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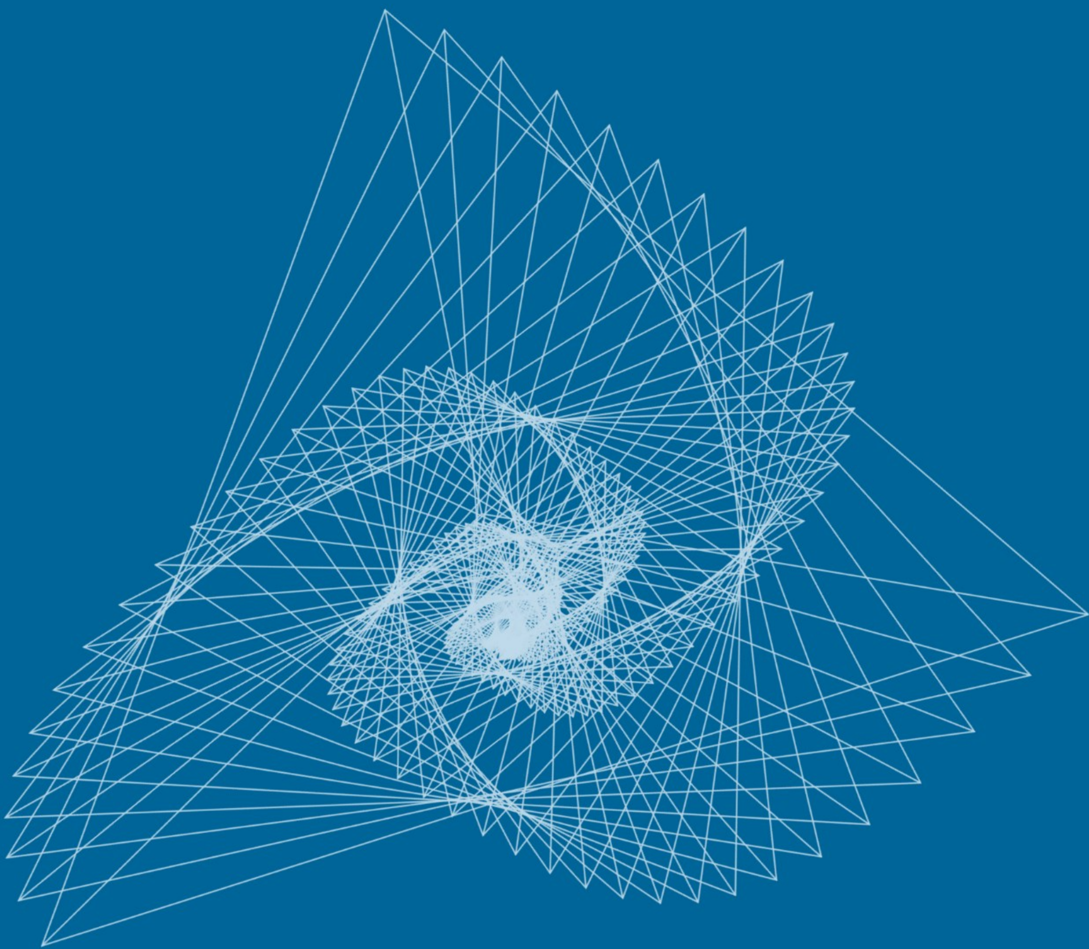

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N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences

Ural Federal University named after the first President of Russia B.N. Yeltsin

Contact Information

16 S. Kovalevskaya str., Ekaterinburg, Russia, 620990
Phone: +7 (343) 375-34-73   Fax: +7 (343) 374-25-81
Email: secretary@umjuran.ru
Web-site: https://umjuran.ru

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MIN-MAX SOLUTIONS FOR PARAMETRIC CONTINUOUS STATIC GAME UNDER ROUGHNESS (PARAMETERS IN THE COST FUNCTION AND FEASIBLE REGION IS A ROUGH SET)

Yousria A. Aboelnaga

Higher Technological Institute,
Tenth of Ramadan City, 44629, Egypt
yousria_naga@yahoo.com

Mai F. Zidan

Faculty of Engineering, Tanta University,
Al-Geish St., Tanta, 31512, Egypt
mai.fathi@f-eng.tanta.edu.eg

Abstract: Any simple perturbation in a part of the game whether in the cost function and/or conditions is a big problem because it will require a game re-solution to obtain the perturbed optimal solution. This is a waste of time because there are methods required several steps to obtain the optimal solution, then at the end we may find that there is no solution. Therefore, it was necessary to find a method to ensure that the game optimal solution exists in the case of a change in the game data. This is the aim of this paper. We first provided a continuous static game rough treatment with Min-Max solutions, then a parametric study for the processing game and called a parametric rough continuous static game (PRCSG). In a Parametric study, a solution approach is provided based on the parameter existence in the cost function that reflects the perturbation that may occur to it to determine the parameter range in which the optimal solution point keeps in the surely region that is called the stability set of the 1^{st} kind. Also the sets of possible upper and lower stability to which the optimal solution belongs are characterized. Finally, numerical examples are given to clarify the solution algorithm.

Keywords: Continuous static game, Rough programming, Non-linear programming, Rough set theory, Parametric linear programming, Parametric non-linear programming.

1. Introduction

Rough programming is introduced in [7, 8, 13, 19]. It is classified into three classes based on the roughness detected [13]; 1^{st} class: the roughness exists in a feasible set with crisp objective function; 2^{nd} class: the roughness exists in the objective function with a crisp feasible set; 3^{rd} class: the roughness exists in both feasible set and objective function. Therefore, the nonlinear programming problem [2, 3, 11] with a rough representation is called a rough nonlinear programming problem (RNPP) and can be defined by roughness, which may be in constraints and/or objective functions. The RNPP has two solution sets: possibly and surely feasible optimal solution sets in the 1^{st} class. Parametric optimization [6] is a helpful universal tool that has recently got many applications in process engineering systems [1, 7, 14, 20]. Given a constrained optimization problem, including a set of variables and bounded parameters, it provides the optimal value of the variables as an explicit function of the parameters without completely enumerating the entire space of the parameters. This means that if the value of the parameter changes within the given bounds, the optimal solution can be obtained by a simple evaluation of the explicit function without having to resolve an optimization problem. This is a powerful result that has been successfully applied in multiobjective optimization [10, 15], stochastic optimization [4, 16], flexibility analysis, hybrid, and robust model-based control [5, 17]. This paper investigates the analysis of basic concepts in a parametric rough continuous static game when parameters are in the cost functions and roughness is in the constraints which are not discussed before. A survey of rough optimality, rough set theory (RST), and rough functions is necessary for parametric studying of RNPP.

This article is structured as follows: Section 2 introduces Min-Max solutions for the rough continuous static game of 1^{st} class with an illustrative example to explain the solution steps. In Section 3, the Rough Continuous Static game parametric study when roughness exists in the constraints and parameters in the objective functions is discussed with a numerical example to clear up the presented concepts. Section 4 covers the conclusion and future studies.

2. 1^{st} class of Rough Continuous Static game (RCSG)

The continuous static game (CSG) [9, 17, 18] is defined as an optimization problem where each player in the game controls a specified subset of the system parameters and seeks to minimize his own cost criterion subject to specified constraints. In these games control notation is used and each player $i = 1, \dots, r$ selects his control vector $u^i \in E^s$ seeking to minimize a scalar valued criterion

$$G_i(x, u), \tag{2.1}$$

subject to n equality constrains

$$g(x, u) = 0, \tag{2.2}$$

where $x \in E^s$ is the state and

$$u = (u^1, \dots, u^r) \in E^s, \quad s = s_1 + \dots + s_r$$

is the composite control. The composite control is required to be an element of a regular control constraint set $\Omega \subseteq E^s$ of the form:

$$\Omega = \{u \in E^s | \, h(x, u) \geq 0\},$$

where $x = \zeta(u)$ is the solution of (2.2) given u. The functions

$$G_i(x, u): \; E^n * E^s \to E^1, \quad g(x, u): E^n * E^s \to E^n, \quad h(x, u): E^n \to E^s \to E^q$$

are assumed to be e^1, with

$$\left| \frac{\partial g(x, u)}{\partial x} \right| \neq 0$$

in a ball around a solution point (x, u). The above problem can be written as:

$$\begin{gathered} \min G_i(x, u), \\ \text{S.T.} \\ M = \{x \in E^n, u \in E^s | \; g(x, u) = 0, \; h(x, u) \geq 0\}, \end{gathered}$$

where G_i is called the cost function for each player $i = 1, \dots, r$, and M is called the feasible set of the problem.

Definition 1. *Let $A(U,R)$ be an approximation space, U be the universe and R be an equivalence relation on U. Let M be a subset of U, i.e. $M \subseteq U$. Then, the 1^{st} class of RCSG can be defined as follows:*

$$\begin{gathered}\min G_i(x,u),\\ S.T.\\ M_* \subseteq M \subseteq M^*,\end{gathered}$$

where M_*, M^* are lower and upper approximation of the feasible set, respectively. The solution sets of the 1^{st} class of RCSG are called a surely optimal solution set and a possibly optimal solution set. To find such solutions, first solve the following problem:

$$\begin{gathered}\min G_i(x,u),\\ S.T.\\ m \in M^*.\end{gathered}$$

The optimal solution set of the above game is

$$O = \big\{\tilde{m} \in M^* |\ G_i(x,\tilde{m}) = \min_{m \in M^*} G_i(x,u)\big\}.$$

The optimal solution value of the cost function on upper approximation set is

$$G^* = \min_{m \in M^*} G_i(x,u).$$

There are now two possibilities: one providing surely optimal solution, possibly optimal solution and another one leads us to resolve the game on the lower approximation as follows:

- If $O_1 = O \cap M_* \neq \emptyset$, then a solution set O_1 is called a surely optimal solution set and $O \sim O_1$ is possibly optimal solution set.
- If $O_1 = O \cap M_* = \emptyset$, then $O \subseteq M_{BN}$ (boundary approximation of feasible region). In this case, the game does not have surely optimal solution set. Therefore, it will be resolved on the lower approximation to obtain O_2, which is an optimal solution set on the lower approximation.

Where, the game on the lower approximation is

$$\min G_i(x,u), \tag{2.3}$$
$$\text{S.T.}$$
$$M_* \subseteq M, \tag{2.4}$$

and the optimal solution value of the cost function on lower approximation set is

$$G_* = \min_{m \in M_*} G_i(x,u).$$

Definition 2. *If $O_1 \neq \emptyset$, then O_1 contains all surely optimal solutions, hence it is called the surely optimal set $O_1 \subseteq M$.*

Definition 3. *$O \sim O_1$ contains possibly optimal solutions, hence it is called the possibly optimal set because $O \sim O_1 \subseteq M_{BN}$.*

2.1. Min-Max solutions for the 1^{st} class of RCSG

For this game player i chooses his control under the assumption that all players have formed a coalition to maximize his cost. In other words, all players expect one cooperate against the remaining player i.

Definition 4. *A point $w_* = (u_{*_i}, v_*)$ is a completely regular lower point if and only if w_* is a regular point of Ω and u_{*_i} is a regular point of U_{*_i} for each $i = 1, \dots, r$. Here u_{*_i} is a lower control vector for each player $i = 1, \dots, r$ and v_* is the composite lower control of the remaining players, $h_*(\cdot)$ defines lower inequality constraints and*

$$U_{*_i} = \left\{u_{*_i} \in E^s \middle| \; h_*[\zeta(u_{*_i}, v_*), u_{*_i}, v_*] \geq 0\right\}. \tag{2.5}$$

Definition 5. *A point $\tilde{u}_* \in \Omega$ is a rough min-max lower point for player i if and only if*

$$G_i[\zeta(\tilde{u}_{*_i}, v_*), \tilde{u}_{*_i}, v_*] \leq G_i[\zeta(\tilde{u}_*), \tilde{u}_*] \leq G_i[\zeta(u_{*_i}, \tilde{v}_*), u_{*_i}, \tilde{v}_*]$$

for all $u_{*_i} \in U_{*_i}$ where $\tilde{u}_* = (\tilde{u}_{*_i}, \tilde{v}_*), U_{*_i}$ is defined by (2.5) and $x_* = \zeta(u_*)$ is the solution to $M_*(x_*, \tilde{u}_*) = 0$. For a local rough min-max lower point replace U_{*_i} by $B_i \cap U_*$ for some ball $B_i \subset E^s$ centered at $\tilde{u}_*$.

Lemma 1. *If $\tilde{u}_* = (u_{i_*}, v_*) \in \Omega$ is a local min-max lower point for player i for the game (2.3)–(2.4), and if $x_* = \zeta(u_*)$ is the solution to $M_*(x_*, \tilde{u}_*) = 0$, then there exists a vector $\gamma_{*_i} \in E^n$ defined by*

$$\frac{\partial J_{*_i}[x_*, \tilde{u}_*, \gamma_{*_i}]}{\partial x_*} = 0, \qquad \frac{\partial J_{*_i}[x_*, \tilde{u}_*, \gamma_{*_i}]}{\partial u_{*_i}} e^i = 0,$$

for all $e^i \in T_{*_i}$ and

$$\frac{\partial J_{*_i}[x_*, \tilde{u}_*, \gamma_{*_i}]}{\partial v_*} e^v = 0,$$

for all $e^v \in T_{*_v}$, where

$$J_{*_i}[x_*, \tilde{u}_*, \gamma_{*_i}] = G_{*_i}(x_*, u_*) - {\gamma_{*_i}}^T g_*(x_*, u_*), \tag{2.6}$$

and the tangent cones T_{*_i} and T_{*_v} are given by

$$T_{*_i} = \left\{e^i \in E^{s_i} \middle| \; \left[\frac{\partial h_*}{\partial u_{*_i}} - \frac{\partial h_*}{\partial x_*}\left[\frac{\partial g_*}{\partial x_*}\right]^{-1} \frac{\partial g_*}{\partial u_{*_i}}\right] e^i\right\} \geq 0,$$
$$T_{*_v} = \left\{e^i \in E^{s-s_i} \middle| \; \left[\frac{\partial h_*}{\partial v_*} - \frac{\partial h_*}{\partial x_*}\left[\frac{\partial g_*}{\partial x_*}\right]^{-1} \frac{\partial g_*}{\partial v_*}\right] e^v\right\} \geq 0,$$

where h_* denotes the active inequality constraints at $u_* = (u_{*_i}, v_*)$.

Proposition 1. *If $\tilde{u}_* = (u_*, v_*) \in \Omega$ is a completely regular rough min-max lower point for the player i and $x_* = \zeta(\tilde{u}_*)$ is the solution to $M_*(x_*, \tilde{u}_*) = 0$, then there exist vectors $\lambda_{*_i} \in E^n$,*

$\bar{\lambda}_{*_i} \in E^n$, $\mu_{*_i} \in E^q$ *and* $\bar{\mu}_{*_i} \in E^q$ *such that:*

$$\frac{\partial L_{*_i}[x_*, \tilde{u}_*, \lambda_{*_i}, \mu_{*_i}]}{\partial x_*} = 0, \tag{2.7}$$

$$\frac{\partial L_{*_i}[x_*, \tilde{u}_*, \bar{\lambda}_{*_i}, \bar{\mu}_{*_i}]}{\partial x_*} = 0, \tag{2.8}$$

$$\frac{\partial L_{*_i}[x_*, \tilde{u}_*, \lambda_{*_i}, \mu_{*_i}]}{\partial u_*} = 0, \tag{2.9}$$

$$\frac{\partial L_{*_i}[x_*, \tilde{u}_*, \bar{\lambda}_{*_i}, \bar{\mu}_{*_i}]}{\partial v_*} = 0, \tag{2.10}$$

$$M_*(x_*, \tilde{u}_*) = 0,$$

$$\mu_{*_i}^T h_*(\tilde{x}, \tilde{u}_*) = 0, \tag{2.11}$$

$$\bar{\mu}_{*_i}^T h_*(x_*, \tilde{u}_*) = 0, \tag{2.12}$$

$$h_*(\tilde{x}, \tilde{u}_*) \geq 0,$$

$$\mu_{*_i} \geq 0, \tag{2.13}$$

$$\bar{\mu}_{*_i} \leq 0, \tag{2.14}$$

where

$$\begin{aligned} L_{*_i}[x_*, \tilde{u}_*, \lambda_{*_i}, \mu_{*_i}] &= G_i(x_*, u_*) - \lambda_{*_i} M_*(x_*, u_{*_i}) - \mu_{*_i} h_*(x_*, u_{*_i}), \\ L_{*_i}[x_*, \tilde{u}_*, \bar{\lambda}_{*_i}, \bar{\mu}_{*_i}] &= G_i(x_*, u_*) - \bar{\lambda}_{*_i} M_*(x_*, u_{*_i}) - \bar{\mu}_{*_i} h_*(x_*, u_{*_i}). \end{aligned} \tag{2.15}$$

P r o o f. For player i consider the cones

$$K_{*_i} = \Big\{ y_{*_i} \in E^{s_i} \big| \; {y_{*_i}}^T = {\mu_{*_i}}^T \Big[\frac{\partial h_*}{\partial u_{*_i}} - \frac{\partial h_*}{\partial x_*} \Big[\frac{\partial g_*}{\partial x_*} \Big]^{-1} \frac{\partial g_*}{\partial u_{*_i}} \Big], \quad {\mu_{*_i}}^T h_* = 0, \; \mu_{*_i} \geq 0 \Big\},$$

$$K_{*_2} = \Big\{ y_{*_2} \in E^{s-s_i} \big| \; {y_{*_2}}^T = {\mu_{*_i}}^T \Big[\frac{\partial h_*}{\partial v_*} - \frac{\partial h_*}{\partial x_*} \Big[\frac{\partial g_*}{\partial x_*} \Big]^{-1} \frac{\partial g_*}{\partial v_*} \Big], \quad {\mu_{*_2}}^T h_* = 0, \; \mu_{*_2} \geq 0 \Big\},$$

and their polars, respectively,

$$\begin{aligned} {K^P}_{*_i} &= \big\{ Z_{*_i} \in E^{s_i} \big| \; {y_{*_i}}^T Z_{*_i} \geq 0 \;\; \forall y_{*_i} \in K_{*_i} \big\}, \\ {K^P}_{*_2 i} &= \big\{ Z_{*_2} \in E^{s-s_i} \big| \; {y_{*_2}}^T Z_{*_2} \geq 0 \;\; \forall y_{*_2} \in K_{*_2} \big\}. \end{aligned}$$

Since u_{*_i} is a regular point of U_{*_i} the tangent cone T_{i_*} to U_{*_i} is given by

$$T_{*_i} = {K^P}_{*_i}, \quad T_{*_v} = {K^P}_{*_2}.$$

From this result and Lemma 1 we have

$$\frac{\partial J_{*_i}[x_*, \tilde{u}_*, \gamma_{*_i}]}{\partial u_{*_i}} \in K_{*_i}, \tag{2.16}$$

$$\frac{\partial J_{*_i}[x_*, \tilde{u}_*, \gamma_{*_i}]}{\partial v_*} \in K_{*_2}, \tag{2.17}$$

where J_{*_i} is defined by (2.6) and γ_{*_i} is defined by

$$\frac{\partial J_{*_i}[x_*, \tilde{u}_*, \gamma_{*_i}]}{\partial x_*} = 0. \tag{2.18}$$

Define

$$\lambda_{*_i}{}^T = \gamma_{*_i}{}^T - \mu_{*_i}^T \frac{\partial h}{\partial x_*}\Big[\frac{\partial g_*}{\partial x_*}\Big]^{-1}, \tag{2.19}$$

$$\tilde{\lambda}_{*_i}^T = \gamma_{*_i}{}^T - \mu_{*_2}^T \frac{\partial h}{\partial x_*}\Big[\frac{\partial g_*}{\partial x_*}\Big]^{-1}, \tag{2.20}$$

with L_{*_i} defined by (2.15) and choosing $\mu_{*_i} - \tilde{\mu}_{*_i} = 0$, equation (2.7) follows from (2.18) by substituting for γ_{*_i} from (2.19). Similarly (2.9), (2.11) and (2.13) follow from (2.16), (2.19) and the definition of k_{*_i}. Choosing $\tilde{\mu}_{*_i} = -\tilde{\mu}_{*_2} \leq 0$ and substituting for γ_{*_i} from (2.20) we see that the equation (2.8) follows from (2.18). Similarly (2.10), (2.12) and (2.14) follow from (2.17) and from the definition of k_{*_2}. □

Definition 6. *A point $w^* = (u_i^*, v^*)$ is a completely regular upper point if and only if w^* is a regular point of Ω and u_i^* is a regular point of U_i^* for each $i = 1, \ldots, r$. Here u_i^* is a lower control vector for each player $i = 1, \ldots, r$ and v^* is the composite lower control of the remaining players, $h^*(\cdot)$ defines lower inequality constraints and*

$$U_i^* = \big\{u_i^* \in E^s |\ h^*[\zeta(u_i^*, v^*), u_i^*, v^*] \geq 0\big\}. \tag{2.21}$$

Definition 7. *A point $\tilde{u}^* \in \Omega$ is a rough min-max upper point for player i if and only if*

$$G_i[\zeta(\tilde{u}_i^*, v^*), \tilde{u}_i^*, v^*] \leq G_i[\zeta(\tilde{u}^*), \tilde{u}^*] \leq G_i[\zeta(u_{*_i}, \tilde{v}_*), u_{*_i}, \tilde{v}_*]$$

For all $u_i^* \in U_i^*$ where $\tilde{u}^* = (\tilde{u}_i^*, \tilde{v}^*), U_i^*$ is defined by (2.21) and $x^* = \zeta(u^*)$ is the solution to $M^*(x^*, \tilde{u}^*) = 0$. For a local rough min-max lower point replace U_i^* by $B_i \cap U^*$ for some ball $B_i \subset E^s$ centered at $\tilde{u}^*$.

Proposition 2. *If $\tilde{u}^* = (u^*, v^*) \in \Omega$ is a completely regular rough min-max upper point for the player i and $x^* = \zeta(\tilde{u}^*)$ is the solution to $M^*(x^*, \tilde{u}^*) = 0$, then there exist vectors $\lambda_i^* \in E^n$, $\bar{\lambda}_i^* \in E^n$, $\mu_i^* \in E^q$ and a vector $\bar{\mu}_i^* \in E^q$ such that:*

$$\frac{\partial L_i^*[x^*, \tilde{u}^*, \lambda_i^*, \mu_i^*]}{\partial x^*}{=}0, \quad \frac{\partial L_i^*[x^*, \tilde{u}^*, \bar{\lambda}_i^*, \bar{\mu}_i^*]}{\partial x^*}{=}0, \quad \frac{\partial L_i^*[x^*, \tilde{u}^*, \lambda_i^*, \mu_i^*]}{\partial u^*}{=}0, \quad \frac{\partial L_i^*[x^*, \tilde{u}^*, \bar{\lambda}_i^*, \bar{\mu}_i^*]}{\partial v^*}{=}0,$$

$$M^*(x^*, \tilde{u}^*) = 0, \quad {\mu_i^*}^T h^*(\tilde{x}, \tilde{u}^*){=}0, \quad {\bar{\mu}_i^*}^T h^*(\tilde{x}, \tilde{u}^*) = 0, \quad h^*(\tilde{x}, \tilde{u}^*) \geq 0, \quad \mu_i^* \geq 0, \quad \bar{\mu}_i^* \leq 0,$$

where

$$L_i^*[x^*, \tilde{u}^*, \lambda_i^*, \mu_i^*] = G_i(x^*, u^*, v^*) - \lambda_i^* M^*(x^*, u^*, v^*) - \mu_i^* h^*(x^*, u^*, v^*),$$
$$L_i^*[x^*, \tilde{u}^*, \lambda_i^*, \mu_i^*] = G_i(x^*, u^*, v^*) - \bar{\lambda}_i^* M^*(x^*, u^*, v^*) - \bar{\mu}_i^* h^*(x^*, u^*, v^*).$$

P r o o f is similar to proof of Proposion 1. □

The numerical example below will explain how we can find the min-max optimal solution point for the 1^{st} class RCSG.

Example 1. Two firms sell substitutable products and seek to maximize their profits through advertising. The steady-state profits of firms 1 and 2 are taken respectively, as

$$H_1(\cdot) = 5x - u - v,$$
$$H_2(\cdot) = 3x - v - 2u.$$

The equilibrium (steady-state) fraction of the market x that firms receive is given by

$$-7x + u - 4xv = 0 \subseteq M(\cdot) \subseteq -2x + v - 2xu = 0, \tag{2.22}$$

where u and v are control vectors of firm 1 and firm 2 respectively.

Solution 1. Two firms seek to maximize $H_1(\cdot)$ and $H_2(\cdot)$. Thus, firms seek to minimize

$$G_1(\cdot) = -5x + u + v, \tag{2.23}$$
$$G_2(\cdot) = -3x + v + 2u. \tag{2.24}$$

Let us solve the problem (*upper approximation*) using Min-Max concept:

For player 1: the game (2.22), (2.23) will be the following

$$\begin{gathered}\min_{u^* \in M^*} G_1(\cdot),\\ \text{S.T.}\\ M^* = -2x^* + v^* - 2x^*u^* = 0.\end{gathered}$$

Here we have

$$\begin{gathered}L_1^* = G_1(\cdot) - \lambda_1^* M^* = -5x^* + u^* + v^* - \lambda_1^*(-2x^* + v^* - 2x^*u^*),\\ \frac{\partial L_1^*}{\partial x^*} = -5 - \lambda_1^*(-2 - 2u^*) = 0 \quad \Rightarrow \quad u^* = \frac{5 - 2\lambda_1^*}{2\lambda_1^*},\\ \frac{\partial L_1^*}{\partial u^*} = 1 - \lambda_1^*(-2x^*) = 0 \quad \Rightarrow \quad x^* = -\frac{1}{2\lambda_1^*},\\ \frac{\partial L_1^*}{\partial v^*} = 1 - \lambda_1^* = 0 \quad \Rightarrow \quad \lambda_1^* = 1 \quad \Rightarrow \quad u^* = \frac{3}{2}, \quad x^* = -\frac{1}{2}.\end{gathered}$$

Since $M^* = 0 \Rightarrow V^* = -5/2$. So, Rough Min-Max upper point is $\tilde{u}^* = (3/2, -5/2)$ and the optimal solution value of the cost function on upper approximation is $G_1^* = 1.5$.

By testing this point, we see that the lower approximation is satisfied. So the solution we have got is a surely optimal solution for player 1 and there is no need for a solution at the lower approximation.

For player 2: the game (2.22), (2.24) will be the following

$$\begin{gathered}\min_{v^* \in M^*} G_2(\cdot),\\ \text{S.T.}\\ M^* = -2x^* + v^* - 2x^*u^* = 0.\end{gathered}$$

Here we have

$$\begin{gathered}L_2^* = G_2(\cdot) - \lambda_2^* M^* = -3x^* + v^* + 2u^* - \lambda_2^*(-2x^* + v^* - 2x^*u^*),\\ \frac{\partial L_2^*}{\partial x^*} = -3 - \lambda_2^*(-2 - 2u^*) = 0 \quad \Rightarrow \quad u^* = \frac{3 - 2\lambda_2^*}{2\lambda_2^*},\\ \frac{\partial L_2^*}{\partial u^*} = 2 - \lambda_2^*(-2x^*) = 0 \quad \Rightarrow \quad x^* = -\frac{1}{\lambda_2^*},\\ \frac{\partial L_2^*}{\partial v^*} = 1 - \lambda_2^* = 0 \quad \Rightarrow \quad \lambda_2^* = 1 \quad \Rightarrow \quad u^* = \frac{1}{2}, \quad x^* = -1.\end{gathered}$$

Since $M^* = 0 \Rightarrow V^* = -3$. So, Rough Min-Max upper point is $\tilde{u}^* = (1/2, -3)$ and the optimal solution value of the cost function on upper approximation is $G_2^* = 1$. By testing this point, the lower approximation is not achieved. So the game for player 2 does not have a surely optimal solution and it will be resolved at the lower approximation.

Let us continue consideration (*lower approximation*).
For player 2: the game is the following

$$\begin{gathered}\min_{v_* \in M_*} G_2(\cdot),\\ S.T.\\ M_* = -7x_* + u_* - 4x_* v_* = 0.\end{gathered}$$

Here we have

$$\begin{aligned}L_{2_*} = G_2(\cdot) - \lambda_{2_*} M_* &= -3x_* + v_* + 2u_* - \lambda_{2_*}(-7x_* + u_* - 4x_* v_*),\\ \frac{\partial L_{2_*}}{\partial x_*} &= -3 - \lambda_{2_*}(-7 - 4v_*) = 0 \quad \Rightarrow \quad v_* = \frac{3 - 7\lambda_{2_*}}{4\lambda_{2_*}},\\ \frac{\partial L_{2_*}}{\partial v_*} &= 1 + \lambda_{2_*}(4x_*) = 0 \quad \Rightarrow \quad x^* = -\frac{1}{4\lambda_{2_*}},\\ \frac{\partial L_{2_*}}{\partial u_*} = 2 - \lambda_{2_*} = 0 \quad &\Rightarrow \quad \lambda_{2_*} = 2 \quad \Rightarrow \quad v_* = -\frac{11}{8}, \quad x_* = -\frac{1}{8}.\end{aligned}$$

Since $M_* = 0 \Rightarrow u_* = -3/16$. So, Rough Min-Max lower point is $\tilde{u}_* = (-3/16, -11/8)$ and the optimal solution value of the cost function on lower approximation is $G_{2_*} = -1.375$.

3. Parametric Rough Continuous Static game

A parametric study of a Rough Continuous Static game often provides a new insight into the Rough Continuous Static game. The parameter existence in Parametric Rough Continuous Static game (PRCSG) results in two cases, the first one: roughness is in the constraints and the parameter is in the cost function and the second case: the parameter is in the constraints and roughness is in the cost function. In this paper, the first case is what our study is about and the basic concepts of the parametric convex programming which are used here are established in Osman papers [5, 12] i.e. the solvability set, the stability set of the first kind.

3.1. The 1^{st} case of Parametric Rough Continuous Static game (PRCSG)

Here we consider the case when the parameters are in the cost function and roughness is in the constraints. It can be defined as:

$$\min G_i(u_i, P_i), \tag{3.1}$$
$$\text{S.T.}$$
$$m \in M, \tag{3.2}$$

where G_i is a cost function for each player $i = 1, \dots, r$ and M is the feasible region of the game which is a rough convex set defined by $M_* \subseteq M \subseteq M^*$, and P_i are real parameters, $i = 1, \dots, r$.

Definition 8. *The solvability set of the game* (3.1), (3.2) *denoted by S is defined by two sets $S_* \subseteq S \subseteq S^*$, where*

$$S_* = \{P_i \in R^r | \min_{m \in M_*} G_i(u_i, P_i) \text{ exists}\}, \quad S^* = \{P_i \in R^r | \min_{m \in M^*} G_i(u_i, P_i) \text{ exists}\}$$

for any $P_i \in S$. If $\bar{m}$ is a surely optimal solution then the stability of the 1^{st} kind can be defined and obtained.

Definition 9. *Suppose that $P_i \in S$ with a corresponding surely optimal solution $\bar{m} \in O_1(P_i)$ where*

$$O_1(P_i) = \{\bar{m} \in M_*|\ G_i(\bar{m}, P_i) = \min_{m \in M^*} G_i(u_i, P_i)\}.$$

*Then **the stability set of the** 1^{st} **kind** for the game (3.1), (3.2) corresponding to $\bar{m}$ denoted by $S_1(\bar{m})$ is defined by*

$$S_1(\bar{m}) = \{P_i \in S|\ G_i(\bar{m}, P_i) = \min_{m \in M^*} G_i(u_i, P_i)\}$$

for any $P_i \in S$. If $\bar{m}$ is a possible optimal solution then the stability of the 3^{rd} kind can be defined and obtained.

Definition 10. *Suppose that $P_i \in S$ with a corresponding lower optimal solution $\bar{m}_* \in O_2(P_i)$, where*

$$O_2(P_i) = \{\bar{m}_* \in M_*|\ G_i(\bar{m}_*, P_i) = \min_{m \in M_*} G_i(u_i, P_i)\}.$$

*Then **the possibly lower stability set** for the game (3.1), (3.2) corresponding to $\bar{m}_*$ denoted by $S_{2_*}(\bar{m}_*)$ is defined by*

$$S_{2_*}(\bar{m}_*) = \{P_i \in S|\ G_i(\bar{m}_*, P_i) = \min_{m \in M_*} G_i(u_i, P_i)\}.$$

Definition 11. *Suppose that $P_i \in S$ with a corresponding upper optimal solution $\bar{m}^* \in O_3(P_i)$ where*

$$O_3(P_i) = \{\bar{m}^* \in M^*|\ G_i(\bar{m}^*, P_i) = \min_{m \in M^*} G_i(u_i, P_i)\}.$$

*Then **the possibly upper stability set** for the game (3.1), (3.2) corresponding to $\bar{m}^*$ denoted by $S_2^*(\bar{m}^*)$ is defined by*

$$S_2^*(\bar{m}^*) = \{P_i \in S|\ G_i(\bar{m}^*, P_i) = \min_{m \in M^*} G_i(u_i, P_i)\}.$$

3.2. Further steps

The steps below lead us to obtain the stability set of the 1^{st} kind, the possibly upper and lower stability sets when the parameters are found in cost function and roughness is in the constraints:

1. Formulate the game and include all the parameters that need to be examined.

2. For player 1, begin with a certain $P_i \in S$ and use Proposition 2 to get the optimal solution at upper approximation.

3. If the point that is obtained from the previous step is a surely optimal solution point then formulate Proposition 2 at it to find the stability set of 1^{st} kind. Otherwise, for a possible optimal solution point use Propositions 1 and 2 to obtain the possibly lower and upper stability sets.

4. Repeat for each one of the remaining players.

Example 2. Consider the game

$$G_1(\cdot) = u^2 + v^2 + P_1 u - v, \tag{3.3}$$
$$G_2(\cdot) = u^2 + v^2 - 5u + P_2 v, \tag{3.4}$$
$$\text{S.T.}$$
$$(u + v \leq 1) \subseteq M(\cdot) \subseteq (u^2 + v^2 \leq 9,\ u + v \leq 5). \tag{3.5}$$

Solution 2 (Upper approximation). Using Min-Max concept we find the following results.

For player 1: Suppose $P_1 = -7$ and rewrite the game (3.3), (3.5) to be

$$\min_{u^* \in M^*} G_1(\cdot),$$
$$\text{S.T.}$$
$$M^* = ({u^*}^2 + {v^*}^2 \leq 9,\ u^* + v^* \leq 5).$$

We have here

$$L_1^* = {u^*}^2 + {v^*}^2 - 7u^* - v^* + \mu_1^*({u^*}^2 + {v^*}^2 - 9) + \mu_2^*(u^* + v^* - 5) = 0,$$
$$\frac{\partial L_1^*}{\partial u^*} = 2u^* - 7 + 2u^*\mu_1^* + \mu_2^* = 0, \quad \frac{\partial L_1^*}{\partial v^*} = 2v^* - 1 + 2v^*\mu_1^* + \mu_2^* = 0,$$
$$\mu_1^*({u^*}^2 + {v^*}^2 - 9) = 0, \quad \mu_2^*(u^* + v^* - 5) = 0.$$

When $\mu_1^* > 0$ and $\mu_2^* = 0$, we conclude

$$2u^* - 7 + 2u^*\mu_1^* = 0 \quad \Rightarrow \quad u^* = \frac{7}{2(1 + \mu_1^*)},$$

$$2v^* - 1 + 2v^*\mu_1^* = 0 \quad \Rightarrow \quad v^* = \frac{1}{2(1 + \mu_1^*)}.$$

Since $\mu_1^* > 0 \Rightarrow {u^*}^2 + {v^*}^2 - 9 = 0$. Then, $\mu_1^* = 0.178511 \Rightarrow (u^*, V^*) = (2.9698, 0.42445)$.

When $\mu_2^* > 0$ and $\mu_1^* = 0$, we conclude

$$2u^* - 7 + \mu_2^* = 0 \quad \Rightarrow \quad u^* = \frac{7 - \mu_2^*}{2},$$
$$2v^* - 1 + \mu_2^* = 0 \quad \Rightarrow \quad v^* = \frac{1 - \mu_2^*}{2}.$$

Since $\mu_2^* > 0 \Rightarrow u^* + v^* - 5 = 0$. Then, $\mu_2^* = -1$. This solution is refused.

So, the upper optimal solution point $(u^*, v^*) = (2.9698, 0.42445)$ is a surely optimal solution point then *the stability of the* 1^{st} *kind* will be obtained as follows:

$$2(2.9698) + P_1 + 2(2.9698)\mu_1^* = 0 \quad \Rightarrow \quad \mu_1^* = \frac{-5.94225 - P_1}{5.94228}.$$

Since $\mu_1^* > 0 \Rightarrow -5.94225 - P_1 > 0 \Rightarrow P_1 < -5.94225$.

