GRAPHS $\Gamma$ OF DIAMETER 4 FOR WHICH $\Gamma_{3,4}$ IS A STRONGLY REGULAR GRAPH WITH $\mu = 4, 6^i$

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Abstract: We consider antipodal graphs $\Gamma$ of diameter 4 for which $\Gamma_{1,2}$ is a strongly regular graph. A.A. Makhnev and D.V. Paduchikh noticed that, in this case, $\Delta = \Gamma_{3,4}$ is a strongly regular graph without triangles. It is known that in the cases $\mu = \mu(\Delta) \in \{2, 4, 6\}$ there are infinite series of admissible parameters of strongly regular graphs with $k(\Delta) = \mu(r+1)+r^2$, where $r$ and $s = -(\mu + r)$ are nonprincipal eigenvalues of $\Delta$. This paper studies graphs with $\mu(\Delta) = 4$ and 6. In these cases, $\Gamma$ has intersection arrays \{r^2 + 4r + 3, r^2 + 4r, 4, 1, 1, 4, r^2 + 4r, r^2 + 4r + 3\} and \{r^2 + 6r + 5, r^2 + 6r, 6, 1, 1, 6, r^2 + 6r, r^2 + 6r + 5\}, respectively. It is proved that graphs with such intersection arrays do not exist.

Keywords: Distance-regular graph, Strongly regular graph, Triple intersection numbers.

1. Introduction

We consider undirected graphs without loops or multiple edges.

Let $\Gamma$ be a connected graph. The distance $d(a, b)$ between two vertices $a$ and $b$ of $\Gamma$ is the length of a shortest path between $a$ and $b$ in $\Gamma$. Given a vertex $a$ in a graph $\Gamma$, we denote by $\Gamma_i(a)$ the subgraph induced by $\Gamma$ on the set of all vertices that are at distance $i$ from $a$. The subgraph $[a] = \Gamma_1(a)$ is called the neighbourhood of the vertex $a$.

Let $\Gamma$ be a graph and $a, b \in \Gamma$. Then the number of vertices in $[a] \cap [b]$ is denoted by $\mu(a,b)$ (by $\lambda(a,b)$) if $a$ and $b$ are at distance 2 (are adjacent) in $\Gamma$. Further, a subgraph induced by $[a] \cap [b]$ is called a $\mu$-subgraph (a $\lambda$-subgraph). Let $\Gamma$ be a graph of diameter $d$ and $i, j \in \{1, 2, 3, \ldots, d\}$. A graph $\Gamma_i$ has the same set of vertices as $\Gamma$ and vertices $u$ and $w$ are adjacent in $\Gamma_i$ if $d_{\Gamma}(u, w) = i$. A graph $\Gamma_{i,j}$ has the same set of vertices as $\Gamma$ and vertices $u$ and $w$ are adjacent in $\Gamma_j$ if $d_{\Gamma}(u, w) \in \{i, j\}$.

If vertices $u$ and $w$ are at distance $i$ in $\Gamma$, then we denote by $b_i(u, w)$ (by $c_i(u, w)$) the number of vertices in the intersection $\Gamma_{i+1}(u)$ ($\Gamma_{i-1}(u)$) with $[w]$. A graph $\Gamma$ of diameter $d$ is called distance-regular with intersection array \{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\} if the values $b_i(u, w)$ and $c_i(u, w)$ are

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Let $\Delta$ be a distance-regular graph with intersection array $\{k, k - a_1 - 1, (r - 1)c_2, 1; 1, c_2, k - a_1 - 1, k\}$.

Consider an antipodal distance-regular graph $\Gamma$ of diameter 4 for which $\Gamma$ is a strongly regular graph. Makhnev and Paduchikh noticed in [3] that, in this case, $\Delta = \Gamma_{3,4}$ is a strongly regular graph without triangles and the antipodality index of $\Gamma$ equals 2. It is known that in the cases $\mu = \mu(\Delta) \in \{2, 4, 6\}$ there arise infinite series of admissible parameters of strongly regular graphs with $k(\Delta) = \mu(r + 1) + r^2$, where $r$ and $s = -(\mu + r)$ are nonprincipal eigenvalues of $\Delta$.

