x \). This implies that \( (x_i) \) is not subset of \( A \cup B \). So, \( (x_i) \) is neither subset of \( A \) nor subset of \( B \). Therefore, \( (x_i) \subset \mathcal{X} \setminus A \) or \( (x_i) \subset \mathcal{X} \setminus B \) which \( \mathcal{K} \)-converges to point \( x \). So, \( x \in \text{cl}_{\mathcal{K}}(\mathcal{X} \setminus A) \) or \( x \in \text{cl}_{\mathcal{K}}(\mathcal{X} \setminus B) \). Since \( \mathcal{X} \setminus A \) and \( \mathcal{X} \setminus B \) are closed sets, then
\[
x \in (\mathcal{X} \setminus A) \cup (\mathcal{X} \setminus B) = \mathcal{X} \setminus A \cup B
\]
holds.

Let \( \{ A_i \} \) be a collection of \( \mathcal{K} \)-open subsets of \( \mathcal{X} \). Then, \( \text{cl}_{\mathcal{K}}(\mathcal{X} \setminus A_i) = \mathcal{X} \setminus A_i \forall i \in \mathbb{N} \). By considering (K2), we have
\[
\cap_{i \in \mathbb{N}}(\mathcal{X} \setminus A_i) \subseteq \text{cl}_{\mathcal{K}}\left( \cap_{i \in \mathbb{N}}(\mathcal{X} \setminus A_i) \right).
\]

Let \( x \in \text{cl}_{\mathcal{K}} \cap_{i \in \mathbb{N}}(\mathcal{X} \setminus A_i) \) be an arbitrary element. Then, \( \exists (x_i) \subset \cap_{i \in \mathbb{N}}(\mathcal{X} \setminus A_i) \) which is \( \mathcal{K} \)-convergent to \( x \). Then, \( (x_i) \subset (\mathcal{X} \setminus A_i) \forall i \in \mathbb{N} \). Since \( \mathcal{X} \setminus A_i \) are closed sets, then \( x \in \mathcal{X} \setminus A_i \) \( \forall i \in \mathbb{N} \). Therefore,
\[
x \in \cap_{i \in \mathbb{N}}(\mathcal{X} \setminus A_i).
\]

Hence, the set \( \mathcal{T}_{\mathcal{K}} \) is a topology and \( (\mathcal{X}, \mathcal{T}_{\mathcal{K}}) \) is a T.S. \( \square \)
\textbf{Definition 14.} The T.S. \((X,T_x)\) is called as \(I^K\)-sequential T.S. For abbreviation we will show it by \(I^K\)-seq.-top. An \(I^K\)-seq.-top. \((X,T_x)\) is said to be \(I^K\)-discrete space if \(T_x = \mathcal{P}(X)\).

\textbf{Theorem 5.} Let \(I, K, I_1, K_1, I_2\) and \(K_2\) stand for ideals of \(\mathbb{N}\) and \((X,T)\) represents a T.S. Let \(I_1 \subset I_2\) and \(K_1 \subset K_2\). Then,
\begin{enumerate}
  \item \(T_{I_2}^{K_2} \subset T_{I_1}^{K_1}\),
  \item \(T_{I_2}^{K_1} \subset T_{I_1}^{K_2}\).
\end{enumerate}

\textbf{Proof.} Let \(v\) be any \(I^{K_2}\)-open subset of \(X\). Then, \(X - v\) is \(I^{K_2}\)-closed and \(\text{cl}_{I^{K_2}}(X - v) = X - v\) hold. To prove \(v\) is \(I^{K_1}\)-open subset of \(X\), we will show that
\[\text{cl}_{I^{K_1}}(X - v) \subset X - v.\]

Let \(x \in \text{cl}_{I^{K_1}}(X - v)\) be any point. Then, there exists \((x_i) \subset X - v\) s.t. \(x_i \xrightarrow{I^{K_1}} x\). Since \(K_1 \subset K_2\), then by Proposition 3.6 in [13], \(x_i \xrightarrow{I^{K_2}} x\). So, \(x \in \text{cl}_{I^{K_2}}(X - v)\). Therefore, \(x \in X - v\). Hence \(X - v\) is \(I^{K_2}\)-closed set and \(v\) is \(I^{K_2}\)-open subset of \(X\).

The second one can be proved by using the fact that if \(I_1 \subset I_2\), then, \(x_i \xrightarrow{I^1} x\) implies \(x_i \xrightarrow{I^2} x\), it easily can be proved. \hfill \Box

\textbf{Theorem 6.} Let \(I\) and \(K\) stand for the ideals of \(\mathbb{N}\) and \((X,T)\) represent a T.S. Then, every \(I^*\)-open set is \(I^K\)-open set.