So *the stability of the* 1^{st} *kind* is:

$$(P_1 < -5.94225).$$

For player 2: Suppose $P_2 = -4$ and rewrite the game (3.4), (3.5) to be

$$\min_{u^* \in M^*} G_2(\cdot),$$
$$\text{S.T.}$$
$$M^* = (u^{*^2} + v^{*^2} \leq 9, u^* + v^* \leq 5).$$

We have here

$$L_1^* = u^{*^2} + v^{*^2} - 5u^* - 4v^* + \mu_1^*(u^{*^2} + v^{*^2} - 9) + \mu_2^*(u^* + v^* - 5) = 0,$$
$$\frac{\partial L_1^*}{\partial u^*} = 2u^* - 5 + 2u^*\mu_1^* + \mu_2^* = 0, \quad \frac{\partial L_1^*}{\partial v^*} = 2v^* - 4 + 2v^*\mu_1^* + \mu_2^* = 0,$$
$$\mu_1^*(u^{*^2} + v^{*^2} - 9) = 0, \quad \mu_2^*(u^* + v^* - 5) = 0.$$

When $\mu_1^* > 0$ and $\mu_2^* = 0$, we have

$$2u^* - 5 + 2u^*\mu_1^* = 0 \Rightarrow u^* = \frac{5}{2(1+\mu_1^*)},$$
$$2v^* - 4 + 2v^*\mu_1^* = 0 \Rightarrow v^* = \frac{4}{2(1+\mu_1^*)}.$$

Since $\mu_1^* > 0 \Rightarrow u^{*^2} + v^{*^2} - 9 = 0$. Then, $\mu_1^* = 0.06718 \Rightarrow (u^*, V^*) = (2.3426, 1.874)$.

When $\mu_2^* > 0$ and $\mu_1^* = 0$, we have

$$2u^* - 5 + \mu_2^* = 0 \Rightarrow u^* = \frac{5 - \mu_2^*}{2},$$
$$2v^* - 4 + \mu_2^* = 0 \Rightarrow v^* = \frac{4 - \mu_2^*}{2}.$$

Since $\mu_2^* > 0 \Rightarrow u^* + v^* - 5 = 0$. Then, $\mu_2^* = -12$. This solution is refused.

So, the upper optimal solution point $(u^*, V^*) = (2.3426, 1.874)$ is a surely optimal solution point then *the stability of the* 1^{st} *kind* will be obtained as follows:

$$2(1.874) + P_2 + 2(1.874)\mu_1^* = 0 \quad \Rightarrow \quad \mu_1^* = \frac{-3.748 - P_2}{3.748}.$$

Since $\mu_1^* > 0 \Rightarrow -3.748 - P_2 > 0 \Rightarrow P_2 < -3.748$.

So *the stability of the* 1^{st} *kind* corresponds to the case

$$(P_2 < -3.748).$$

4. Conclusion

This paper presented a parametric study of the rough continuous static game when parameters exist in the crisp cost function and constraints are rough sets. Furthermore, the paper provided new concepts including the solvability set and the stability set of the 1^{st} kind and the possibly stability set in a rough environment. Also, the surely and possibly optimal solution sets of RCSG in the 1^{st} class are introduced. Future work may focus on the parametric study of the PRCSG when parameters are found in the constraints and the cost function is a rough function.

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HAHN'S PROBLEM WITH RESPECT TO SOME PERTURBATIONS OF THE RAISING OPERATOR $X - c$

Baghdadi Aloui

University of Gabes, Higher Institute of Industrial Systems of Gabes
Salah Eddine Elayoubi Str., 6033 Gabes, Tunisia

Baghdadi.Aloui@fsg.rnu.tn

Jihad Souissi

University of Gabes, Faculty of Sciences of Gabes
Erriadh Str., 6072 Gabes, Tunisia

jihadsuissi@gmail.com

Abstract: In this paper, we study the Hahn's problem with respect to some raising operators perturbed of the operator $X - c$, where c is an arbitrary complex number. More precisely, the two following characterizations hold: up to a normalization, the q-Hermite (resp. Charlier) polynomial is the only $H_{\alpha,q}$-classical (resp. $\mathcal{S}_\lambda$-classical) orthogonal polynomial, where $H_{\alpha,q} := X + \alpha H_q$ and $\mathcal{S}_\lambda := (X+1) - \lambda\tau_{-1}$.

Keywords: Orthogonal polynomials, Linear functional, $\mathcal{O}$-classical polynomials, Raising operators, q-Hermite polynomials, Charlier polynomials.

1. Introduction

Let $\mathcal{O}$ be a linear operator acting on the space of polynomials which sends polynomials of degree n to polynomials of degree $n + n_0$, where n_0 is a fixed integer ($n \geq 0$ if $n_0 \geq 0$ and $n \geq |n_0|$ if $n_0 < 0$). We call a sequence $\{P_n\}_{n\geq 0}$ of orthogonal polynomials $\mathcal{O}$-classical if $\{\mathcal{O}P_n\}_{n\geq 0}$ is also orthogonal.

In particular, if $\mathcal{O} = D$, the standard derivative, we recover the know family of classical orthogonal polynomials (Hermite, Laguerre, Bessel and Jacobi). This characterization is called Hahn's characterization (see [11, 18]) of the classical orthogonal polynomials. If $\mathcal{O} = H_q$, where

$$H_q f(x) = \frac{h_q f(x) - f(x)}{(q-1)x}, \quad q \neq 1, \quad h_q f(x) = f(qx),$$

we recover the so-called H_q-classical polynomials (for more details, see [12]). We can also cite [14], where the authors described the all D_ω-classical orthogonal polynomials, with

$$D_\omega f(x) := \frac{\tau_{-\omega} f(x) - f(x)}{w}, \quad \omega \neq 0, \quad \tau_{-\omega} f(x) = f(x+\omega).$$

The literature on these topics is extremely vast. See further examples in [1–5, 7, 8, 11, 12, 14].

In this paper we consider some *raising operators* related to the operator X. It is easy to see that the orthogonality is not preserved by X, then we can consider and study some perturbed operators. Here we consider the following two operators ($c = 0$ or $c = 1$):

$$H_{\alpha,q} := X + \alpha H_q \tag{1.1}$$

$$\mathcal{S}_\lambda := (X+1) - \lambda\tau_{-1}, \tag{1.2}$$

and we study the same problem, called Hahn's problem. More precisely, we find all orthogonal polynomial sequences $\{P_n\}_{n\geq 0}$ such that $\{\mathcal{O}P_n\}_{n\geq 0}$, $\mathcal{O} = H_{\alpha,q}$ or $\mathcal{S}_\lambda$, are also orthogonal. As a result, we conclude that the q-Hermite polynomial sequence is the only $H_{\alpha,q}$-classical sequence and the Charlier polynomial sequence is the only $\mathcal{S}_\lambda$-classical sequence.

The structure of the paper is the following. In Section 2, a basic background about forms of orthogonal polynomials is given. In Section 3, we show that, up to a dilatation, the q-Hermite (resp. Charlier) polynomial is the only $H_{\alpha,q}$-classical (resp. $\mathcal{S}_\lambda$-classical) orthogonal polynomial. In Section 4, we give a conclusion and describe some prospects.

2. Preliminaries

Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients and $\mathbb{P}'$ be its dual space, whose elements are *forms*. We denote by $\langle u, p\rangle$ the action of $u \in \mathbb{P}'$ on $p \in \mathbb{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the moments of u. Let us define the following operations in $\mathbb{P}'$. For any form u, any polynomial f, and any $(a, b, c) \in \mathbb{C}\backslash\{0\}\times\mathbb{C}^2$, let $Du = u'$, fu, $(x-c)^{-1}u$, $\tau_{-b}u$ and $h_a u$ be the forms defined by duality, [16]:

$$\langle fu, p\rangle := \langle u, fp\rangle, \quad \langle u', p\rangle := -\langle u, p'\rangle, \quad (fu)' = f'u + fu',$$
$$\langle h_a u, p\rangle := \langle u, p(ax)\rangle, \quad \langle \tau_{-b}u, p\rangle := \langle u, p(x-b)\rangle,$$
$$\langle (x-c)^{-1}u,\ p\rangle := \Big\langle u,\ \frac{p(x)-p(c)}{x-c}\Big\rangle, \quad p \in \mathbb{P}.$$

A form u is called *normalized* if it satisfies $(u)_0 = 1$. We assume that the forms used in this paper are normalized.

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials (MPS) with $\deg P_n = n$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathbb{P}'$, defined by $\langle u_n, P_m\rangle = \delta_{n,m}$, $n,\ m \geq 0$. Notice that u_0 is said to be the canonical functional associated with the MPS $\{P_n\}_{n\geq 0}$. The sequence $\{P_n\}_{n\geq 0}$ is called symmetric when $P_n(-x) = (-1)^n P_n(x)$, $n \geq 0$.

Let us recall the following result [17].

Lemma 1. *For any $u \in \mathbb{P}'$ and any integer $m \geq 1$, the following statements are equivalent*:

(i) $\langle u, P_{m-1}\rangle \neq 0$, $\langle u, \quad P_n\rangle = 0$, $n \geq m$.

(ii) $\exists \lambda_\nu \in \mathbb{C}$, $0 \leq \nu \leq m-1$, $\lambda_{m-1} \neq 0$ *such that* $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.

As a consequence, the dual sequence $\{u_n^{[1]}\}_{n\geq 0}$ of $\{P_n^{[1]}\}_{n\geq 0}$ where

$$P_n^{[1]}(x) := (n+1)^{-1} P'_{n+1}(x), \quad n \geq 0,$$

is given by

$$Du_n^{[1]} = -(n+1)u_{n+1}, \quad n \geq 0.$$

Similarly, the dual sequence $\{\tilde{u}_n\}_{n\geq 0}$ of $\{\tilde{P}_n\}_{n\geq 0}$, where

$$\tilde{P}_n(x) := a^{-n} P_n(ax+b)$$

with $(a, b) \in \mathbb{C}\backslash\{0\}\times\mathbb{C}$, is given by

$$\tilde{u}_n = a^n (h_{a^{-1}} \circ \tau_{-b}) u_n,\ n \geq 0.$$

The form u is called *regular* if we can associate with it a sequence $\{P_n\}_{n\geq 0}$ such that

$$\langle u, P_n P_m\rangle = r_n \delta_{n,m}, \quad n,\ m \geq 0, \quad r_n \neq 0, \quad n \geq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is then called a monic *orthogonal* polynomial sequence (MOPS) with respect to u. Note that $u=(u)_0u_0$, with $(u)_0\neq 0$. When u is regular, let F be a polynomial such that $Fu=0$. Then $F=0$, [16].

Proposition 1 [16]. *Let $\{P_n\}_{n\geq 0}$ be a MPS with $\deg P_n = n$, $n\geq 0$, and let $\{u_n\}_{n\geq 0}$ be its dual sequence. The following statements are equivalent.*

(i) $\{P_n\}_{n\geq 0}$ *is orthogonal with respect to* u_0.

(ii) $u_n = \langle u_0, P_n^2\rangle^{-1} P_n u_0, \quad n\geq 0.$

(iii) $\{P_n\}_{n\geq 0}$ *satisfies the three-term recurrence relation*

$$\begin{cases} P_0(x)=1, \quad P_1(x)=x-\beta_0, \\ P_{n+2}(x)=(x-\beta_{n+1})P_{n+1}(x)-\gamma_{n+1}P_n(x), \quad n\geq 0, \end{cases} \tag{2.1}$$

where $\beta_n=\langle u_0, xP_n^2\rangle\langle u_0,P_n^2\rangle^{-1}$, $n\geq 0$ *and* $\gamma_{n+1}=\langle u_0,P_{n+1}^2\rangle\langle u_0,P_n^2\rangle^{-1}\neq 0, \quad n\geq 0.$

If $\{P_n\}_{n\geq 0}$ is a MOPS with respect to the regular form u_0, then $\{\tilde{P}_n\}_{n\geq 0}$ is a MOPS with respect to the regular form $\tilde{u}_0=(h_{a^{-1}}\circ\tau_{-b})u_0$, and satisfies [15]

$$\begin{cases} \tilde{P}_0(x)=1, \quad \tilde{P}_1(x)=x-\tilde{\beta}_0, \\ \tilde{P}_{n+2}(x)=(x-\tilde{\beta}_{n+1})\tilde{P}_{n+1}(x)-\tilde{\gamma}_{n+1}\tilde{P}_n(x), \quad n\geq 0, \end{cases}$$

where $\tilde{\beta}_n=a^{-1}(\beta_n-b)$ and $\tilde{\gamma}_{n+1}=a^{-2}\gamma_{n+1}$.

A MOPS $\{p_n\}_{n\geq 0}$ is called D-classical, if $\{Dp_n\}_{n\geq 0}$ is also orthogonal (*Hermite, Laguerre, Bessel or Jacobi*), [10, 11]. Moreover, if $\{p_n\}_{n\geq 0}$ is orthogonal with respect to u_0, then there exists a monic polynomial ϕ with $\deg\phi\leq 2$ and a polynomial ψ with $\deg\psi=1$ such that u_0 satisfies a *Pearson's equation* (PE) [15]

$$D(\phi u_0)+\psi u_0=0.$$

Any shift leaves invariant the D-classical character. Indeed, the shifted linear functional $\tilde{u}=(h_{a^{-1}}\circ\tau_{-b})u$ fulfills the equation

$$(\widetilde{\Phi}\tilde{u})'+\widetilde{\Psi}\tilde{u}=0,$$

where (see [15, 16])

$$\widetilde{\Phi}(x)=a^{-t}\Phi(ax+b) \quad \text{and} \quad \widetilde{\Psi}(x)=a^{1-t}\Psi(ax+b).$$

3. Hahn's problem with respect to some perturbations of the raising operator $X-c$

Clearly, the orthogonality is not preserved by the operator $X-c$, which is given by

$$(X-c)\big(f(x)\big)=(x-c)f(x), \quad f\in\mathbb{P}.$$

Our goal, in this section is to describe all $\mathcal{O}$-classical orthogonal polynomials. More precisely, we find all orthogonal polynomial sequences $\{P_n\}_{n\geq 0}$ such that $\{\mathcal{O}P_n\}_{n\geq 0}$ are also orthogonal, where $\mathcal{O}=H_{\alpha,q}$ or $\mathcal{O}=\mathcal{S}_\lambda$ are the operators defined by (1.1) and (1.2). This operators are two perturbations of the operator $X-c$ where $c=0$ and $c=1$.

3.1. Orthogonal polynomials via raising operator $X - \alpha H_q$

Let us introduce the following lemma.

Lemma 2 [12]. *The following properties hold*

$$H_q(fg)(x) = f(x)(H_q g)(x) + g(x)(H_q f)(x) + (q-1)x(H_q f)(x)(H_q g)(x), \quad f,\, g \in \mathcal{P},$$
$$H_q(fu) = (h_{q^{-1}} f) H_q u + q^{-1}(H_{q^{-1}} f)u, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}'.$$

where

$$H_q f(x) = \frac{h_q f(x) - f(x)}{(q-1)x}, \quad q \neq 1 \quad \textit{and} \quad h_q f(x) = f(qx).$$

Now, recall the operator

$$\begin{aligned} H_{\alpha,q} : \mathbb{P} &\longrightarrow \mathbb{P}, \\ f &\longmapsto H_{\alpha,q}(f) := xf + \alpha H_q(f). \end{aligned}$$

Definition 1. *We call a sequence $\{P_n\}_{n\geq 0}$ of orthogonal polynomials $H_{\alpha,q}$-classical if there exists a sequence $\{Q_n\}_{n\geq 0}$ of orthogonal polynomials such that $H_{\alpha,q} P_n = Q_{n+1},\ n \geq 0$.*

For any MPS $\{P_n\}_{n\geq 0}$ we define the MPS $\{Q_n\}_{n\geq 0}$, given by

$$Q_{n+1}(x) := H_{\alpha,q} P_n(x),\ n \geq 0,$$

or equivalently

$$Q_{n+1}(x) := xP_n(x) + \alpha(H_q P_n)(x), \quad n \geq 0, \tag{3.1}$$

with initial value $Q_0(x) = 1$.

Our next goal is to describe all the $H_{\alpha,q}$-classical polynomial sequences. Note that, we need $\alpha \neq 0$ to ensure that $\{Q_n\}_{n\geq 0}$ is an orthogonal sequence. Indeed, if we suppose that $\alpha = 0$, the relation (3.1) becomes, for $x = 0$, $Q_{n+1}(0) = 0,\ n \geq 0$, which contradicts the orthogonality of $\{Q_n\}_{n\geq 0}$.

Clearly, the operator $H_{\alpha,q}$ raises the degree of any polynomial. Such operator is called *raising operator* [9, 13, 19]. By transposition of the operator $H_{\alpha,q}$, we get

$${}^t H_{\alpha,q} = X - \alpha H_q. \tag{3.2}$$

Denote by $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ the dual basis in $\mathbb{P}'$ corresponding to $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively. Then, according to Lemma 1 and (3.2), the relation

$$xv_{n+1} - \alpha H_q(v_{n+1}) = u_n, \quad n \geq 0, \tag{3.3}$$

holds. Assume that $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are MOPS satisfying

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad \gamma_{n+1} \neq 0, \quad n \geq 0, \end{cases} \tag{3.4}$$

$$\begin{cases} Q_0(x) = 1,\ Q_1(x) = x - \rho_0, \\ Q_{n+2}(x) = (x - \rho_{n+1})Q_{n+1}(x) - \varrho_{n+1}Q_n(x), \quad \varrho_{n+1} \neq 0, \quad n \geq 0. \end{cases} \tag{3.5}$$

Next, a first result will be deduced as a consequence of the relations (3.1), (3.4) and (3.5).

Proposition 2. *The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ satisfy the following finite type relation*

$$P_n(x) + (q-1)xH_q(P_n)(x) = q^n Q_n(x), \quad n \geq 0.$$

P r o o f. Using (3.4), we obtain

$$H_q(P_{n+2})(x) = H_q\big((x-\beta_{n+1})P_{n+1}\big)(x) - \gamma_{n+1}H_q(P_n)(x), \quad n \geq 0.$$

According to the Lemma 2, we obtain for $n \geq 0$

$$H_q(P_{n+2})(x) = (x-\beta_{n+1})H_q(P_{n+1})(x) + P_{n+1}(x) + (q-1)xH_q(P_{n+1})(x) - \gamma_{n+1}H_q(P_n)(x),$$

or equivalently

$$xP_{n+2}(x) + \alpha(H_qP_{n+2})(x) = Q_{n+3}(x), \quad n \geq 0,$$

which gives us for $n \geq 0$

$$(x-\beta_{n+1})xP_{n+1}(x)+\alpha(qx-\beta_{n+1})(H_qP_{n+1})(x)-\gamma_{n+1}\big(xP_n(x)+\alpha(H_qP_n)(x)\big)+\alpha P_{n+1}(x)=Q_{n+3}(x).$$

We use (3.1) and the last equation becomes for $n \geq 0$

$$(x-\beta_{n+1})Q_{n+2}(x) + \alpha(q-1)x(H_qP_{n+1})(x) - \gamma_{n+1}Q_{n+1}(x) + \alpha P_{n+1}(x) = Q_{n+3}(x). \tag{3.6}$$

Inserting (3.5) in (3.6), we obtain

$$\alpha P_{n+1}(x) + \alpha(q-1)x(H_qP_{n+1})(x) = \big(\beta_{n+1}-\rho_{n+2}\big)Q_{n+2}(x) + \big(\gamma_{n+1}-\varrho_{n+2}\big)Q_{n+1}(x), \quad n \geq 0.$$

In fact, this result is valid for $n+1$ replaced by n. More precisely, we have for all $n \geq 0$

$$\alpha P_n(x) + \alpha(q-1)x(H_qP_n)(x) = \big(\beta_n-\rho_{n+1}\big)Q_{n+1}(x) + \big(\gamma_n-\varrho_{n+1}\big)Q_n(x),$$

with the convention $\gamma_0 = 0$. By comparing the degrees in the previous equation, we get $\beta_n = \rho_{n+1}$, $n \geq 0$ and $\alpha q^n = \gamma_n - \varrho_{n+1}$, $n \geq 0$. Hence the desired result is proven. □

Note that, for $n = 0$ the relation (3.1) gives $\rho_0 = 0$, for $n = 1$ the Proposition 2 gives

$$(x-\beta_0) + (q-1)x = qx - \rho_0 = qx,$$

then $\beta_0 = \rho_1 = 0$. Now we establish, in the next lemma, an algebraic relation between the forms u_0 and v_0.

Lemma 3. *The forms u_0 and v_0 satisfy the following relation*

$$v_0 - (q-1)H_q(xv_0) = u_0. \tag{3.7}$$

P r o o f. According to Proposition 2 we obtain

$$\big\langle v_0 - (q-1)H_q(xv_0),\, P_n\big\rangle = 0, \quad n \geq 1. \tag{3.8}$$

On the other hand,

$$\big\langle v_0 - (q-1)H_q(xv_0),\, P_0\big\rangle = 1,$$

since $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to the form v_0, where v_0 is supposed normalized. According to Lemma 1 and using (3.8), we obtain the desired result. □

Based on the last lemma, we can state the following theorem.

Theorem 1. *The form v_0 satisfies the following Pearson's equation*

$$(H_q v_0) - \frac{1}{\alpha} x v_0 = 0, \tag{3.9}$$

and then the scaled q-Hermite polynomial sequence is the only $H_{\alpha,q}$-classical sequence.

P r o o f. According to Proposition 1 (ii), the relation (3.3) can be written as follows

$$x Q_{n+1}(x) v_0 - \alpha H_q(Q_{n+1} v_0) = \lambda_n P_n(x) u_0, \quad n \geq 0, \tag{3.10}$$

where

$$\lambda_n := \langle v_0, Q_{n+1}^2 \rangle \langle u_0, P_n^2 \rangle^{-1}, \quad n \geq 0.$$

Making $n = 0$ in (3.10), we get

$$x^2 v_0 - \alpha H_q(x v_0) = -\alpha u_0, \quad \big(Q_1(x) = x, \quad \varrho_1 = -\alpha\big).$$

Substituting this relation in (3.7), we obtain

$$q H_q(x v_0) - \frac{1}{\alpha}(x^2 + \alpha) v_0 = 0.$$

Note that we have $q H_q(x v_0) = x(H_q v_0) + v_0$, then

$$(H_q v_0) - \frac{1}{\alpha} x v_0 = 0, \tag{3.11}$$

which gives

$$\Big((H_q v_0) - \frac{1}{\alpha} x v_0\Big)_{n+1} = 0, \quad n \geq 0,$$

and then

$$(v_0)_{n+2} = -\alpha [n]_q (v_0)_n, \quad n \geq 0.$$

Moreover, $(v_0)_1 = \rho_1 = 0$, hence $(v_0)_{2n+1} = 0,\ n \geq 0$. We can conclude that $\{Q_n\}_{n \geq 0}$ is symmetric. Using the Proposition 2, we obtain

$$Q_n(x) = q^{-n} P_n(qx), \quad n \geq 0.$$

Then we also conclude that $\{P_n\}_{n \geq 0}$ is symmetric. Moreover, the relation (3.11) corresponds to a Pearson's equation of q-Hermite linear functional, hence $Q_n(x)$ is the q-Hermite polynomial. In addition, we have $Q_n(x) = q^{-n} P_n(qx),\ n \geq 0$, then $P_n(x)$ is the scaled q-Hermite polynomial. □

3.2. Orthogonal polynomials via raising operator $(X+1) - \lambda \tau_{-1}$

In this part, we use the following lemma.

Lemma 4 [1]. *The following properties hold*

$$D_w(fg)(x) = f(x)(D_w g)(x) + g(x)(D_w f)(x) + w(D_w f)(x)(D_w g)(x), \quad f, g \in \mathcal{P},$$
$$D_{-w}(fu) = g(D_{-w} u) + (D_{-w} g)(\tau_w u), \quad f \in \mathcal{P}, \quad u \in \mathcal{P}',$$
$$\tau_b \circ D_w = D_w \circ \tau_b \text{ in } \mathcal{P} \text{ and } \mathcal{P}', \quad b \in \mathbb{C},$$

where

$$D_\omega f(x) := \frac{\tau_{-\omega} f(x) - f(x)}{w}, \quad \omega \neq 0 \quad \text{and} \quad \tau_{-\omega} f(x) = f(x + \omega).$$

Recall the operator

$$\begin{aligned} \mathcal{S}_\lambda : \mathbb{P} &\longrightarrow \mathbb{P}, \\ f &\longmapsto \mathcal{S}_\lambda(f) = (x+1)(f) - \lambda \tau_{-1} f. \end{aligned}$$

Definition 2. *We call a sequence $\{P_n\}_{n\geq 0}$ of orthogonal polynomials $\mathcal{S}_\lambda$-classical if there exists a sequence $\{Q_n\}_{n\geq 0}$ of orthogonal polynomials such that $\mathcal{S}_\lambda P_n = Q_{n+1},\ n \geq 0$.*

For any MPS $\{P_n\}_{n\geq 0}$ we define the MPS $\{Q_n\}_{n\geq 0}$, given by

$$Q_{n+1}(x) := \mathcal{S}_\lambda P_n(x),\ n \geq 0, \tag{3.12}$$

or equivalently

$$Q_{n+1}(x) := (x+1)P_n(x) - \lambda P_n(x+1),\ n \geq 0, \tag{3.13}$$

with initial value $Q_0(x) = 1$.

Our next goal is to describe all the $\mathcal{S}_\lambda$-classical polynomial sequences. Note that, we need $\lambda \neq 0$ to ensure that $\{Q_n\}_{n\geq 0}$ is an orthogonal sequence. Indeed, if we suppose that $\lambda = 0$, the relation (3.13) becomes, for $x = -1$, $Q_{n+1}(-1) = 0,\ n \geq 0$, which contradicts the orthogonality of $\{Q_n\}_{n\geq 0}$.

Clearly, the operator $\mathcal{S}_\lambda$ raises the degree of any polynomial. Such operator is called a *raising operator* [9, 13, 19]. By transposition of the operator $\mathcal{S}_\lambda$, we get

$${}^t\mathcal{S}_\lambda = (X+1) - \lambda \tau_1. \tag{3.14}$$

Denote by $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ the dual basis in $\mathbb{P}'$ corresponding to $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively. Then, according to Lemma 1 and (3.14), the relation

$$(x+1)v_{n+1} - \lambda \tau_1 v_{n+1} = u_n, \quad n \geq 0,$$

holds. Assume that $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are MOPS satisfying

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad \gamma_{n+1} \neq 0, \quad n \geq 0, \end{cases} \tag{3.15}$$

$$\begin{cases} Q_0(x) = 1, \quad Q_1(x) = x - \rho_0, \\ Q_{n+2}(x) = (x - \rho_{n+1})Q_{n+1}(x) - \varrho_{n+1}Q_n(x), \quad \varrho_{n+1} \neq 0, \quad n \geq 0. \end{cases} \tag{3.16}$$

Next, a first result will be deduced as a consequence of the relations (3.13), (3.15) and (3.16).

Proposition 3. *The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ satisfy the following finite type relation*

$$Q_n(x) = \tau_{-1}P_n(x), \quad n \geq 0,$$

with

$$\begin{aligned} \rho_{n+1} &= \beta_n, \quad n \geq 0, \\ \varrho_{n+1} &= \gamma_n + \lambda, \quad n \geq 0, \end{aligned}$$

and with the convention $\gamma_0 = 0$.

P r o o f. Multiplying (3.15) by $x+1$, we obtain

$$(x+1)P_{n+2}(x) = (x-\beta_{n+1})(x+1)P_{n+1}(x) - \gamma_{n+1}(x+1)P_n(x), \quad n \geq 0.$$

Applying $\lambda\tau_{-1}$ to the (3.15) and taking the difference between the two resulting equations, we obtain

$$\begin{aligned}(x+1)P_{n+2}(x) - \lambda(\tau_{-1}P_{n+2})(x) = (x-\beta_{n+1})\big((x+1)P_{n+1}(x) - \lambda(\tau_{-1}P_{n+1})(x)\big)\\ -\gamma_{n+1}\big((x+1)P_n(x) - \lambda(\tau_{-1}P_n)(x)\big) - \lambda P_{n+1}(x+1).\end{aligned}$$

Substituting (3.13) in the last equation, we get

$$Q_{n+3}(x) = (x-\beta_{n+1})Q_{n+2}(x) - \gamma_{n+1}Q_{n+1}(x) - \lambda P_{n+1}(x+1), \quad n \geq 0.$$

Using the three-term recurrence relation (3.16), we get

$$\lambda P_{n+1}(x+1) = \big(\rho_{n+2} - \beta_{n+1}\big)Q_{n+2}(x) + \big(\varrho_{n+2} - \gamma_{n+1}\big)Q_{n+1}(x), \quad n \geq 0.$$

In fact, this result is valid for $n+1$ replaced by n. Then, by comparing the degrees in the previous equation, we get $\rho_{n+1} = \beta_n$ and $\varrho_{n+1} = \gamma_n + \lambda$, $n \geq 0$, and $Q_n(x) = \tau_{-1}P_n(x)$, $\quad n \geq 0$, with the convention $\gamma_0 = 0$. □

The following result is a straightforward consequence of Proposition 3.

Lemma 5. *The forms u_0 and v_0 satisfy the following relation*

$$\tau_1 v_0 = u_0.$$

According to Lemma 5, and based on some characterizations of Charlier polynomials [1], we can state the following theorem.

Theorem 2. *The Charlier polynomial sequence $\{C_n^\lambda(x)\}_{n\geq 0}$ where $\lambda > 0$, is the only $\mathcal{S}_\lambda$-classical orthogonal sequence. More precisely, we have for $n \geq 0$:*

$$P_n(x) = C_n^\lambda(x), \tag{3.17}$$

$$Q_n(x) = C_n^\lambda(x+1). \tag{3.18}$$

P r o o f. Assume that $\{P_n\}_{n\geq 0}$ is a monic $\mathcal{S}_\lambda$-classical orthogonal sequence. Then there exists a monic orthogonal sequence $\{Q_n\}_{n\geq 0}$ satisfying (3.13), which gives by transposition the following system

$$\langle v_0, (x+1)P_n(x) - \lambda P_n(x+1)\rangle = \langle v_0, Q_{n+1}(x)\rangle = 0, \quad n \geq 0.$$

But the left hand side reads as

$$\langle (x+1)v_0 - \lambda\tau_1 v_0, P_n(x)\rangle = 0, \quad n \geq 0.$$

In other words,

$$(x+1)v_0 - \lambda\tau_1 v_0 = 0.$$

Applying the operator τ_{-1}, we obtain

$$(x+2)\tau_{-1}v_0 - \lambda v_0 = 0.$$

Equivalently,

$$(x+1)\tau_{-1}v_0 + \tau_{-1}v_0 - (x+1)v_0 + (x+1)v_0 - \lambda v_0 = 0,$$

which also gives

$$(x+1)[\tau_{-1}v_0 - v_0] + \tau_{-1}v_0 + (x+1)v_0 - \lambda v_0 = 0,$$

or equivalently

$$(x+1)D_1 v_0 + \tau_{-1}v_0 + (x+1)v_0 - \lambda v_0 = 0.$$

By using Lemma 4, the last relation becomes

$$D_1\big(x(\tau_1 v_0)\big) + (x-\lambda)(\tau_1 v_0) = 0,$$

which means that $v_0 = \tau_{-1}C(\lambda)$, where $C(\lambda)$ is the Charlier form with $\lambda > 0$. In addition, using the Proposition 3, we obtain that $P_n(x) = C_n^\lambda(x)$ are the monic Charlier polynomials and then

$$Q_n(x) = C_n^\lambda(x+1), \quad n \geq 0.$$

□

4. Conclusion and prospects

We described Hahn's problem for some perturbed raising operators of the operator $X - c$ using the Pearson equation, which is satisfied by the corresponding linear functionals. Indeed, we have proved that the q-Hermite (resp. Charlier) polynomial is the only $H_{\alpha,q}$-classical (resp. $\mathcal{S}_\lambda$-classical) orthogonal polynomial, where $H_{\alpha,q} := X + \alpha H_q$ and $\mathcal{S}_\lambda := (X+1) - \lambda\tau_{-1}$.

Now, using (3.17), (3.18) and (3.12), we obtain

$$\mathcal{S}_\lambda C_n^\lambda(x) = C_{n+1}^\lambda(x+1), \quad n \geq 0,$$

which gives, by induction, the following formula

$$\mathcal{S}_\lambda^{(m)} C_n^\lambda(x) = C_{n+m}^\lambda(x+m), \quad n \geq 0, \tag{4.1}$$

where $\mathcal{S}_\lambda^{(m)} = \mathcal{S}_\lambda^{(m)} \circ \cdots \circ \mathcal{S}_\lambda^{(m)}$.

Making $n = 0$ in (4.1) we get

$$\mathcal{S}_\lambda^{(m)}(1) = C_m^\lambda(x+m), \quad m \geq 0.$$

For prospects, we can replace the operator H_q in Subsection 3.1 by the Dunkl operator ($T_\mu := D + 2\mu H_{-1}$, see [6]) and study the same problem. Indeed, we have [6]

$$\Big(X - \frac{1}{2}T_\mu\Big)H_n^\mu(x) = \frac{\gamma_\mu(n+1)}{2\gamma_\mu(n)(n+1)}H_{n+1}^\mu(x), \quad n \geq 0, \tag{4.2}$$

where $H_n^\mu(x)$ is the monic generalized Hermite polynomial and where $\gamma_\mu(n)$ is defined by

$$\gamma_\mu(2m) = \frac{2^{2m} m!\,\Gamma(m+\mu+1/2)}{\Gamma(\mu+1/2)}, \quad \text{and} \quad \gamma_\mu(2m+1) = \frac{2^{2m+1} m!\,\Gamma(m+\mu+1/2)}{\Gamma(\mu+3/2)}.$$

In view of (4.2), we can say that $\{H_n^\mu\}_{n\geq 0}$ is an $\mathcal{O}$-classical polynomial sequence, since it fulfills Hahn's property relatively to the raising operator

$$\mathcal{O} := X - \frac{1}{2}T_\mu,$$

i.e., it is an orthogonal polynomial sequence whose sequence of $\mathcal{O}$-derivatives is also orthogonal.

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GENERALIZED ORDER (α, β) ORIENTED SOME GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTIONS

Tanmay Biswas

Rajbari, Rabindrapally,
R.N. Tagore Road, P.O.-Krishnagar, Dist-Nadia, PIN-741101, West Bengal, India
tanmaybiswas_math@rediffmail.com

Chinmay Biswas

Department of Mathematics, Nabadwip Vidyasagar College,
Nabadwip, Dist.-Nadia, PIN-741302, West Bengal, India
chinmay.shib@gmail.com

Abstract: In this paper we establish some results relating to the growths of composition of two entire functions with their corresponding left and right factors on the basis of their generalized order (α, β) and generalized lower order (α, β) where α and β are continuous non-negative functions on $(-\infty, +\infty)$.

Keywords: Entire function, Growth, Composition, Generalized order (α, β), Generalized lower order (α, β).

1. Introduction, definitions and notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ and the maximum term $\mu_f(r)$ of

$$f = \sum_{n=0}^{\infty} a_n z^n$$

on $|z| = r$ are defined as

$$M_f = \max_{|z|=r} |f(z)|, \quad \mu_f(r) = \max_{n \geq 0} (|a_n| r^n)$$

respectively. We use the standard notations and definitions of the theory of entire functions which are available in [11] and [12], and therefore we do not explain those in details. For $x \in [0, \infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ be the set of all positive integers, define iterations of the exponential and logarithmic functions as

$$\exp^{[k]} x = \exp(\exp^{[k-1]} x), \quad \log^{[k]} x = \log(\log^{[k-1]} x),$$

with convention that

$$\log^{[0]} x = x, \quad \log^{[-1]} x = \exp x, \quad \exp^{[0]} x = x, \quad \exp^{[-1]} x = \log x.$$

Now considering this, let us recall that Juneja et al. [5] defined the (p, q)-th order and (p, q)-th lower order of an entire function, respectively, as follows:

Definition 1 [5]. *The (p,q)-th order and (p,q)-th lower order of an entire function f are defined as*:

$$\begin{matrix} \rho^{(p,q)}(f) \\ \lambda^{(p,q)}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

where p and q always denote positive integers with $p \geq q$.

Extending the notion (p,q)-th order, recently Shen et al. [6] introduced the new concept of $[p,q]-\varphi$ order of an entire function where $p \geq q$. Later on, combining the definition of (p,q)-order and $[p,q]-\varphi$ order, Biswas (see, e.g., [1]) redefined the (p,q)-order of an entire function without restriction $p \geq q$.

However the above definition is very useful for measuring the growth of entire functions. If $p=l$ and $q=1$ then we write $\rho^{(l,1)}(f)=\rho^{(l)}(f)$ and $\lambda^{(l,1)}(f)=\lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire function f. For details about generalized order one may see [8]. Also for $p=2$ and $q=1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire function f. Recently, Chyzhykov et al. [3] showed that both the definitions of generalized order and (p,q)-order have the disadvantage that they do not cover arbitrary growth (see [3, Example 1.4]).

Taking this into account, let L be a class of continuous non-negative on $(-\infty,+\infty)$ function α such that

$$\alpha(x)=\alpha(x_0)\geq 0, \quad \text{for} \quad x\leq x_0 \quad \text{with} \quad \alpha(x)\uparrow+\infty \quad \text{as} \quad x\to+\infty$$

and

$$\alpha\big((1+o(1))x\big)=(1+o(1))\alpha(x) \quad \text{as} \quad x\to+\infty.$$

We say that $\alpha\in L^0$, if $\alpha\in L$ and

$$\alpha(cx)=(1+o(1))\alpha(x) \quad \text{as} \quad x_0\leq x\to+\infty$$

for each $c\in(0,+\infty)$, i.e., α is slowly increasing function. Clearly $L^0\subset L$.