In the present paper, we consider graphs with $\mu(\Delta) = 4$ and 6. In these cases, $\Gamma$ has intersection arrays

$$\{r^2 + 4r + 3, r^2 + 4r, 4, 1; 1, 4, r^2 + 4r, r^2 + 4r + 3\}$$

and

$$\{r^2 + 6r + 5, r^2 + 6r, 6, 1; 1, 6, r^2 + 6r, r^2 + 6r + 5\},$$

respectively.

If $\mu(\Delta) = 4$, then $\Delta$ has parameters $(v, r^2 + 4r + 4, 0, 4)$, where

$$v = 1 + (r^2 + 4r + 4) + \frac{(r^2 + 4r + 4)(r^2 + 4r + 3)}{4}.$$ 

Further, $\Delta$ has nonprincipal eigenvalues $r$ and $-(r + 4)$, and the multiplicity of $r$ is equal to $(r + 3)(r + 2)(r^2 + 5r + 8)/8$.

**Theorem 1.** A distance-regular graph with intersection array

$$\{r^2 + 4r + 3, r^2 + 4r, 4, 1; 1, 4, r^2 + 4r, r^2 + 4r + 3\}$$

does not exist.

If $\mu(\Delta) = 6$, then $\Delta$ has parameters $(v, r^2 + 6r + 6, 0, 6)$, where

$$v = 1 + (r^2 + 6r + 6) + (r^2 + 6r + 6)(r^2 + 6r + 5)/6.$$ 

Further, $\Delta$ has nonprincipal eigenvalues $r$ and $-(r + 6)$, and the multiplicity of $r$ is equal to $(r + 5)(r^2 + 6r + 6)(r + 4)/12$. Therefore, $r$ is even or congruent to 3 modulo 4.

**Theorem 2.** A distance-regular graph with intersection array

$$\{r^2 + 6r + 5, r^2 + 6r, 6, 1; 1, 6, r^2 + 6r, r^2 + 6r + 5\}$$

does not exist.

**Corollary 1.** Distance-regular graphs with intersection arrays

$$\{32, 27, 6, 1; 1, 6, 27, 32\}, \quad \{45, 40, 6, 1; 1, 6, 40, 45\}, \quad \{77, 72, 6, 1; 1, 6, 72, 77\}, \quad \{96, 91, 6, 1; 1, 6, 91, 96\}, \quad \{117, 112, 6, 1; 1, 6, 112, 117\}$$

do not exist.
2. Triple intersection numbers

Let $\Gamma$ be a distance-regular graph of diameter $d$. If $u_1$, $u_2$, and $u_3$ are vertices of the graph $\Gamma$ and $r_1, r_2,$ and $r_3$ are nonnegative integers not greater than $d$, then $\{u_1u_2u_3 \mid r_1r_2r_3\}$ is the set of vertices $w \in \Gamma$ such that

$$d(w, u_i) = r_i, \quad \left\{u_1u_2u_3 \mid r_1r_2r_3\right\} = \left\{u_1u_2u_3 \mid r_1r_2r_3\right\}.$$ 

The numbers $\left[u_1u_2u_3 \mid r_1r_2r_3\right]$ are called triple intersection numbers. For a fixed triple $u_1, u_2, u_3$ of vertices, we will write $[r_1r_2r_3]$ instead of $\left[u_1u_2u_3 \mid r_1r_2r_3\right]$.

Unfortunately, there are no general formulas for numbers $[r_1r_2r_3]$. However, [2] suggests a method for calculating some numbers $[r_1r_2r_3]$.