\textbf{Proof.} If we take \(K = \text{Fin}\) then \(I^*\)-open set will be \(I^K\)-open set. \hfill \Box

\textbf{Theorem 7.} Let \(I\) and \(K\) stand for the ideals of \(\mathbb{N}\) and \((X,T)\) represent a T.S. Then, every \(I^K\)-open set is \(K\)-open set.

\textbf{Proof.} Let \(v\) be an arbitrary \(I^K\)-open subset of \(X\). Then, \(X - v\) is \(I^K\)-closed and
\[\text{cl}_{I^K}(X - v) = X - v.\]

To prove \(v\) is \(K\)-open, it is sufficient to show that \(X - v\) is \(K\)-closed, i.e,
\[X - v = \overline{X - v}^K.\]

It is clear that \(X - v \subset \overline{X - v}^K\). Let \(x \in \overline{X - v}^K\) be an arbitrary element s.t. \(\exists(x_i) \subset X - v\) satisfying \(x_i \xrightarrow{K} x\).

Then, by Lemma 3.5 in [13] we have \(x_i \xrightarrow{I^K} x\). So, \(x \in \text{cl}_{I^K}(X - v) = X - v\). Hence, the theorem proved. \hfill \Box

\textbf{Proposition 2.} Let \(I\) and \(K\) stand for the ideals of \(\mathbb{N}\) and \((X,T)\) represent a T.S. Then, the following statements are true:
\begin{enumerate}
  \item If \(K \subset I\), then, each \(I\)-open set is \(I^K\)-open set.
  \item If the space \(X\) is a first countable space and the ideal \(I\) has additive property with respect to \(K\) (see Definition 3.10 in [13]), then, each \(I^K\)-open set is \(I\)-open set.
  \item If \(I \subset K\), then every \(K\)-open set is \(I^K\)-open.
\end{enumerate}

\textbf{Proof.} The proof is obvious from Proposition 3.7 and Theorem 3.11 of [13]. \hfill \Box
5. $I^K$-continuity of functions

In this section we will define $I^K-$continuous and sequential $I^K-$continuous functions. We will prove that in any $I^K$-sequential T.S. these two concepts coincide. Also, we will state some theorems that give the definition of $I^K$-continuous function in different words and ways. At the end of this section we will see that the combination of $I^K$-continuous functions is $I^K$-continuous.

**Definition 15.** Let $I$ and $K$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, T_X)$ $(\mathcal{Y}, T_Y)$ represent $I^K-$seq.-top. spaces. A function $f$, from $\mathcal{X}$ to $\mathcal{Y}$ is said to be

(i) $I^K$-continuous which provides that inverse image of any $I^K$-open subset of $\mathcal{Y}$ is $I^K$-open in $\mathcal{X}$.

(ii) Sequentially $I^K$-continuous which provides that $f(x_i) \xrightarrow{I^K} f(x) \forall (x_i) \subset \mathcal{X}$ with $x_i \xrightarrow{I^K} x$.

**Theorem 8.** Let $I$ and $K$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, T_X)$ $(\mathcal{Y}, T_Y)$ represent $I^K$-seq.-top. spaces; and $f$, from $\mathcal{X}$ to $\mathcal{Y}$ be a function. Then, $f$ is $I^K$-continuous iff it is sequentially $I^K$-continuous.

**Proof.** Let $f$ be an $I^K$-continuous function. Then, inverse image of any $I^K$-open subset of $\mathcal{Y}$ is $I^K$-open subset in $\mathcal{X}$. Let $(x_i) \subset \mathcal{X}$ be a sequence with $x_i \xrightarrow{I^K} x$. Then, there exists $M \in \mathcal{F}(I)$ s.t. the following sequence

$$t_i := \begin{cases} x_i, & i \in M, \\ x, & i \notin M \end{cases}$$

is $K$-convergent to $x$. That is, for each neighborhood $v$ of $x$ we have

$$\{i \in \mathbb{N} : t_i \in v\} \in \mathcal{F}(K).$$

Let $V$ be any $I^K$-open neighborhood of $f(x)$. Then, $f^{-1}(V)$ is $I^K$-open subset of $\mathcal{X}$ which contains the point $x$. So, it is a neighborhood of $x$. Therefore,

$$\{i \in \mathbb{N} : f(t_i) \in V\} \in \mathcal{F}(K),$$

implies that $\{i \in \mathbb{N} : f(t_i) \in V\} \in \mathcal{F}(K)$. Hence, the sequence

$$f(t_i) := \begin{cases} f(x_i), & i \in M, \\ f(x), & i \notin M \end{cases}$$

is $K$-convergent to $f(x)$. So, $f(x_i) \xrightarrow{I^K} f(x)$. Hence, $f$ is sequentially $I^K$-continuous function.