Further we assume that throughout the present paper α, α_1, α_2, β, β_1 and β_2 always denote the functions belonging to L^0.

Considering this, the value

$$\rho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log M_f(r))}{\beta(\log r)} \quad (\alpha\in L, \quad \beta\in L)$$

is called [7] the generalized order (α,β) of an entire function f. For details about the generalized order (α,β) one may see [7]. During the past decades, several authors made close investigations on the properties of entire functions related to the generalized order (α,β) in some different direction. For the purpose of further applications, Biswas et al. [2] rewrite the definition of the generalized order (α,β) of entire function in the following way after giving a minor modification to the original definition (e.g. see, [7]) which considerably extend the definition of φ-order of entire function introduced by Chyzhykov et al. [3]:

Definition 2 [2]. *The generalized order (α,β) and the generalized lower order (α,β) of an entire function f are defined as*:

$$\begin{matrix} \rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(M_f(r))}{\beta(r)}.$$

Definition 1 is a special case of Definition 2 for $\alpha(r) = \log^{[p]} r$ and $\beta(r) = \log^{[q]} r$.
Since for $0 \le r < R$,

$$\mu_f(r) \le M_f(r) \le \frac{R}{R-r}\mu_f(R) \quad \text{(cf. [10])},$$

it is easy to see that

$$\begin{matrix} \rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\mu_f(r))}{\beta(r)} \quad \text{(also see [2])}.$$

In the paper we would like to establish some newly developed results based on the comparative growth of composite entire functions on the basis of their generalized order (α, β) and generalized lower order (α, β).

2. Known results

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [4]. *Let f and g are any two entire functions with $g(0) = 0$. Also let b satisfy $0 < b < 1$ and $c(b) = (1-b)^2/(4b)$. Then for all sufficiently large values of r, we have*

$$M_f(c(b)M_g(br)) \le M_{f\circ g}(r) \le M_f(M_g(r)).$$

In addition if $b = 1/2$, then for all sufficiently large values of r, the inequality is true

$$M_{f\circ g}(r) \ge M_f\Big(\frac{1}{8}M_g\left(\frac{r}{2}\right)\Big).$$

Lemma 2 [9]. *Let f and g be entire functions. Then for every $\delta > 1$ and $0 < r < R$, we have*

$$\mu_{f\circ g}(r) \le \frac{\delta}{\delta-1}\mu_f\Big(\frac{\delta R}{R-r}\mu_g(R)\Big).$$

Lemma 3 [9]. *If f and g are any two entire functions. Then for all sufficiently large values of r, the estimate is true*

$$\mu_{f\circ g}(r) \ge \frac{1}{2}\mu_f\Big(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\Big).$$

3. Main results

In this section we present the main results of the paper.

Theorem 1. *Let f and g be any two entire functions such that*

$$0 < \lambda_{(\alpha_1,\beta_1)}[f] \le \rho_{(\alpha_1,\beta_1)}[f] < +\infty, \quad \textit{and} \quad \rho_{(\alpha_2,\beta_2)}[g] > 0.$$

If $\alpha_2(\beta_1^{-1}(r)) \in L_1$, then we have the estimate

$$\limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r)))))}{\alpha_1(\mu_f(\beta_1^{-1}(r)))} \ge \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}.$$

P r o o f. From the definition of $\rho_{(\alpha_1,\beta_1)}[f]$, we get for all sufficiently large values of r that

$$\alpha_1(\mu_f(\beta_1^{-1}(r))) \leq (\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon)r. \tag{3.1}$$

Further in view of the first part of Lemma 3, it follows for all sufficiently large values of r that

$$\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r))) \geq (1 + o(1))(\lambda_{(\alpha_1,\beta_1)}[f] - \varepsilon)\beta_1\Big(\mu_g\Big(\frac{\beta_2^{-1}(r)}{4}\Big)\Big).$$

Since $\alpha_2(\beta_1^{-1}(r)) \in L_1$, we obtain from above for a sequence of values of r tending to infinity that

$$\begin{aligned} \alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r))))) &\geq (1 + o(1))\alpha_2\Big(\mu_g\Big(\frac{\beta_2^{-1}(r)}{4}\Big)\Big) \\ i.e.,\ \alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r))))) &\geq (1 + o(1))(\rho_{(\alpha_2,\beta_2)}[g] - \varepsilon)r. \end{aligned}$$

Now combining (3.1) and above inequalities we get that

$$\limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r)))))}{\alpha_1(\mu_f(\beta_1^{-1}(r)))} \geq \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}.$$

Hence the theorem follows. □

Theorem 2. *Let f and g be any two entire functions such that*

$$0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < +\infty \quad \textit{and} \quad \lambda_{(\alpha_2,\beta_2)}[g] > 0.$$

If $\alpha_2(\beta_1^{-1}(r)) \in L_1$, then

$$\liminf_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r)))))}{\alpha_1(\mu_f(\beta_1^{-1}(r)))} \geq \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}.$$

Theorem 3. *Let f and g be any two entire functions such that*

$$0 < \lambda_{(\alpha_1,\beta_1)}[f] < +\infty \quad \textit{and} \quad \lambda_{(\alpha_2,\beta_2)}[g] > 0.$$

If $\alpha_2(\beta_1^{-1}(r)) \in L_1$, then

$$\limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r)))))}{\alpha_1(\mu_f(\beta_1^{-1}(r)))} \geq \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

The proofs of Theorem 2 and Theorem 3 would run parallel to that of Theorem 1. We omit the details.

Theorem 4. *Let f and g be any two entire functions such that*

$$0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < +\infty \quad \textit{and} \quad \rho_{(\alpha_2,\beta_2)}[g] < +\infty.$$

If $\alpha_2(\beta_1^{-1}(r)) \in L_1$, then

$$\limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r)))))}{\alpha_1(\mu_f(\beta_1^{-1}(r)))} \leq \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

P r o o f. From the definition of $\lambda_{(\alpha_1,\beta_1)}[f]$, we get for all sufficiently large values of r that

$$\alpha_1(\mu_f(\beta_1^{-1}(r))) \geq (\lambda_{(\alpha_1,\beta_1)}[f] - \varepsilon)r. \tag{3.2}$$

Further taking $R = 2r$ in Lemma 2 we obtain for all sufficiently large values of r that

$$\alpha_1(\mu_{f\circ g}(r)) \leq (1 + o(1))(\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon)\beta_1(\mu_g(2r)). \tag{3.3}$$

Since $\alpha_2(\beta_1^{-1}(r)) \in L_1$, we obtain from above for all sufficiently large values of r that

$$\begin{aligned} \alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r))))g) &\leq (1 + o(1))\alpha_2(\mu_g(2\beta_2^{-1}(r))) \\ i.e.,\ \alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r))))) &\leq (1 + o(1))(\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon)r. \end{aligned}$$

Now combining (3.2) and above inequalities we get that

$$\limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r)))))}{\alpha_1(\mu_f(\beta_1^{-1}(r)))} \leq \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Hence the theorem follows. □

Theorem 5. *Let f and g be any two entire functions such that*

$$0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < +\infty \quad \text{and} \quad \lambda_{(\alpha_2,\beta_2)}[g] < +\infty.$$

If $\alpha_2(\beta_1^{-1}(r)) \in L_1$, then we have

$$\liminf_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r)))))}{\alpha_1(\mu_f(\beta_1^{-1}(r)))} \leq \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Theorem 6. *Let f and g be any two entire functions such that*

$$0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < +\infty \quad \text{and} \quad \rho_{(\alpha_2,\beta_2)}[g] < +\infty.$$

If $\alpha_2(\beta_1^{-1}(r)) \in L_1$, then we have

$$\liminf_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(r)))))}{\alpha_1(\mu_f(\beta_1^{-1}(r)))} \leq \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]}.$$

The proofs of Theorem 5 and Theorem 6 would run parallel to that of Theorem 4. We omit the details.

Theorem 7. *Let f, g, h and k be four entire functions such that*

$$\lambda_{(\alpha_3,\beta_3)}[h] > 0, \quad \lambda_{(\alpha_4,\beta_4)}[k] > 0 \quad \text{and} \quad \rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_4,\beta_4)}[k].$$

Also let C and D be any two positive constants.

(*i*) *Any one of the following four conditions are assumed to be satisfied:*

(*a*) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\beta_3(r) = D\exp(\alpha_4(r))$;

(*b*) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\beta_3(r) > \exp(\alpha_4(r))$;

(*c*) $\exp(\alpha_2(r)) > \beta_1(r)$ *and* $\beta_3(r) = D\exp(\alpha_4(r))$;

(*d*) $\exp(\alpha_2(r)) > \beta_1(r)$ *and* $\beta_3(r) > \exp(\alpha_4(r))$;

then

$$\lim_{r\to+\infty}\frac{\alpha_3(\mu_{hok}(\beta_4^{-1}(\log r)))}{\alpha_1(\mu_{fog}(\beta_2^{-1}(\log r)))}=\infty.$$

(*ii*) *Any one of the following two conditions are assumed to be satisfied*:

(*a*) $\beta_1(r)=C(\exp(\alpha_2(r)))$ *and* $\alpha_4(\beta_3^{-1}(r))\in L^0$;

(*b*) $\beta_3(r)>\exp(\alpha_4(r))$ *and* $\alpha_4(\beta_3^{-1}(r))\in L^0$;

then

$$\lim_{r\to+\infty}\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_{hok}(\beta_4^{-1}(\log r))))))}{\alpha_1(\mu_{fog}(\beta_2^{-1}(\log r)))}=\infty.$$

(*iii*) *Any one of the following two conditions are assumed to be satisfied*:

(*a*) $\beta_3(r)=D\exp(\alpha_4(r))$ *and* $\alpha_2(\beta_1^{-1}(r))\in L^0$;

(*b*) $\beta_3(r)>\exp(\alpha_4(r))$ *and* $\alpha_2(\beta_1^{-1}(r))\in L^0$;

then

$$\lim_{r\to+\infty}\frac{\alpha_3(\mu_{hok}(\beta_4^{-1}(\log r)))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{fog}(\beta_2^{-1}(\log r))))))}=\infty.$$

(*iv*) *If* $\alpha_2(\beta_1^{-1}(r))\in L_1$ *and* $\alpha_4(\beta_3^{-1}(r))\in L^0$, *then*

$$\lim_{r\to+\infty}\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_{hok}(\beta_4^{-1}(\log r))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{fog}(\beta_2^{-1}(\log r))))))}=\infty.$$

Proof. In view of (3.3) we get for all sufficiently large values of r that

$$\alpha_1(\mu_{fog}(\beta_2^{-1}(\log r)))\le (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\mu_g(2\beta_2^{-1}(\log r))). \tag{3.4}$$

Case I. Let $\beta_1(r)=C(\exp(\alpha_2(r)))$. Then we have from (3.4) for all sufficiently large values of r that

$$\alpha_1(\mu_{fog}(\beta_2^{-1}(\log r)))\le C(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.5}$$

Case II. Let $\exp(\alpha_2(r))>\beta_1(r)$. Then we have from (3.4) for all sufficiently large values of r that

$$\alpha_1(\mu_{fog}(\beta_2^{-1}(\log r)))\le (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.6}$$

Case III. Let $\alpha_2(\beta_1^{-1}(r))\in L^0$. Then we get from(3.4) for all sufficiently large values of r that

$$\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{fog}(\beta_2^{-1}(\log r))))))\le r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.7}$$

Further it follows from Lemma 3 for all sufficiently large values r that

$$\begin{aligned}\alpha_3(\mu_{hok}(\beta_4^{-1}(\log r))) &\ge (1+o(1))\alpha_3\Big(\mu_h\Big(\frac{1}{16}\mu_k\Big(\frac{\beta_4^{-1}(\log r)}{4}\Big)\Big)\Big)\\ \text{i.e., }\ \alpha_3(\mu_{hok}(\beta_4^{-1}(\log r))) &\ge (1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]-\varepsilon)\beta_3\Big(\mu_k\Big(\frac{\beta_4^{-1}(\log r)}{4}\Big)\Big).\end{aligned} \tag{3.8}$$

Case IV. Let $\beta_3(r)=D\exp(\alpha_4(r))$. Then from (3.8) it follows for all sufficiently large values of r that

$$\alpha_3(\mu_{hok}(\beta_4^{-1}(\log r)))\ge D(1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}. \tag{3.9}$$

Case V. Let $\beta_3(r) > \exp(\alpha_4(r))$. Now from (3.8) it follows for all sufficiently large values of r that

$$\alpha_3(\mu_{h \circ k}(\beta_4^{-1}(\log r))) > \ (1 + o(1))(\lambda_{(\alpha_3,\beta_3)}[h] - \varepsilon) r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}. \tag{3.10}$$

Case VI. Let $\alpha_4(\beta_3^{-1}(r)) \in L^0$. Then from (3.8) we obtain for all sufficiently large values of r that

$$\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_{h \circ k}(\beta_4^{-1}(\log r)))))) \geq r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}. \tag{3.11}$$

Since $\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_4,\beta_4)}[k]$ we can choose $\varepsilon(> 0)$ in such a way that

$$\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon < \lambda_{(\alpha_4,\beta_4)}[k] - \varepsilon. \tag{3.12}$$

Now combining (3.5) of Case I and (3.9) of Case IV it follows for all sufficiently large values of r that

$$\frac{\alpha_3(\mu_{h \circ k}(\beta_4^{-1}(\log r)))}{\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r)))} \geq \frac{D(1 + o(1))(\lambda_{(\alpha_3,\beta_3)}[h] - \varepsilon) r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{C(1 + o(1))(\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon) r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

So from (3.12) and above we obtain that

$$\liminf_{r \to +\infty} \frac{\alpha_3(\mu_{h \circ k}(\beta_4^{-1}(\log r)))}{\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r)))} = \infty. \tag{3.13}$$

Similarly combining (3.5) of Case I and (3.10) of Case V we get that

$$\liminf_{r \to +\infty} \frac{\alpha_3(\mu_{h \circ k}(\beta_4^{-1}(\log r)))}{\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r)))} = \infty. \tag{3.14}$$

Analogously combining (3.6) of Case II and (3.9) of Case IV, we obtain that

$$\lim_{r \to +\infty} \frac{\alpha_3(\mu_{h \circ k}(\beta_4^{-1}(\log r)))}{\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r)))} = \infty. \tag{3.15}$$

Likewise combining (3.6) of Case II and (3.10) of Case V it follows that

$$\lim_{r \to +\infty} \frac{\alpha_3(\mu_{h \circ k}(\beta_4^{-1}(\log r)))}{\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r)))} = \infty. \tag{3.16}$$

Hence the first part of the theorem follows from (3.13), (3.14), (3.15) and (3.16).

Again combining (3.5) of Case I and (3.11) of Case VI we obtain for all sufficiently large values of r that

$$\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_{h \circ k}(\beta_4^{-1}(\log r))))))}{\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r)))} \geq \frac{r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{C(1 + o(1))(\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon) r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

So from (3.12) and above we obtain that

$$\lim_{r \to +\infty} \frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_{h \circ k}(\beta_4^{-1}(\log r))))))}{\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r)))} = \infty.$$

Similarly combining (3.6) of Case II and (3.11) of Case VI we also get the same conclusion. Therefore the second part of the theorem is established.

Again combining (3.7) of Case III and (3.9) of Case IV it follows for all sufficiently large values of r that

$$\frac{\alpha_3(\mu_{h\circ k}(\beta_4^{-1}(\log r)))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))))))} \geq \frac{D(1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}. \tag{3.17}$$

Now in view of (3.12) we obtain from (3.17) that

$$\lim_{r\to+\infty} \frac{\alpha_3(\mu_{h\circ k}(\beta_4^{-1}(\log r)))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))))))} = \infty. \tag{3.18}$$

Similarly combining (3.7) of Case III and (3.10) of Case V we get that

$$\lim_{r\to+\infty} \frac{\alpha_3(\mu_{h\circ k}(\beta_4^{-1}(\log r)))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))))))} = \infty. \tag{3.19}$$

Hence the third part of the theorem follows from (3.18) and (3.19).

Further combining (3.7) of Case III and (3.11) of Case VI we obtain for all sufficiently large values of r that

$$\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_{h\circ k}(\beta_4^{-1}(\log r))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))))))} \geq \frac{r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

Now in view of (3.12) we obtain from above that

$$\lim_{r\to+\infty} \frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_{h\circ k}(\beta_4^{-1}(\log r))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))))))} = \infty.$$

This proves the fourth part of the theorem. □

This implies the following theorem.

Theorem 8. *Let f and g be any two entire functions such that*

$$\rho_{(\alpha_1,\beta_1)}[f\circ g] < \infty \quad \text{and} \quad \lambda_{(\alpha_3,\beta_3)}[g] > 0.$$

Then

$$\lim_{r\to\infty} \frac{\{\alpha_1(\mu_{f\circ g}(\beta_1^{-1}(\log r)))\}^2}{\alpha_3(\mu_g(\beta_3^{-1}(\log r)))\cdot\alpha_3(\mu_g(\beta_3^{-1}(r)))} = 0.$$

P r o o f. For arbitrary positive ε we have for all sufficiently large values of r that

$$\alpha_1(\mu_{f\circ g}(\beta_1^{-1}(\log r))) \leq (\rho_{(\alpha_1,\beta_1)}[f\circ g]+\varepsilon)\log r. \tag{3.20}$$

Again for all sufficiently large values of r we get

$$\alpha_3(\mu_g(\beta_3^{-1}(\log r))) \geq (\lambda_{(\alpha_3,\beta_3)}[g]-\varepsilon)\log r. \tag{3.21}$$

Similarly for all sufficiently large values of r we have

$$(\alpha_3(\mu_g(\beta_3^{-1}(r)))) \geq (\lambda_{(\alpha_3,\beta_3)}[g]-\varepsilon)r. \tag{3.22}$$

From (3.20) and (3.21) we have for all sufficiently large values of r that

$$\frac{\alpha_1(\mu_{f\circ g}(\beta_1^{-1}(\log r)))}{\alpha_3(\mu_g(\beta_3^{-1}(\log r)))} \leq \frac{(\rho_{(\alpha_1,\beta_1)}[f\circ g]+\varepsilon)\log r}{(\lambda_{(\alpha_3,\beta_3)}[g]-\varepsilon)\log r}.$$

As $\varepsilon > 0$ is arbitrary we obtain from above that

$$\limsup_{r\to+\infty}\frac{\alpha_1(\mu_{f\circ g}(\beta_1^{-1}(\log r)))}{\alpha_3(\mu_g(\beta_3^{-1}(\log r)))} \le \frac{\rho_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_3,\beta_3)}[g]}. \tag{3.23}$$

Again from (3.20) and (3.22) we get for all sufficiently large values of r that

$$\frac{\alpha_1(\mu_{f\circ g}(\beta_1^{-1}(\log r)))}{\alpha_3(\mu_g(\beta_3^{-1}(r)))} \le \frac{(\rho_{(\alpha_1,\beta_1)}[f\circ g]+\varepsilon)\log r}{(\lambda_{(\alpha_3,\beta_3)}[g]-\varepsilon)r}.$$

Since $\varepsilon > 0$ is arbitrary it follows from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(\mu_{f\circ g}(\beta_1^{-1}(\log r)))}{\alpha_3(\mu_g(\beta_3^{-1}(r)))} = 0. \tag{3.24}$$

Thus the theorem follows from (3.23) and (3.24). □

Remark 1. Theorem 1 to Theorem 8 can also be deduced in terms of maximum modulus of entire functions with the help of Lemma 1.

Theorem 9. *Let f and g be any two entire functions such that*

$$\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f] \le \rho_{(\alpha_1,\beta_1)}[f].$$

Also let C be any positive constant and $\beta_1 \in L_2$.

(*i*) *Any one of the following two conditions are assumed to be satisfied*:

(*a*) $\beta_1(r) = C(\exp(\alpha_2(r)))$;

(*b*) $\exp(\alpha_2(r)) > \beta_1(r)$;

then

$$\limsup_{r\to+\infty}\frac{\{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))\}^2}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))\cdot\beta_1(\mu_g(2\beta_2^{-1}(\log r)))} = 0.$$

(*ii*) *If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then*

$$\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))))))\cdot\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))\cdot\beta_1(\mu_g(2\beta_2^{-1}(\log r)))} = 0.$$

P r o o f. From the definition of generalized lower order (α_1,β_1) of f we have for arbitrary positive ε and for all sufficiently large values of r that

$$\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r)))) \ge r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}. \tag{3.25}$$

As $\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f]$ we can choose $\varepsilon > 0$ in such a way that

$$\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon < \lambda_{(\alpha_1,\beta_1)}[f] - \varepsilon. \tag{3.26}$$

Now combining (3.5) of Case I and (3.25) we have for all large positive numbers of r,

$$\frac{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))} \le \frac{C(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}}.$$

In view of (3.26) we get from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))}=0. \tag{3.27}$$

Again combining (3.6) of Case II and (3.25) we get for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))}\le\frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}}.$$

Now in view of (3.26) we obtain from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))}=0. \tag{3.28}$$

Further combining (3.7) of Case III and (3.25) we get for all sufficiently large positive numbers of r that

$$\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))}\le\frac{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}}.$$

So in view of (3.26) we obtain from above that

$$\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))}=0. \tag{3.29}$$

Now from (3.4) we get that

$$\limsup_{r\to+\infty}\frac{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}\le\rho_{(\alpha_1,\beta_1)}[f]. \tag{3.30}$$

From (3.27) and (3.30) we obtain for all sufficiently large values of r that

$$\begin{aligned}&\limsup_{r\to+\infty}\frac{\{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))\}^2}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))\cdot\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}\\ =&\lim_{r\to+\infty}\frac{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))}\cdot\limsup_{r\to+\infty}\frac{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}\le 0\cdot\rho_{(\alpha_1,\beta_1)}[f]=0.\end{aligned} \tag{3.31}$$

Similarly from (3.28) and (3.30) we obtain that

$$\limsup_{r\to+\infty}\frac{\{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))\}^2}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))\cdot\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}=0.$$

Therefore the first part of the theorem follows from (3.31) and above.

Again from (3.29) and (3.30) we get for all large values of r that

$$\begin{aligned}&\limsup_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))))))\cdot\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))\cdot\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}\\ =&\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))}\cdot\limsup_{r\to+\infty}\frac{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}\le 0\cdot\rho_{(\alpha_1,\beta_1)}[f]=0,\end{aligned}$$

$$i.e.,\ \lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))))))\cdot\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(\mu_f(\beta_1^{-1}(\log r))))\cdot\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}=0.$$

Thus the second part of the theorem is established. □

In the line of Theorem 9 and with the help of Lemma 1, one can easily prove the following theorem and therefore its proof is omitted:

Theorem 10. *Let f and g be any two entire functions such that*

$$\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f].$$

Also let C be any positive constant and $\beta_1 \in L_2$.

(i) *Any one of the following two conditions are assumed to be satisfied*:

(a) $\beta_1(r) = C(\exp(\alpha_2(r)))$;

(b) $\exp(\alpha_2(r)) > \beta_1(r)$;

then

$$\limsup_{r\to+\infty} \frac{\{\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))\}^2}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r)))) \cdot \beta_1(M_g(\beta_2^{-1}(\log r)))} = 0.$$

(ii) *If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))))) \cdot \alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r)))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\log r)))) \cdot \beta_1(M_g(\beta_2^{-1}(\log r)))} = 0.$$

Theorem 11. *Let f and g be any two entire functions such that*

$$\lambda_{(\alpha_1,\beta_1)}[f] < \infty \quad \text{and} \quad \rho_{(\alpha_3,\beta_3)}[f\circ g] < \infty$$

where $\alpha_2, \beta_1 \in L_2$. Then

$$\limsup_{r\to+\infty} \frac{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))) \cdot \alpha_3(\mu_{f\circ g}(\beta_3^{-1}(r)))}{\beta_1(\mu_g(2\beta_2^{-1}(\log r))) \cdot \alpha_2(\mu_g(\beta_2^{-1}(r)))} \leq \frac{\rho_{(\alpha_3,\beta_3)}[f\circ g] \cdot \rho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

P r o o f. For all sufficiently large values of r we have

$$\alpha_3(\mu_{f\circ g}(\beta_3^{-1}(r))) \leq (\rho_{(\alpha_3,\beta_3)}[f\circ g] + \varepsilon)r. \tag{3.32}$$

Again for all sufficiently large values of r it follows that

$$\alpha_2(\mu_g(\beta_2^{-1}(r))) \geq (\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon)r. \tag{3.33}$$

Now combining (3.32) and (3.33) we have for all sufficiently large values of r that

$$\frac{\alpha_3(\mu_{f\circ g}(\beta_3^{-1}(r)))}{\alpha_2(\mu_g(\beta_2^{-1}(r)))} \leq \frac{\rho_{(\alpha_3,\beta_3)}[f\circ g] + \varepsilon}{\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon}.$$

As $\varepsilon > 0$ is arbitrary we get from above that

$$\limsup_{r\to+\infty} \frac{\alpha_3(\mu_{f\circ g}(\beta_3^{-1}(r)))}{\alpha_2(\mu_g(\beta_2^{-1}(r)))} \leq \frac{\rho_{(\alpha_3,\beta_3)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]}. \tag{3.34}$$

Now from (3.30) and (3.34) we obtain that

$$\limsup_{r\to+\infty} \frac{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r))) \cdot \alpha_3(\mu_{f\circ g}(\beta_3^{-1}(r)))}{\beta_1(\mu_g(2\beta_2^{-1}(\log r))) \cdot \alpha_2(\mu_g(\beta_2^{-1}(r)))}$$

$$\leq \limsup_{r\to+\infty} \frac{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\beta_1(\mu_g(2\beta_2^{-1}(\log r)))} \cdot \limsup_{r\to+\infty} \frac{\alpha_3(\mu_{f\circ g}(\beta_3^{-1}(r)))}{\alpha_2(\mu_g(\beta_2^{-1}(r)))} \leq \frac{\rho_{(\alpha_3,\beta_3)}[f\circ g] \cdot \rho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Hence the theorem follows. □

In the line of Theorem 11 and with the help of Lemma 1, one can easily proof the following theorem and therefore its proof is omitted:

Theorem 12. *Let f and g be any two entire functions such that*

$$\lambda_{(\alpha_1,\beta_1)}[f] < \infty \quad \textit{and} \quad \rho_{(\alpha_3,\beta_3)}[f \circ g] < \infty$$

where $\alpha_2, \beta_1 \in L_2$. Then

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f\circ g}(\beta_2^{-1}(\log r))) \cdot \alpha_3(M_{f\circ g}(\beta_3^{-1}(r)))}{\beta_1(M_g(\beta_2^{-1}(\log r))) \cdot \alpha_2(M_g(\beta_2^{-1}(r)))} \le \frac{\rho_{(\alpha_3,\beta_3)}[f\circ g] \cdot \rho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Theorem 13. *Let f and g be any two entire functions such that*

$$\rho_{(\alpha_1,\beta_1)}[f] < \infty \quad \textit{and} \quad \lambda_{(\alpha_3,\beta_3)}[f \circ g] = \infty.$$

Then

$$\lim_{r\to+\infty} \frac{\alpha_3(\mu_{f\circ g}(r))}{\alpha_1(\mu_f(\beta_1^{-1}(\beta_3(r))))} = \infty.$$

P r o o f. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of r tending to infinity

$$\alpha_3(\mu_{f\circ g}(r)) \le \Delta \cdot \alpha_1(\mu_f(\beta_1^{-1}(\beta_3(r)))). \tag{3.35}$$

Again from the definition of $\rho_{(\alpha_1,\beta_1)}[f]$, it follows for all sufficiently large values of r that

$$\alpha_1(\mu_f(\beta_1^{-1}(\beta_3(r)))) \le (\rho_{(\alpha_1,\beta_1)}[f] + \epsilon)\beta_3(r). \tag{3.36}$$

Thus from (3.35) and (3.36), we have for a sequence of values of r tending to infinity that

$$\alpha_3(\mu_{f\circ g}(r)) \le \Delta(\rho_{(\alpha_1,\beta_1)}[f] + \epsilon)\beta_3(r)$$

$$i.e., \ \frac{\alpha_3(\mu_{f\circ g}(r))}{\beta_3(r)} \le \frac{\Delta(\rho_{(\alpha_1,\beta_1)}[f] + \epsilon)\beta_3(r)}{\beta_3(r)}$$

$$i.e., \ \liminf_{r+\infty} \frac{\alpha_3(\mu_{f\circ g}(r))}{\beta_3(r)} = \lambda_{(\alpha_3,\beta_3)}[f\circ g] < \infty.$$

This is a contradiction. Thus the theorem follows. □

Remark 2. Theorem 13 is also valid with "limit superior" instead of "limit" if $\lambda_{(\alpha_3,\beta_3)}[f\circ g] = \infty$ is replaced by $\rho_{(\alpha_3,\beta_3)}[f \circ g] = \infty$ while the other conditions remain the same.

Analogously one may also state the following theorem without its proof as it may be carried out in the line of Theorem 13.

Theorem 14. *Let f and g be any two entire functions such that*

$$\rho_{(\alpha_1,\beta_1)}[g] < \infty \quad \textit{and} \quad \rho_{(\alpha_3,\beta_3)}[f \circ g] = \infty.$$

Then

$$\limsup_{r\to+\infty} \frac{\alpha_3(\mu_{f\circ g}(r))}{\alpha_1(\mu_g(\beta_1^{-1}(\beta_3(r))))} = \infty.$$

Remark 3. Theorem 14 is also valid with "limit" instead of "limit superior" if $\rho_{(\alpha_3,\beta_3)}[f\circ g] = \infty$ is replaced by $\lambda_{(\alpha_3,\beta_3)}[f \circ g] = \infty$ and the other conditions remain the same.

Remark 4. Theorem 13, Theorem 14, Remark 2 and Remark 3 can also be deduced in terms of maximum modulus of entire functions.

4. Conclusion

Actually this paper deals with the extension of the researches on the growth properties of composite entire functions on the basis of their generalized order (α, β) where α and β are continuous non-negative functions on $(-\infty, +\infty)$. This assumption can also be modified by the treatment of the ideas of generalized type (α, β). Moreover, some extensions of the same may be done in the light of generalized relative order (α, β). Furthermore, the concept of generalized order (α, β) and generalized type (α, β) should have a broad range of applications in complex dynamics, factorization theory of entire functions of single complex variable, the solution of complex differential equations etc. which may be an ample scope of further research.

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OPEN PACKING NUMBER FOR SOME CLASSES OF PERFECT GRAPHS

K. Raja Chandrasekar

Amrita College of Engineering and Technology,
Amritagiri, Erachakulam Post, Nagercoil-629902, Tamil Nadu, India
rajmath84@gmail.com

S. Saravanakumar

Kalasalingam Academy of Research and Education,
Anand Nagar, Krishnankoil-626126, Tamil Nadu, India
alg.ssk@gmail.com

Abstract: Let G be a graph with the vertex set $V(G)$. A subset S of $V(G)$ is an *open packing set* of G if every pair of vertices in S has no common neighbor in G. The maximum cardinality of an open packing set of G is the *open packing number* of G and it is denoted by $\rho^o(G)$. In this paper, the exact values of the open packing numbers for some classes of perfect graphs, such as split graphs, $\{P_4, C_4\}$-free graphs, the complement of a bipartite graph, the trestled graph of a perfect graph are obtained.

Keywords: Open packing number, 2-packing number, Perfect graphs, Trestled graphs.

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology, we refer to the book by Chartrand and Lesniak [4]. For a vertex v in $V(G)$, the *open neighborhood* of v and the *closed neighborhood* of v are defined by $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. Similarly for a subset S of $V(G)$, the *open* and *closed neighborhoods* of S are defined by $N(S) = \cup_{v\in S}N(v)$ and $N[S] = \cup_{v\in S}N[v]$. For any two sets A and B, define $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$. In a graph G, a vertex of degree 1 is a *leaf* and the vertex adjacent to a leaf is a *support vertex* of G. For a set S of vertices of G, the *induced subgraph* is the maximal subgraph of G with vertex set S and is denoted by $\langle S\rangle$. Thus two vertices of S are adjacent in $\langle S\rangle$ if and only if they are adjacent in G.

A subset S of $V(G)$ is *independent* if no two vertices in S are adjacent in G. The *independence number* $\beta_0(G)$ is the maximum cardinality of an independent set in G. A subset M of $E(G)$ is *independent* if no two edges in M are adjacent in G. A set of independent edges in G is a *matching* of G. The *edge independence number* $\beta_1(G)$ is the maximum cardinality of a matching in G.

A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The *chromatic number* $\chi(G)$ is the minimum number of colors required for a proper coloring of G. A *clique* in G is a complete subgraph of G. The maximum order of a clique in G is the *clique number* of G and is denoted by $\omega(G)$. A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

Perfect graphs were introduced by Berge [2], who conjectured that a graph G is perfect if and only if $\overline{G}$ is perfect. This Perfect Graph Conjecture was verified by Lovász [10]. Since the chromatic number and clique number of an odd cycle of length at least 5 are not equal, it follows that if an induced subgraph of a graph G is an odd cycle of length at least 5, then the graphs G and $\overline{G}$ are not perfect. Berge [3] conjectured that every graph that is not perfect contains either an induced odd cycle of length at least 5 or its complement graph $\overline{G}$ contains such a cycle. This famous conjecture (sometimes referred to as the Strong Perfect Graph Conjecture) was verified by Chudnovsky et al. [5]. It is well known that $\chi(G) \geq \omega(G)$, for every graph G. Hence if G contains a triangle, then $\chi(G) \geq 3$. It may be surprising that Mycielski [13] has proved that there exist triangle free graphs with large chromatic number. Several classes of perfect graphs and their properties are given in [9].

A subset S of $V(G)$ is an *open packing set* of G if every pair of vertices in S has no common neighbor in G. The maximum cardinality of an open packing set of G is the *open packing number* of G and is denoted by $\rho^o(G)$. An open packing set of cardinality $\rho^o(G)$ is a ρ^o*-set* of G. Clearly every open packing set contains at most one vertex in every open neighborhood $N(v)$ of a vertex v in G. A subset S' of $V(G)$ is a 2*-packing set* of G if every pair of vertices in S' is of distance at least 3 in G. The maximum cardinality of a 2-packing set is the 2*-packing number* and is denoted by $\rho(G)$. We observe that for any connected graph G, every 2-packing set of G is an open packing set of G and hence $\rho(G) \leq \rho^o(G)$.

The concepts of 2-packing and open packing sets in graphs were introduced respectively by Meir and Moon [11] and Henning et.al [8]. Fisher et al. [7] proved that for any connected graph G, $\rho(G) = \rho(\mu(G))$, where $\mu(G)$ is the Mycielskian of the graph G. Henning and Slater [8] proved that for any graph G of order n $\rho^o(G) \leq 2n/3$. They also obtained the characterization of all graphs G for which $\rho^o(G) = 2n/3$. In [14], the authors have proved that $\rho^o(G) \leq n/\delta(G)$. Moreover, the characterization of all connected graphs for which the equality holds was settled in [12].

In this paper, we obtain the exact values of the open packing number for some families of perfect graphs, such as, split graphs, $\{P_4, C_4\}$-free graphs, the complement of a bipartite graph and the trestled graph of a perfect graph.

2. Split graphs and $\{\mathbf{P_4}, \mathbf{C_4}\}$-free graphs

Definition 1. *A graph G is a $\{P_4, C_4\}$-free graph if neither P_4 nor C_4 is an induced subgraph of G.*

We know that every $\{P_4, C_4\}$-free graph is a perfect graph, see [15]. We now determine the exact value of the open packing number for $\{P_4, C_4\}$-free graphs. The following assertion is used to prove the theorem.

Assertion 1. *Let S be an open packing set of G. Then every component of the induced subgraph $\langle S\rangle$ is isomorphic to either K_1 or K_2.*

Theorem 1. *Let G be a $\{P_4, C_4\}$-free graph of order $n \geq 2$. Then $\rho^o(G)$ is either 1 or 2. Further, $\rho^o(G) = 2$ if and only if $\Delta(G) = n-1$ and $\delta(G) = 1$.*

P r o o f. If diam$\,(G) \geq 3$, then G contains a P_4 as an induced subgraph, which is a contradiction. Hence diam$\,(G)$ is either 1 or 2. If diam$\,(G) = 1$, then G is a complete graph K_n and hence $\rho^o(G)$ is either 1 or 2 depending on whether $n \geq 3$ or $n = 2$.

Suppose diam$\,(G) = 2$. Let S be a ρ^o-set of G. If $|S| \geq 3$, then by Assertion 1 there exist two vertices u and v in G such that $u, v \in S$ and $uv \notin E(\langle S\rangle)$. Consequently $uv \notin E(G)$. Since diam$\,(G) = 2$, there exists a vertex x in $V(G)$ such that x is adjacent to u and v in G, which is a contradiction to u and v are in a ρ^o-set S of G. Hence $\rho^o(G)$ is either 1 or 2.

We claim that $\rho^o(G) = 2$ if and only if $\Delta(G) = n-1$ and $\delta(G) = 1$. Suppose $\rho^o(G) = 2$ and let $S' = \{u, v\}$ be a ρ^o-set of G. It follows from the above argument that $uv \in E(G)$.