Assume that $u, v,$ and $w$ are vertices of the graph $\Gamma$, $W = d(u, v)$, $U = d(v, w)$, and $V = d(u, w)$. Since there is exactly one vertex $x = u$ such that $d(x, u) = 0$, then the number $[0jh]$ is 0 or 1. Hence, $[0jh] = \delta_{jW}\delta_{hV}$. Similarly, $[i0h] = \delta_{iW}\delta_{hU}$ and $[ij0] = \delta_{U}\delta_{jV}$.

Another set of equations can be obtained by fixing the distance between two vertices from $\{u, v, w\}$ and counting the number of vertices located at all possible distances from the third. Then, we get

$$\sum_{l=1}^{d} [ijh] = p_{jh}^U - [0jh], \quad \sum_{l=1}^{d} [i lh] = p_{lh}^V - [i0h], \quad \sum_{l=1}^{d} [ijl] = p_{ijl}^W - [ij0]. \tag{2.1}$$

At the same time, some triples disappear. If $|i - j| > W$ or $i + j < W$, then $p_{ij}^W = 0$; therefore, $[ijh] = 0$ for all $h \in \{0, \ldots, d\}$. Define

$$S_{ijh}(u, v, w) = \sum_{r,s,t=0}^{d} Q_{r_1}Q_{s_j}Q_{t_h} \left[uvw \mid r_st\right].$$

If Krein’s parameter $q_{ijh}^h$ is 0, then $S_{ijh}(u, v, w) = 0$.

3. A distance-regular graph with intersection array

$$\{r^2 + 4r + 3, r^2 + 4r, 4, 1, 4, r^2 + 4r, r^2 + 4r + 3\}\}

In this section, $\Gamma$ is a distance-regular graph with intersection array

$$\{r^2 + 4r + 3, r^2 + 4r, 4, 1, 4, r^2 + 4r, r^2 + 4r + 3\}\}.$$

Then, $\Gamma$ has $1 + (r^2 + 4r + 3) + (r^2 + 4r + 3)(r^2 + 4r)/4 + (r^2 + 4r + 3) + 1$ vertices and the spectrum

$$(r + 3)(r + 1) \quad \text{of multiplicity} \quad 1,$$

$r + 3 \quad \text{of multiplicity} \quad \frac{(r^2 + 5r + 8)(r^2 + 3r + 4)(r + 1)}{16(r + 2)}$, 

$r - 1 \quad \text{of multiplicity} \quad \frac{(r^2 + 5r + 8)(r + 4)(r + 3)(r + 1)}{16(r + 2)}$, 

$-(r + 1) \quad \text{of multiplicity} \quad \frac{(r^2 + 5r + 8)(r^2 + 3r + 4)(r + 3)}{16(r + 2)}$, 

$-(r + 5) \quad \text{of multiplicity} \quad \frac{(r^2 + 3r + 4)(r + 3)(r + 1)r}{16(r + 2)}$. 

The multiplicity of \( r + 3 \) is equal to
\[
\frac{(r^2 + 5r + 8)(r^2 + 3r + 4)(r + 1)}{16(r + 2)}.
\]

Further,
\[
(r^2 + 5r + 8, r + 2) = (3r + 8, r + 2)
\]
divides 2 and \((r + 2, r^2 + 3r + 4) = (r + 2, r + 4)\) divides 2; therefore \( r + 2 \) divides 4. Consequently, \( r = 2 \), a contradiction with the fact that the multiplicity of \( r + 3 \) is equal to
\[
(r^2 + 5r + 8)(r^2 + 3r + 4)(r + 1)/(16(r + 2)) = 22 \times 14 \times 3/64.
\]

Theorem 1 is proved.