Conversely, let the function $f$ be sequentially $I^K$-continuous and $v$ is any $I^K$-open subset of $\mathcal{Y}$. Assume that $f^{-1}(v)$ is not $I^K$-open subset of $\mathcal{X}$. Then, $\mathcal{X} - f^{-1}(v)$ is not $I^K$-closed subset of $\mathcal{X}$. So,

$$\exists(x_i) \subset \mathcal{X} - f^{-1}(v) \text{ s.t. } x_i \xrightarrow{I^K} x \text{ and } x \notin \mathcal{X} - f^{-1}(v),$$

i.e. $x_i \notin f^{-1}(v) \forall n$ and $x_i \xrightarrow{I^K} x$ which means $x \in f^{-1}(v)$. Since $f$ is $I^K$-sequentially continuous function then $f(x_i) \xrightarrow{I^K} f(x)$. So, $f(x) \in v$ and $f(x_i) \notin v \forall n$. This is a contradiction. \hfill $\Box$

**Lemma 5.** Let $I$ and $K$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, T_X)$ $(\mathcal{Y}, T_Y)$ represent $I^K$-seq.-top. spaces and $f$, from $\mathcal{X}$ to $\mathcal{Y}$ be an $I^K$-continuous function. If $(y_i) \subset \mathcal{Y}$ be a sequence s.t. $y_i \xrightarrow{I^K} y$, then $f^{-1}(y_i) \xrightarrow{I^K} f^{-1}(y)$. 
Proof. Let $f$ be an $\mathcal{I}^K$-continuous function. Let $y_i \xrightarrow{\mathcal{I}} y$ then $\exists M \in \mathcal{F}(\mathcal{I})$ s.t. the sequence

$$s_n = \begin{cases} y_i, & i \in M, \\ y, & i \notin M \end{cases}$$

is $\mathcal{K}$-convergent to $y$. So, for each neighborhood $v$ of $\mathcal{Y}$,

$$\{i \in \mathbb{N} : y_i \in v\} \in \mathcal{F}(\mathcal{K}).$$

Since $f$ is $\mathcal{I}^K$-continuous function, then inverse image of any $\mathcal{I}^K$-open set in $\mathcal{Y}$ is $\mathcal{I}^K$-open in $\mathcal{X}$, $f^{-1}(v)$ is open neighborhood of $x$ in $\mathcal{X}$. Then

$$\{i \in \mathbb{N} : f^{-1}(y_i) \in f^{-1}(v)\} \in \mathcal{F}(\mathcal{K}).$$

Therefore,

$$f^{-1}(s_n) = \begin{cases} f^{-1}(y_i), & i \in M, \\ f^{-1}(y), & i \notin M, \end{cases}$$

is $\mathcal{K}$-convergent to $f^{-1}(y)$ and hence $f^{-1}(y_i) \xrightarrow{\mathcal{I}^K} f^{-1}(y)$.

Theorem 9. Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^K})$, $(\mathcal{Y}, \mathcal{T}_{\mathcal{I}^K})$ represent $\mathcal{I}^K$-seq.-top. spaces. Then the function $f$, from $\mathcal{X}$ to $\mathcal{Y}$ is $\mathcal{I}^K$-continuous iff

$$\text{cl}_{\mathcal{I}^K}(f^{-1}(B)) = f^{-1}(\text{cl}_{\mathcal{I}^K}(B))$$

holds $\forall B \subset \mathcal{Y}$.

Proof. Assume that function $f$, from $\mathcal{X}$ to $\mathcal{Y}$ is $\mathcal{I}^K$-continuous function. Let

$$x \in \text{cl}_{\mathcal{I}^K}(f^{-1}(B)).$$

Then, $\exists (x_i) \subset f^{-1}(B)$ s.t. $x_i \xrightarrow{\mathcal{I}^K} x$. Since $f$ is $\mathcal{I}^K$-continuous so,

$$f(x_i) \xrightarrow{\mathcal{I}^K} f(x).$$

In another hand $(x_i) \subset B$, so $f(x) \in \text{cl}_{\mathcal{I}^K}(B)$ and $x \in f^{-1}(\text{cl}_{\mathcal{I}^K}(B))$.