Let $D_1 = N_G(u) \setminus \{v\}$ and $D_2 = N_G(v) \setminus \{u\}$. Since u and v have no common neighbor in G, it follows that $D_1 \cap D_2 = \emptyset$.

If $D_1 = D_2 = \emptyset$, then $G = K_2$, and we are done. Hence $D_1 \neq \emptyset$ or $D_2 \neq \emptyset$. Suppose $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$ and let $x \in D_1$ and $y \in D_2$. Then $uy \notin E(G)$ and $vx \notin E(G)$ and hence the induced subgraph $\langle\{u, v, x, y\}\rangle$ is isomorphic to either P_4 or C_4 depending on $xy \notin E(G)$ or $xy \in E(G)$, which is a contradiction to G is a $\{P_4, C_4\}$-free graph. Hence we may assume without loss of generality that $D_1 \neq \emptyset$ and $D_2 = \emptyset$. Then $\deg(v) = 1$ and u is the support vertex adjacent to v in G.

Now we claim that $V(G) = N[u]$. If $z \in V(G) \setminus N[u]$, then $d(z, v) \geq 3$, which is a contradiction to $\operatorname{diam}(G) \leq 2$. Hence $V(G) = N[u]$. It follows that $\deg(u) = n - 1$ and $\deg(v) = 1$. Thus $\Delta(G) = n - 1$ and $\delta(G) = 1$.

Conversely, suppose $\Delta(G) = n - 1$ and $\delta(G) = 1$. Let $x, y \in V(G)$ be such that $\deg(x) = n - 1$ and $\deg(y) = 1$. Clearly $S = \{x, y\}$ is an open packing set of G and hence $\rho^o(G) \geq |S| = 2$. Moreover, any open packing set of G contains at most one vertex in $N(x) = V(G) \setminus \{x\}$, it follows that $\rho^o(G) = 2$. □

Definition 2. *A split graph G is a graph whose vertex set can be partitioned into two sets K and I, where the vertices in K form a complete graph and the vertices in I are independent. The partition (K, I) is a split partition of the split graph G.*

Clearly every split graph is a perfect graph, see [9]. We now determine the open packing number of a split graph G in terms of the 2-packing number $\rho(G)$.

Theorem 2. *Let G be a connected split graph of order n with split partition (K, I) and $\Delta(G) < n - 1$. Then $\rho^o(G) = \rho(G)$.*

P r o o f. Let S be a ρ^o-set of the split graph G. Since no two vertices in S have a common neighbor in G and the induced subgraph $\langle K\rangle$ is complete, it follows that $|S \cap K| \leq 2$. We consider the following three cases depending on $|S \cap K|$ is 0, 1, or 2.

Case i. $|S \cap K| = 0$.

Then $S \subseteq I$ and hence the distance between any pair of vertices in S is exactly 3. Consequently S is a 2-packing set of G and hence $\rho^o(G) = |S \cap I| \leq \rho(G)$. Thus $\rho^o(G) = \rho(G)$ as $\rho(G) \leq \rho^o(G)$.

Case ii. $|S \cap K| = 1$.

Let $S \cap K = \{u\}$. Now we claim that $|S|$ is either 2 or 1 depending on whether u is a support vertex of G or not. Let $v' \in I$ be such that $v'u \notin E(G)$, the existence of v' is guaranteed by the assumption that $\Delta(G) < n - 1$. Let v be a neighbor of v' in K. Then $v' \notin S$, otherwise u and v' in S have a common neighbor, namely v, in G, which is a contradiction. Further, $N(u) \cap I$ contains at most one vertex of S, it follows that $|S \cap I|$ is either 0 or 1. If $|S \cap I| = 0$, then $|S| = 1$ and hence S is a 2-packing set of G. Consequently $\rho^o(G) = \rho(G) = 1$.

Suppose $|S \cap I| = 1$ and let $S \cap I = \{u'\}$. Then u' must be a leaf neighbor of u in G, otherwise u and u' will have a common neighbor in G, which is a contradiction. Hence $S = \{u, u'\}$, where $deg\,(u') = 1$ and $uu' \in E(G)$. Now consider the set $S' = (S \setminus \{u\}) \cup \{v'\}$, where $v' \notin N(u)$. Clearly S' is a 2-packing set of G and hence $|S'| = 2 = \rho^o(G) \leq \rho(G)$. Thus $\rho(G) = \rho^o(G)$.

Case iii. $|S \cap K| = 2$.

Let $S \cap K = \{u, v\}$. We claim that $|S \cap I| = 0$. Suppose $z' \in S \cap I$ and let z be a neighbor of z' in K. If $z = u$, then z' and v in S have a common neighbor, namely u, in G, which is a contradiction. Similarly, if $z = v$, then z' and u in S have a common neighbor in G, which is a contradiction. Finally, suppose $z \notin \{u, v\}$. Then z' and u have a common neighbor, namely z, in G, which is again a contradiction. Thus $|S \cap I| = 0$. Since G is a connected split graph and $\Delta(G) < n - 1$, it follows that $\operatorname{diam}(G) = 3$. Let $u', v' \in V(G)$ be such that $d(u', v') = 3$. Then

$\{u', v'\} \subseteq I$ and the set $S' = \{u', v'\}$ is a 2-packing set of G. Hence $2 = \rho^o(G) \leq \rho(G)$. Thus $\rho^o(G) = \rho(G)$. □

Remark 1. If G is a split graph of order n with $\Delta(G) = n - 1$, then $\rho(G) = 1$ and

$$\rho^o(G) = \begin{cases} 2 & \text{if } \delta(G) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

3. Complement of a bipartite graph and the trestled graph of a graph

In this section, we determine the exact values of the open packing number of the complement of a bipartite graph and the trestled graph of a graph.

Theorem 3. *Let G be a connected bipartite graph which is not a complete bipartite graph with bipartition (X, Y) and $2 \leq |X| \leq |Y|$. Then $\rho^o(\overline{G})$ is either 1 or 2. Further, $\rho^o(\overline{G}) = 2$ if and only if either $|X| = 2$ or, there exist two vertices $x \in X$ and $y \in Y$ such that $N_G(x) \supseteq Y \setminus \{y\}$ and $N_G(y) \supseteq X \setminus \{x\}$.*

P r o o f. We first claim that $\rho^o(\overline{G})$ is either 1 or 2. Since G is not a complete bipartite graph, let $x \in X$ and $y \in Y$ be such that $xy \notin E(G)$. Then $xy \in E(\overline{G})$.

Let $N_x = (X \setminus \{x\}) \cup \{y\}$ and $N_y = (Y \setminus \{y\}) \cup \{x\}$. Clearly $N_x \subseteq N_{\overline{G}}(x)$, $N_y \subseteq N_{\overline{G}}(y)$ and $N_x \cup N_y = V(\overline{G})$ as $xy \in E(\overline{G})$ and, $\langle X \rangle$ and $\langle Y \rangle$ are complete subgraphs in $\overline{G}$. Let S be a ρ^o-set in $\overline{G}$. Then $|S \cap N_{\overline{G}}(x)| \leq 1$ and $|S \cap N_{\overline{G}}(y)| \leq 1$. Hence $|S \cap N_x| \leq 1$ and $|S \cap N_y| \leq 1$ as $N_x \subseteq N_{\overline{G}}(x)$ and $N_y \subseteq N_{\overline{G}}(y)$. Consequently

$$|S \cap (N_x \cup N_y)| = \left|S \cap V(\overline{G})\right| = |S| = \rho^o(\overline{G}) \leq 2.$$

Thus $\rho^o(\overline{G})$ is either 1 or 2.

Suppose $\rho^o(\overline{G}) = 2$. Let $S_1 = \{u', v'\}$ be a ρ^o-set of $\overline{G}$. Now we consider the following three cases depending on u' and v' are in X or, Y or, X and Y, respectively, in $\overline{G}$.

Case i. $u', v' \in X$.

If $|X| \geq 3$, then there exists a vertex, say w', in X such that w' is adjacent to u' and v' in $\overline{G}$, which is a contradiction. Suppose $|X| = 2$. Then $X = \{u', v'\}$. If there exists a vertex, say z', in Y such that z' is adjacent to u' and v' in $\overline{G}$, then z' is an isolated vertex, a vertex of degree 0, in G and hence G is disconnected, which is a contradiction. Thus every vertex of Y is adjacent to at most one vertex in X. Consequently X is an open packing set in $\overline{G}$ and hence $\rho^o(\overline{G}) = |X| = 2$.

Case ii. $u', v' \in Y$.

It follows from the similar argument of Case i that $\rho^o(\overline{G}) = |Y| = 2$. Since $2 \leq |X| \leq |Y| = \rho^o(\overline{G}) = 2$, it follows that $\rho^o(\overline{G}) = |X| = 2$.

Case iii. $u' \in X$ and $v' \in Y$.

If $N_{\overline{G}}(u') \cap (Y \setminus \{v'\}) \neq \emptyset$ or $N_{\overline{G}}(v') \cap (X \setminus \{u'\}) \neq \emptyset$, then there exists a vertex in $\overline{G}$ which is commonly adjacent to u' and v', which is a contradiction to $S_1 = \{u', v'\}$ is an open packing set in $\overline{G}$. Hence $N_{\overline{G}}(u') \cap (Y \setminus \{v'\}) = \emptyset$ and $N_{\overline{G}}(v') \cap (X \setminus \{u'\}) = \emptyset$. Consequently $N_G(u') \supseteq Y \setminus \{v'\}$ and $N_G(v') \supseteq X \setminus \{u'\}$.

The converse is obvious.

Remark 2. If G is a complete bipartite graph $K_{m,n}$, $2 \leq m \leq n$, then

$$\rho^o(\overline{G}) = \rho^o(K_m \cup K_n) = \begin{cases} 4 & \text{if } m = n = 2, \\ 3 & \text{if } m = 2 \text{ and } n \geq 3, \\ 2 & \text{otherwise.} \end{cases}$$

Definition 3. Let $G = (V, E)$ be a graph and let k be any positive integer. The *trestled graph* $T_k(G)$ *of index* k is the graph obtained from G by adding k copies of K_2 to each edge uv of G and joining u and v to the respective end vertices of each K_2.

The trestled graph $T_2(C_5)$ is given in Fig. 1. Some basic algorithmic results on trestled graphs are given in [1] and [6]. Clearly $\chi(T_k(G)) = \chi(G)$ and $\omega(T_k(G)) = \omega(G)$ and hence if G is a perfect graph, then $T_k(G)$ is also a perfect graph.

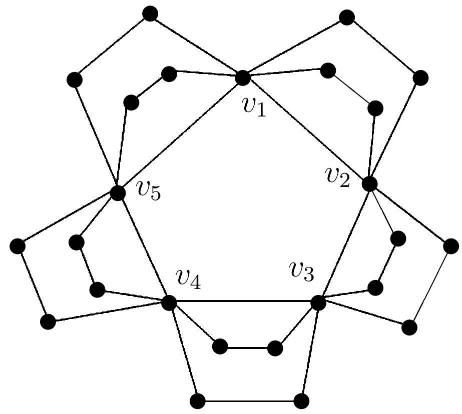

Figure 1. The trestled graph $T_2(C_5)$

Theorem 4. *Let G be a connected graph of order n and let k be any positive integer. Then $\rho^o(T_k(G)) = n$.*

P r o o f. Let $V(G) = \{v_1, v_2, \dots, v_n\}$. Let $N_1 = N_G(v_1)$ and, for every i, $2 \le i \le n$, $N_i = N_G(v_i) \setminus (\bigcup_{j=1}^{i-1} N_G(v_j))$. Clearly $N_1, N_2, \dots, N_n$ are disjoint, $\bigcup_{i=1}^{n} N_i = V(G)$ and, for every i, $1 \le i \le n$, $N_i \subseteq N_G(v_i)$. Now we consider the partition of $V(T_k(G)) \setminus V(G)$ into n disjoint subsets, say $N_1', N_2', \dots, N_n'$, as follows: for every i, $1 \le i \le n$, let $N_i' = N_{T_k(G)}(v_i) \setminus N_G(v_i)$. Clearly $V(T_k(G)) = \bigcup_{i=1}^{n}(N_i \cup N_i')$ and, for every i, $1 \le i \le n$, $N_i \cup N_i' \subseteq N_{T_k(G)}(v_i)$.

Now we claim that $\rho^o(T_k(G)) = n$. Let S be a ρ^o-set of $T_k(G)$. Then for every i, $1 \le i \le n$, $|S \cap N_{T_k(G)}(v_i)| \le 1$ and hence $|S \cap (N_i \cup N_i')| \le 1$ as $N_i \cup N_i' \subseteq N_{T_k(G)}(v_i)$. Since $V(T_k(G)) = \bigcup_{i=1}^{n}(N_i \cup N_i')$, it follows that $\rho^o(T_k(G)) = |S| \le n$. Also for each i, $1 \le i \le n$, choose $x_i \in N_i'$. Clearly $S' = \{x_1, x_2, \dots, x_n\}$ is an open packing set of $T_k(G)$ as no two $x_i'^s$ have a common neighbor in $T_k(G)$. Hence $\rho^o(T_k(G)) \ge n$. Thus $\rho^o(T_k(G)) = n$. $\square$

4. Conclusion and scope

In this paper we have determined the exact value of open packing number for some families of perfect graphs. Designing efficient algorithms for computing $\rho^o(G)$ for other classes of perfect graphs such as, interval graphs, circular arc graphs, bipartite graphs are some interesting problems for further investigation. In particular, finding the open packing number for trees is a challenging open problem for further study.

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ON ROUTING PROBLEM WITH STARTING POINT OPTIMIZATION[1]

Alexander G. Chentsov[a),b),†], **Pavel A. Chentsov**[b)]

[a)]Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, 620108, Russia

[b)]Ural Federal University,
19 Mira Str., Ekaterinburg, 620002, Russia

[†]chentsov@imm.uran.ru

Abstract: One problem focused on engineering applications is considered. It is assumed that sequential visits to megacities have been implemented. After all visits have been made, it is required to return to the starting point (a more complex dependence on the starting point is also considered). But the last requirement is not strict: some weakening of the return condition is acceptable. Under these assumptions, it is required to optimize the choice of starting point, route, and specific trajectory. The well-known dynamic programming (DP) is used for the solution. But when using DP, significant difficulties arise associated with the dependence of the terminal component of the criterion on the starting point. Starting point enumeration is required. We consider the possibility of reducing the enumeration associated with applied variants of universal (relative to the starting point) dynamic programming. Of course, this approach requires some transformation of the problem.

Keywords: Dynamic programming, Precedence conditions, Route.

Introduction

This study addresses the routing problem with precedence conditions and complicated cost functions. Besides, it is required to implement a return to the neighborhood of the starting point (more general variants are also considered). This condition may be related to the peculiarities of applied problems. We keep in mind the cutting of sheets and dismantling in the nuclear power industry. Of course, the well-known Traveling Salesman Problem (TSP) is a natural prototype for this problem. But in our setting, many new difficulties arise. We will mention just a few related to the TSP investigations; see [1, 7, 9–12, 16–18].

In engineering applications, the problem of visiting megacities often arises. This is due to the possible multivariance of the permutations. So, in the control problem when cutting sheets on CNC machines, these megacities are realized when digitizing the contours of parts; this discretization sampling is required for computer applications. Now let us note the precedence conditions. In addition, in control problems when cutting a sheet, these conditions arise, in particular, for the following reasons: for each part, cutting the inner contours must precede the cutting of the outer contour. Of course, there are other specific reasons for using precedence conditions. Among other restrictions, we note the requirements for thermal conductivity. It is useful to note that these requirements are dynamic in nature: they arise depending on the tasks being performed.

We emphasize the importance of starting point optimization. When dismantling radioactive elements, at a step of moving from the starting point, the performer is under the influence of all

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radioactive elements to be dismantled. So this step is very important. Therefore, a rational choice of starting point is important. The requirement to return to a neighborhood of the starting point may be related to reasons for sufficient proximity to the transport tool at the starting point. So, the constraints used arise from the needs of actual applied problems.

Of course, without taking into account the above restrictions, a very difficult extremal problem arises. This problem requires serious formalization and the development of theoretical methods. Therefore, this paper provides a detailed exposition of general mathematical concepts. Besides, we use fairly complex constructions of admissible solutions with a choice of basic components: starting point, route, and trajectory. This hierarchical construction of the solution is important.

We use dynamic programming (DP) as the main solution method. But, in our complete problem, the necessity of enumeration of starting points arises. More precisely, its own DP procedure is required for every starting point (in fact, this procedure is attached to the starting point). Apparently, the enumeration of starting points when employing DP is unavoidable. However, we can try to reduce this enumeration. For this, we use auxiliary DP procedures that are universal relative to the starting point; more precisely, we follow [6]. We construct minorant and majorant procedures using DP. In terms of these procedures realized more simply, we aim at the required reduction of the enumeration. Such a goal is attained by weakening the closed routing problem (but, we use our own method and in more general cases). In this case, we obtain a simpler solution to our complete problem. Also, we use an approach of [6] under more general conditions on movements when visiting megacities. This generalization is related to applied problems (for example, such a construction is required in sheet cutting problems).

1. General notions and designations

We use the standard set-theoretical notation (quantifiers and logical connectives), $\varnothing$ stands for the empty set and $\triangleq$ for equality by definition. A family is a set whose elements are also sets. For any objects x and y, we denote by $\{x;y\}$ an unordered pair of x and y: $x \in \{x;y\}$, $y \in \{x;y\}$, and $(z=x) \vee (z=y)$ for every $z \in \{x;y\}$. If s is an object, then $\{s\} \triangleq \{s;s\}$ is a singleton containing $s:\ s \in \{s\}$. Also, sets are objects. For any objects x and y, the family $(x,y) \triangleq \{\{x\};\{x;y\}\}$ is the ordered pair (see [14, Ch. II, Sect. 2]) with the first element x and the second element y. If h is an ordered pair, then $\mathrm{pr}_1(h)$ and $\mathrm{pr}_2(h)$ are the first and the second elements of h, respectively. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are objects, then $(\mathbf{a},\mathbf{b},\mathbf{c}) \triangleq ((\mathbf{a},\mathbf{b}),\mathbf{c})$ (see [8, Ch. 1]). If A, B, and C are sets, then $A \times B \times C \triangleq (A \times B) \times C$ (see [8, Ch. 1]). For a set H, we denote by $\mathcal{P}(H)$ and $\mathcal{P}'(H)$ the families of all subsets and all nonempty subsets of H, repectively; thus, $\mathcal{P}'(H) = \mathcal{P}(H) \setminus \{\varnothing\}$. Denote by $\mathrm{Fin}(H)$ the family of all finite nonempty subsets of the set H, $\mathrm{Fin}(H) \subset \mathcal{P}'(H)$. If H is a finite set, then $\mathrm{Fin}(H) = \mathcal{P}'(H)$.

If A, B, C, and D are nonempty sets, and g is a mapping from $A \times B \times C$ to D, then $(x,y) \in A \times B \times C$ for $x \in A \times B$ and $y \in C$, and the value $g(x,y) \in D$ is well defined; we also write this value as $g(x_1,x_2,y)$, where $x_1 = \mathrm{pr}_1(x) \in A$ and $x_2 = \mathrm{pr}_2(x) \in B$.

As usual, $\mathbb{R}$ denotes the real line,

$$\mathbb{R}_+ \triangleq \{\, \xi \in \mathbb{R} | 0 \le \xi\}, \quad \mathbb{N} \triangleq \{1;2;\ldots\}, \quad \mathbb{N}_0 \triangleq \{0\} \cup \mathbb{N} = \{0;1;2;\ldots\} \in \mathcal{P}'(\mathbb{R}_+);$$
$$\overline{p,q} \triangleq \{\, k \in \mathbb{N}_0 | (p \le k) \& (k \le q)\} \quad \forall p \in \mathbb{N}_0 \quad \forall q \in \mathbb{N}_0.$$

Of course, $\overline{1,0} = \varnothing$ and $\overline{1,s} = \{k \in \mathbb{N} | k \le s\}$ for $s \in \mathbb{N}$. For a nonempty finite set K, denote by $|K| \in \mathbb{N}$ the cardinality of K and by $(\mathrm{bi})[K]$ the set of all bijections from $\overline{1,|K|}$ onto K. Let $|\varnothing| \triangleq 0$. For a nonempty set S, denote by $\mathcal{R}_+[S]$ the set of all nonnegative real-valued functions on S.

2. The problem setting

Fix a nonempty set X and a set $X^0 \in \mathrm{Fin}(X)$. We consider X as a comprehending set and X^0 as the set of all possible starting points. Let $N \in \mathbb{N}$, $N \geq 2$. Let

$$M_1 \in \mathrm{Fin}(X), \quad \ldots, \quad M_N \in \mathrm{Fin}(X). \tag{2.1}$$

We consider the sets from (2.1) as megacities. These megacities are visiting objects. Suppose that

$$\big(X^0 \cap M_j = \varnothing \quad \forall j \in \overline{1,N}\big)\&\big(\, M_p \cap M_q = \varnothing \quad \forall p \in \overline{1,N} \quad \forall q \in \overline{1,N} \setminus \{p\}\big). \tag{2.2}$$

Conditions (2.2) are typical for routing problems. Finally, we fix (nonempty) relations

$$\mathbb{M}_1 \in \mathcal{P}'(M_1 \times M_1), \quad \ldots, \quad \mathbb{M}_N \in \mathcal{P}'(M_N \times M_N). \tag{2.3}$$

An ordered pair of $\mathbb{M}_j$, $j \in \overline{1,N}$, defines possible variants of works connected with visiting M_j. We call these works internal. For every megacity, we introduce arrival points and departure points: $\mathfrak{M}_j \triangleq \{\, \mathrm{pr}_1(z) :\ z \in \mathbb{M}_j\}$ and $\mathbf{M}_j \triangleq \{\, \mathrm{pr}_2(z) :\ z \in \mathbb{M}_j\}$ for $j \in \overline{1,N}$; of course, $\mathfrak{M}_j \in \mathcal{P}'(M_j)$ and $\mathbf{M}_j \in \mathcal{P}'(M_j)$. Moreover, we obtain that

$$\Big(\mathbb{X} \triangleq X^0 \cup (\bigcup_{i=1}^{N} \mathfrak{M}_i) \in \mathrm{Fin}(X)\Big)\&\Big(\mathbf{X} \triangleq X^0 \cup (\bigcup_{i=1}^{N} \mathbf{M}_i) \in \mathrm{Fin}(X)\Big).$$

Let $\mathfrak{N} \triangleq \mathcal{P}'(\overline{1,N})$, and let $\mathfrak{N}^{(j)} \triangleq \{\, K \in \mathfrak{N} | j \in K\}$ for $j \in \overline{1,N}$. We fix N mappings

$$A_1 : (\mathbf{X} \setminus \mathbf{M}_1) \times \mathfrak{N}^{(1)} \longrightarrow \mathcal{P}'(M_1), \quad \ldots, \quad A_N : (\mathbf{X} \setminus \mathbf{M}_N) \times \mathfrak{N}^{(N)} \longrightarrow \mathcal{P}'(M_N) \tag{2.4}$$

with the following property:

$$A_j(x, K) \cap \mathfrak{M}_j \neq \varnothing \quad \forall j \in \overline{1,N} \quad \forall x \in \mathbf{X} \setminus \mathbf{M}_j \quad \forall K \in \mathfrak{N}^{(j)}. \tag{2.5}$$

The mappings (2.4) are used for constraints representation; (2.5) is a compatibility condition. We note that our construction is similar to [2, Sect. 2]. But our mappings (2.4) are defined on smaller sets as compared with analogous mappings from [2, Sect. 2]. Note that our mappings (2.4) can be extended to analogous mappings from [2] (the corresponding variant was considered in [2, p. 215]). In this extension of our definition, conditions (2.5) turn into [2, (3)]. So, this extension is an unessential operation.

In what follows, $\mathbb{P} \triangleq (\mathrm{bi})[\overline{1,N}]$; elements of $\mathbb{P}$ are complete routes (index permutations). If $\alpha \in \mathbb{P}$, then $\alpha^{-1} \in \mathbb{P}$ is the inverse of α:

$$\alpha(\alpha^{-1}(k)) = \alpha^{-1}(\alpha(k)) = k \quad \forall k \in \overline{1,N}.$$

A specific choice of $\alpha \in \mathbb{P}$ may be restricted by precedence conditions. For their introduction, we suppose that a set $\mathbf{K} \in \mathcal{P}(\overline{1,N} \times \overline{1,N})$ is given. Let

$$\forall \mathbf{K}_0 \in \mathcal{P}'(\mathbf{K}) \quad \exists z_0 \in \mathbf{K}_0 :\ \mathrm{pr}_1(z_0) \neq \mathrm{pr}_2(z) \quad \forall z \in \mathbf{K}_0. \tag{2.6}$$

Specific cases of (2.6) were discussed in [3, Ch. 2]. Let

$$\begin{aligned} \mathbf{A} \triangleq \{\, \alpha \in \mathbb{P} |\ \forall t_1 \in \overline{1,N} \quad \forall t_2 \in \overline{1,N} \ \big((\alpha(t_1), \alpha(t_2)) \in \mathbf{K}\big) \Longrightarrow (t_1 < t_2)\} \\ = \{\, \alpha \in \mathbb{P} |\ \alpha^{-1}(\mathrm{pr}_1(z)) < \alpha^{-1}(\mathrm{pr}_2(z)) \ \ \forall z \in \mathbf{K}\} \neq \varnothing \end{aligned}$$

be the set of all routes (permutations) admissible by precedence. We consider the following processes:

$$(x \in X^0) \longrightarrow (x_1^{(1)} \in A_{\alpha(1)}(x, \overline{1,N}) \rightsquigarrow x_2^{(1)} \in \mathbf{M}_{\alpha(1)}) \longrightarrow \big(x_1^{(2)} \in A_{\alpha(2)}(x_2^{(1)}, \overline{1,N} \setminus \{\alpha(1)\}) \\ \rightsquigarrow x_2^{(2)} \in \mathbf{M}_{\alpha(2)}\big) \longrightarrow \cdots \longrightarrow \big(x_1^{(N)} \in A_{\alpha(N)}(x_2^{(N-1)}, \{\alpha(N)\}) \rightsquigarrow x_2^{(N)} \in \mathbf{M}_{\alpha(N)}\big) \tag{2.7}$$

where $\alpha \in \mathbf{A}$ and $(x_1^{(j)}, x_2^{(j)}) \in \mathbb{M}_{\alpha(j)}$ for $j \in \overline{1,N}$ (here, we suppose that the number N is sufficiently great). From (2.7), it is obvious that a trajectory coordinated with the route α is used. Let us introduce the corresponding definition. Let $\mathbb{Z}$ be the set of all mappings from $\overline{0,N}$ to $\mathbb{X} \times \mathbf{X}$. So, elements of $\mathbb{Z}$ are tuples

$$(z_i)_{i\in\overline{0,N}} : \overline{0,N} \longrightarrow \mathbb{X} \times \mathbf{X}$$

and only they. If $x \in X^0$ and $\alpha \in \mathbb{P}$, then

$$\mathcal{Z}_\alpha[x] \stackrel{\triangle}{=} \{ (z_t)_{t\in\overline{0,N}} \in \mathbb{Z} | (z_0 = (x,x)) \& (z_t \in \mathbb{M}_{\alpha(t)} \ \forall t \in \overline{1,N}) \\ \&(\mathrm{pr}_1(z_s) \in A_{\alpha(s)}(\mathrm{pr}_2(z_{s-1}), \{\alpha(l) : \ l \in \overline{s,N}\}) \ \forall s \in \overline{1,N})\}. \tag{2.8}$$

It is easy to verify that (2.8) corresponds to [2, (4)] (for $x = x^0$ in [2, (4)]). By analogy with [2], we give a natural extension of (2.8) to the case when the index set $\overline{1,N}$ is replaced by $K \in \mathfrak{N}$. For $K \in \mathfrak{N}$, we introduce the set $\mathbb{Z}_K$ of all tuples

$$(z_t)_{t\in\overline{0,|K|}} : \overline{0,|K|} \longrightarrow (\mathbb{X} \cup \mathbf{X}) \times \mathbf{X}.$$

Then, for $x \in \mathbf{X}$, $K \in \mathfrak{N}$, and $\alpha \in (\mathrm{bi})[K]$, we suppose that

$$\mathcal{Z}(x,K,\alpha) \stackrel{\triangle}{=} \big\{(z_t)_{t\in\overline{0,|K|}} \in \mathbb{Z}_K | (z_0 = (x,x)) \& (z_t \in \mathbb{M}_{\alpha(t)} \ \forall t \in \overline{1,|K|}) \\ \&(\mathrm{pr}_1(z_s) \in A_{\alpha(s)}(\mathrm{pr}_2(z_{s-1}), \{\alpha(l) : \ l \in \overline{s,|K|}\}) \ \forall s \in \overline{1,|K|})\big\}; \tag{2.9}$$

see [2, (5)]. It is verified by induction that (2.9) is a nonempty set. Besides, relations (2.3) are finite. Therefore,

$$\mathcal{Z}(x,K,\alpha) \in \mathrm{Fin}(\mathbb{Z}_K) \quad \forall x \in \mathbf{X} \quad \forall K \in \mathfrak{N} \quad \forall \alpha \in (\mathrm{bi})[K]. \tag{2.10}$$

Moreover, $\mathcal{Z}_\alpha[x] = \mathcal{Z}(x, \overline{1,N}, \alpha) \ \forall x \in X \ \forall \alpha \in \mathbb{P}$. As a result, by (2.10),

$$\mathcal{Z}_\alpha[x] \in \mathrm{Fin}(\mathbb{Z}) \quad \forall x \in X^0 \quad \forall \alpha \in \mathbb{P}.$$

Now, we introduce a modification of mappings (2.4). Namely, for $j \in \overline{1,N}$, $x \in \mathbf{X} \setminus \mathbf{M}_j$, and $\tilde{K} \in \mathfrak{N}^{(j)}$, we suppose that

$$\mathbb{A}_j(x, \tilde{K}) \stackrel{\triangle}{=} \{ z \in \mathbb{M}_j | \ \mathrm{pr}_1(z) \in A_j(x, \tilde{K})\}. \tag{2.11}$$

Then, by (2.9) and (2.11), we get that

$$\mathcal{Z}(x,K,\alpha) = \big\{(z_t)_{t\in\overline{0,|K|}} \in \mathbb{Z}_K | \ (z_0 = (x,x)) \\ \&(z_t \in \mathbb{A}_{\alpha(t)}(\mathrm{pr}_2(z_{t-1}), \{\alpha(l) : \ l \in \overline{t,|K|}\}) \ \forall t \in \overline{1,|K|})\big\} \tag{2.12}$$

for $x \in \mathbf{X}$, $K \in \mathfrak{N}$, and $\alpha \in (\mathrm{bi})[K]$. As a particular case of (2.12), the following representation holds for $x \in X^0$ and $\alpha \in \mathbb{P}$:

$$\mathcal{Z}_\alpha[x] = \big\{(z_t)_{t\in\overline{0,N}} \in \mathbb{Z} | \ (z_0 = (x,x)) \& (z_t \in \mathbb{A}_{\alpha(t)}(\mathrm{pr}_2(z_{t-1}), \{\alpha(l) : \ l \in \overline{t,N}\}) \ \forall t \in \overline{1,N})\big\}.$$

By analogy with [6, (2.11)], for $x \in X^0$, we set

$$\tilde{D}[x] \triangleq \{(\alpha, \mathbf{z}) \in \mathbf{A} \times \mathbb{Z} | \mathbf{z} \in \mathcal{Z}_\alpha[x]\}. \tag{2.13}$$

Of course, $\tilde{D}[x] \in \mathrm{Fin}(\mathbf{A} \times \mathbb{Z})\ \ \forall x \in X^0$. Finally,

$$\mathbf{D} \triangleq \{(\alpha, \mathbf{z}, x) \in \mathbf{A} \times \mathbb{Z} \times X^0 | (\alpha, \mathbf{z}) \in \tilde{D}[x]\} \in \mathrm{Fin}(\mathbf{A} \times \mathbb{Z} \times X^0). \tag{2.14}$$

In what follows, we suppose that $\mathbf{M} \triangleq \bigcup_{i=1}^{N} \mathbf{M}_i$; of course, $\mathbf{M} \in \mathrm{Fin}(\mathbf{X})$.

Cost functions. We fix the following $N + 2$ functions:

$$\mathbf{c} \in \mathcal{R}_+[\mathbf{X} \times \mathbb{X} \times \mathfrak{N}], \quad c_1 \in \mathcal{R}_+[\mathbb{M}_1 \times \mathfrak{N}], \quad \ldots, \quad c_N \in \mathcal{R}_+[\mathbb{M}_1 \times \mathfrak{N}], \quad f \in \mathcal{R}_+[\mathbf{M} \times X^0]. \tag{2.15}$$

We use the function $\mathbf{c}$ to estimate the (exterior) permutations between megacities and from X^0 to megacities. We use the functions c_j, $j \in \overline{1, N}$, to estimate the (interior) works connected with visiting M_j. Finally, f estimates the terminal state of our process (the point $x_2^{(N)}$ in (2.7)). In what follows, we consider only an additive criterion. For $x \in X^0$, $\alpha \in \mathbb{P}$, and $(z_t)_{t \in \overline{0,N}} \in \mathcal{Z}_\alpha[x]$, we consider

$$\begin{gathered}\mathfrak{C}_\alpha[(z_t)_{t \in \overline{0,N}} | x] \triangleq \sum_{t=1}^{N} \big[\mathbf{c}(\mathrm{pr}_2(z_{t-1}), \mathrm{pr}_1(z_t), \{\alpha(k) :\ k \in \overline{t, N}\}) + c_{\alpha(t)}(z_t, \{\alpha(k) :\ k \in \overline{t, N}\})\big] \\ + f(\mathrm{pr}_2(z_N), x) \in \mathbb{R}_+\end{gathered} \tag{2.16}$$

as a base for the criterion in an x-problem (x is a starting point). For $x \in X^0$, we obtain the following x-problem:

$$\mathfrak{C}_\alpha[\mathbf{z}|x] \longrightarrow \min, \quad (\alpha, \mathbf{z}) \in \tilde{D}[x]; \tag{2.17}$$

for this problem, the define the extremum $V[x] \in \mathbb{R}_+$ as the smallest of the numbers $\mathfrak{C}_\alpha[\mathbf{z}|x]$, $(\alpha, \mathbf{z}) \in \tilde{D}[x]$. Moreover, for $x \in X^0$,

$$(\mathrm{SOL})[x] \triangleq \{(\alpha, \mathbf{z}) \in \tilde{D}[x] |\ \mathfrak{C}_\alpha[\mathbf{z}|x] = V[x]\} \in \mathcal{P}'(\tilde{D}[x]) \tag{2.18}$$

is the set of all optimal solutions of problem (2.17).

Now, we introduce the following complete problem:

$$\mathfrak{C}_\alpha[\mathbf{z}|x] \longrightarrow \min, \quad (\alpha, \mathbf{z}, x) \in \mathbf{D}. \tag{2.19}$$

For this problem, the global extremum is defined as

$$\mathbb{V} \triangleq \min_{(\alpha, \mathbf{z}, x) \in \mathbf{D}} \mathfrak{C}_\alpha[\mathbf{z}|x] \in \mathbb{R}_+ \tag{2.20}$$

and the set of all optimal solutions is

$$\mathbf{SOL} \triangleq \{(\alpha, \mathbf{z}, x) \in \mathbf{D} |\ \mathfrak{C}_\alpha[\mathbf{z}|x] = \mathbb{V}\} \in \mathrm{Fin}(\mathbf{D}).$$

The following representation of $\mathbb{V}$ (2.20) is also useful:

$$\mathbb{V} \triangleq \min_{x \in X^0} V[x]. \tag{2.21}$$

In this paper, we consider only the possibilities of DP as a method for investigating problems (2.17) and (2.19). We keep in mind a variant of DP, which is a development of a scheme from [1]. Using representation (2.21), we consider the problem

$$V[x] \longrightarrow \min, \quad x \in X^0.$$

This problem allows us to obtain (see (2.21)) important properties of solutions to problem (2.19) in terms of x-problems (2.17), $x \in X^0$.