4. A distance-regular graph with intersection array
\[
\{r^2 + 6r + 5, r^2 + 6r, 6; 1, 6, r^2 + 6r, r^2 + 6r + 5\}
\]

In this section, \( \Gamma \) is a distance-regular graph with intersection array
\[
\{r^2 + 6r + 5, r^2 + 6r, 6; 1, 6, r^2 + 6r, r^2 + 6r + 5\}.
\]

Then, \( \Gamma \) has
\[
1 + (r^2 + 6r + 5) + (r^2 + 6r + 5)(r^2 + 6r)/6 + (r^2 + 6r + 5) + 1
\]
vertices, the spectrum
\[
(r + 5)(r + 1) \quad \text{of multiplicity} \quad 1,
\]
\[
r + 5 \quad \text{of multiplicity} \quad f = (r + 4)(r + 3)(r + 2)(r + 1)/24,
\]
\[
r - 1 \quad \text{of multiplicity} \quad (r + 6)(r + 5)(r + 4)(r + 1)/24,
\]
\[
-(r + 1) \quad \text{of multiplicity} \quad (r + 5)(r + 4)(r + 3)(r + 2)/24,
\]
\[
-(r + 7) \quad \text{of multiplicity} \quad (r + 5)(r + 2)(r + 1)r/24,
\]

and the matrix \( Q \) (see [1]) of dual eigenvalues

\[
\begin{pmatrix}
1 & f & f(r + 6)(r + 5) & f(r + 5) & f(r + 5)r \\
1 & r + 1 & (r + 2)(r + 3)(r + 1) & r + 1 & (r + 4)(r + 3)(r + 1) \\
1 & 0 & r/2 - 2 & 0 & r/2 + 1 \\
1 & -f & (r + 6)(r - 1) & f & f(r + 7)r \\
1 & -f & (r + 2)(r + 3)(r + 1) & f(r + 5) & f(r + 5)r
\end{pmatrix}.
\]

Lemma 1. The intersection numbers are
\[
\begin{align*}
p_{11}^1 &= 4, & p_{11}^2 &= r^2 + 6r, & p_{12}^1 &= r^2 + 6r, & p_{12}^2 &= r^4/6 + 2r^3 + 29r^2/6 - 7r, & p_{13}^1 &= 0, & p_{14}^1 &= 1; \\
p_{11}^2 &= 6, & p_{12}^2 &= r^2 + 6r - 7, & p_{13}^2 &= 6, & p_{22}^1 &= r^4/6 + 2r^3 + 29r^2/6 - 7r + 12, & p_{22}^2 &= 1, & p_{23}^2 &= 2; & \\
p_{12}^3 &= r^2 + 6r, & p_{13}^3 &= 4, & p_{14}^3 &= 1, & p_{22}^3 &= 1, & p_{23}^3 &= 3, & p_{33}^3 &= 0; & \\
p_{13}^4 &= r^2 + 6r + 5, & p_{22}^4 &= r^4/6 + 2r^3 + 41r^2/6 + 5r.
\end{align*}
\]
Proof. Direct calculations using formulas from [1, Lemma 4.1.7]. □

Fix vertices \( u, v, \) and \( w \) of the graph \( \Gamma \) and define

\[
\{ijh\} = \left\{ \frac{uvw}{ijh} \right\}, \quad [ijh] = \left[ \frac{uvw}{ijh} \right].
\]

Let \( \Delta = \Gamma_2(u) \), and let \( \Lambda \) be a graph with vertices from \( \Delta \) in which two vertices are adjacent if they are at distance 2 in \( \Gamma \). Then \( \Lambda \) is a regular graph of degree

\[
p_{22}^2 = r^4/6 + 2r^3 + 29r^2/6 - 7r + 12
\]
on vertices.

**Lemma 2.** Let \( d(u, v) = d(u, w) = 2 \) and \( d(v, w) = 1 \). Then, the triple intersection numbers are

\[
[111] = r_4, \quad [112] = [121] = -r_4 + 6, \quad [122] = r_3 + r_4 + r^2 + 6r - 19, \quad [123] = [132] = -r_3 + 6; \quad [211] = -r_3 - r_4 + 4, \quad [212] = [221] = r_3 + r_4 + r^2 + 6r - 12,
\]

\[
[222] = r^4/6 + 2r^3 + 17r^2/6 - 19r + 36, \quad [223] = [232] = r_3 + r_4 + r^2 + 6r - 12, \quad [233] = -r_3 - r_4 + 4, \quad [234] = [243] = 1; \quad [311] = r_3, \quad [312] = [321] = -r_3 + 6, \quad [322] = r_3 + r_4 + r^2 + 6r - 19, \quad [323] = [332] = -r_4 + 6;
\]

\[
[333] = r_4, \quad [422] = 1,
\]

where \( r_3 + r_4 \leq 4 \).