Now, let $x \in f^{-1}(\text{cl}_{\mathcal{I}^K}(B))$, i.e. $f(x) \in \text{cl}_{\mathcal{I}^K}(B)$. Therefore, $\exists (y_i) \subset B$ s.t. $x_i \xrightarrow{\mathcal{I}^K} x$. Then, by Lemma 5 there exists $(x_i) = (f^{-1}(y_i)) \subset f^{-1}(B)$ s.t. $x_i \xrightarrow{\mathcal{I}^K} x$, where $x = f^{-1}(y)$ holds. So, $x \in \text{cl}_{\mathcal{I}^K}(f^{-1}(B))$. Hence,

$$\text{cl}_{\mathcal{I}^K}(f^{-1}(B)) = f^{-1}(\text{cl}_{\mathcal{I}^K}(B)).$$

Conversely, let

$$\text{cl}_{\mathcal{I}^K}(f^{-1}(B)) = f^{-1}(\text{cl}_{\mathcal{I}^K}(B)), \quad \forall B \in \mathcal{P}(\mathcal{Y}).$$

Let $v$ be $\mathcal{I}^K$-open subset of $\mathcal{Y}$ then

$$\text{cl}_{\mathcal{I}^K}(\mathcal{Y} - B) = \mathcal{Y} - B.$$ 

Let $B = \mathcal{Y} - v$, then

$$\text{cl}_{\mathcal{I}^K}(f^{-1}(\mathcal{Y} - v)) = f^{-1}(\text{cl}_{\mathcal{I}^K}(\mathcal{Y} - v)) = f^{-1}(\mathcal{Y} - v).$$

This shows that $f^{-1}(\mathcal{Y} - v)$ is $\mathcal{I}^K$-closed. Hence, the following equality

$$f^{-1}(\mathcal{Y} - v) = \mathcal{X} - f^{-1}(v)$$

implies that $\mathcal{X} - f^{-1}(v)$ is $\mathcal{I}^K$-closed. Therefore $f^{-1}(v)$ is $\mathcal{I}^K$-open set. 

$\square$
Corollary 2. Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, T_{\mathcal{X}^\mathcal{K}})$ $(\mathcal{Y}, T_{\mathcal{Y}^\mathcal{K}})$ represent $\mathcal{I}^\mathcal{K}$-seq.-top. spaces. A function $f$, from $\mathcal{X}$ to $\mathcal{Y}$ is $\mathcal{I}^\mathcal{K}$-continuous iff
\[
\text{int}_{\mathcal{X}^\mathcal{K}}(f^{-1}(B)) = f^{-1}(\text{int}_{\mathcal{X}^\mathcal{K}}(B)) \quad \forall B \subset \mathcal{Y}.
\]

Definition 16. Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, T_{\mathcal{X}^\mathcal{K}})$ $(\mathcal{Y}, T_{\mathcal{Y}^\mathcal{K}})$ represent $\mathcal{I}^\mathcal{K}$-seq.-top. spaces and $f$, from $\mathcal{X}$ to $\mathcal{Y}$ be a function. The function $f$ is $\mathcal{I}^\mathcal{K}$-continuous at a point $x \in \mathcal{X}$ if inverse image of any neighborhood of $f(x)$ is a neighborhood of $x$ in $\mathcal{X}$.

Corollary 3. Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, T_{\mathcal{X}^\mathcal{K}})$ $(\mathcal{Y}, T_{\mathcal{Y}^\mathcal{K}})$ represent $\mathcal{I}^\mathcal{K}$-seq.-top. spaces. Then, the function $f$, from $\mathcal{X}$ to $\mathcal{Y}$ is $\mathcal{I}^\mathcal{K}$-continuous iff it is $\mathcal{I}^\mathcal{K}$-continuous at every point $x \in \mathcal{X}$.

Definition 17. Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, T_{\mathcal{X}^\mathcal{K}})$ $(\mathcal{Y}, T_{\mathcal{Y}^\mathcal{K}})$ represent $\mathcal{I}^\mathcal{K}$-seq.-top. spaces and $f$, from $\mathcal{X}$ to $\mathcal{Y}$ be a function, $f$ is said to be $\mathcal{I}^\mathcal{K}$-closure preserving if
\[
f(\text{cl}_{\mathcal{X}^\mathcal{K}}(A)) = \text{cl}_{\mathcal{X}^\mathcal{K}}(f(A)) \quad \forall A \subset \mathcal{X}.
\]

Theorem 10. The function $f$, from $\mathcal{X}$ to $\mathcal{Y}$ is $\mathcal{I}^\mathcal{K}$-continuous iff it is $\mathcal{I}^\mathcal{K}$-closure preserving.