3. Dynamic programming

For the DP procedure developing an approach of [1], the dependence of the terminal function f on the starting point is an essentially complicated circumstance. Indeed, its own version of DP is required for every $x \in X^0$ (we keep in mind the search of optimal solutions to problem (2.19)). Therefore, we first consider a simpler case when the terminal function is independent of points from X^0. More precisely, in this section, we fix

$$\mathbf{f} \in \mathcal{R}_+[\mathbf{M}]. \tag{3.1}$$

Now, we introduce analogs of problems (2.17) and (2.19) corresponding to the change $f \longrightarrow \mathbf{f}$. For this, we replace (2.16) by the following expression for $x \in X^0$, $\alpha \in \mathbb{P}$, and $(z_t)_{t\in\overline{0,N}} \in \mathcal{Z}_\alpha[x]$:

$$\begin{aligned}\hat{\mathfrak{C}}_\alpha[(z_t)_{t\in\overline{0,N}}|\mathbf{f}] \triangleq \sum_{t=1}^{N} \big[\mathbf{c}(\mathrm{pr}_2(z_{t-1}), \mathrm{pr}_1(z_t), \{\alpha(k) :\ k \in \overline{t,N}\}) + c_{\alpha(t)}(z_t, \{\alpha(k) :\ k \in \overline{t,N}\})\big] \\ + \mathbf{f}(\mathrm{pr}_2(z_N)) \in \mathbb{R}_+.\end{aligned} \tag{3.2}$$

Then, (2.17) becomes the following problem. For $x \in X^0$, we consider the following (auxiliary) x-problem:

$$\hat{\mathfrak{C}}_\alpha[\mathbf{z}|\mathbf{f}] \longrightarrow \min, \quad (\alpha, \mathbf{z}) \in \tilde{D}[x]; \tag{3.3}$$

for problem (3.3), we introduce the extremum $\hat{V}[x|\mathbf{f}]$ as the smallest of the numbers $\hat{\mathfrak{C}}_\alpha[\mathbf{z}|\mathbf{f}]$, $(\alpha, \mathbf{z}) \in \tilde{D}[x]$. Similarly, we replace (2.19) by the following (complete) problem:

$$\hat{\mathfrak{C}}_\alpha[\mathbf{z}|\mathbf{f}] \longrightarrow \min, \quad (\alpha, \mathbf{z}, x) \in \mathbf{D}. \tag{3.4}$$

For this problem, we consider the extremum

$$\hat{\mathbb{V}}[\mathbf{f}] \triangleq \min_{(\alpha,\mathbf{z},x)\in\mathbf{D}} \hat{\mathfrak{C}}_\alpha[\mathbf{z}|\mathbf{f}] \in \mathbb{R}_+. \tag{3.5}$$

Moreover, by (2.15) and (3.5), we get that

$$\hat{\mathbb{V}}[\mathbf{f}] = \min_{x\in X^0} \hat{V}[x|\mathbf{f}].$$

Now, for $x \in X^0$, we introduce the set

$$\hat{\mathbb{S}}[x|\mathbf{f}] \triangleq \big\{(\alpha, \mathbf{z}) \in \tilde{D}[x] |\ \hat{\mathfrak{C}}_\alpha[\mathbf{z}|\mathbf{f}] = \hat{V}[x|\mathbf{f}]\big\} \in \mathrm{Fin}(\tilde{D}[x]) \tag{3.6}$$

of all optimal solutions to problem (3.3). Finally,

$$\hat{\mathbf{S}}[\mathbf{f}] \triangleq \big\{(\alpha, \mathbf{z}, x) \in \mathbf{D} |\ \hat{\mathfrak{C}}_\alpha[\mathbf{z}|\mathbf{f}] = \hat{\mathbb{V}}[\mathbf{f}]\big\} \in \mathrm{Fin}(\mathbf{D})$$

is the set of all optimal solutions to problem (3.4).

Returning to (2.12), we introduce partial routes admissible by deletion (tasks from a list). Namely, we introduce a mapping $\mathbf{I}$ operating in $\mathfrak{N}$ by the following rule for $K \in \mathfrak{N}$:

$$\mathbf{I}(K) \triangleq K \setminus \{\mathrm{pr}_2(z) :\ z \in \Xi[K]\}, \tag{3.7}$$

where $\Xi[K] \triangleq \big\{z \in \mathbf{K} |\ (\mathrm{pr}_1(z) \in K) \& (\mathrm{pr}_2(z) \in K)\big\}$. Suppose that, for $\mathbb{K} \in \mathfrak{N}$,

$$(\mathbf{I}-\mathrm{bi})[\mathbb{K}] \triangleq \big\{\alpha \in (\mathrm{bi})[\mathbb{K}] |\ \alpha(s) \in \mathbf{I}(\{\alpha(t) :\ t \in \overline{s, |\mathbb{K}|}\}) \ \ \forall s \in \overline{1, |\mathbb{K}|}\big\}.$$

Then, by statements of [3, Part 2], $(\mathbf{I}-\mathrm{bi})[K] \neq \varnothing$ for $K \in \mathfrak{N}$; moreover,

$$\begin{gathered}\mathbf{A} = (\mathbf{I}-\mathrm{bi})[\overline{1,N}] = \big\{\alpha \in \mathbb{P}|\ (\alpha(1) \in \mathbf{I}(\overline{1,N})) \\ \&(\alpha(k) \in \mathbf{I}(\overline{1,N} \setminus \{\alpha(l):\ l \in \overline{1,k-1}\})\ \forall k \in \overline{2,N})\big\}.\end{gathered} \tag{3.8}$$

Thus, according to (3.1), the admissibility by precedence and the admissibility by deletion are identical for complete routes. Now, for $x \in \mathbf{X}$ and $K \in \mathfrak{N}$, we consider the corresponding partial routing problem. First, we introduce a partial criterion. If $x \in \mathbf{X}$, $K \in \mathfrak{N}$, $\alpha \in (\mathrm{bi})[K]$, and $(z_t)_{t\in\overline{0,|K|}} \in \mathcal{Z}(x,K,\alpha)$, then

$$\begin{gathered}\hat{\mathfrak{C}}^*_\alpha[(z_t)_{t\in\overline{0,|K|}}|\mathbf{f};K] \triangleq \sum_{s=1}^{|K|}\big[\mathbf{c}(\mathrm{pr}_2(z_{s-1}),\mathrm{pr}_1(z_s),\{\alpha(t):\ t\in\overline{s,|K|}\}) \\ +c_{\alpha(s)}(z_s,\{\alpha(t):\ t\in\overline{s,|K|}\})\big] + \mathbf{f}(\mathrm{pr}_2(z_{|K|})).\end{gathered} \tag{3.9}$$

Of course, the case $K = \overline{1,N}$ is possible; so, for $\alpha \in \mathbb{P}$ and $(z_t)_{t\in\overline{0,N}} \in \mathcal{Z}_\alpha[x]$, the number $\hat{\mathfrak{C}}^*_\alpha[(z_t)_{t\in\overline{0,N}}|\mathbf{f};\overline{1,N}] \in \mathbb{R}_+$ is defined. In addition, by (3.2) and (3.9), for $x \in \mathbf{X}$, $K \in \mathfrak{N}$, $\alpha \in \mathbb{P}$, and $\mathbf{z} \in \mathcal{Z}_\alpha[x]$, we have

$$\hat{\mathfrak{C}}_\alpha[\mathbf{z}|\mathbf{f}] = \hat{\mathfrak{C}}^*_\alpha[\mathbf{z}|\mathbf{f};\overline{1,N}]. \tag{3.10}$$

By analogy with (2.13), for $x \in \mathbf{X}$ and $K \in \mathfrak{N}$, we set

$$\hat{D}^*(x,K) \triangleq \big\{(\alpha,\mathbf{z}) \in (\mathbf{I}-\mathrm{bi})[K]\times\mathbb{Z}_K|\ \mathbf{z}\in\mathcal{Z}(x,K,\alpha)\big\} \in \mathrm{Fin}((\mathbf{I}-\mathrm{bi})[K]\times\mathbb{Z}_K).$$

For $x \in \mathbf{X}$ and $K \in \mathfrak{N}$, consider the following problem:

$$\hat{\mathfrak{C}}^*_\alpha[\mathbf{z}|\ \mathbf{f};K] \longrightarrow \min, \quad (\alpha,\mathbf{z}) \in \hat{D}^*(x,K);$$

$v_\mathbf{f}(x,K)$ denotes the smallest of the numbers $\hat{\mathfrak{C}}^*_\alpha[\mathbf{z}|\ \mathbf{f};K]$, $(\alpha,\mathbf{z}) \in \hat{D}^*(x,K)$. Note that, by (2.13) and (3.8), we have

$$\tilde{D}[x] = \hat{D}^*(x,\overline{1,N}) \quad \forall x \in X^0. \tag{3.11}$$

From (3.10) and (3.11), we get that

$$\hat{V}[x|\mathbf{f}] = v_\mathbf{f}(x,\overline{1,N}) \quad \forall x \in X^0. \tag{3.12}$$

Finally, we set

$$v_\mathbf{f}(x,\varnothing) \triangleq \mathbf{f}(x) \quad \forall x \in \mathbf{M}. \tag{3.13}$$

Now we construct a function defined on $(\mathbf{X}\times\mathfrak{N})\cup(\mathbf{M}\times\{\varnothing\})$. Namely,

$$v_\mathbf{f} \in \mathcal{R}_+[(\mathbf{X}\times\mathfrak{N})\cup(\mathbf{M}\times\{\varnothing\})]$$

is defined by the following conditions:

$$\big(v_\mathbf{f}(x,K) \triangleq \min_{(\alpha,\mathbf{z})\in\hat{D}^*(x,K)} \hat{\mathfrak{C}}^*_\alpha[\mathbf{z}|\mathbf{f};K] \ \ \forall (x,K)\in\mathbf{X}\times\mathfrak{N}\big)\&(v_\mathbf{f}(x,\varnothing) \triangleq \mathbf{f}(x) \ \ \forall x\in\mathbf{M}) \tag{3.14}$$

(we use the obvious equality $\mathcal{P}(\overline{1,N}) = \mathfrak{N}\cup\{\varnothing\}$). By (3.12), we define the value function

$$\hat{V}[\cdot|\mathbf{f}] \triangleq (\hat{V}[x|\mathbf{f}])_{x\in X^0} \in \mathcal{R}_+[X^0]. \tag{3.15}$$

Theorem 1. *If $x \in \mathbf{X}$ and $K \in \mathfrak{N}$, then*

$$v_\mathbf{f}(x,K) = \min_{j\in\mathbf{I}(K)}\ \min_{z\in\mathbb{A}_j(x,K)} \big[\mathbf{c}(x,\mathrm{pr}_1(z),K) + c_j(z,K) + v_\mathbf{f}(\mathrm{pr}_2(z),K\setminus\{j\})\big]. \tag{3.16}$$

Theorem 1 is extracted from [2, Theorem 1]. Now, we discuss only some peculiarities (note that [2, Theorem 1] was proved by analogy with [4, Theorem 5.1]). In [2, Theorem 1], the case of fixed starting point was considered. However, the Bellman function from [2, Sect. 3] can be defined on the set $\mathbf{X} \times \mathcal{P}(\overline{1,N})$ (note that, in [4], not the whole Bellman function was used): we follow a scheme of [2, Sect. 3] for every starting point from X^0. In addition, the definition of the Bellman function from [2, Sect. 3] corresponds to (3.14). Thus, Theorem 1 from [2] is true in our case (also note [13, Theorem 1] where this question was also considered). We obtain (3.16). As a particular case, we note the corresponding analog of [2, (18)]: if $x \in X^0$, then, by (3.12) and (3.16),

$$\hat{V}[x|\mathbf{f}] = \min_{j \in \mathbf{I}(\overline{1,N})} \min_{z \in \mathbb{A}_j(x,\overline{1,N})} \big[\mathbf{c}(x, \mathrm{pr}_1(z), \overline{1,N}) + c_j(z, \overline{1,N}) + v_{\mathbf{f}}(\mathrm{pr}_2(z), \overline{1,N} \setminus \{j\})\big]. \tag{3.17}$$

As in [2–4, 6], to reduce computational complexity, we will only constructbuild special layers of our Bellman function. First, we introduce special subsets of $\overline{1,N}$. We consider these subsets as substantial task lists. Let

$$\mathcal{G} \triangleq \big\{K \in \mathfrak{N} |\ \forall z \in \mathbf{K}\ \ (\mathrm{pr}_1(z) \in K) \Longrightarrow (\mathrm{pr}_2(z) \in K)\big\}.$$

Let also $\mathcal{G}_s \triangleq \{\ K \in \mathcal{G} |\ s = |K|\}\ \forall s \in \overline{1,N}$. Then $\{\mathcal{G}_1; \ldots; \mathcal{G}_N\}$ is a partition of $\mathcal{G}$; $\mathcal{G}_N = \{\overline{1,N}\}$ and $\mathcal{G}_1 = \big\{\{t\} :\ t \in \overline{1,N} \setminus \mathbf{K}_1\big\}$, where $\mathbf{K}_1 \triangleq \{\mathrm{pr}_1(z) :\ z \in \mathbf{K}\}$. Finally (see [2–4, 6]),

$$\mathcal{G}_{s-1} = \{\ K \setminus \{t\} :\ K \in \mathcal{G}_s,\ t \in \mathbf{I}(K)\}. \tag{3.18}$$

So, we can implement the procedure $\mathcal{G}_N \longrightarrow \mathcal{G}_{N-1} \longrightarrow \cdots \longrightarrow \mathcal{G}_1$ (we use (3.18)). Further, we construct sets $\mathcal{D}_0,\ \mathcal{D}_1, \ldots, \mathcal{D}_N$. Let $\mathcal{D}_N \triangleq \{\ (x, \overline{1,N}) :\ x \in X^0\}$ and

$$\mathcal{D}_0 \triangleq \{\ (x, \varnothing) :\ x \in \bigcup_{i \in \overline{1,N} \setminus \mathbf{K}_1} \mathbf{M}_i\}. \tag{3.19}$$

For $s \in \overline{1, N-1}$ and $K \in \mathcal{G}_s$, we successively construct

$$\mathcal{J}_s(K) \triangleq \big\{j \in \overline{1,N} \setminus K |\ \{j\} \cup K {\in} \mathcal{G}_{s+1}\big\}, \quad \mathcal{M}_s[K] \triangleq \bigcup_{j \in \mathcal{J}_s(K)} \mathbf{M}_j,\ \mathbb{D}_s[K] \triangleq \big\{(x, K) :\ x {\in} \mathcal{M}_s[K]\big\}$$

(all these sets are nonempty; see [3, Sect. 4.9]). Finally, for $s \in \overline{1, N-1}$, we set

$$\mathcal{D}_s \triangleq \bigcup_{K \in \mathcal{G}_s} \mathbb{D}_s[K].$$

So, all layers $\mathcal{D}_0,\ \mathcal{D}_1, \ldots, \mathcal{D}_N$ are constructed. We recall that (see [2, (3.6)])

$$\big(\mathrm{pr}_2(z), K \setminus \{j\}\big) \in \mathcal{D}_{s-1} \quad \forall s \in \overline{1,N} \quad \forall (x, K) \in \mathcal{D}_s \quad \forall j \in \mathbf{I}(K) \quad \forall z \in \mathbb{M}_j. \tag{3.20}$$

Now, we construct the Bellman function layers using Theorem 1 and (3.20). More precisely, we keep in mind the functions

$$v_{\mathbf{f}}^{(0)} \in \mathcal{R}_+[\mathcal{D}_0], \quad v_{\mathbf{f}}^{(1)} \in \mathcal{R}_+[\mathcal{D}_1], \quad \ldots, \quad v_{\mathbf{f}}^{(N)} \in \mathcal{R}_+[\mathcal{D}_N]. \tag{3.21}$$

Using (3.13) and (3.19), we set

$$v_{\mathbf{f}}^{(0)}(x, \varnothing) \triangleq \mathbf{f}(x)\ \ \forall x \in \bigcup_{i \in \overline{1,N} \setminus \mathbf{K}_1} \mathbf{M}_i. \tag{3.22}$$

In general, we define $v_{\mathbf{f}}^{(s)}$ for $s \in \overline{0, N}$ by the following rule:

$$v_{\mathbf{f}}^{(s)}(x, K) \triangleq v_{\mathbf{f}}(x, K) \quad \forall (x, K) \in \mathcal{D}_s \tag{3.23}$$

((3.22) is a particular case of (3.23)). Of course, (3.23) is a mathematical definition. Now, we introduce a recurrence procedure for immediate construction of all functions (3.21). Namely, $v_{\mathbf{f}}^{(0)}$ is known (see (3.22)). For $s \in \overline{1, N}$, the transformation $v_{\mathbf{f}}^{(s-1)} \longrightarrow v_{\mathbf{f}}^{(s)}$ is implemented by the rule

$$v_{\mathbf{f}}^{(s)}(x, K) = \min_{j \in \mathbf{I}(K)} \min_{z \in \mathbb{A}_j(x,K)} \big[\mathbf{c}(x, \mathrm{pr}_1(z), K) + c_j(z, K) + v_{\mathbf{f}}^{(s-1)}(\mathrm{pr}_2(z), K \setminus \{j\})\big] \quad \forall (x, K) \in D_s. \tag{3.24}$$

So, (3.24) defines the following recurrence procedure:

$$v_{\mathbf{f}}^{(0)} \longrightarrow v_{\mathbf{f}}^{(1)} \longrightarrow \cdots \longrightarrow v_{\mathbf{f}}^{(N)}.$$

Note that, by (3.23), $v_{\mathbf{f}}^{(N)}(x, K) = v_{\mathbf{f}}(x, K)$ for $(x, K) \in \mathcal{D}_N$. Using (3.12) and the representation of $\mathcal{D}_N$, we get that

$$\hat{V}[x|\mathbf{f}] = v_{\mathbf{f}}(x, \overline{1, N}) = v_{\mathbf{f}}^{(N)}(x, \overline{1, N}) \quad \forall x \in X^0. \tag{3.25}$$

Here, we note an obvious corollary of (3.17). Namely, by (3.20), we get that, $(\mathrm{pr}_2(z), \overline{1, N} \setminus \{j\}) \in \mathcal{D}_{N-1}$ for $x \in X^0$, $j \in \mathbf{I}(\overline{1, N})$, and $z \in \mathbb{M}_j$. Therefore, by (3.17), we have

$$\hat{V}[x|\mathbf{f}] = \min_{j \in \mathbf{I}(\overline{1,N})} \min_{z \in \mathbb{A}_j(x,\overline{1,N})} [\mathbf{c}(x, \mathrm{pr}_1(z), \overline{1, N}) + c_j(z, \overline{1, N}) + v_{\mathbf{f}}^{(N-1)}(\mathrm{pr}_2(z), \overline{1, N} \setminus \{j\})] \quad \forall x \in X^0$$

(we use (2.11) and (3.23)). So, we can construct $\hat{V}[\cdot|\mathbf{f}]$ (3.15). As a corollary, we can find $\hat{\mathbb{V}}[\mathbf{f}]$ (3.8) and a point $x^0 \in X^0$ for which $\hat{V}[x^0|\mathbf{f}] = \hat{\mathbb{V}}[\mathbf{f}]$. So, x^0 is an optimal starting point in the problem with the terminal function $\mathbf{f}$.

Now, we will build an optimal solution to problem (3.7). We fix a point x^0 with this optimality property. Let $\mathbf{z}^{(0)} \triangleq (x^0, x^0)$. Using (3.24), we choose $\eta_1 \in \mathbf{I}(\overline{1, N})$ and $\mathbf{z}^{(1)} \in \mathbb{A}_{\eta_1}(x^0, \overline{1, N})$ for which

$$\hat{V}[x^0|\mathbf{f}] = \mathbf{c}(x^0, \mathrm{pr}_1(\mathbf{z}^{(1)}, \overline{1, N}) + c_{\eta_1}(\mathbf{z}^{(1)}, \overline{1, N}) + v_{\mathbf{f}}^{(N-1)}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1, N} \setminus \{\eta_1\}) \tag{3.26}$$

(we follow a procedure of [2, Sect. 4]). Then, $(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1, N} \setminus \{\eta_1\}) \in \mathcal{D}_{N-1}$, and therefore (see (3.24))

$$\begin{aligned} v_{\mathbf{f}}^{(N-1)}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1, N} \setminus \{\eta_1\}) = \min_{j \in \mathbf{I}(\overline{1,N}\setminus\{\eta_1\})} \min_{z \in \mathbb{A}_j(\mathrm{pr}_2(\mathbf{z}^{(1)}, \overline{1,N}\setminus\{\eta_1\})} \big[\mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \mathrm{pr}_1(z), \overline{1, N} \setminus \{\eta_1\}) \\ + c_j(z, \overline{1, N} \setminus \{\eta_1\}) + v_{\mathbf{f}}^{(N-2)}(\mathrm{pr}_2(z), \overline{1, N} \setminus \{\eta_1; j\})\big]; \end{aligned} \tag{3.27}$$

of course, we take into account that, by (3.20),

$$(\mathrm{pr}_2(z), \overline{1, N} \setminus \{\eta_1; j\}) = (\mathrm{pr}_2(z), (\overline{1, N} \setminus \{\eta_1\}) \setminus \{j\}) \in \mathcal{D}_{N-2} \quad \forall j \in \mathbf{I}(\overline{1, N} \setminus \{\eta_1\}) \quad \forall z \in \mathbb{M}_j.$$

Now, using (3.27), we choose $\eta_2 \in \mathbf{I}(\overline{1, N} \setminus \{\eta_1\})$ and $\mathbf{z}^{(2)} \in \mathbb{A}_{\eta_2}(\mathrm{pr}_2(\mathbf{z}^{(1)}, \overline{1, N} \setminus \{\eta_1\})$ for which

$$\begin{aligned} v_{\mathbf{f}}^{(N-1)}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \overline{1, N} \setminus \{\eta_1\}) = \mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \mathrm{pr}_1(\mathbf{z}^{(2)}), \overline{1, N} \setminus \{\eta_1\}) + c_{\eta_2}(\mathbf{z}^{(2)}, \overline{1, N} \setminus \{\eta_1\}) \\ + v_{\mathbf{f}}^{(N-2)}(\mathrm{pr}_2(\mathbf{z}^{(2)}), \overline{1, N} \setminus \{\eta_1; \eta_2\}). \end{aligned} \tag{3.28}$$

From (3.26) and (3.28), we get that

$$\hat{V}[x^0|\mathbf{f}] = \mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(0)}), \mathrm{pr}_1(\mathbf{z}^{(1)}), \overline{1,N}) + \mathbf{c}(\mathrm{pr}_2(\mathbf{z}^{(1)}), \mathrm{pr}_1(\mathbf{z}^{(2)}), \overline{1,N} \setminus \{\eta_1\}) + c_{\eta_1}(\mathbf{z}^{(1)}, \overline{1,N})$$
$$+ c_{\eta_2}(\mathbf{z}^{(2)}, \overline{1,N} \setminus \{\eta_1\}) + v_{\mathbf{f}}^{(N-2)}(\mathrm{pr}_2(\mathbf{z}^{(2)}), \overline{1,N} \setminus \{\eta_1; \eta_2\}).$$

Of course, $\eta_1 \neq \eta_2$. Further, procedures similar to (3.26) and (3.28) must be continued until $\overline{1,N}$ is exhausted. As a result (see [2, Sect. 4]), $\eta \triangleq (\eta_j)_{j \in \overline{1,N}} \in \mathbf{A}$ and $(\mathbf{z}^{(j)})_{j \in \overline{0,N}} \in \mathcal{Z}_\eta[x^0]$ with the property

$$\hat{\mathfrak{C}}_\eta[(\mathbf{z}^{(j)})_{j \in \overline{0,N}}|\mathbf{f}] = \hat{V}[x^0|\mathbf{f}] \tag{3.29}$$

will be build. Using the optimality of x^0, we get from (3.29) that

$$(\eta, (\mathbf{z}^{(j)})_{j \in \overline{0,N}}, x^0) \in \hat{\mathbf{S}}[\mathbf{f}] \tag{3.30}$$

(indeed, by (3.6) and (3.29), $(\eta, (\mathbf{z}^{(j)})_{j \in \overline{0,N}}) \in \hat{\mathbb{S}}[x^0|\mathbf{f}]$). So, for problem (3.4), we found the global extremum $\hat{\mathbb{V}}[\mathbf{f}]$ and an optimal solution. In what follows, we consider (3.4) as an auxiliary problem.

4. Individual dynamic programming

Now, we return to problem (2.19) for which the terminal component of our criterion corresponds to (2.16). In this case, the DP procedure is "attached" to the starting point. More precisely, its own DP procedure is required for every point $x \in X^0$. This procedure corresponds to [4, 5]. Therefore, we consider it very briefly. We use $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N$ and layers $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_{N-1}$ from Section 3. For $x \in X^0$, we fix $D_N(x) \triangleq \{(x, \overline{1,N})\}$ and obtain a singleton attached to the starting point x. To universal notation, we set

$$\big(D_j(x) \triangleq \mathcal{D}_j \ \forall j \in \overline{0, N-1}\big) \& \big(D_N(x) \triangleq \{(x, \overline{1,N})\}\big), \tag{4.1}$$

where $x \in X^0$. By (4.1), we have only one singular layer (corresponding to the index N). Recall (see [5]) that, for $x \in X^0$,

$$(\mathrm{pr}_2(z), K \setminus \{j\}) \in D_{s-1}(x) \quad \forall s \in \overline{1,N} \quad \forall (y,K) \in D_s(x) \quad \forall j \in \mathbf{I}(K) \quad \forall z \in \mathbb{M}_j. \tag{4.2}$$

So, in (4.2), we have a natural analog of (3.20).

Note that, to solve the x-problem for $x \in X^0$, we can use a DP procedure from [2] (a DP procedure from [5] is a particular case of that from [2]). Now, we will restrict ourselves to the algorithmic version of the presentation.

For more concise notation, we fix $x \in X^0$ unless otherwise stated. Recall that $D_j(x) \neq \varnothing$ $\forall j \in \overline{0,N}$. Now, we introduce a recurrence procedure for constructing layers of the Bellman function

$$\mathbf{v}_0[x] \in \mathcal{R}_+[D_0(x)], \quad \mathbf{v}_1[x] \in \mathcal{R}_+[D_1(x)], \quad \ldots, \quad \mathbf{v}_N[x] \in \mathcal{R}_+[D_N(x)]. \tag{4.3}$$

Let

$$\mathbf{v}_0[x](y, \varnothing) \triangleq f(y,x) \ \forall y \in \bigcup_{j \in \overline{1,N} \setminus \mathbf{K}_1} \mathbf{M}_j; \tag{4.4}$$

we use the obvious equality $D_0(x) = \mathcal{D}_0$, see (3.19). Further, for $s \in \overline{1,N}$, transformation of $\mathbf{v}_{s-1}[x] \in \mathcal{R}_+[D_{s-1}(x)]$ to $\mathbf{v}_s[x] \in \mathcal{R}_+[D_s(x)]$ is defined by the following rule for $(y,K) \in D_s(x)$:

$$\mathbf{v}_s[x](y,K) \triangleq \min_{j \in \mathbf{I}(K)} \min_{z \in \mathbb{A}_j(y,K)} \big[\mathbf{c}(y, \mathrm{pr}_1(z), K) + c_j(z,K) + \mathbf{v}_{s-1}[x](\mathrm{pr}_2(z), K \setminus \{j\})\big] \tag{4.5}$$

(we use (4.2)). In this section, we suppose that functions (4.3) are implemented by the procedure

$$\mathbf{v}_0[x] \longrightarrow \mathbf{v}_1[x] \longrightarrow \cdots \longrightarrow \mathbf{v}_N[x]. \tag{4.6}$$

More precisely, we have $\mathbf{v}_0[x]$ (see (4.4)) and construct $\mathbf{v}_1[x]$ using (4.5) for $s = 1$ and so on. Here, there exists a unique Bellman function (see [2, Sect. 3]) for which functions (4.3) are implemented as a contraction system; this property is similar to (3.23). But now we restrict ourselves to the recurrence procedure (4.6).

The function $\mathbf{v}_N[x]$ is defined by the unique value

$$\mathbf{v}_N[x](x, \overline{1,N}) = V[x] \tag{4.7}$$

similar to (3.25); in this connection, see [2, (2.3)]. We obtain an extremum of the x-problem (2.17). Now we will very briefly consider the procedure that implements an element of (SOL)$[x]$ (2.18).

Suppose that $\mathbf{h}^{(0)} \triangleq (x, x)$. Using an analog of [2, (23)], we choose $\zeta_1 \in \mathbf{I}(\overline{1,N})$ and $\mathbf{h}^{(1)} \in \mathbb{A}_{\zeta_1}(x, \overline{1,N})$ for which

$$V[x] = \mathbf{c}(x, \mathrm{pr}_1(\mathbf{h}^{(1)}), \overline{1,N}) + c_{\zeta_1}(\mathbf{h}^{(1)}, \overline{1,N}) + \mathbf{v}_{N-1}[x](\mathrm{pr}_2(\mathbf{h}^{(1)}), \overline{1,N} \setminus \{\zeta_1\}); \tag{4.8}$$

by (4.2), we have $(\mathrm{pr}_2(\mathbf{h}^{(1)}), \overline{1,N} \setminus \{\zeta_1\}) \in D_{N-1}(x)$. From (4.5), we get that

$$\begin{aligned}\mathbf{v}_{N-1}[x](\mathrm{pr}_2(\mathbf{h}^{(1)}), \overline{1,N} \setminus \{\zeta_1\}) = \min_{j \in \mathbf{I}(\overline{1,N} \setminus \{\zeta_1\})} \ \min_{z \in \mathbb{A}_j(\mathrm{pr}_2(\mathbf{h}^{(1)}), \overline{1,N} \setminus \{\zeta_1\})} \big[\mathbf{c}(\mathrm{pr}_2(\mathbf{h}^{(1)}), \mathrm{pr}_1(z), \\ \overline{1,N} \setminus \{\zeta_1\}) + c_j(z, \overline{1,N} \setminus \{\zeta_1\}) + \mathbf{v}_{N-2}[x](\mathrm{pr}_2(z), \overline{1,N} \setminus \{\zeta_1; j\})\big].\end{aligned} \tag{4.9}$$

Using (4.9), we choose $\zeta_2 \in \mathbf{I}(\overline{1,N} \setminus \{\zeta_1\})$ and $\mathbf{h}^{(2)} \in \mathbb{A}_{\zeta_2}(\mathrm{pr}_2(\mathbf{h}^{(1)}), \overline{1,N} \setminus \{\zeta_1\})$ for which

$$\begin{aligned}\mathbf{v}_{N-1}[x](\mathrm{pr}_2(\mathbf{h}^{(1)}), \overline{1,N} \setminus \{\zeta_1\}) = \mathbf{c}(\mathrm{pr}_2(\mathbf{h}^{(1)}), \mathrm{pr}_1(\mathbf{h}^{(2)}), \overline{1,N} \setminus \{\zeta_1\}) \\ + c_{\zeta_2}(\mathbf{h}^{(2)}, \overline{1,N} \setminus \{\zeta_1\}) + \mathbf{v}_{N-2}[x](\mathrm{pr}_2(\mathbf{h}^{(2)}), \overline{1,N} \setminus \{\zeta_1; \zeta_2\}).\end{aligned} \tag{4.10}$$

By (4.2), we have $(\mathrm{pr}_2(\mathbf{h}^{(2)}), \overline{1,N} \setminus \{\zeta_1; \zeta_2\}) = (\mathrm{pr}_2(\mathbf{h}^{(2)}), (\overline{1,N} \setminus \{\zeta_1\}) \setminus \{\zeta_2\}) \in D_{N-2}(x)$. From (4.8) and (4.10), we obtain the following equality:

$$\begin{aligned}V[x] = \mathbf{c}(x, \mathrm{pr}_1(\mathbf{h}^{(1)}), \overline{1,N}) + \mathbf{c}(\mathrm{pr}_2(\mathbf{h}^{(1)}), \mathrm{pr}_1(\mathbf{h}^{(2)}), \overline{1,N} \setminus \{\zeta_1\}) + c_{\zeta_1}(\mathbf{h}^{(1)}, \overline{1,N}) \\ + c_{\zeta_2}(\mathbf{h}^{(2)}, \overline{1,N} \setminus \{\zeta_1\}) + \mathbf{v}_{N-2}[x](\mathrm{pr}_2(\mathbf{h}^{(2)}), \overline{1,N} \setminus \{\zeta_1; \zeta_2\}).\end{aligned} \tag{4.11}$$

Further, procedures similar to (4.8) and (4.10) must be continued until $\overline{1,N}$ is exhausted. As a result, we get (see [2, Sect. 4]) that

$$\zeta \triangleq (\zeta_j)_{j \in \overline{1,N}} \in \mathbf{A} : \ (\mathbf{h}^{(j)})_{j \in \overline{0,N}} \in \mathcal{Z}_\zeta[x]; \tag{4.12}$$

in addition, $\mathfrak{C}_\zeta[(\mathbf{h}^{(j)})_{j \in \overline{0,N}}|x] = V[x]$ (the latter equality for $N = 2$ follows from (4.11)). Thus, $(\zeta, (\mathbf{h}^{(j)})_{j \in \overline{0,N}}) \in (\mathrm{SOL})[x]$.

Remark 1. Now, we return to procedure (4.6). If our goal is only to define $V[x]$ (4.7), then we can use the following analog of a procedure from [15]. Namely, we consider the issue of some memory savings. We have $\mathbf{v}_0[x]$. Let $s \in \overline{1,N}$ and $\mathbf{v}_{s-1}[x]$ be known. Then, by (4.5), we construct $\mathbf{v}_s[x]$. If $s = N$, then our procedure is complete. If $s < N$, then $\mathbf{v}_{s-1}[x]$ is annihilated and replaced by $\mathbf{v}_s[x]$. So, in the computer memory for this scheme there situated only one layer of the Bellman function. As a result, we obtain (4.7). This procedure must be implemented for every $x \in X^0$. In addition, we can determine all values $V[x]$, $x \in X^0$ without using (4.8)–(4.11). As a result, we can find $\mathbb{V}$ (2.21) and $x^0 \in X^0$ for which $V[x^0] = \mathbb{V}$.

Further, we implement scheme (4.8)–(4.11) for $x = x^0$. Since $(\zeta, (\mathbf{h}^{(j)})_{j\in\overline{0,N}}, x^0)$ (see (4.12)), we obtain an optimal solution to problem (2.19). Our scheme includes solutions to all x-problems (2.17), although using (4.8)–(4.11) is required once. We get a laborious procedure. Therefore, below we will consider an approach in which it is assumed that the enumeration of not all points $x \in X^0$ is realized. Namely, we introduce special majorizing and minorazing problems for which simpler versions of DP procedures can be used.

5. Auxiliary routing problems and enumeration problem

First, we introduce a special variant of the problem from Section 3. Consider a function $\mathbf{O} \in \mathcal{R}_+[\mathbf{M}]$ such that

$$\mathbf{O}(x) \stackrel{\triangle}{=} 0 \quad \forall x \in \mathbf{M}. \tag{5.1}$$

Now, we implement the constructions from Section 3 for

$$\mathbf{f} = \mathbf{O}. \tag{5.2}$$

But, in this implementation, we first restrict ourselves to the construction of the function

$$\hat{V}[\cdot|\mathbf{O}] = (\hat{V}[x|\mathbf{O}])_{x\in X^0} \in \mathcal{R}_+[X^0]. \tag{5.3}$$

For this, we use a variant of DP similar to that in Remark 1. More precisely, we use the layers $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_N$ from Section 3 with property (3.20).

Further, we construct functions (3.21) for (5.2). Namely, $v_{\mathbf{O}}^{(0)} \in \mathcal{R}_+]\mathcal{D}_0]$ is defined by the rule $v_{\mathbf{O}}^{(0)}(x, K) = 0 \ \forall (x, K) \in \mathcal{D}_0$; thus, $v_{\mathbf{O}}^{(0)}$ is identically equal to zero. If $s \in \overline{1,N}$ and $v_{\mathbf{O}}^{(s-1)} \in \mathcal{R}_+[\mathcal{D}_{s-1}]$ is known, then we define $v_{\mathbf{O}}^{(s)} \in \mathcal{R}_+[\mathcal{D}_s]$ by the rule

$$v_{\mathbf{O}}^{(s)}(x, K) = \min_{j\in\mathbf{I}(K)} \min_{z\in\mathbb{A}_j(x,K)} [\mathbf{c}(x, \mathrm{pr}_1(z), K) + c_j(z, K) + v_{\mathbf{O}}^{(s-1)}(\mathrm{pr}_2(z), K \setminus \{j\})] \quad \forall (x, K) \in D_s. \tag{5.4}$$

If $s = N$, then $\hat{V}[x|\mathbf{O}] = v_{\mathbf{O}}^{(s)}(x, \overline{1,N}) \quad \forall x \in X^0$ (we obtain function (5.3)); our procedure is complete. If $s < N$, then $v_{\mathbf{O}}^{(s-1)}$ is annihilated and replaced by $v_{\mathbf{O}}^{(s)}$. The layer $v_{\mathbf{O}}^{(s)}$ is used for constructing $v_{\mathbf{O}}^{(s+1)}$. As a result, we obtain $v_{\mathbf{O}}^{(N)} \in \mathcal{R}_+[\mathcal{D}_N]$ for which

$$\hat{V}[\cdot|\mathbf{O}] = v_{\mathbf{O}}^{(N)}. \tag{5.5}$$

Further, we use (5.5) to construct a majorizing function on X^0. In addition, by (2.16) and (3.2) for $x \in X^0$, $\alpha \in \mathbb{P}$, and $(z_t)_{t\in\overline{0,N}} \in \mathcal{Z}_\alpha[x]$, we have

$$\mathfrak{C}_\alpha[(z_t)_{t\in\overline{0,N}}|x] = \hat{\mathfrak{C}}_\alpha[(z_t)_{t\in\overline{0,N}}|\mathbf{O}] + f(\mathrm{pr}_2(z_N), x) \geq \hat{\mathfrak{C}}_\alpha[(z_t)_{t\in\overline{0,N}}|\mathbf{O}]. \tag{5.6}$$

As a result, from (2.17), (3.3), and (5.6), we get that

$$\hat{V}[x|\mathbf{O}] \leq V[x] \quad \forall x \in X^0. \tag{5.7}$$

So, for now, we have a lower bound for $V[\cdot]$. To construct an upper bound, we need a procedure of type (3.26)–(3.30). First, we implement the construction procedure for all functions

$$v_{\mathbf{O}}^{(0)}, \quad v_{\mathbf{O}}^{(1)}, \quad \ldots, \quad v_{\mathbf{O}}^{(N)} \tag{5.8}$$

using the above variant from the present section with the following correction: all functions (5.8) are preserved in computer memory. So, we do not rewrite layers of the Bellman function.