Proof. Simplification of formulas (2.1). □

By Lemma 2, we have

\[
r^4/6 + 2r^3 + 17r^2/6 - 19r + 28 \leq [222] = -2r_3 - 2r_4 + r^4/6 + 2r^3 + 17r^2/6 - 19r + 36 \leq r^4/6 + 2r^3 + 17r^2/6 - 19r + 36.
\]

**Lemma 3.** Let \( d(u, v) = d(u, w) = 2 \) and \( d(v, w) = 3 \). Then, the triple intersection numbers are

\[
[112] = -r_{11} + 6, \quad [113] = r_{11}, \quad [121] = -r_{12} + 6, \quad [122] = r_{11} + r_{12} + r^2 + 6r - 19, \quad [123] = -r_{11} + 6, \quad [132] = -r_{12} + 6; \quad [212] = [221] = r_{11} + r_{12} + r^2 + 6r - 12, \quad [213] = [231] = -r_{11} - r_{12} + 4, \quad [214] = [241] = 1,
\]

\[
[222] = -2r_3 - 2r_4 + r^4/6 + 2r^3 + 17r^2/6 - 19r + 36, \quad [223] = [232] = r_{11} + r_{12} + r^2 + 6r - 12; \quad [312] = -r_{12} + 6, \quad [313] = r_{12}, \quad [321] = -r_{11} + 6, \quad [322] = r_{11} + r_{12} + r^2 + 6r - 19,
\]

\[
[323] = -r_{12} + 6, \quad [331] = r_{11}, \quad [332] = -r_{11} + 6; \quad [422] = 1,
\]

where \( r_{11} + r_{12} \leq 4 \).
\textbf{Proof.} Simplification of formulas (2.1). □

By Lemma 3, we have
\[ r^4/6 + 2r^3 + 17r^2/6 - 19r + 28 \leq [222] = -2r_3 - 2r_4 + r^4/6 + 2r^3 + 17r^2/6 - 19r + 36 \leq r^4/6 + 2r^3 + 17r^2/6 - 19r + 36. \]

\textbf{Lemma 4.} Let \( d(u, v) = d(u, w) = 2 \) and \( d(v, w) = 4 \). Then, the triple intersection numbers are
\[
[113] = [131] = 6, \quad [122] = r^2 + 6r - 7;
\]
\[
[213] = [231] = r^2 + 6r - 7, \quad [222] = r^4/6 + 2r^3 + 29r^2/6 - 7r + 12;
\]
\[
[313] = [331] = 6, \quad [322] = r^2 + 6r - 7;
\]
\[
[422] = 1.
\]

\textbf{Proof.} Simplification of formulas (2.1). □

By Lemma 4, we have
\[ [222] = r^4/6 + 2r^3 + 29r^2/6 - 7r + 12. \]

Recall that
\[ p_{12}^2 = r^2 + 6r - 7, \quad p_{22}^2 = r^4/6 + 2r^3 + 29r^2/6 - 7r + 12, \quad p_{23}^2 = r^2 + 6r - 7, \quad p_{24}^2 = 1. \]