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ be an $\mathcal{I}^\mathcal{K}$-continuous function. Then, for any subset $B$ of $\mathcal{Y}$
\[
\text{cl}_{\mathcal{X}^\mathcal{K}}(f^{-1}(B)) = f^{-1}(\text{cl}_{\mathcal{X}^\mathcal{K}}(B))
\]
holds. Consider a set $A \subset \mathcal{X}$ s.t. $f(A)$ is subset of $\mathcal{Y}$. So,
\[
\text{cl}_{\mathcal{X}^\mathcal{K}}(f^{-1}(f(A))) = f^{-1}(\text{cl}_{\mathcal{X}^\mathcal{K}}(f(A)))
\]
holds and it implies that $f(\text{cl}_{\mathcal{X}^\mathcal{K}}(A)) = \text{cl}_{\mathcal{X}^\mathcal{K}}(f(A)) \forall A \subset \mathcal{X}$ holds.

Conversely, let $f$ be $\mathcal{I}^\mathcal{K}$-closure preserving function, then
\[
\text{cl}_{\mathcal{X}^\mathcal{K}}(f(A)) = \text{cl}_{\mathcal{X}^\mathcal{K}}(f(f(A))) \quad \forall A \subset \mathcal{X}.
\]

Let $v$ be any subset of $\mathcal{Y}$, then $f^{-1}(v)$ is subset of $\mathcal{X}$ and
\[
f(\text{cl}_{\mathcal{X}^\mathcal{K}}(f^{-1}(v))) = \text{cl}_{\mathcal{X}^\mathcal{K}}(f(f^{-1}(v)) = \text{cl}_{\mathcal{X}^\mathcal{K}}(v)
\]
holds. So
\[
\text{cl}_{\mathcal{X}^\mathcal{K}}(f^{-1}(v)) = f^{-1}(\text{cl}_{\mathcal{X}^\mathcal{K}}(v))
\]
and by Theorem 9 the function $f$ is $\mathcal{I}^\mathcal{K}$-continuous. $\square$

Theorem 11. Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be $\mathcal{I}^\mathcal{K}$-seq.-top. spaces. Let $f$, from $\mathcal{X}$ to $\mathcal{Y}$ and $g$, from $\mathcal{Y}$ to $\mathcal{Z}$ be $\mathcal{I}^\mathcal{K}$-continuous functions. Then $g \circ f : \mathcal{X} \to \mathcal{Z}$ is $\mathcal{I}^\mathcal{K}$-continuous functions.

Proof. Let $v$ be any $\mathcal{I}^\mathcal{K}$-open subset of $\mathcal{Z}$. Since $g$ is $\mathcal{I}^\mathcal{K}$-continuous function then $g^{-1}(v)$ is $\mathcal{I}^\mathcal{K}$-open subset of $\mathcal{Y}$ and because $f$ is $\mathcal{I}^\mathcal{K}$-continuous function therefore $f^{-1}(g^{-1}(v))$ is $\mathcal{I}^\mathcal{K}$-open subset of $\mathcal{X}$ hence $(g \circ f)^{-1}(v)$ is $\mathcal{I}^\mathcal{K}$-open subset of $\mathcal{X}$. $\square$
6. Subspace of \( \mathcal{I}^K \)-seq.-top. space

In this section subspaces of the \( \mathcal{I}^K \)-seq.-top. space and its properties under an \( \mathcal{I}^K \)-continuous function will be discussed.

**Definition 18.** Let \( (\mathcal{X}, \mathcal{T}_{\mathcal{I}^K}) \) be an \( \mathcal{I}^K \)-seq.-top. space and \( \mathcal{Y} \subset \mathcal{X} \). Then
\[
C_Y : \mathcal{P}(\mathcal{Y}) \to \mathcal{P}(\mathcal{Y}), \quad C_Y(A) = \mathcal{Y} \cap \text{cl}_{\mathcal{I}^K}(A)
\]
is a Kuratowski operator. Define a T.S. as
\[
\mathcal{T}^Y_{\mathcal{I}^K} = \{ U \cap \mathcal{Y}, Y \in \mathcal{T}_{\mathcal{I}^K} \} \subset \mathcal{P}(\mathcal{Y}).
\]
This T.S. is called \( \mathcal{I}^K \)-subspace of \( \mathcal{X} \).