As a result, we obtain all layers (5.8). It is important that all these layers are constructed by one DP procedure: we have a DP procedure universal relative to the starting point. As in Section 4, this procedure is based on (5.4). But in this procedure, we accumulate our own knowledge about (5.8) (when defining only (5.5), we do not accumulate this knowledge).

Further, by means of layers (5.8), for every $x \in X^0$, we use (4.8)–(4.11) for $\mathbf{f} = \mathbf{O}$. So, for every $x \in X^0$, we find the set $\hat{\mathbb{S}}[x|\mathbf{O}]$ (see (3.6)) of all optimal solutions to problem (3.3).

Further, for $x \in X^0$ and $(\alpha, \mathbf{z}) \in \hat{\mathbb{S}}[x|\mathbf{O}]$, we set

$$\tilde{v}(\alpha, \mathbf{z}, x) \triangleq \hat{V}[x|\mathbf{O}] + f(\mathrm{pr}_2(\mathbf{z}(N)), x). \tag{5.9}$$

We obtain the following new dependence for $x \in X^0$:

$$(\alpha, \mathbf{z}) \longmapsto \tilde{v}(\alpha, \mathbf{z}, x) : \hat{\mathbb{S}}[x|\mathbf{O}] \longrightarrow \mathbb{R}_+.$$

Consider the number

$$\hat{\mathbb{V}}[x] \triangleq \min_{(\alpha,\mathbf{z})\in\hat{\mathbb{S}}[x|\mathbf{O}]} \tilde{v}(\alpha, \mathbf{z}, x) = \hat{V}[x|\mathbf{O}] + \min_{(\alpha,\mathbf{z})\in\hat{\mathbb{S}}[x|\mathbf{O}]} f(\mathrm{pr}_2(\mathbf{z}(N)), x). \tag{5.10}$$

Of course, (5.10) defines a function $\hat{\mathbb{V}}[\cdot]$, $x \longmapsto \hat{\mathbb{V}}[x] : X^0 \longrightarrow \mathbb{R}_+$.

Proposition 1. *If $x \in X^0$, then $V[x] \le \tilde{v}(\alpha, \mathbf{z}, x) \quad \forall (\alpha, \mathbf{z}) \in \hat{\mathbb{S}}[x|\mathbf{O}]$.*

P r o o f. Fix $x^* \in X^0$ and $(\alpha^*, \mathbf{z}^*) \in \hat{\mathbb{S}}[x|\mathbf{O}]$. Then, by (5.9), we have

$$\tilde{v}(\alpha^*, \mathbf{z}^*, x^*) = \hat{V}[x^*|\mathbf{O}] + f(\mathrm{pr}_2(\mathbf{z}^*(N)), x^*). \tag{5.11}$$

In addition, by (3.2) and (5.1), we obtain the equality

$$\hat{\mathfrak{C}}_{\alpha^*}[\mathbf{z}^*|\mathbf{O}] = \sum_{t=1}^{N} \big[\mathbf{c}(\mathrm{pr}_2(\mathbf{z}^*(t-1)), \mathrm{pr}_1(\mathbf{z}^*(t)), \{\alpha^*(k) : k \in \overline{t, N}\}) + c_{\alpha^*(t)}(\mathbf{z}^*(t), \{\alpha^*(k) : k \in \overline{t, N}\})\big].$$

By the choice of $(\alpha^*, \mathbf{z}^*)$ and (3.6), the following equality holds:

$$\hat{\mathfrak{C}}_{\alpha^*}[\mathbf{z}^*|\mathbf{O}] = \hat{V}[x^*|\mathbf{O}]. \tag{5.12}$$

By (2.16) and (5.11)–(5.12), we get that

$$\tilde{v}(\alpha^*, \mathbf{z}^*, x^*) = \hat{\mathfrak{C}}_{\alpha^*}[\mathbf{z}^*|\mathbf{O}] + f(\mathrm{pr}_2(\mathbf{z}^*(N)), x^*) = \sum_{t=1}^{N} \big[\mathbf{c}(\mathrm{pr}_2(\mathbf{z}^*(t-1)), \mathrm{pr}_1(\mathbf{z}^*(t)), \{\alpha^*(k) : k \in \overline{t, N}\})$$
$$+ c_{\alpha^*(t)}(\mathbf{z}^*(t), \{\alpha^*(k) : k \in \overline{t, N}\})\big] + f(\mathrm{pr}_2(\mathbf{z}^*(N)), x^*) = \mathfrak{C}_{\alpha^*}[\mathbf{z}^*|x^*]. \tag{5.13}$$

In addition, $(\alpha^*, \mathbf{z}^*) \in \tilde{D}[x^*]$ (see (3.6)). Therefore (see (2.17)), $V[x^*] \le \mathfrak{C}_{\alpha^*}[\mathbf{z}^*|x^*]$. The required inequality $\tilde{v}(\alpha^*, \mathbf{z}^*, x^*) \ge V[x^*]$ follows from (5.13). Since x^* and $(\alpha^*, \mathbf{z}^*)$ were chosen arbitrarily, our proposition is established. □

Corollary 1. *The inequality $V[x] \le \hat{\mathbb{V}}[x]$ holds for $x \in X^0$.*

P r o o f. The proof is an immediate combination of (5.10) and Proposition 1. □

So, we get that $\hat{\mathbb{V}}[\cdot] \triangleq (\hat{\mathbb{V}}[x])_{x \in X^0} \in \mathcal{R}_+[X^0]$ is a majorant for $V[\cdot]$:

$$V[x] \le \hat{\mathbb{V}}[x] \quad \forall x \in X^0. \tag{5.14}$$

In this connection, we introduce the number

$$\mathbf{V} \triangleq \min_{x \in X^0} \hat{\mathbb{V}}[x] \in \mathbb{R}_+. \tag{5.15}$$

From (2.21), (5.14), and (5.15), we obtain the following estimate:

$$\mathbb{V} \leq \mathbf{V}. \tag{5.16}$$

For $x \in X^0$, we consider

$$\tilde{v}(\alpha, \mathbf{z}, x) \longrightarrow \min, \quad (\alpha, \mathbf{z}) \in \hat{\mathbb{S}}[x|\mathbf{O}],$$

as a majorant problem. Now, we consider a variant of a minorant problem. In this construction, we aim to make (5.7) more precise. For this, we introduce the function

$$\varphi \triangleq (\min_{y \in X^0} f(x, y))_{x \in \mathbf{M}} \in \mathcal{R}_+]\mathbf{M}].$$

Now, we consider the scheme from Section 3 for $\mathbf{f} = \varphi$. In this case, we implement layers (3.21); as a result, we obtain the functions

$$v_\varphi^{(0)} \in \mathcal{R}_+[\mathcal{D}_0], \quad v_\varphi^{(1)} \in \mathcal{R}_+[\mathcal{D}_1], \quad \ldots, \quad v_\varphi^{(N)} \in \mathcal{R}_+[\mathcal{D}_N].$$

We have a successive implementation $v_\varphi^{(0)} \longrightarrow v_\varphi^{(1)} \longrightarrow \cdots \longrightarrow v_\varphi^{(N)}$. In addition,

$$v_\varphi^{(0)}(x, \varnothing) = \varphi(x) \quad \forall x \in \bigcup_{i \in \overline{1,N} \setminus \mathbf{K}_1} \mathbf{M}_i.$$

If $s \in \overline{1,N}$, then the transformation of $v_\varphi^{(s-1)}$ to $v_\varphi^{(s)}$ is defined by (3.24) for $\mathbf{f} = \varphi$. In this construction, we use a procedure with rewriting layers: for $s < N$, we use $v_\varphi^{(s-1)}$ to construct $v_\varphi^{(s)}$ by the above variant of (3.24) and annihilate $v_\varphi^{(s-1)}$ thereafter; using $v_\varphi^{(s)}$, we construct $v_\varphi^{(s+1)}$ and so on. From (3.25), we get that

$$\hat{V}[x|\varphi] = v_\varphi(x, \overline{1,N}) = v_\varphi^{(N)}(x, \overline{1,N}) \quad \forall x \in X^0. \tag{5.17}$$

By (5.17), we obtain the function $\hat{V}[\cdot|\varphi]$. Now, by analogy with [6], we introduce an estimating set

$$X_0 \triangleq \{x \in X^0 | \hat{V}[x|\varphi] \leq \mathbf{V}\}. \tag{5.18}$$

The set $X^0_{\text{opt}} \triangleq \{x \in X^0 | V[x] = \mathbb{V}\}$ satisfies the inclusion

$$X^0_{\text{opt}} \subset X_0. \tag{5.19}$$

The proof is similar to that of [6, Proposition 5.1].

Remark 2. For completeness, we verify (5.19). Let $y \in X^0_{\text{opt}}$. Then $y \in X^0$ and $V[y] = \mathbb{V}$. Using (2.18), we choose $(\alpha', \mathbf{z}') \in (\mathrm{SOL})[y]$. By the choice of y, we have

$$\mathfrak{C}_{\alpha'}[\mathbf{z}'|y] = V[y] = \mathbb{V}. \tag{5.20}$$

In addition, $\varphi(\mathrm{pr}_2(\mathbf{z}'(N))) \leq f(\mathrm{pr}_2(\mathbf{z}'(N), y)$. As a corollary, by (3.18) and (3.2), we obtain

$$\begin{aligned} \hat{\mathfrak{C}}_{\alpha'}[\mathbf{z}'|\varphi] = \sum_{t=1}^{N} \big[& \mathbf{c}(\mathrm{pr}_2(\mathbf{z}'(t-1)), \mathrm{pr}_1(\mathbf{z}'(t)), \{\alpha'(k) : \ k \in \overline{t,N}\}) \\ & + c_{\alpha'(t)}(\mathbf{z}'(t), \{\alpha'(k) : \ k \in \overline{t,N}\})\big] + \varphi(\mathrm{pr}_2(\mathbf{z}'(N)) \leq \mathfrak{C}_{\alpha'}[\mathbf{z}'|y]. \end{aligned} \tag{5.21}$$

Since $(\alpha', \mathbf{z}') \in \tilde{D}[y]$, we get that $\hat{V}[y|\varphi] \leq \hat{\mathfrak{C}}_{\alpha'}[\mathbf{z}'|\varphi]$. By (5.21), $\hat{V}[y|\varphi] \leq \mathfrak{C}_{\alpha'}[\mathbf{z}'|y]$. Using (5.16) and (5.20), we get that

$$\hat{V}[y|\varphi] \leq \mathbf{V}. \tag{5.22}$$

By (5.18) and (5.22), the inclusion $y \in X_0$ holds. Since y was chosen arbitrarily, the inclusion (5.19) is established. □

Note that $X^0_{\text{opt}} \neq \varnothing$ (see (2.21)) and, as a corollary, according to (5.19), $X_0 \neq \varnothing$. So, $X_0 \in \mathcal{P}'(X^0)$; by the definition from Section 2,

$$\mathbb{V} = \min_{x \in X_0} V[x]. \tag{5.23}$$

Now, by analogy with [6, Sect. 5], we obtain the following scheme for solving problem (2.19). Let us list basic steps.

(1) Determine the upper bound $\mathbf{V}$ (5.15) by solving the majorizing problem by means of a universal (relative to points from X^0) variant of DP.

(2) Solve the minorating problem constructing $\hat{V}[\cdot|\varphi] = v^{(N)}_{\varphi}$ by the scheme with rewriting layers of the Bellman function (it is a universal (relative to points from X^0) variant of DP).

(3) Construct the set X_0.

(4) Solve all x-problems (2.17), $x \in X_0$, by means of individual variants of DP; for this, determine $V[x]$, $x \in X_0$, and implement $\mathbb{V}$ by (5.23) is effective. Moreover, for the obtained point $x^0 \in X^0_{\text{opt}}$, by a procedure similar to (4.8)–(4.11), an x^0-optimal solution $(\alpha^0, (z^0_t)_{t \in \overline{0,N}}) \in \tilde{D}[x^0]$ is found. As a result, $(\alpha^0, (z^0_t)_{t \in \overline{0,N}}, x^0) \in \mathbf{SOL}$.

Note that step (4) can be implemented as follows. First, for every $x \in X_0$, we implement a DP procedure with rewriting layers of the Bellman function (we keep in mind an individual DP procedure). As a result, we obtain $V[x]$, $x \in X_0$. Further, we find $x^0 \in X_0$ for which (see (5.23)) $V[x^0] = \mathbb{V}$ (more precisely, we find the point of minimum of $V[x]$, $x \in X_0$). Then, $x^0 \in X^0_{\text{opt}}$. After that, we implement an individual variant of DP for the case $x = x^0$ (see (4.8)–(4.12)).

6. Weakening of the closed routing problem

In this section, we will take a very short look at one traditional variant of the closed routing problem, as well as its natural weakening. In our construction, we are oriented to [6, Sect. 6]. Let $\rho \in \mathcal{R}_+[X \times X]$ be a metric on the set X. So, (X, ρ) is a metric space. Suppose that

$$f(\tilde{x}, x) = \rho(\tilde{x}, x) \quad \forall \tilde{x} \in \mathbf{M} \quad \forall x \in X^0. \tag{6.1}$$

In fact, this requirement means that we consider the routing problem with return to the starting point (see (2.16)) and the latter means the distance (this interpretation is more natural for metric routing problems where other components of additive criterion mean the distance). In this connection, we recall the known closed TSP (see [7, 11]).

Remark 3. Note that, in the general setting in Section 2, we can consider solutions with terminal permutation to the starting point.

In applied problems, the requirement of return can often be weakened. Consider one variant of such weakening. Let

$$B^0_{\rho}(x, \varepsilon) \triangleq \{\, y \in X^0 |\ \rho(x, y) \leq \varepsilon \}$$

for $x \in X^0$ and $\varepsilon \in \mathbb{R}_+$, $\varepsilon > 0$. We fix this number ε, $\varepsilon > 0$. In what follows, we replace f (6.1) using another definition. For $x \in X$ and $A \in \mathcal{P}'(X)$, let $\rho(x; A) \stackrel{\triangle}{=} \inf(\{\, \rho(x,y) :\ y \in A\})$; $\rho(x; A) \in \mathbb{R}_+$. Suppose that

$$f(\tilde{x}, x) \stackrel{\triangle}{=} \rho(\tilde{x}; B^0_\rho(x, \varepsilon)) \quad \forall \tilde{x} \in \mathbf{M} \quad \forall x \in X^0. \tag{6.2}$$

Recall that $\varphi \in \mathcal{R}_+[\mathbf{M}]$ is defined by the rule

$$\varphi(x) = \min_{y \in X^0} f(x,y) = \min_{y \in X^0} \rho(x; B^0_\rho(y, \varepsilon)) \ \ \forall x \in \mathbf{M}. \tag{6.3}$$

Proposition 2. *The equality $\varphi(x) = \rho(x; X^0)$ holds for $x \in \mathbf{M}$ in the case* (6.2).

P r o o f. The corresponding scheme is similar to [6, Proposition 6.1]. Let us describe it for completeness. Fix $x_* \in \mathbf{M}$. Then

$$\rho(x_*; X^0) = \min_{y \in X^0} \rho(x_*, y). \tag{6.4}$$

For every $y \in X^0$, the inclusion $y \in B^0_\rho(y, \varepsilon))$ holds; therefore, by (6.3), we have the inequality $f(x_*, y) \le \rho(x_*, y)$. As a result, $\varphi(x_*) \le \rho(x_*, y)$ for $y \in X^0$. Therefore, by (6.4), we have

$$\varphi(x_*) \le \rho(x_*; X^0). \tag{6.5}$$

Since $B^0_\rho(y, \varepsilon)) \subset X^0$, we have $\rho(x_*; X^0) \le \rho(x_*; B^0_\rho(y, \varepsilon)) = f(x_*, y)$ for $y \in X^0$. In view of (6.3), the inequality $\rho(x_*; X^0) \le \varphi(x_*)$ is established. Using (6.5), we obtain the equality $\varphi(x_*) = \rho(x_*; X^0)$. Since x_* was chosen arbitrarily, the required statement is obtained. □

So, we get that (in our case) $\varphi = (\rho(x; X^0))_{x \in \mathbf{M}}$ and our minorant problem coincides with the simplest variant of Consider the simplest version of the problem that implements the lower estimate. But, using (6.2) (instead of (6.1)), we decrease $\mathbf{V}$ somewhat with respect to (6.1). Indeed, $\rho(\tilde{x}; B^0_\rho(x, \varepsilon)) \le \rho(\tilde{x}, x)$ for $\tilde{x} \in \mathbf{M}$ and $x \in X^0$.

We can consider the replacement (6.1) $\longrightarrow$ (6.2) as a weakening of the initial problem with the terminal function (6.1). For this weakening, it is required to successfully achieve a “reduction” of the set X_0 compared to X^0.

7. Computational experiment

In this section, we will consider examples related to the engineering problem of cutting sheets on CNC machines. It is assumed that a sheet cutting plan already exists. More precisely, cutting should be done along the equidistances of the contours. For each equidistance, a corresponding sampling is made. So, in fact, we have a “discrete equidistance.” For each point of such a discrete equidistance, from the outside to the part, there are a piercing point and a switch-off point of the tool. Besides, each pair of these points corresponds to the starting point of the contour cut.

We consider the set of all such points as a megacity (we mean the set of all points of the two types listed above). The tool moves to the piercing point in idle (fast) mode. Upon reaching this point, the tool enters the working mode and begins to pierce the metal. After piercing, the tool moves in cutting mode to the starting point of the contour cutting. It then performs a contour cut with finishing at the start point of the cut and travels to the switch-off point, where the tool shuts off and starts idling to the next contour of the finish point. We do not affect the contour cut. Each contour cut must be made once. So, we exclude this process from consideration. But

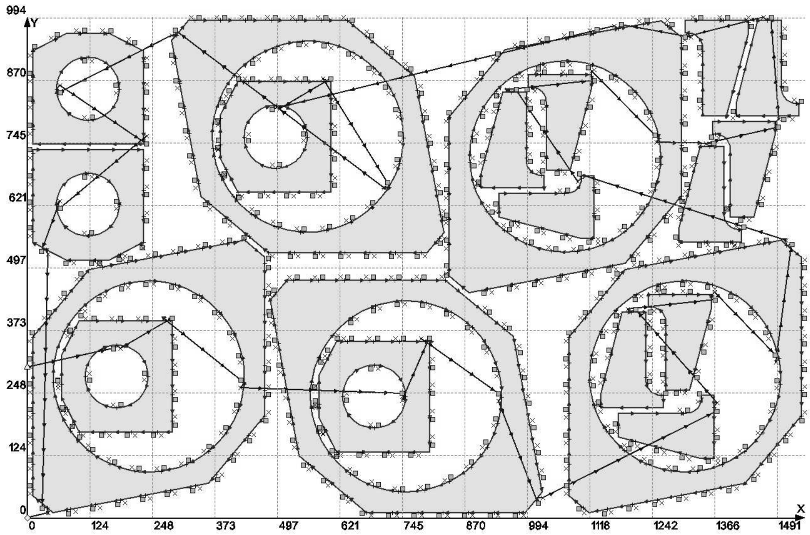

Figure 1. Calculation results for Example 1.

other movements can be chosen to minimize the overall tool time. It is required to regulate the visiting process. Besides, we have to select a sequence of piercing points (and the corresponding contour cut start points and tool switch-off points). Finally, we must choose a starting point for our process.

Let us recall the restrictions. The precedence condition (in particular) is associated with the following requirement: cutting the inner contours must precede the outer ones (there are other options). In connection with other restrictions, we note the thermal tolerance (these restrictions will be considered in the second example). These constraints lead to the use of task-list-dependent cost functions (in our model).

In the first example, we will investigate the case considered in Section 3 (see (3.1)). Define the sets

$$A_j(x, K), \quad j \in \overline{1, N}, \quad x \in \mathbf{X} \setminus \mathbf{M}_j, \quad K \in \mathfrak{N},$$

for reasons of maximin of the Euqlidian distance with respect to the cut out contours. In the second example, we will suppose that $A_j(x, K) = \mathfrak{M}_j$ for all $j \in \overline{1, N}$, $x \in \mathbf{X} \setminus \mathbf{M}_j$, and $K \in \mathfrak{N}^{(j)}$ (so, here we can select any piercing point).

Example 1. The calculations for the first example were performed on a computer with an Intel Xeon CPU E5-2620 processor, 8 GB of memory, and a Windows 10 (64-bit) operating system. The program was developed in C++ using the Qt library to build a user interface.

The number of contours is 30. The number of ordered pairs is 20.

The starting point was chosen from a rectangle with corners (0 mm, 0 mm), (0 mm, 1000 mm), (1550 mm, 1000 mm), and (1550 mm, 0 mm). The step of point checking was 100 mm.

The result is 74.507. The starting point is (0 mm, 300 mm). The terminal point is (0 mm, 0 mm). The duration of calculation is 32 h 29 min 55 sec. The counting results are shown in Fig. 1.

Example 2. This example used a computer with an Intel i7-2630QM processor, 8 GB of memory, and a Windows 7 (64-bit) operating system. The same language and libraries were used.

This example uses the cost functions from [5]. These functions are dependent on the list of visited megacities and are related to the technical limitations of CNC metal cutting plants. This allows thermal restrictions to be taken into account. There should be enough metal to provide a quality of the cut around the finish cut segment. The dependence on the task list allows us to

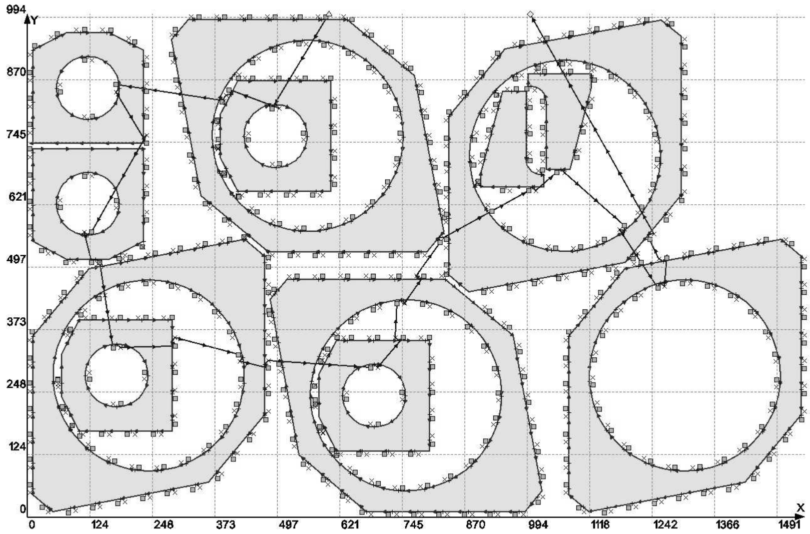

Figure 2. Calculation results for Example 2.

account for the cut-out contours to fix the voids in the metal located near the sections of the cut completion.

The length of the finish cut area 300 mm (see [5]). The width of the finish cut area is 150 mm. A 1,000,000 penalty was used if 25% (or more) of the finish cut area was covered with holes in the metal or outside the sheet space.

The starting point was chosen from a rectangle with corners (0 mm, 0 mm), (0 mm, 1000 mm), (1550 mm, 1000 mm), and (1550 mm, 0 mm). The step of point checking was 100 mm. Starting and finishing points can be different. The maximum range from start to finish point must be less than 500 mm.

First, a count of all start-finish points was made. 51 calculations were made. The obtained result is 50,145. The start point is (600 mm, 1000 mm), and the finish point is (1000 mm, 1000 mm). The duration of the calculations is 34 min 52 sec. The results are shown in Fig. 2.

Then the upper estimate of the result was found with a value of 50 475. The calculation time was 40 sec. The number of start-finish points was reduced from 51 to 34. For these 34 points, calculations were made with a counting time of 23 min 5 sec. The result obtained, of course, is the same as in the case of using all start-finish points. So, the total time spent on the process with counting the estimates and reducing the number of points is 24 min 25 sec. This is less than the total calculation time for 51 points (34 min 52 sec).

8. Conclusion

The paper discusses an "additive" routing problem with constraints and cost functions. depending on the task list. The well-known DP solution method is applied. The optimal choice of the starting point, route, and specific trajectory has been implemented. The settings are investigated with the requirement to return to a neighborhood of the starting point and without this requirement. Thus, individual (to the starting point) and universal (relative to the starting point) DP procedures arise. In addition, the option of using a universal DP is proposed for the application of an individual DP procedure.

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DISTANCE-REGULAR GRAPH WITH INTERSECTION ARRAY $\{27, 20, 7; 1, 4, 21\}$ DOES NOT EXIST[1]

Konstantin S. Efimov

Ural State University of Economics,
62 March 8th Str., Ekaterinburg, 620144, Russia

Ural Federal University,
19 Mira Str., Ekaterinburg, 620002, Russia

konstantin.s.efimov@gmail.com

Alexander A. Makhnev

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, 620108, Russia

Ural Federal University,
19 Mira Str., Ekaterinburg, 620002, Russia

makhnev@imm.uran.ru

Abstract: In the class of distance-regular graphs of diameter 3 there are 5 intersection arrays of graphs with at most 28 vertices and noninteger eigenvalue. These arrays are $\{18, 14, 5; 1, 2, 14\}$, $\{18, 15, 9; 1, 1, 10\}$, $\{21, 16, 10; 1, 2, 12\}$, $\{24, 21, 3; 1, 3, 18\}$, and $\{27, 20, 7; 1, 4, 21\}$. Automorphisms of graphs with intersection arrays $\{18, 15, 9; 1, 1, 10\}$ and $\{24, 21, 3; 1, 3, 18\}$ were found earlier by A.A. Makhnev and D.V. Paduchikh. In this paper, it is proved that a graph with the intersection array $\{27, 20, 7; 1, 4, 21\}$ does not exist.

Keywords: Distance-regular graph, Graph Γ with strongly regular graph Γ_3, Automorphism.

1. Introduction

We consider undirected graphs without loops and multiple edges. For given vertex a of a graph Γ, we denote by $\Gamma_i(a)$ the subgraph of Γ induced by the set of all vertices at distance i from a. The subgraph $[a] = \Gamma_1(a)$ is called the *neighbourhood of the vertex* a.

Let Γ be a graph of diameter d and $i \in \{1, 2, 3, \ldots, d\}$. The graph Γ_i have the same set of vertices, and vertices u and w are adjacent in Γ_i if $d_\Gamma(u, w) = i$.

If vertices u and w are at distance i in Γ, then denote by $b_i(u, w)$ (by $c_i(u, w)$) the number of vertices in the intersection of $\Gamma_{i+1}(u)$ ($\Gamma_{i-1}(u)$) with $[w]$. A graph Γ of diameter d is called a *distance-regular graph with intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if the values $b_i(u, w)$ and $c_i(u, w)$ are independent of the choice of the vertices u and w at distance i in Γ for any $i = 0, \ldots, d$. For such graph and for $0 \le i, j, h \le d$, the number $p^h_{ij} = \begin{bmatrix} uv \\ ij \end{bmatrix}$ is independent of u and v for all vertices $u, v \in \Gamma$ with $d(u, v) = h$. The constants p^h_{ij} are called the intersection numbers of Γ [1].

The incidence system with set of points P and set of lines $\mathcal{L}$ is called α*-partial geometry of order* (s, t) (and is denoted by $pG_\alpha(s, t)$) if every line contains exactly $s + 1$ points, every point lies

[1]This work was supported by RFBR and NSFC (project No. 20-51-53013).

exactly on $t+1$ lines, any two points lie on at most one line, and, for any antiflag $(a,l) \in (P,\mathcal{L})$, there is exactly α lines passing through a and intersecting l. If $\alpha = t+1$, then the geometry is called a dual 2-scheme; and if $\alpha = t$, then the geometry is called a net.

The *point graph* of a geometry of points and lines is a graph whose vertices are points of the geometry, and two different vertices are adjacent if they lie on a common line. It is easy to understand that the point graph of a partial geometry $pG_\alpha(s,t)$ is strongly regular with parameters

$$v = (s+1)(1+st/\alpha), \quad k = s(t+1), \quad \lambda = (s-1)+(\alpha-1)t, \quad \mu = \alpha(t+1).$$

A strongly regular graph having these parameters for some positive integers α, s, t is called a *pseudogeometric graph* for $pG_\alpha(s,t)$.

In the class of distance-regular graphs Γ of diameter 3, there are 5 hypothetical graphs with at most 28 vertices and non-integer eigenvalues. They have intersection arrays $\{18,14,5;1,2,14\}$, $\{18,15,9;1,1,10\}$, $\{21,16,10;1,2,12\}$, $\{24,21,3;1,3.18\}$, and $\{27,20,7;1,4,21\}$. Earlier, automorphisms of graphs with intersection arrays $\{18,15,9;1,1,10\}$ and $\{24,21,3;1,3,18\}$ were found by A.A. Makhnev and D.V. Paduchikh [4], [5].

In this paper, we study the properties of a hypothetical distance-regular graph with intersection array $\{27,20,7;1,4,21\}$ and prove the following theorem.

Theorem 1. *A distance-regular graph with intersection array* $\{27,20,7;1,4,21\}$ *does not exist.*

2. Preliminary results

In the proof of Theorem 1, we use triple intersection numbers [2].

Let Γ be a distance-regular graph of diameter d. If u_1, u_2, and u_3 are vertices of Γ and r_1, r_2, and r_3 are non-negative integers not greater than d, then $\left\{ \begin{matrix} u_1u_2u_3 \\ r_1r_2r_3 \end{matrix} \right\}$ is the set of vertices $w \in \Gamma$ such that

$$d(w,u_i) = r_i, \qquad \begin{bmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{bmatrix} = \left| \left\{ \begin{matrix} u_1u_2u_3 \\ r_1r_2r_3 \end{matrix} \right\} \right|.$$

The numbers $\begin{bmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{bmatrix}$ are called triple intersection numbers. For a fixed triple of vertices u_1, u_2, u_3, we will write $[r_1r_2r_3]$ instead of $\begin{bmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{bmatrix}$. Unfortunately, there are no general formulas for the numbers $[r_1r_2r_3]$. However, a method for calculating some numbers $[r_1r_2r_3]$ was suggested in [2].

Assume that u, v, and w are vertices of the graph Γ, $W = d(u,v)$, $U = d(v,w)$, and $V = d(u,w)$. Since there is exactly one vertex $x = u$ such that $d(x,u) = 0$, the number $[0jh]$ is either 0 or 1. Hence, $[0jh] = \delta_{jW}\delta_{hV}$. Similarly, $[i0h] = \delta_{iW}\delta_{hU}$ and $[ij0] = \delta_{iU}\delta_{jV}$.

Another set of equations can be obtained by fixing the distance between two vertices from $\{u,v,w\}$ and counting the number of vertices located at all possible distances from the third:

$$\sum_{l=1}^{d}[ljh] = p^U_{jh} - [0jh], \quad \sum_{l=1}^{d}[ilh] = p^V_{ih} - [i0h], \quad \sum_{l=1}^{d}[ijl] = p^W_{ij} - [ij0]. \tag{2.1}$$

At the same time, some triplets disappear. For $|i-j| > W$ or $i+j < W$, we have $p^W_{ij} = 0$; therefore, $[ijh] = 0$ for all $h \in \{0,\dots,d\}$.

Let

$$S_{ijh}(u,v,w) = \sum_{r,s,t=0}^{d} Q_{ri}Q_{sj}Q_{th} \begin{bmatrix} uvw \\ rst \end{bmatrix}.$$

If Krein's parameter $q^h_{ij} = 0$, then $S_{ijh}(u, v, w) = 0$.

We fix vertices u, v, and w of a distance-regular graph Γ of diameter 3 and put

$$\{ijh\} = \left\{ \begin{matrix} uvw \\ ijh \end{matrix} \right\}, \quad [ijh] = \left[\begin{matrix} uvw \\ ijh \end{matrix} \right], \quad [ijh]' = \left[\begin{matrix} uwv \\ ihj \end{matrix} \right], \quad [ijh]^* = \left[\begin{matrix} vuw \\ jih \end{matrix} \right], \quad [ijh]^\sim = \left[\begin{matrix} wvu \\ hji \end{matrix} \right].$$

In the cases $d(u, v) = d(u, w) = d(v, w) = 2$ or $d(u, v) = d(u, w) = d(v, w) = 3$, the calculation of the numbers

$$[ijh]' = \left[\begin{matrix} uwv \\ ihj \end{matrix} \right], \quad [ijh]^* = \left[\begin{matrix} vuw \\ jih \end{matrix} \right], \quad [ijh]^\sim = \left[\begin{matrix} wvu \\ hji \end{matrix} \right]$$

(symmetrizing an array of triple intersection numbers) can give new relations for the prove of the nonexistence of the graph.

3. Proof of Theorem 1

In this section, we prove Theorem 1.

Let Γ be a distance-regular graph with intersection array $\{27, 20, 7; 1, 4, 21\}$. Then Γ has $1 + 27 + 135 + 45 = 208$ vertices, the spectrum $27^1, (2 + \sqrt{13})^{45}, -1^{117}, (5 - 2\sqrt{13})^{45}$, and the dual matrix Q of eigenvalues

$$\begin{pmatrix} 1 & 45 & 117 & 45 \\ 1 & \frac{10}{3}\sqrt{13} + \frac{5}{3} & -13/3 & -\frac{10}{3}\sqrt{13} + \frac{5}{3} \\ 1 & -\frac{2}{3}\sqrt{13} + \frac{5}{3} & -13/3 & \frac{2}{3}\sqrt{13} + \frac{5}{3} \\ 1 & -7 & 13 & -7 \end{pmatrix}.$$

By [3, Lemma 3], the complement of Γ_3 is a pseudo-geometric graph for $pG_{21}(27, 5)$.

Lemma 1. *The intersection numbers of the graph Γ are*:

(1) $p^1_{11} = 6$, $p^1_{21} = 20$, $p^1_{32} = 35$, $p^1_{22} = 80$, $p^1_{33} = 10$;

(2) $p^2_{11} = 4$, $p^2_{12} = 16$, $p^2_{13} = 7$, $p^2_{22} = 90$, $p^2_{23} = 28$, $p^2_{33} = 10$;

(3) $p^3_{12} = 21$, $p^3_{13} = 6$, $p^3_{22} = 84$, $p^3_{23} = 30$, $p^3_{33} = 8$.

P r o o f. The lemma is proved by direct calculations. □

We fix vertices u, v, and w of the graph Γ and put

$$\{ijh\} = \left\{ \begin{matrix} uvw \\ ijh \end{matrix} \right\}, \quad [ijh] = \left[\begin{matrix} uvw \\ ijh \end{matrix} \right].$$

Let $\Delta = \Gamma_2(u)$ and $\Lambda = \Delta_2$. Then Λ is a regular graph of degree 90 on 135 vertices.

Lemma 2. *Let $d(u, v) = d(u, w) = 2$ and $d(v, w) = 1$. Then the triple intersection numbers are*:

(1) $[111] = r_4$, $[112] = [121] = -r_4 + 4$, $[122] = -r_1 + r_3 + r_4 + 5$; $[123] = [132] = r_1 - r_3 + 7$, $[133] = -r_1 + r_3$;

(2) $[211] = -r_2 - r_4 + 6$, $[212] = [221] = r_2 + r_4 + 9$, $[222] = r_1 - r_2 - r_4 + 53$, $[223] = [232] = -r_1 + 28$, $[233] = r_1$;

(3) $[311] = r_2$, $[312] = [321] = -r_2 + 7$, $[322] = r_2 - r_3 + 21$, $[323] = [332] = r_3$, $[333] = -r_3 + 10$,

where $r_1, r_3 \in \{0, 1, \ldots, 10\}$, $r_2 \in \{0, 1, \ldots, 6\}$, *and* $r_4 \in \{0, 1, \ldots, 4\}$.

P r o o f. Let $[111] = r_4$. Then $[113] = 0$ and $[111] + [112] = c_2 = 4$; thus, $[112] = -r_4 + 4$. Similarly, $[121] = -r_4 + 4$.

Let $[311] = r_2$. Then $[313] = 0$ and $[311] + [312] = p^2_{13} = 7$; thus, $[312] = -r_2 + 7$.

Using formulas (2.1), we obtain all the equalities. □

By Lemma 2, we have $43 \le [222] = r_1 - r_2 - r_4 + 53 \le 63$. Since $\{v, w\} \cup \Lambda(v) \cup \Lambda(w)$ contains $182 - [222]$ vertices, we have $182 - [222] \le 135$; hence, $47 \le [222] \le 63$ and $-r_1 + r_2 + r_4 \le 6$.