Let \( v \) and \( w \) be vertices from \( \Lambda \). Then the number \( d \) of edges between \( \Lambda(v) \) and \( \Lambda - (\{v\} \cup \Lambda(v)) \) is
\[ d = p_{12}^2 \frac{[uwx]}{221} + p_{23}^2 \frac{[uwy]}{223} + p_{24}^2 \frac{[uwz]}{224}, \]
where \( x, y, \) and \( z \) are vertices from \( \{uw\} \) for \( i = 1, 3, \) and \( 4 \), respectively. Now, \( d \) satisfies the inequalities
\[
(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 56) + r^4/6 + 2r^3 + 29r^2/6 - 7r + 12 \leq d
\]
\[
\leq (r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 72) + r^4/6 + 2r^3 + 29r^2/6 - 7r + 12.
\]

On the other hand,
\[ d = \sum_{w \in \Lambda(v)} (p_{22}^2 - 1 - \lambda_\Lambda(v, w)) = k_\Lambda \left( p_{22}^2 - 1 - \frac{\sum_{w \in \Lambda(v)} \lambda_\Lambda(v, w)}{k_\Lambda} \right). \]

So,
\[ d = (r^4/6 + 2r^3 + 29r^2/6 - 7r + 12)(r^4/6 + 2r^3 + 29r^2/6 - 7r + 11 - \lambda), \]
where \( \lambda \) is the average value of degree of the vertex \( w \) in the graph \( \Lambda \). Consequently,
\[
\frac{(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 56) + 1}{r^4/6 + 2r^3 + 29r^2/6 - 7r + 12} + 1 \leq \frac{r^4}{6} + 2r^3 + \frac{29r^2}{6} - 7r + 11 - \lambda
\]
\[
\leq \frac{(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 72)}{r^4/6 + 2r^3 + 29r^2/6 - 7r + 12} + 1
\]
and
\[
\frac{r^4}{6} + 2r^3 + \frac{29r^2}{6} - 7r + 10 - \frac{(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 72)}{r^4/6 + 2r^3 + 29r^2/6 - 7r + 12} \leq \lambda
\]
\[
\leq \frac{r^4}{6} + 2r^3 + \frac{29r^2}{6} - 7r + 10 - \frac{(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 56)}{r^4/6 + 2r^3 + 29r^2/6 - 7r + 12}.
\]
Lemma 5. Let \( d(u, v) = d(u, w) = d(v, w) = 2 \). Then, the triple intersection numbers are

\[
[111] = r_9, \quad [112] = -r_7 - r_9 + 6, \quad [113] = r_7, \quad [121] = -r_10 - r_9 + 6,
\]
\[
[122] = r_7 + r_8 + r_9 + r_{10} + r^2 + 6r - 19, \quad [123] = -r_7 - r_8 + 6,
\]
\[
[131] = r_{10}, \quad [132] = -r_{10} - r_8 + 6, \quad [133] = r_8;
\]
\[
[211] = -r_8 - r_9 + 6, \quad [212] = [221] = r_7 + r_8 + r_9 + r_{10} + r^2 + 6r - 19,
\]
\[
[213] = [231] = -r_{10} - r_7 + 6, \quad [222] = -2r_7 - 2r_8 - 2r_9 - 2r_{10} + r^4/6 + 2r^3 + 17r^2/6 - 19r + 48,
\]
\[
[223] = [232] = r_7 + r_8 + r_9 + r_{10} + r^2 + 6r - 19, \quad [224] = [242] = 1, \quad [233] = -r_8 - r_9 + 6;
\]
\[
[311] = r_8, \quad [312] = -r_{10} - r_8 + 6, \quad [313] = r_{10}, \quad [321] = -r_7 - r_8 + 6,
\]
\[
[322] = r_7 + r_8 + r_9 + r_{10} + r^2 + 6r - 19, \quad [323] = -r_{10} - r_9 + 6,
\]
\[
[331] = r_7, \quad [332] = -r_7 - r_9 + 6, \quad [333] = r_9; \quad [422] = 1,
\]

where

\[
r_9 + r_7, \quad r_9 + r_{10}, \quad r_7 + r_8, \quad r_{10} + r_8, \quad r_8 + r_9, \quad r_7 + r_{10} \leq 6.
\]

\[\square\]

Proof. Simplification of formulas (2.1).