**Lemma 6.** Let \( \mathcal{Y} \) be an \( \mathcal{I}^K \)-subspace of \( \mathcal{I}^K \)-seq.-top. space \( \mathcal{X} \). If set \( A \) is \( \mathcal{I}^K \)-open subset of \( \mathcal{Y} \) and \( \mathcal{Y} \) is an \( \mathcal{I}^K \)-subset of \( \mathcal{X} \). Then \( A \) is \( \mathcal{I}^K \)-open subset of \( \mathcal{X} \).

**Proof.** Let \( A \) be \( \mathcal{I}^K \)-open subset of \( \mathcal{Y} \). Then \( \exists U \in \mathcal{T}_{\mathcal{I}^K} \text{ s.t. } A = \mathcal{Y} \cap U \). Since \( \mathcal{Y} \) is an \( \mathcal{I}^K \)-open subset of \( \mathcal{X} \). Then \( A \in \mathcal{T}_{\mathcal{I}^K} \).

**Proposition 3.** Let \( (\mathcal{X}, \mathcal{T}_{\mathcal{I}^K}) \) and \( (\mathcal{Y}, \mathcal{T}_{\mathcal{I}^K}') \) be \( \mathcal{I}^K \)-sequential spaces, \( f : \mathcal{X} \to \mathcal{Y} \) be \( \mathcal{I}^K \)-continuous function and \( A \subset \mathcal{X} \) is \( \mathcal{I}^K \)-subspace of \( \mathcal{X} \). Then \( f/A : A \to \mathcal{Y} \), the restriction \( f \) over \( A \) is \( \mathcal{I}^K \)-continuous function.

**Proof.** Let \( U \) be an \( \mathcal{I}^K \)-open subset of \( \mathcal{Y} \). Since \( f \) is \( \mathcal{I}^K \)-continuous function then \( f^{-1}(U) \) is \( \mathcal{I}^K \)-open subset of \( \mathcal{X} \). That is \( f^{-1}(U) \in \mathcal{T}_{\mathcal{I}^K} \).

In other hand \( f^{-1}(U) = A \cap f^{-1}(U) \). So \( f^{-1}(U) \) is \( \mathcal{I}^K \)-open subset of subspace \( A \). Hence \( f/A \) is \( \mathcal{I}^K \)-continuous function.

**Lemma 7.** If \( A \) is \( \mathcal{I}^K \)-subspace of \( \mathcal{I}^K \)-sequential T.S. \( \mathcal{X} \). Then the inclusion map \( j : A \to \mathcal{X} \) is \( \mathcal{I}^K \)-continuous.

**Proof.** If \( U \) is \( \mathcal{I}^K \)-open in \( \mathcal{X} \) then \( j^{-1}(U) = U \cap A \) is \( \mathcal{I}^K \)-open in subspace \( \mathcal{Y} \) hence \( j \) is \( \mathcal{I}^K \)-continuous.

**Proposition 4.** Let \( (\mathcal{X}, \mathcal{T}_{\mathcal{I}^K}) \) and \( (\mathcal{Y}, \mathcal{T}_{\mathcal{I}^K}') \) be \( \mathcal{I}^K \)-sequential spaces, \( B \subset \mathcal{Y} \) be subspace of \( \mathcal{Y} \) and \( f : \mathcal{X} \to B \) be \( \mathcal{I}^K \)-continuous function. Then, \( h : \mathcal{X} \to \mathcal{Y} \) obtained by expanding the range of \( f \) is \( \mathcal{I}^K \)-continuous.

**Proof.** To show \( h : \mathcal{X} \to \mathcal{Y} \) is \( \mathcal{I}^K \)-continuous function, if \( B \) as subspace of \( \mathcal{Y} \) then note that \( h \) is the composition of the map \( f : \mathcal{X} \to B \) and \( j : B \to \mathcal{Y} \).
7. Conclusion

In this article we defined the notion of $I^K$-closed (resp. $I^K$-open) set in a T.S. $(X, T)$ and established some important results concerning this notion. Furthermore, we defined the $I^K$-seq.-top., which is a generalized form of the $I^*$-sequential space. We also talked about $I^K$-continuity of functions and saw that in $I^K$-seq.-top. space the notion of continuity and sequential continuity are the same. And in the last section of the paper, subspace of $I^K$-sequential space have been studied and some important results established.

REFERENCES