Lemma 3. *Let $d(u, v) = d(u, w) = 2$ and $d(v, w) = 3$. Then the triple intersection numbers are*:

(1) $[113] = r_5 + r_6 + r_7 + r_8 - r_9 - 26$, $[121] = -r_5 - r_6 - r_7 - r_8 + r_{10} + 30$, $[122] = r_5 + r_6 + r_7 + r_8 - r_9 - r_{10} - 14$, $[123] = r_9$, $[131] = r_5 + r_6 + r_7 + r_8 - r_{10} - 26$, $[132] = r_{10}$, $[133] = -r_5 - r_6 - r_7 - r_8 + 33$;

(2) $[212] = r_5 + r_7 + r_8 - r_9 - 9$, $[213] = -r_5 - r_7 - r_8 + r_9 + 25$, $[221] = r_5 + r_6 + r_8 - r_{10} - 9$, $[222] = -r_5 - r_6 - r_7 - 2r_8 + r_9 + r_{10} + 97$, $[223] = r_7 + r_8 - r_9 + 2$, $[231] = -r_5 - r_6 - r_8 + r_{10} + 25$, $[232] = r_6 + r_8 - r_{10} + 2$, $[233] = r_5$;

(3) $[312] = r_6$, $[313] = -r_6 + 7$, $[321] = r_7$, $[322] = r_8$, $[323] = -r_7 - r_8 + 28$, $[331] = -r_7 + 7$, $[332] = -r_6 - r_8 + 28$, $[333] = r_6 + r_7 + r_8 - 25$,

where $r_5 \in \{0, 1, \ldots, 8\}$, $r_6, r_7 \in \{1, 2, \ldots, 7\}$, $r_8 \in \{11, 12, \ldots, 27\}$, *and* $r_9, r_{10} \in \{0, 1, \ldots, 7\}$.

P r o o f. Using (2.1), we arrive at relations (1)–(3) of the Lemma 3. □

By Lemma 3, we have $47 \le [222] = -r_5 - r_6 - r_7 - 2r_8 + r_9 + r_{10} + 97 \le 90$.

Consider the appropriate symmetrization. Let $d(u, v) = d(u, w) = 2$ and $d(v, w) = 3$. Then the following equalities are true: $[123] = r_9 = [132]' = r'_{10}$, $[233] = r_5 = r'_5$, $r_6 = [312] = [321]' = r'_7$, $[322] = r_8 = r'_8$. Further, $r_7 + r_8 - r_9 + 2 = [223] = [232]' = r'_6 + r'_8 - r'_{10} + 2$.

Lemma 4. *Let $d(u, v) = d(u, w) = d(v, w) = 2$. Then the triple intersection numbers are*:

(1) $[111] = r_9 + r_{10} - r_{11} - 24$, $[112] = [121] = r_{15}$, $[113] = [131] = r_{11}$, $[122] = -r_{10} - r_{15} + 16$, $[123] = [132] = r_{10}$, $[133] = 7 - r_{11} - r_{10}$;

(2) $[211] = r_{15}$, $[212] = [221] = -r_{10} - r_{15} + 16$, $[213] = r_{10}$, $[222] = 2r_9 + 2r_{10} - 11$, $[223] = [232] = 28 - r_9 - r_{10}$, $[231] = r_{10}$, $[233] = r_9$;

(3) $[311] = r_{11}$, $[312] = [321] = r_{10}$, $[313] = [331] = 7 - r_{11} - r_{10}$, $[322] = -r_{10} - r_{15} + 16$, $[323] = [332] = r_9$, $[333] = r_{11} + r_{10} + 3$,

where $r_{11} + 24 \le r_9 + r_{10} \le 28$, $r_{11} + r_{10} \le 7$, $r_{10} + r_{15} \le 16$, *and* $r_{12} \le 22$.

P r o o f. Using formulas (2.1), we get the equalities:

$[111] = -r_{11} - r_{12} + 4$, $[112] = r_{15}$, $[113] = r_{11}$, $[121] = r_{10} + r_{12} + r_{15} + r_{16} - 28$, $[122] = -r_{10} - r_{15} + 16$, $[123] = -r_{12} - r_{16} + 28$, $[131] = -r_{10} + r_{11} - r_{12} - r_{16} + 28$, $[132] = r_{10}$, $[133] = -r_{11} + r_{12} + r_{16} - 21$;

$[211] = r_{12} + r_{13} + r_{14} + r_{15} - 28$, $[212] = -r_{13} - r_{15} + 16$, $[213] = -r_{12} - r_{14} + 28$, $[221] = -r_9 - r_{10} - r_{12} - r_{13} - r_{15} + 44$, $[222] = r_9 + r_{10} + r_{13} + r_{14} + 45$, $[223] = r_{12}$, $[231] = r_9 + r_{10} - r_{14}$, $[232] = -r_9 - r_{10} + 28$, $[233] = r_{14}$;

$[311] = r_{11} - r_{12} - r_{13} - r_{14} + 28$, $[312] = r_{13}$, $[313] = -r_{11} + r_{12} + r_{14} - 21$, $[321] = r_9 + r_{13} - r_{16}$, $[322] = -r_9 - r_{13} + 28$, $[323] = r_{16}$, $[331] = -r_9 - r_{11} + r_{12} + r_{14} + r_{16} - 21$, $[332] = r_9$, $[333] = r_{11} - r_{12} - r_{14} - r_{16} + 31$.

Now consider symmetrization. The following equalities are true:

$[112] = r_{15} = r^*_{15}$, $[113] = r_{11} = r^*_{11}$, $[223] = r_{12} = r^*_{12}$, $[233] = r_{14} = r'_{14}$, $[323] = r_{16} = r^{\sim}_{16}$, $[332] = r_9 = r^*_9$, $r_{10} = [132] = [312]^* = r^*_{13}$.

Further, $r_9 + r_{10} + r_{13} + r_{14} + 45 = [222] = [222]^* = r^*_9 + r^*_{10} + r^*_{13} + r^*_{14} + 45 = r_9 + r_{13} + r_{10} + r^*_{14} + 45$; therefore, $[233] = r_{14} = r^*_{14} = [323] = r_{16}$.

We have $[111] = -r_{11} - r_{12} + 4$; hence $r_{11} + r_{12} = r'_{11} + r'_{12} = r^{\sim}_{11} + r^{\sim}_{12}$. Similarly, $[122] = -r_{10} - r_{15} + 16$; therefore, $r_{10} + r_{15} = r'_{10} + r'_{15}$, $[123] = -r_{12} - r_{16} + 28$, and $r_{12} + r_{16} = r'_{12} + r'_{16}$.

Finally, $[133] = -r_{11} + r_{12} + r_{16} - 21 = -r'_{11} + r'_{12} + r'_{16} - 21$; thus, $r_{11} = r'_{11}$, $r_{12} = r'_{12}$, and $r_{16} = r'_{16}$. Hence $r_{11} = [113] = [131] = -r_{10} + r_{11} - r_{12} - r_{16} + 28$ and $r_{10} + r_{12} + r_{16} = 28$. Further, $r_{12} = [223] = [232] = -r_9 - r_{10} + 28$, $r_{12} + r_9 + r_{10} = 28$, and $r_9 = r_{16}$.

The equalities $[113] = [131] = r_{11}$, $r_{11} = r^*_{11}$, and $[311] = r_{11} + r_{10} - r_{13}$ imply that $r_{10} = r_{13}$. Hence, we obtain the equalities from the conclusion of the lemma. □

By Lemma 4, we have $r_{11} + 24 \le r_9 + r_{10} \le 28$; hence $45 \le [222] = 2r_9 + 2r_{10} - 11 \le 56 - 11 = 45$. Thus, Λ is an edge-regular graph with parameters (135,90,45).

In view of Lemmas 2 and 3, the following inequalities hold for the number of edges e between $\Lambda(w)$ and $\Lambda - (\{w\} \cup \Lambda(w))$:

$$2068 = 47 \cdot 16 + 47 \cdot 28 \le e = 63 \cdot 16 + 90 \cdot 28 \le 3528.$$

Contrariwise, we have $e = 90 \cdot 89 - \sum_i [222]^i$; therefore, $2068 \le e = 90 \cdot 89 - \sum_i [222]^i \le 3528$, $4482 \le \sum_i [222]^i \le 5942$, and $49.8 \le \sum_i [222]^i / 90 \le 66.03$.

The resulting contradiction completes the proof of Theorem 1.

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POSITIONAL IMPULSE AND DISCONTINUOUS CONTROLS FOR DIFFERENTIAL INCLUSION[1]

Ivan A. Finogenko

Matrosov Institute for System Dynamics and Control Theory,
Siberian Branch of Russian Academy of Sciences,
134 Lermontova Str., Irkutsk, 664033, Russia

fin@icc.ru

Alexander N. Sesekin

Ural Federal University,
19 Mira Str., Ekaterinburg, 620002, Russia

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, 620108, Russia

a.n.sesekin@urfu.ru

Abstract: Nonlinear control systems presented in the form of differential inclusions with impulse or discontinuous positional controls are investigated. The formalization of the impulse-sliding regime is carried out. In terms of the jump function of the impulse control, the differential inclusion is written for the ideal impulse-sliding regime. The method of equivalent control for differential inclusion with discontinuous positional controls is used to solve the question of the existence of a discontinuous system for which the ideal impulse-sliding regime is the usual sliding regime. The possibility of the combined use of the impulse-sliding and sliding regimes as control actions in those situations when there are not enough control resources for the latter is discussed.

Keywords: Impulse position control, Discontinuous position control, Differential inclusion, Impulse-sliding regime, Sliding regime.

Introduction

Impulse-sliding regimes for differential equations arise in problems of impulse optimal control when the system is affected by perturbations. The formalization of impulse-sliding regimes for differential equations was done in [8]. When describing the motions of systems subject to perturbations, the right-hand side can also be not uniquely defined. Therefore, under the action of perturbations on the system, it is natural to describe the motion of the system using differential inclusions and impulse control (see [3, 5]). In [5], the formalization of the impulse-sliding regime for systems of this type is given. In this paper, we investigate the properties of impulse-sliding regimes. In addition, an equivalent control method is applied to systems of this type [6, 7]. We also discuss the issue of the combined use of impulse-sliding and sliding regimes.

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1. Description of impulse-sliding regime

We study a dynamical system

$$\dot{x}(t) \in F(t, x(t)) + B(t, x(t))u, \quad t \in I = [t_0, \vartheta], \tag{1.1}$$

with the initial condition $x(t_0) = x_0$. Here, $F(\cdot, \cdot)$ is a multivalued function with convex compact values in $\mathbb{R}^n$, the matrix function $B(\cdot, \cdot)$ of dimension $n \times m$ is continuous in the set of variables in the considered domain, and $u = (u_1, \ldots, u_m)^T$ is a function that describes some control action on the system.

For $F(\cdot, \cdot)$, we write the following basic assumptions.

(B1) For almost all $t \in \mathbb{R}$, the mapping $F(t, x)$ is upper semicontinuous in x. This means that, for arbitrary $\varepsilon > 0$, there exists $\delta = \delta(t, x, \varepsilon) > 0$ such that $F(t, x') \subset F^\varepsilon(t, x)$ for all $x' \in W_\delta(x)$, where $F^\varepsilon(t, x)$ is the ε-neighborhood of the set $F(t, x)$ and $W_\delta(x)$ is the δ-neighborhood of the point x.

(B2) For any x, the multivalued mapping $t \to F(t, x)$ has a measurable selector, i.e., there is a measurable function $f(t) \in F(t, x)$ for almost all $t \in I$.

(B3) The multivalued mapping $F(t, x)$ satisfies the condition of sublinear growth: the inequality $\|w\| \leq l(1 + \|x\|)$ holds for any $(t, x) \in \mathbb{R}^{n+1}$ and $w \in F(t, x)$.

Under these assumptions, the differential inclusion

$$\dot{x} \in F(t, x) \tag{1.2}$$

has a solution $x(t)$, which can be extended to the entire number axis $\mathbb{R}^1$ (see [1]). It is assumed that the matrix $B(t, x)$ satisfies the Frobenius condition

$$\sum_{\nu=1}^{n} \frac{\partial b_{ij}(t, x)}{\partial x_\nu} b_{\nu l}(t, x) = \sum_{\nu=1}^{n} \frac{\partial b_{il}(t, x)}{\partial x_\nu} b_{\nu j}(t, x),$$

which will provide the unique reaction of system (1.1) on the control u in the case when u is an impulse action on this system (see [9]). By impulse positional control, we mean some abstract operator $(t, x) \longrightarrow U(t, x)$ that maps the space of variables t, x into the space m of vector distributions [8] according to the rule

$$U(t, x) = r(t, x(t))\, \delta_t,$$

where $r(t, x)$ is a vector function with values in $\mathbb{R}^m$ and δ_t is the Dirac impulse function concentrated at the point t. "Running impulse" $r(t, x(t))\, \delta_t$ as a generalized function does not make sense. An impulse control of this type is understood as a discrete implementation of a "running impulse" in the form of a sequence of correcting impulses concentrated at the points of some partition $h\colon t_0 < t_1 < \ldots < t_N = \theta$ of the segment I. The result of such a sequential correction is a discontinuous curve $x^h(\cdot)$, here called "Euler's broken line" or impulse-sliding regime. Let us describe more precisely the impulse-sliding regime. Let us define a network of "Euler's broken lines" $x^h(\cdot)$ corresponding to the set of partitions directed in magnitude

$$d(h) = \max(t_{k+1} - t_k), \quad h\colon t_0 < t_1 < \cdots < t_p = \vartheta$$

of the segment I. For this purpose, we first define the jump function by means of the equations

$$S(t, x, r(t, x)) = z(1) - z(0), \quad \dot{z}(\xi) = B(t, z(\xi))r(t, x), \quad z(0) = x.$$

Here, we take into account that, in fact, the dependence $z = z(\xi, t, x, r(t, x))$ takes place. Note also that the jump function is a vector function $S = (S^1, \ldots, S^n)$.

The jumps of the "Euler's broken lines" at the points of the partitions h of the segment I are determined by the equations

$$S\big(t_i, x^h(t_i), r(t_i, x^h(t_i))\big) = z(1) - z(0), \quad \dot{z}(\xi) = B(t_{t_i}, z(\xi))r(t_i, x^h(t_i))$$

with initial conditions $z(0) = x^h(t_i)$. The "Euler's broken line" $x^h(\cdot)$ is constructed as a function of bounded variation, which coincides with the solution of the differential inclusion (1.2) on each interval $(t_i, t_{i+1}]$ with initial conditions

$$x(t_i) = x^h(t_i) + S(t_i, x^h(t_i), r(t_i, x^h(t_i))), \quad x(t_0) = x_0, \quad i = 0, \ldots, p-1.$$

We will assume that the following equality is valid for all admissible t and x:

$$r\big(t, x + S(t, x, r(t, x))\big) = 0, \tag{1.3}$$

which means that, after an impulsive action on the system at time t, the phase point $x(t)$ will be on the manifold (target set)

$$\Phi = \big\{(t, x) : r(t, x) = 0\big\}.$$

Note that the definition of the jump function and condition (1.3) imply the relation

$$S = 0 \Leftrightarrow r = 0,$$

which is further used without reservation. It is also assumed that the functions $S(t, x, r)$ and $r(t, x)$ are continuously differentiable.

Under some additional assumptions, the sequence of "Euler's broken lines" has a convergent subsequence, the limit of which will be on the surface Φ. It is called the ideal sliding mode. The purpose of the impulse control is to keep the phase point on the manifold Φ. In [5], the differential inclusion of an ideal pulse-sliding mode is obtained in the form

$$\begin{gathered}\dot{x} \in \frac{\partial S(t, x, r(t, x))}{\partial t} + \frac{\partial S(t, x, r(t, x))}{\partial r}\frac{\partial r(t, x)}{\partial t} \\ + \left(E + \frac{\partial S(t, x, r(t, x))}{\partial x} + \frac{\partial S(t, x, r(t, x))}{\partial r}\frac{\partial r(t, x)}{\partial x}\right) F(t, x), \\ x(t_0 + 0) = x(t_0) + S\big(t_0, x(t_0), r(t_0, x(t_0))\big).\end{gathered} \tag{1.4}$$

Controls of the positional-impulse type were used to solve various problems of game theory and control, in particular, when constructing positional impulse controls in degenerate linear-quadratic optimal control problems. Note also that "Euler's broken lines" for the same positional impulse control may differ in the way of constructing jumps. One of them is listed above. Another one can be found in [4]. Accordingly, the equations of ideal sliding-impulse modes will differ.

In literature, you can find other methods for constructing jumps of impulse control, where the term "impulse-sliding regime" is used in a broader sense. As for processes of "sliding" type, to a greater extent, they are an attribute of controlled systems with discontinuous positional controls (feedbacks) and the theory of discontinuous systems in general, where such movements are called sliding regime. In this paper, a differential inclusion with discontinuous positional controls with constraints on control resources is constructed for which the ideal "impulse-sliding regime" of inclusion (1.1) is the usual sliding regime in the sense of the theory of discontinuous systems. It is the main mode of functioning of a discontinuous controlled system and allows solving such problems as stabilization, complete controllability, and tracking (movement along a predetermined trajectory). A huge number of works are devoted to these questions.

2. Multivalued equivalent controls

We will consider a controlled differential inclusion

$$\dot{x} \in F(t,x) + \tilde{u}, \tag{2.1}$$

where $\tilde{u} = (\tilde{u}_i, \dots, \tilde{u}_n)$, $\tilde{u}_i(t,x) = -H_i(t,x)\operatorname{sgn} S^i$, $H_i(t,x) \geq 0$ are some continuous functions, and S^i is the ith component of the jump function, $i = 1, \dots, n$.

If $S^i = 0$, then denote by $\tilde{U}_i(t,x)$ the segment $[-H_i(t,x), H_i(t,x)]$ and if $S^i \neq 0$, then $\tilde{U}_i(t,x) = \tilde{u}_i$. Let $\tilde{U}(t,x) = \tilde{U}_1(t,x) \times \cdots \times \tilde{U}_n(t,x)$. Under a solution to problem (2.1), we mean a solution to the differential inclusion

$$\dot{x} \in F(t,x) + \tilde{U}(t,x), \tag{2.2}$$

those, absolutely continuous function satisfying (2.2) almost everywhere on the considered segment I.

We will represent inclusion (2.2) in the form of a controlled system

$$\begin{cases} \dot{x} \in F(t,x) + \tilde{u}, \\ \tilde{u} \in \tilde{U}(t,x). \end{cases} \tag{2.3}$$

A solution to problem (2.3), defined on the segment I, is a pair $(x(t), \tilde{u}(t))$ consisting of an absolutely continuous function $x(t)$ (trajectory) and a measurable function $\tilde{u}(t)$ (control) satisfying inclusions (2.3) almost everywhere on I.

Lemma 1. *Let the multivalued mapping $F(t,x)$ satisfy conditions* (B1)–(B3)*, and let the functions $r(t,x)$ and $H_i(t,x)$ be continuous. Then, for any initial conditions $x(t_0) = x_0$, there exists a solution to inclusion* (2.2) *and, for any solution $x(t)$ to inclusion* (2.2)*, there exists a measurable function $\tilde{u}(t)$ such that the pair $(x(t), \tilde{u}(t))$ is a solution to problem* (2.3).

P r o o f. It is easy to check that the multivalued mapping $\tilde{U}(t,x)$ is upper semicontinuous and locally bounded. Then the right-hand side of inclusion (2.2) is upper semicontinuous, as the algebraic sum of two upper semicontinuous multivalued mappings. In addition, it is easy to check that the right-hand side of inclusion (2.2) possesses property (B1) and is integrally bounded. Then there exist a solution to inclusion (2.2) (see [1]).

Let $x(t)$ be a solution to inclusion (2.2). Then

$$\dot{x}(t) \in F(t, x(t)) + \tilde{U}(t, x(t))$$

for almost all $t \in I$ and Filippov's implicit function lemma (see [1, Theorem 1.5.15]) implies the existence of a measurable function $\tilde{u}(t) \in \tilde{U}(t, x(t))$ such that $\dot{x}(t) \in F(t, x(t)) + \tilde{u}(t)$ for almost all $t \in I$. Then the pair $(x(t), \tilde{u}(t))$ is a solution to the controlled system (2.3) and the lemma is proved.

We consider sliding regimes to inclusion (2.1) in relation to the surface

$$\Gamma = \{(t,x) : S(t,x,r(t,x)) = 0\}$$

or, which is equivalent, to the surface Φ.

A solution $x(t)$ to inclusion (2.1) satisfying the condition $(t, x(t)) \in \Phi$ will be called the sliding regime. One of the main ways to obtain equations of sliding regimes of discontinuous control systems is the method of equivalent controls (see [2]). The controls should be chosen so that the velocity vector $\dot{x}(t)$ at the points $(t, x(t))$ of the discontinuity surface lies in the tangent plane to this surface. Such controls $\tilde{u}^{eq}$ are called equivalent if they satisfy the given constraints. In the problem under consideration, these constraints have the form $\tilde{u}^{eq} \in \tilde{U}(t, x)$.

We denote by S_t the partial derivative of the mapping $t \to S(t, x, r(t, x))$ with respect to the variable t and by S_x the Jacobi matrix of the mapping $x \to S(t, x, r(t, x))$ with respect to the variable x. Let

$$\tilde{U}^{eq}(t, x) = S_t + S_x F(t, x).$$

Define a multivalued analogue of equivalent control for differential inclusion (2.1) in the form

$$\tilde{U}^{*eq}(t, x) = \tilde{U}^{eq}(t, x) \cap \tilde{U}(t, x).$$

Theorem 1. *Let $x(t)$ be a sliding regime of inclusion (2.1) and*

$$S_x = -E_n \tag{2.4}$$

for any $(t, x) \in \Gamma$, where E_n is an $n \times n$ identity matrix. Then

$$\tilde{U}^{*eq}\big(t, x(t)\big) \neq \emptyset \tag{2.5}$$

for almost all t and the function $x(t)$ is the trajectory of the controlled system

$$\begin{cases} \dot{x} \in F(t, x) + \tilde{u}, \\ \tilde{u} \in \tilde{U}^{*eq}(t, x). \end{cases} \tag{2.6}$$

P r o o f. Since the function $x(t)$ is a solution to inclusion (2.2), according to Lemma 1, there is a measurable function $\tilde{u}(t) \in \tilde{U}(t, x(t))$ such that the inclusion $\dot{x}(t) \in F(t, x(t)) + \tilde{u}(t)$ holds for almost all t. Since $(t, x(t))$ is a sliding regime, we have $(t, x(t)) \in \Gamma$ and, from the condition $S_x(t, x(t), r(t, x(t)) = -E_n$, we get

$$0 \in S_t(t, x(t), r)t, x(t)) + S_x(t, x(t), r(t, x(t))F(t, x(t)) - \tilde{u}(t). \tag{2.7}$$

It follows from (2.7) that $\tilde{u}(t) \in \tilde{U}^{eq}(t, x(t))$ for almost all t. Hence, $\tilde{u}(t) \in \tilde{U}^{*eq}(t, x(t))$ for almost all $t \in I$, condition (2.5) holds, and the pair $(x(t), \tilde{u}(t))$ is a solution to the controlled system (2.6). The theorem is proved.

Theorem 1 gives a necessary condition for the existence of a sliding mode for a differential inclusion (2.1).

We investigate sufficient conditions for the existence of sliding regimes S using the function

$$V(t, x) = \frac{1}{2}\langle S, S \rangle,$$

where $\langle \cdot, \cdot \rangle$ stand for the scalar product.

For any $\delta > 0$, we use the notation

$$W_\delta(t, x) = \big\{(t', x') \colon \|x' - x\| < \delta,\ |t - t'| < \delta\big\}.$$

Theorem 2. *Let condition (2.4) hold and, for every point $(t, x) \in \Gamma$, there exist $\varepsilon > 0$ and a neighborhood $W_\delta(t, x)$ such that*

$$\max_{w \in F(t', x')} |S_t^i + w_i| < H_i(t, x) - \varepsilon \tag{2.8}$$

for all indices $i = 1, \dots, n$ and all $(t', x') \in W_\delta(t, x)$.

Then the following statements are true.

(1) *For any solution to inclusion* (2.1) *with initial conditions* $(t_0, x_0) \in \Gamma$, *there holds* $(t, x(t)) \in \Gamma$ *for all points* $t \geq t_0$ *at which this solution exists.*

(2) *For any initial conditions* $(t_0, x_0) \in \Gamma$, *there is a sliding regime of inclusion* (2.1) *defined as a solution to inclusion* (2.2), *and any solution* $x(t)$ *with initial condition* $(t_0, x_0) \in \Gamma$ *is a sliding regime if and only if it is a trajectory of the controlled system* (2.3) *with the same initial condition.*

Theorem 2 follows from statements 1 and 3 of Theorem 3 from [4] with the replacement of the function $\sigma(t, x)$ by the function $S(t, x, r(t, x))$ and the use of Lemma 1, condition (2.4), and inequality (2.8).

3. Impulse-sliding and sliding regimes of differential inclusions

It follows immediately from the definitions that the differential inclusion (1.4) of the ideal impulse-sliding regime is written as

$$\dot{x} \in F(t, x) + \tilde{U}^{eq}(t, x). \tag{3.1}$$

Then the results of the previous section can be applied to it.

Theorem 3. *Let conditions* (B1)–(B3) *be satisfied, and let* (2.4), (1.3), *and inequality*

$$\|S(\tau, y, r(\tau, y)) - S(t, x, r(t, x))\| \leq L(|\tau - t| + \|y - x\|), \tag{3.2}$$

also hold for all admissible t, τ, x, *and* y. *Then:*

(1) *For inclusion* (1.1), *any sequence of "Euler's broken lines" has a subsequence uniformly converging to the ideal impulse-sliding mode, any ideal impulse-sliding regime* $\tilde{x}(t)$ *satisfies the condition* $S(t, \tilde{x}, r(t, \tilde{x})) = 0$ *and is a solution to the discontinuous system* (2.2) *and the trajectory of the controlled system*

$$\begin{cases} \dot{x} \in F(t, x) + \tilde{u}, \\ \tilde{u} \in \tilde{U}^{eq}(t, x) \end{cases} \tag{3.3}$$

with the initial condition $\tilde{x}(t_0 + 0) = x_0 + S(t_0, x_0, r(t_0, x_0))$.

(2) *If, in addition, inequalities* (2.8) *hold, then any ideal impulse-sliding regime* $\tilde{x}(t)$ (1.1) *is a sliding regime* (2.2) *with discontinuous positional control* $\tilde{u}$.

Note that the controlled system (3.3) and the differential inclusion (3.1) are equivalent in the sense that any trajectory from the pair $(x(t), \tilde{u}(t))$ is a solution to inclusion (3.1) and any solution to this inclusion is a trajectory of system (3.3).

Note also that the sliding mode in Theorem 3 is stable with respect to the target set Φ. If this is not the case (outside the scope of Theorem 2), then the usual sliding mode can be terminated and the solution can be kept on the target set using the impulse-sliding regime.

4. Example

Consider a controlled system

$$\begin{cases} \dot{x}_1(t) = -\operatorname{sgn}(x_2(t) - 1) + x_1(t)u_1, \\ \dot{x}_2(t) = -\operatorname{sgn}(x_1(t) - 1) + x_2(t)u_2. \end{cases} \tag{4.1}$$

It is required to organize a sliding mode on the set $x_1 \equiv 1$, $x_2 \equiv 1$. In this case, impulse control can be omitted. We put $u_1 \equiv u_1 \equiv 0$. Then the trajectory of the system (4.1) in the space x_1, x_2, t reaches the plane $x_1 = 1$ or $x_2 = 1$. After that, moving along this plane, it reaches the straight line $x_1 = x_2 = 1$ and will stay on this straight line in sliding regime.

If we consider the system

$$\begin{cases} \dot{x}_1(t) = \operatorname{sgn}(x_2(t) - 1) + x_1(t)u_1, \\ \dot{x}_2(t) = \operatorname{sgn}(x_1(t) - 1) + x_2(t)u_2, \end{cases} \tag{4.2}$$

then it is possible to provide sliding on the set $x_1 = x_2 = 1$ only with the help of the impulse-sliding regime.

The vector function $r = (r_1, r_2)^T$ is defined by the equalities

$$r_1(t,x) = -\ln x_1, \quad r_2(t,x) = -\ln x_2.$$

The control u has the form

$$U(t, x(t)) = r(t,x)\delta_t.$$

The problem for the impulse control u is to keep the phase point at the intersection of the straight lines $x_1 = 1$ and $x_2 = 1$, which are determined from the conditions $\ln x_1 = 0$ and $\ln x_2 = 0$. The jump function $S(t,x,r)$ has the form

$$S(t,x,r) = \begin{cases} x_1(e^{r_1} - 1), \\ x_2(e^{r_2} - 1). \end{cases}$$

The multivalued function $F(x)$ on the right-hand side of system (4.2) is defined as follows:

$$F_i = \begin{cases} 1, & x_i > 0, \\ -1, & x_i < 0, \qquad i = 1, 2. \\ [-1, 1], & x_i = 0, \end{cases}$$

This corresponds to the simplest convex extension of the right-hand side of the discontinuous equation (4.2) in Filippov's sense.

The fulfillment of conditions (1.3) and (2.4) for these functions $r(t,x)$ and $S(t,x,r(t,x))$ is verified directly.

The impulse-sliding regime is described by the equations $\dot{x}_1 = 0$ with the initial condition $x_1(0+) = 1$ and $\dot{x}_2 = 0$ with the initial condition $x_2(0+) = 1$. In order, in accordance with Theorem 3, to write a differential inclusion of the form (2.2), it is necessary to specify the coefficients H_1 and H_2 satisfying inequalities (2.8). It is easy to see that these can be any numbers exceeding one.

5. Conclusion

The impulse control that transfers the manipulator from a given position to its final position is constructed in the work. A computational experiment showing the efficiency of the proposed algorithm is presented. The proposed algorithm is simulated in the case when the ideal impulse is approximated by the usual bounded control.

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SOME NOTES ABOUT THE MARTINGALE REPRESENTATION THEOREM AND THEIR APPLICATIONS

Reza Habibi

Iran Banking Institute, Central Bank of Iran,
Pasdaran Ave., 8-th Negarestan, Tehran, Iran
rezahabibi2681@yahoo.com

Abstract: An important theorem in stochastic finance field is the martingale representation theorem. It is useful in the stage of making hedging strategies (such as cross hedging and replicating hedge) in the presence of different assets with different stochastic dynamics models. In the current paper, some new theoretical results about this theorem including derivation of serial correlation function of a martingale process and its conditional expectations approximation are proposed. Applications in optimal hedge ratio and financial derivative pricing are presented and sensitivity analyses are studied. Throughout theoretical results, simulation-based results are also proposed. Two real data sets are analyzed and concluding remarks are given. Finally, a conclusion section is given.

Keywords: Conditional expectation, Derivative pricing, Martingale representation theorem, Optimal hedge ratio, Sensitivity analysis, Serial correlation, Simulation, Stochastic dynamic.

1. Introduction

The martingale representation theorem states that any martingale adapted with respect to a Brownian motion can be expressed as a stochastic integral with respect to the same Brownian motion. It has many applications in construction hedging strategies for various types of assets with different stochastic dynamics, see [1]. In the current note, the time series features of this important theorem are proposed. Before going future, an important lemma is proposed. Let B_t be the standard Brownian motion on $(0,\infty)$ and $\mathfrak{F}_t$ is the sigma-field constructed by history of B_s, $s \le t$, i.e. $\mathfrak{F}_t = \sigma\{B_s | s \le t\}$. Hence, if $s \le t$; then $\mathfrak{F}_s \subseteq \mathfrak{F}_t$. Indeed, $\mathfrak{F}_t$ is the augmented filtration generated by standard Brownian motion B_t. Also, assume that X, Y are the future values of two stochastic processes at some known future time T. According to the martingale representation theorem, it is necessary to assume that both of X, Y are squared integrable random variables with respect to $\mathfrak{F}_\infty$ to use this theorem for X and Y (see, [1]). These assumptions are kept fixed for all further discussions of the paper.

Lemma 1. *Sentences* (a) *and* (b) *are correct*:

(a) *The correlation* ρ_{xy} *between* X, Y, *is given as follows*

$$\rho_{xy} = \frac{\displaystyle\int_0^T E(u_s v_s) ds}{\sqrt{\displaystyle\int_0^T E(u_s^2) ds \int_0^T E(v_s^2) ds}},$$

here u_s *and* v_s *are two predictable processes used in martingale representation theorem applied to* X, Y, *respectively.*

(b) *Suppose that* $E(X|Y=y)=ay+b$, $E(Y|X=x)=cx+d$. *Then,*

$$\begin{cases} \mu_x = a\mu_y + b, \quad \mu_y = c\mu_x + d, \\ \rho_{xy}^2 = ac, \quad \dfrac{\sigma_x^2}{\sigma_y^2} = \dfrac{a}{c}, \end{cases}$$

where μ_A *and* σ_A^2 *are the mean and variance of* $A = X, Y$, *respectively.*

P r o o f. (a) The martingale representation theorem implies that (see [2]) there exist two predictable processes u_t, v_t such that

$$\begin{cases} X = E(X) + \displaystyle\int_0^T u_s dB_s, \\ Y = E(Y) + \displaystyle\int_0^T v_s dB_s. \end{cases}$$

For a review in stochastic calculus, see [7]. It is easy to see that

$$E\Big[\int_0^T u_s dB_s\Big] = E\Big[\int_0^T v_s dB_s\Big] = 0.$$

By multiplying above two equations and taking expectation, it is seen that

$$E(XY) = E(X)E(Y) + E(X)E\Big[\int_0^T v_s dB_s\Big] + E(Y)E\Big[\int_0^T u_s dB_s\Big] + E\int_0^T\int_0^T u_s v_t dB_s dB_t.$$

Notice that [7]

$$E\int_0^T\int_0^T u_s v_t dB_s dB_t = \int_0^T E(u_s v_s)\, ds.$$

Hence, it is seen that covariance between X and Y, i.e., $\sigma_{xy} = \operatorname{cov}(X,Y)$ is given by

$$\sigma_{xy} = \int_0^T E(u_s v_s)\, ds.$$

Also, using the Ito isometric lemma (see [7]), it is seen that

$$E\left(X^2\right) = (E(X))^2 + \int_0^T E\left(u_s^2\right) ds.$$

Therefore,

$$\sigma_x^2 = \operatorname{var} X = \int_0^T E\left(u_s^2\right) ds.$$

Similarly,

$$\sigma_y^2 = \int_0^T E\left(v_s^2\right) ds.$$

Thus, the proof is complete.

(b) The proof is straightforward by using the iterated expectation law (see [2]). Therefore it is omitted. □

Example 1. Here, to give an example for assumption of part (b), a special case is considered. For a specific $t, h > 0$, let X be a martingale with respect to $\mathfrak{F}_t$, then according to the martingale representation theorem, we have

$$X = X_t = \int_0^t u_s dB_s + E(X).$$

Here, $E(X)$ is constant and independent of t. Let $Y = X_{t+h}$. For special case, suppose that u_s is a deterministic real-valued function. Then,

$$\Gamma = Y - X = \int_t^{t+h} u_s dB_s.$$

Clearly, X_t is an independent increment process and Γ has a normal distribution with zero mean and variance $\int_t^{t+h} u_s^2 ds$. Notice that $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a linear combination of $\begin{pmatrix} X \\ \Gamma \end{pmatrix}$ as follows

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ X + \Gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ \Gamma \end{pmatrix},$$

and since $\begin{pmatrix} X \\ \Gamma \end{pmatrix}$ has a joint normal distribution with mean vector $\begin{pmatrix} E(X) \\ 0 \end{pmatrix}$ and covariance matrix

$$\begin{pmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_\Gamma^2 \end{pmatrix},$$

therefore, $\begin{pmatrix} X \\ Y \end{pmatrix}$ has a joint distribution with mean vector $\begin{pmatrix} E(X) \\ E(X) \end{pmatrix}$ and covariance matrix

$$\begin{pmatrix} \sigma_X^2 & \sigma_X^2 \\ \sigma_X^2 & \sigma_X^2 + \sigma_\Gamma^2 \end{pmatrix}.$$

Here,

$$\sigma_X^2 = \int_0^t u_s^2 ds, \quad \sigma_\Gamma^2 = \int_t^{t+h} u_s^2 ds.$$

The correlation between X, Y is

$$\rho_{xy} = \frac{\sigma_X^2}{\sqrt{\sigma_X^2(\sigma_X^2 + \sigma_\Gamma^2)}} = \left(1 + \frac{\sigma_\Gamma^2}{\sigma_X^2}\right)^{-0.5}.$$

Thus (see [6])

$$E(X \mid Y) = E(X_t \mid X_{t+h}) = E(X) + \rho_{xy}\frac{\sigma_X}{\sigma_Y}(Y - E(X)) = E(X) + \left(1 + \frac{\sigma_\Gamma^2}{\sigma_X^2}\right)^{-1}(Y - E(X)).$$

Also, notice that $E(X_{t+h} \mid X_t) = X_t$. Hence, the parameters of a, b, c, and d of the theorem are

$$a = \left(1 + \frac{\sigma_\Gamma^2}{\sigma_X^2}\right)^{-1}, \quad b = (1 - a)E(X), \quad c = 1, \quad d = 0.$$

The rest of the paper is organized as follows. In the next section the application of above Lemma 1 in deriving optimal hedge ratio is discussed. Section 3 uses the (b) of Lemma 1 to approximate the conditional mean and it is applied to financial derivative pricing. Simulation results is given throughout theoretical sections. Real data sets analysis are given in Section 4. Finally, a Conclusion section is given.