By Lemma 5, we have

\[
\frac{r^4}{6} + 2r^3 + \frac{17r^2}{6} - 19r + 24 \leq [222] = -2r_7 - 2r_8 - 2r_9 - 2r_{10} + \frac{r^4}{6} + 2r^3 + \frac{17r^2}{6} - 19r + 48
\]
\[
\leq \frac{r^4}{6} + 2r^3 + \frac{17r^2}{6} - 19r + 48.
\]

Let \( d(u, v) = 2 \).

Let us count the number \( e_2 \) of pairs of vertices \((s, t)\) at distance 2, where \( s \in \{ uu \} \) and \( t \in \{ uu \} \).

On the one hand, by Lemma 2, we have

\[
\frac{r^4}{6} + 2r^3 + 17r^2/6 - 19r + 28 \leq [222] \leq \frac{r^4}{6} + 2r^3 + 17r^2/6 - 19r + 36,
\]

so,

\[
(r^2 + 6r - 7)\left(\frac{r^4}{6} + 2r^3 + \frac{17r^2}{6} - 19r + 28\right) \leq e_2 \leq (r^2 + 6r - 7)\left(\frac{r^4}{6} + 2r^3 + \frac{17r^2}{6} - 19r + 36\right).
\]

On the other hand, by Lemma 5, we have

\[
[212] = r_7 + r_8 + r_9 + r_{10} + r^2 + 6r - 19
\]

and

\[
(r^2 + 6r - 7)\left(\frac{r^4}{6} + 2r^3 + \frac{17r^2}{6} - 19r + 28\right) \leq e_2
\]
\[
= -\sum_i (r_i^2 + r_i^3 + r_i^9 + r_{10}^2) + (r^2 + 6r - 19)\left(\frac{r^4}{6} + 2r^3 + \frac{29r^2}{6} - 7r + 12\right)
\]
\[
\leq (r^2 + 6r - 7)\left(\frac{r^4}{6} + 2r^3 + \frac{17r^2}{6} - 19r + 36\right).
\]
In this way,

\[(r^2 + 6r - 19)\left(\frac{r^4}{6} + 2r^3 + \frac{29r^2}{6} - 7r + 12\right) - (r^2 + 6r - 7)\left(\frac{r^4}{6} + 2r^3 + \frac{17r^2}{6} - 19r + 36\right) \leq (r_7^i + r_8^i + r_9^i + r_{10}^i) \]

\[\leq (r^2 + 6r - 19)\left(\frac{r^4}{6} + 2r^3 + \frac{29r^2}{6} - 7r + 12\right) - (r^2 + 6r - 7)\left(\frac{r^4}{6} + 2r^3 + \frac{17r^2}{6} - 19r + 28\right).\]

Consequently, 

\[(r_7^i + r_8^i + r_9^i + r_{10}^i) \leq -145r^3/6 - 16r^2 - 96r - 12,
\]
a contradiction.

Theorem 2 is proved. \(\square\)

The corollary follows from Theorems 1 and 2.

So, we have shown the nonexistence of graphs with intersection arrays

\[\{r^2 + 4r + 3, r^2 + 4r, 4, 1; 1, 4, r^2 + 4r, r^2 + 4r + 3\}\]

and

\[\{r^2 + 6r + 5, r^2 + 6r, 6, 1; 1, 6, r^2 + 6r, r^2 + 6r + 5\}\]

In particular, distance-regular graphs with intersection arrays

\[\{32, 27, 6, 1; 1, 6, 27, 32\}, \quad \{45, 40, 6, 1; 1, 6, 40, 45\}, \quad \{77, 72, 6, 1; 1, 6, 72, 77\}\]

\[\{96, 91, 6, 1; 1, 6, 91, 96\}, \quad \{117, 112, 6, 1; 1, 6, 112, 117\}\]
do not exist.

REFERENCES