2. Optimal hedge ratio

Here, the application of above discussion in portfolio management is discussed. The cross hedging procedure is the construction of an almost riskless portfolio by using one unit of the first asset X in long position and h units of Y in short position (at the maturity) (see [4]).

Let $Z = X - hY$ be the value of portfolio at maturity T. The variance of Z is given by

$$\sigma_z^2 = \sigma_x^2 + h^2\sigma_y^2 - 2h\sigma_{xy}.$$

By minimizing σ_z^2 with respect to h, it is seen that the optimum hedge ratio h_{opt} is given by

$$h_{opt} = \frac{\sigma_{xy}}{\sigma_y^2} = \frac{\displaystyle\int_0^T E\left(u_s v_s\right) ds}{\displaystyle\int_0^T E\left(v_s^2\right) ds}.$$

Hence, the optimum value of portfolio at maturity is

$$Z = X - h_{opt}Y = E\left(X\right) - h_{opt}E\left(Y\right) + \int_0^T \left(u_s - h_{opt}v_s\right) dB_s.$$

The variance of Z at maturity is $\sigma_x^2(1-\rho_{xy}^2)$.

Next, suppose that the risk free interest rate is zero, then the value of X, Y at maturity T is the following (see [2])

$$\begin{cases} X_t = E(X|\mathfrak{F}_t) = E\left(X\right) + \displaystyle\int_0^t u_s dB_s, \\ Y_t = E(Y|\mathfrak{F}_t) = E\left(Y\right) + \displaystyle\int_0^t v_s dB_s. \end{cases}$$

Consider the self-financed portfolio $Z_t = X_t - H_tY_t$ (see [7]). Assume that $H_t = u_t/v_t$. Notice that

$$dZ = dX - HdY = \left(u - Hv\right) dB.$$

Then, $dZ = 0$. Thus, Z is constant. Indeed, $Z = E\left(X\right) - HE\left(Y\right)$. Hence,

$$X_t - H_tY_t = E\left(X\right) - H_tE\left(Y\right).$$

Hence,

$$H_t = \frac{X_t - E\left(X\right)}{Y_t - E\left(Y\right)}.$$

The following proposition summarizes the above discussion.

Proposition 1. *Sentences* (a) *and* (b) *are correct.*

(a) *Under the martingale representation, the optimum hedge ratio for cross hedging* X *by* Y*, is given by*

$$h_{opt} = \frac{\sigma_{xy}}{\sigma_y^2} = \frac{\displaystyle\int_0^T E\left(u_s v_s\right) ds}{\displaystyle\int_0^T E\left(v_s^2\right) ds}.$$

(b) *The replicating ratio for rebalancing portfolio the dynamic hedging portfolio is*

$$H_t = \frac{X_t - E\left(X\right)}{Y_t - E\left(Y\right)}.$$

P r o o f. See the above discussions. □

Next, consider the martingale representation theorem as follows

$$X_t = E(X|\mathfrak{F}_t) = E(X) + \int_0^t u_s dB_s.$$

Let $G(t) = E(u_t^2)$ and $g(t) = \log(G(t))$ and g' be its first derivative. According to Lemma 1, (a) and Ito isometric lemma, the correlation coefficient $\rho_t(h)$ between X_{t+h}, X_t is given by

$$\rho_t(ht) = \sqrt{\frac{G(t)}{G(t+h)}} = \frac{1}{\sqrt{1 + h/G(t) \times \big(G(t+h) - G(t)\big)/h}}.$$

As $h \to 0^+$, then $\rho_t(h)$ is well-approximated by $1/\sqrt{1+hg'(t)}$. The second term $g''(\mathrm{t})$ could be added to mentioned approximation, which is not necessary in practice.

3. Conditional mean approximation

Here, using the second part of Lemma 1, the conditional mean of $E(X_t|X_{t+h})$ is approximated and then its financial application is seen.

3.1. Approximation

Notice that one can see that X_t is a martingale with respect to filtration $\dot{F}_t$, the σ-field generated by X_s, $s \le t$. Next, assume that the conditional expectation of $E(X_t|X_{t+h})$ is well-approximated by linear combination $aX_{t+h} + b$. Then, using the Lemma 1, (b), it is seen that $\rho_t^2(h) = a$, $b = \mu(1 - \rho_t^2(h))$, where $\mu = E(X) = E(X_t)$. Therefore,

$$E(X_t \mid X_{t+h}) = \mu + \rho_t^2(h)(X_{t+h} - \mu),$$

where $\rho_t^2(h) = 1/(1 + hg'(t))$. The following proposition summarizes the above discussion.

Proposition 2. *Assuming $E(X_t|X_{t+h})$ is well-approximated by a linear function of X_{t+h}, then*

$$E(X_t|X_{t+h}) = \mu + \rho_t^2(h)(X_{t+h} - \mu),$$

where $\mu = E(X) = E(X_t)$ and $\rho_t(h) = 1/\sqrt{1+hg'(t)}$. Here, $G(t) = E(u_t^2)$ and $g(t) = \log(G(t))$ and g' be its first derivative.

P r o o f. See the above discussions. □

Example 1 (cont.). Here, it is shown that the formula of example 1 corresponds to the approximation of Proposition 2, as $h \to 0$. Define

$$\kappa(t) = \int_0^t u_s^2 ds.$$

Thus, $\sigma_X^2 = \kappa(t)$ and $\sigma_I^2 = \kappa(t+h) - \kappa(t) \approx h\kappa'(t)$. Hence,

$$\rho_{xy} = \frac{1}{\sqrt{1 + h\kappa'(t)/\kappa(t)}},$$

this is exactly equal to the approximation formula of Proposition 2.

Remark 1. Here some sensitivity analysis are discussed. Indeed, we have the following properties.

(a) As $h \to 0$, then $\rho_t\,(h) \to 1$ which is clear (since the correlation of each variable with its-self is one). As $h \to \infty$, then $\rho_t\,(h) \to 0$ which is clear since X_{t+h} and X_t are enough far from each others. Also,

$$\frac{\partial \rho}{\partial h} = -0.5g'\,(t)\,(1 + hg'\,(t))^{-3/2}$$

which converges to the $-0.5g'\,(t)$, as $h \to 0$. When $h \to \infty$, then $\partial\rho/\partial h$ goes to zero which is clear since the variation of $\rho_t\,(h)$ is too small at infinity.

(b) It is easy to see that

$$\frac{\partial \rho}{\partial t} = -0.5hg''(t)(1 + hg'(t))^{-3/2}.$$

Example 2. Let $u_s = 2B_s$, then $X_t = B_t^2 - t$ is a martingale. Indeed,

$$G\,(t) = 4t, \quad g'\,(t) = \frac{1}{t}, \quad \rho_t\,(h) = \sqrt{\frac{t}{t+h}}.$$

For example, when $t = 0.1,\ 0.5$, the following Fig. 1 shows the behavior of

$$\rho_{0.5}\,(h) = \sqrt{\frac{1}{1+2h}}, \quad \rho_{0.1}\,(h) = \sqrt{\frac{1}{1+10h}}, \quad h \in (0, 1),$$

respectively. As more $t \to 0$, then the curvature of $\rho_t\,(h)$ is more close to the horizontal axis. Notice that

$$\frac{\partial \rho}{\partial t} = \frac{h}{\sqrt{t}(t+h)^2} \to 0 \quad \text{as} \quad t \to \infty.$$

It is clear because as $t \to \infty$ or $t \to 0$, then $\rho_t\,(h) \to 1$ and its variation is too small. This is an interesting phenomena that as t gets large, then correlation B_t^2 with its future values is large for each h. For special case, when $h = t$, then $\rho_t\,(h) = \sqrt{2}/2$.

Also, let $h = q(t)$, for some real valued function q, and suppose that $q(t)/t$ converges to α (β) as $t \to 0$ $(t \to \infty)$, then $\rho_t\,(h)$ tends to $1/\sqrt{1+\alpha}$ $(1/\sqrt{1+\beta})$. Fig. 1 shows the behavior of $\rho_{0.1}\,(h)$ and $\rho_{0.5}\,(h)$ which verifies the above discussion. For another example, as extension of Brownian motion, consider the Ornstein–Uhlenbeck process U_t defined by

$$dU = -\alpha U dt + \sigma dB.$$

The Ito lemma implies that $X = X_t = e^{\alpha t}U_t$ satisfies the stochastic differential equation

$$dX = \sigma e^{\alpha t} dB$$

which is martingale with respect to $\mathfrak{F}_t$. Using the Example 1, it is seen that

$$\sigma_x^2 = \int_0^t \sigma^2 e^{2\alpha s} ds = \frac{\sigma^2}{2\alpha}(e^{2\alpha(t+h)} - 1)$$

and

$$\sigma_\Gamma^2 = \frac{\sigma^2}{2\alpha}(e^{2\alpha(t+h)} - e^{2\alpha t}).$$

It is seen that

$$E(X_t | X_{t+h}) = aX_{t+h} + b$$

where

$$a = \left(1 + \frac{\sigma_\Gamma^2}{\sigma_X^2}\right)^{-1}, \quad b = (1-a)\,E(X).$$

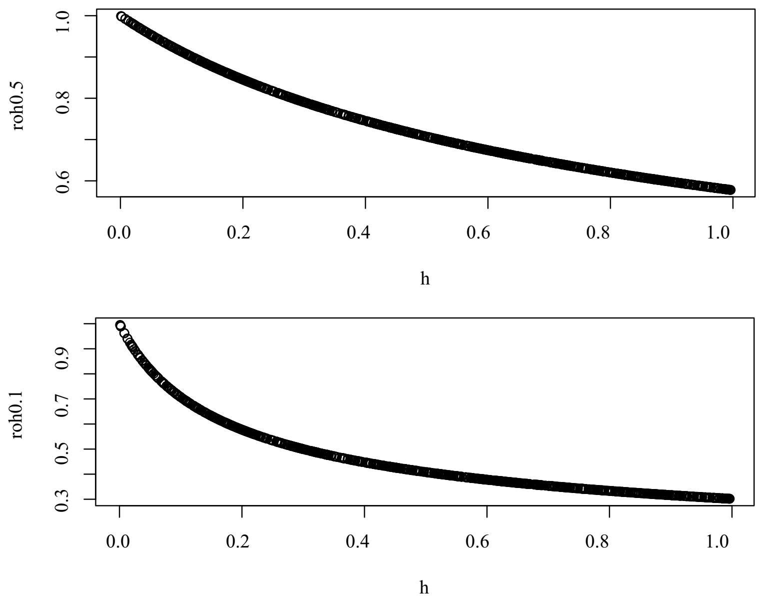

Figure 1. Plots of $\rho_{0.1}(h)$ and $\rho_{0.5}(h)$.

Equivalently,

$$E(U_t|U_{t+h}) = ae^{\alpha h}U_{t+h} + b, \quad E(U_{t+h}|U_t) = e^{-\alpha h}U_t.$$

From Dambis, Dubins–Schwarz (DDS) theorem (see [5, p. 204]), it is seen that

$$U_t = e^{-\alpha h}\tilde{B}\left(\frac{\sigma^2(e^{2\alpha t}-1)}{2\alpha}\right),$$

where $\tilde{B}$ is another Brownian motion. Again, using the results of the previous example, the same results are obtained. Using the results of Remark 1, part (b), it is seen that

$$g(t) = \log\left(\frac{\sigma^2}{2\alpha}\right) + \log(e^{2\alpha t}-1).$$

Then,

$$g''(t) = \frac{-4\alpha^2 e^{2\alpha t}}{(e^{2\alpha t}-1)^2} \to 0, \quad \text{as} \quad t \to \infty.$$

Hence,

$$\frac{\partial \rho}{\partial t} \to 0 \quad \text{as} \quad t \to \infty.$$

3.2. Pricing

In this section, the application of above approximation in pricing of financial derivative is studied. Consider the price of financial derivative f at time t which expires at maturity T $(t \leq T)$ written on a given underlying financial asset. Then,

$$f_t = e^{-r(T-t)}E_Q(f_T|\mathfrak{F}_t)$$

where r, Q, T, and $\mathfrak{F}_t$ are risk free rate, risk neutral probability measure, maturity of financial derivative and the σ-field of price time series $s_u, u \leq t$, (see [7]). Here, under the risk neutral probability measure, the dynamic of price of underlying asset is given by

$$ds = rsdt + \sigma dB$$

at which σ is volatility of price. According to the Black–Scholes formula, the price of financial derivative satisfies the partial differential equation

$$\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 f}{\partial s^2} = rf.$$

Let $X_t = e^{-rt} f_t$. Then, using the Ito lemma, it is seen that

$$dX = e^{-rt}\frac{\partial f}{\partial s}\sigma s dB.$$

Then,

$$X = E_Q\left(e^{-rT} f_T\right) + \int_0^t e^{-ru}\frac{\partial f}{\partial s}\sigma s dB_u.$$

Thus,

$$c_t = e^{-rt}\frac{\partial f}{\partial s}\sigma s = \sigma e^{-rt} s\varDelta,$$

where $\varDelta$ is the Greek letter delta representing the sensitivity parameter of financial derivative with respect to variation of s. Notice that

$$G(t) = e^{-2rt}\sigma^2 E_Q(\varDelta^2 s^2)$$

and

$$g(t) = -2rt + \log\left(\sigma^2\right) + l(t)$$

where $l(t) = \log\left(E_Q\left(\varDelta^2 s^2\right)\right)$ and $g'(t) = -2r + l'(t)$ and

$$\rho_t(h) = \frac{1}{\sqrt{1 + h(-2r + l'(t))}}.$$

In practice, the quantity $E_Q\left(\varDelta^2 s^2\right)$ is approximated using a Monte Carlo simulation. The following proposition summarizes the above discussion.

Proposition 3. *For the financial derivative with price f_t then the correlation coefficient $\rho_t(h)$ between f_{t+h}, f_t is given by*

$$\rho_t(h) = \frac{1}{\sqrt{1 + h(-2r + l'(t))}},$$

where $l(t) = \log\left(E_Q\left(\varDelta^2 s^2\right)\right)$ and $g'(t) = -2r + l'(t)$.

P r o o f. The result is a direct consequence of previous discussions. □

Remark 2. Hereafter, the sensitivity analysis of $\rho_t(h)$ to its parameters σ, h is verified. For trivial derivative we have $f = s$, then $\varDelta = 1$, and under the risk neutral measure Q, we have

$$ds = rsdt + \sigma sdB.$$

The solution is

$$s_t = s_0 e^{\left(r - \sigma^2/2\right)t + \sigma B_t}.$$

Thus, $E_Q\left(\varDelta^2 s^2\right) = E_Q\left(s^2\right) = s_0^2 e^{2rt + \sigma^2 t}$ and $g'(t) = \sigma^2$ and $\rho_t(h) = 1/\sqrt{1 + h\sigma^2}$ independent of t. As $\sigma \to \infty$ (0), then $\rho_t(h)$ tends to the 0 (1). If $h = 1/\sigma^2$, so $\rho_t(h) = \sqrt{2}/2$.

4. Real data sets

In this section, throughout real data sets the computational aspects of above theoretical results are studied.

Example 3. In this example, the application of the formula for backward forecasting of daily stock price of *Apple co.* for period of 3 December 2019 to 2 December 2020 (including 254 observations) is studied. Backward forecasting is useful for checking the correctness of guess of traders about future price of a specified share (see [3]). According to the Proposition 2, the backward forecasting in a martingale process is given as follows:

$$E\left(X_t \mid X_{t+h}\right) = \mu + \rho_t^2\left(h\right)\left(X_{t+h} - \mu\right).$$

As follows, error analyses is given to verify the accuracy of the above formula. Using the first 80 percent of data set (i.e., 202 observations, dated from 3 December 2019 to 21 September 2020), the following Ito process if fitted to the *Apple co.* stock price,

$$ds = 0.003sdt + 0.0305sdB,$$

which has solution

$$s = 64.31e^{0.00254t+0.0305B}.$$

Here,

$$X_t = e^{0.0305B-0.0305^2 t/2}$$

is martingale, and

$$s = 64.31X_t e^{0.003t}.$$

Hence, substituting this equation to the conditional mean approximation, we see

$$E\left(s_t \mid s_{t+h}\right) = 64.31\mu e^{0.003t}\left(1 - \rho_t^2\left(h\right)\right) + e^{-0.003t}\rho_t^2\left(h\right)s_{t+h},$$

where

$$\mu = 1, \quad u_t = \sigma e^{0.0305B-0.0305^2 t/2}, \quad G\left(t\right) = (0.0305)^2 e^{(0.0305)^2 t},$$
$$g'(t) = (0.0305)^2, \quad \rho_t(h) = \frac{1}{\sqrt{1+h(0.0305)^2}}.$$

Next, assuming observations 21 September 2020 to 2 December 2020 are known this is the assumption of the trader about the future, the available data (data for 3 December 2019 to 21 September 2020) are forecasted, backwardly. Here, we used the remaining 20 percent of data, as trader conjecture about future. However, in practice, he may used own dada obtained by his techniques for fundamental analysis. The following Fig. 2 gives the error obtained by different actual and backward forecast for period of 3 December 2019 to 21 September 2020. It is seen that trader guess about future is true.

Example 4. In this example, the daily stock prices of *Amazon co.* for period of 2 October 2017 to 30 September 2019 (including 502 observations) are studied. It is seen that $\sigma = 0.0199$, $r = 0.05$ per year and $s_0 = 959.19$. Consider a call option with strike price $k = 970$, with maturity $T = 1$ (12 months) and European type. The delta parameter is

$$\varDelta = \varPhi\left(d_1\right), \quad d_1 = \frac{\log\left(s/k\right) + \left(r + 0.5\sigma^2\right)\left(T - t\right)}{\sigma\sqrt{T-t}}.$$

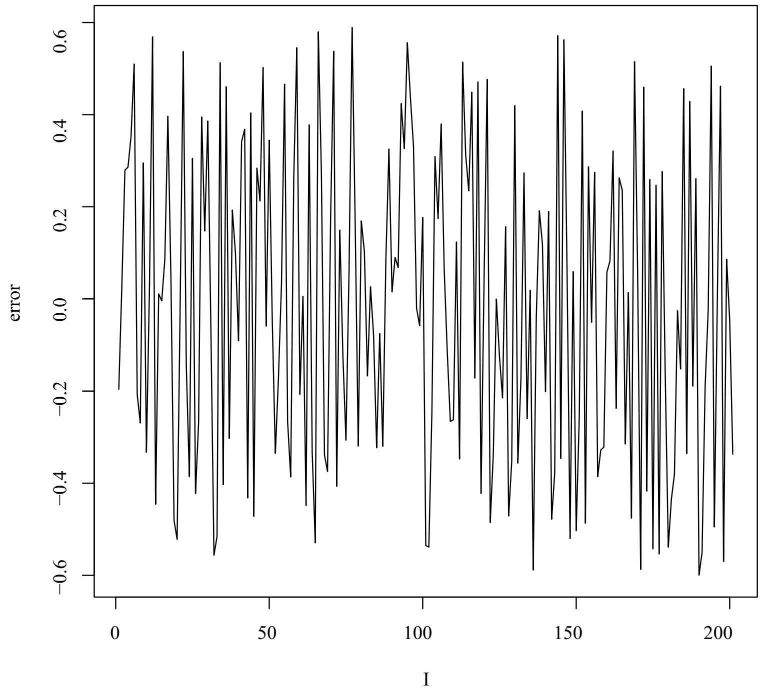

Figure 2. Time series plot of error.

Here, Φ is the normal standard distribution function. The following Fig. 3 shows the $\rho_t\,(0.2)$, $\rho_t\,(2)$ for various values of t. To simulate $l(t)$, a Monte Carlo simulation with 1000 repetitions is performed. Also, the variance reduction method is applied. It is seen that as h becomes large then, naturally, $\rho_t\,(h)$ becomes small. As follows, the Black-Scholes (BS) price of a call option is compared with the approximate price. Also, $\varDelta$ is an important sensitivity Greek letter to obtain a riskless portfolio. Then, actual $\varDelta$ is compared with its approximation. Based on these comparisons, the following table is derived. Here, min, $q_i,\ i = 1, 2, 3$, and max are the minimum, the first, second, third quartiles and the maximum of errors (differences between BS price and $\varDelta$, with their approximations), respectively. It is seen that the approximation works well.

Table 1. Measures of errors.

	min	q_1	q_2	q_3	max
Price	-2.59	-1.31	0.88	1.27	2.68
$\varDelta$	-0.35	-0.12	-0.01	0.17	0.25

5. Conclusion

In this paper, first, the correlation between two stochastic processes, satisfying the martingale representation theorem format, are derived. This correlation is used to obtain the optimal hedge ratio in a portfolio where two assets have the above mentioned stochastic process behaviors. Then, the results are developed to the serial correlation between a stochastic process and its lags. Then, this serial correlation is approximated. Sensitivity analyses of serial correlation to the time and lags and the parameters of underlying stochastic processes are studied and some interesting results about the relationship of process to its lags in long term (when t tends to ∞) are proposed. Using

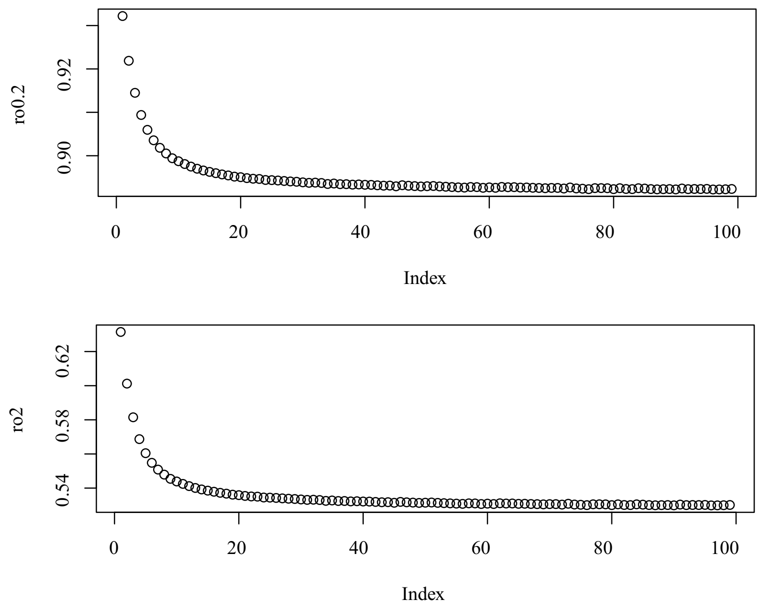

Figure 3. Plots of $\rho_t\,(0.2)$, $\rho_t\,(2)$.

the serial correlation, the backward forecast of price of financial assets such as share, equity, stocks or financial derivatives are presented. Forecasts are done using the backward conditional which is well approximated. Throughout, simulated examples and real data sets applicability of proposed methods are seen.

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INEQUALITIES FOR ALGEBRAIC POLYNOMIALS ON AN ELLIPSE[1]

Tatiana M. Nikiforova

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, 620990, Russia

Ural Federal University,
19 Mira str., Ekaterinburg, 620002, Russia

t.m.nikiforova@yandex.ru

Abstract: The paper presents new solutions to two classical problems of approximation theory. The first problem is to find the polynomial that deviates least from zero on an ellipse. The second one is to find the exact upper bound of the uniform norm on an ellipse with foci ± 1 of the derivative of an algebraic polynomial with real coefficients normalized on the segment $[-1,1]$.

Keywords: Polynomial, Chebyshev polynomials, Ellipse, Segment, Derivative of a polynomial, Uniform norm.

Introduction

Denote by $\mathcal{P}_n^1$ the set of algebraic polynomials of degree n with the unit leading coefficient:

$$p(z) = z^n + c_{n-1}z^{n-1} + \ldots + c_0, \quad c_k \in \mathbb{C}.$$

Consider the ellipse $E = \{z = a\cos t + ib\sin t \mid t \in [0, 2\pi];\ a > b > 0\}$ centered at the origin. Let $\|p\|_E = \max\limits_{z\in E} |p(z)|$. Recall that a polynomial $p_n^* \in \mathcal{P}_n^1$ is called a polynomial least deviating from zero on E (in $C(E)$) if

$$\min_{p\in\mathcal{P}_n^1} \|p\|_E = \|p_n^*\|_E.$$

V.I. Smirnov and N.A. Lebedev showed in 1964 [1, p. 331–333] that the normalized Chebyshev polynomial of the first kind T_n is the polynomial least deviating from zero on an ellipse with foci ± 1. Smirnov and Lebedev considered the ellipse as an image of a circle under the Joukowsky transform

$$f(\omega) = (\omega + 1/\omega)/2.$$

Thus, instead of polynomials, they studied functions of the following form defined on circles:

$$q_n(\omega) = R_n(\omega) + \sqrt{\omega^2 - 1}\cdot Q_{n-1}(\omega),$$

where $R_n(\omega)$ and $Q_{n-1}(\omega)$ are polynomials of degree at most n and $n-1$, respectively. We will give another solution of this problem. Our proof is based on Kolmogorov's theorem about the best uniform approximation of arbitrary continuous function by generalized polynomials.

[1]The work was performed as a part of research conducted in the Ural Mathematical Center.

The second problem considered in the paper is related to inequalities for derivatives of algebraic polynomials. Let

$$E_1 = \{z = a\cos t + bi\sin t \mid t \in [0, 2\pi];\ a^2 - b^2 = 1\}$$

be an ellipse with foci ± 1, and let $\mathcal{P}_n^{[-1,1]}$ be the set of algebraic polynomials of degree n with real coefficients and the unit uniform norm on $[-1, 1]$. In 1986, J.H.B. Kemperman showed [3, Theorem 1.2] that, for any $p_n \in \mathcal{P}_n^{[-1,1]}$,

$$\|p_n'\|_{E_1} \le \frac{n}{2b}\big((a+b)^n - (a-b)^n\big) = \|T_n'\|_{E_1}.$$

We obtained the proof of this fact independently of Kemperman. Our idea and Kemperman's one are similar, but our proof is shorter and easier to understand, as we consider objects in the problem from a slightly different point of view.

1. Polynomial that deviates least from zero on an ellipse

Let us give a solution to the problem about a polynomial $p_n^*(z)$ deviating least from zero on E. We will obtain an explicit formula for $p_n^*(z)$, a value of its norm $\|p_n^*\|_E$, and a recurrence relation between $p_{n+1}^*(z)$, $p_n^*(z)$, and $p_{n-1}^*(z)$.

Lemma 1. *For any positive integer n,*

$$(a\cos t + ib\sin t)^n = A_n\cos(nt) + iB_n\sin(nt) + \sum_{k=0}^{n-1}\alpha_k\cos(kt) + i\beta_k\sin(kt), \tag{1.1}$$

where $A_n, B_n, \alpha_k, \beta_k \in \mathbb{R}$ and

$$A_n = \frac{aA_{n-1} + bB_{n-1}}{2} = \frac{(a+b)^n + (a-b)^n}{2^n}, \quad B_n = \frac{bA_{n-1} + aB_{n-1}}{2} = \frac{(a+b)^n - (a-b)^n}{2^n}.$$

P r o o f. The proof of (1.1) is by induction on n. It is clear that the statement holds for $n = 1$. Assume that the statement holds for $n - 1$, and prove it for n. We have

$$(a\cos t + ib\sin t)^n = (a\cos t + ib\sin t)\cdot\Big(A_{n-1}\cos\big((n-1)t\big) + iB_{n-1}\sin\big((n-1)t\big) + \\ + \sum_{k=0}^{n-2}\widetilde{\alpha}_k\cos(kt) + i\widetilde{\beta}_k\sin(kt)\Big)$$

with real A_{n-1}, B_{n-1}, $\widetilde{\alpha}_k$, and $\widetilde{\beta}_k$. The imaginary unit will only appear in the products of the form

$$i\sin(\ell t)\cos(kt) = i\big(\sin\big((k+\ell)t\big) - \sin\big((k-\ell)t\big)\big)/2.$$

Therefore, all the coefficients A_n, B_n, α_k, and β_k in (1.1) are real.

The proof of the recurrence relations for A_n and B_n is straightforward. Indeed, by removing parentheses in the product, we obtain

$$(a\cos t + ib\sin t)\cdot\Big(A_{n-1}\cos\big((n-1)t\big) + iB_{n-1}\sin\big((n-1)t\big)\Big) = \\ = \frac{aA_{n-1} + bB_{n-1}}{2}\cos(nt) + \frac{bA_{n-1} + aB_{n-1}}{2}i\sin(nt) + \dots$$

Let us show that

$$A_n - B_n = \frac{(a-b)^n}{2^{n-1}}, \quad A_n + B_n = \frac{(a+b)^n}{2^{n-1}}, \quad n \in \mathbb{N}. \tag{1.2}$$

These equations obviously imply explicit formulas for A_n and B_n. Equations (1.2) for $n = 1$ are obvious. We now proceed by induction. Assume that the statement holds for $n-1$, and prove that it holds for n. We have

$$A_n + B_n = \frac{aA_{n-1} + bB_{n-1}}{2} + \frac{bA_{n-1} + aB_{n-1}}{2} =$$

$$= \frac{1}{2}\big((a+b)A_{n-1} + (a+b)B_{n-1}\big) = \frac{(a+b)}{2} \cdot \frac{(a+b)^{n-1}}{2^{n-2}} = \frac{(a+b)^n}{2^{n-1}}.$$

Similarly, we obtain the formula for $A_n - B_n$. □

Corollary 1. *For any $p \in \mathcal{P}_n^1(\mathbb{C})$, its restriction to the ellipse E is represented as*

$$p(z) = A_n \cos(nt) + iB_n \sin(nt) + \sum_{k=0}^{n-1} \alpha_k \cos(kt) + \beta_k \sin(kt),$$

where $\alpha_k, \beta_k \in \mathbb{C}$, $A_n, B_n \in \mathbb{R}$ and $A_n \geq B_n$.

We will not change the meaning of symbols A_n and B_n.

Lemma 2. *Let $\varphi = \big\{1, (a\cos t + ib\sin t), \ldots, (a\cos t + ib\sin t)^{n-1}\big\}$, and let Φ_{n-1} be the set of generalized polynomials from φ with complex coefficients. Then, for $f(t) = A_n\cos(nt) + iB_n\sin(nt)$,*

$$\inf_{p\in\Phi_{n-1}} \|f - p\|_{C[0,2\pi]} = \|f\|_{C[0,2\pi]} = A_n;$$

i.e., $p(t) \equiv 0$ is the best uniform approximation for f by generalized polynomials from φ.

P r o o f. Let $\varepsilon = \big\{t \in [0, 2\pi] \colon |f(t)| = \|f\|_{C[0,2\pi]} = A_n\big\}$. Since the set of values of f is the ellipse with the major horizontal semiaxis of length A_n and the vertical semiaxis of length B_n, we have

$$\varepsilon = \big\{t_k = \pi k/n,\ k = 0, \ldots, 2n\big\}.$$

By Kolmogorov's theorem [2, Theorem 1; 8, p. 47, Theorem 1], to prove the lemma it suffices to show that

$$\min_{t_k} \Re\{p(t_k)\overline{f(t_k)}\} \leq 0 \tag{1.3}$$

for any $p \in \Phi_{n-1}$. Let us substitute $f(t_k)$ into (1.3):

$$\Re\{p(t_k)\overline{f(t_k)}\} = A_n \Re\{(-1)^k p(t_k)\}, \quad k = 0, \ldots, 2n.$$

Note that, by the definition of φ, it follows that $p(t)$ is a trigonometric polynomial of degree $n-1$. If $p(t_k)$ vanishes at least at one point t_k, then inequality (1.3) holds.

Assume that

$$\min_{t_k} \big\{(-1)^k p(t_k)\big\} > 0, \quad k = 0, \ldots, 2n.$$

From our assumption, it follows that $p(t)$ takes values of different signs at $2n+1$ points on $[0, 2\pi]$. Since $p(t)$ is a continuous function, it has at least $2n$ zeros on $[0, 2\pi]$. But this is impossible, as any trigonometric polynomial of degree $n-1$ has at most $2(n-1)$ zeros on $[0, 2\pi)$. Therefore,

$$\min_{t_k} \big\{(-1)^k p(t_k)\big\} \leq 0, \quad k = 0, \ldots, 2n.$$

The statement is proved. □

Theorem 1. *Consider the ellipse* $E = \{z = a\cos t + ib\sin t,\ t\in[0,2\pi]\}$, $a > b > 0$. *The following polynomial deviates least from zero on* E:

$$p_n^*(z) = \frac{1}{2^{n-1}}\sum_{k=0}^{[n/2]} C_n^{2k} z^{n-2k}(z^2-c^2)^k = \frac{1}{2^n}\left((z+\sqrt{z^2-c^2})^n + (z-\sqrt{z^2-c^2})^n\right),$$

where $c^2 = a^2 - b^2$. *Moreover,* $\|p_n^*\|_{C(E)} = \dfrac{(a+b)^n + (a-b)^n}{2^n}$ *and the following recurrence relation holds*:

$$p_{n+1}^*(z) = z\,p_n^*(z) - \frac{c^2}{4}p_{n-1}^*(z).$$

P r o o f. Let us show that $A_n\cos(nt) + iB_n\sin(nt)$ is the restriction to E of an algebraic polynomial from $\mathcal{P}_n^1$. Lemma 2 implies that this polynomial is a solution to the problem

$$A_n\cos(nt) + iB_n\sin(nt) = \frac{A_n}{2}\left(e^{int}+e^{-int}\right) + \frac{iB_n}{2i}\left(e^{int}-e^{-int}\right) =$$
$$= \frac{A_n}{2}\left[(\cos t + i\sin t)^n + (\cos t - i\sin t)^n\right] + \frac{B_n}{2}\left[(\cos t + i\sin t)^n - (\cos t - i\sin t)^n\right] =$$
$$= \frac{(a+b)^n}{2^n}(\cos t + i\sin t)^n + \frac{(a-b)^n}{2^n}(\cos t - i\sin t)^n =$$
$$= \frac{1}{2^n}\Big(\big[(a\cos t + bi\sin t) + (b\cos t + ai\sin t)\big]^n + \big[(a\cos t + bi\sin t) - (b\cos t + ai\sin t)\big]^n\Big).$$

Since

$$z^2 = a^2\cos^2 t + 2abi\cos t\sin t - b^2\sin^2 t = b^2\cos^2 t + 2abi\sin t\cos t - a^2\sin^2 t + (a^2-b^2),$$

we have

$$(b\cos t + ai\sin t)^2 = z^2 - (a^2-b^2). \tag{1.4}$$

Hence,

$$A_n\cos(nt) + iB_n\sin(nt) =$$
$$= \frac{1}{2^n}\left\{\sum_{k=0}^{n} C_n^k z^{n-k}(z^2-(a^2-b^2))^{k/2} + \sum_{k=0}^{n} C_n^k z^{n-k}(-1)^k(z^2-(a^2-b^2))^{k/2}\right\} =$$
$$= \frac{1}{2^n}\sum_{0\le k\le n,\, k/2\in\mathbb{Z}} 2C_n^k z^{n-k}(z^2-(a^2-b^2))^{k/2} = \frac{1}{2^{n-1}}\sum_{0\le k\le n,\, k/2\in\mathbb{Z}} C_n^k z^{n-k}(z^2-(a^2-b^2))^{k/2}.$$

It is obvious that

$$\|p_n^*\|_{C(E)} = A_n = \frac{(a+b)^n + (a-b)^n}{2^n}.$$

Let us prove the recurrence relation. We have

$$(a\cos t + ib\sin t)\big(A_n\cos(nt) + iB_n\sin(nt)\big) - \frac{(a^2-b^2)}{4}\big[A_{n-1}\cos((n-1)t) + iB_{n-1}\sin((n-1)t)\big] =$$
$$= \frac{aA_n + bB_n}{2}\cos((n+1)t) + \frac{bA_n + aB_n}{2}i\sin((n+1)t) +$$
$$+ \left(\frac{aA_n - bB_n}{2} - \frac{(a^2-b^2)A_{n-1}}{4}\right)\cos((n-1)t) + \left(\frac{aB_n - bA_n}{2} - \frac{(a^2-b^2)B_{n-1}}{4}\right)i\sin((n-1)t).$$

It can be easily checked that the coefficients at $\cos\big((n-1)t\big)$ and $\sin\big((n-1)t\big)$ are equal to zero:

$$\frac{aA_n - bB_n}{2} - \frac{(a^2-b^2)A_{n-1}}{4} = \frac{a(aA_{n-1}+bB_{n-1}) - b(bA_{n-1}+aB_{n-1}) - A_{n-1}(a^2-b^2)}{4} = 0.$$

As above, the coefficient at $\sin((n-1)t)$ is equal to zero. The recurrence relation is proved. $\square$

Note that $p_n^*(z) = z^n$ for $a=b=1$ and $p_n^*(z) = T_n(z)/2^{n-1}$ for $a=1$ and $b=0$. It corresponds to well-known results for the unit circle and the interval $[-1,1]$.

2. Inequality for the derivative of an algebraic polynomial with real coefficients on an ellipse

Let us give a solution to the problem of finding the best constant M in the inequality

$$\|p_n'\|_{E_1} \le M, \quad p_n \in \mathcal{P}_n^{[-1,1]}.$$

Theorem 2. *The following inequality holds for every polynomial $p_n \in \mathcal{P}_n^{[-1,1]}$:*

$$\|p_n'\|_{E_1} \le \frac{n}{2b}\big[(a+b)^n - (a-b)^n\big].$$

The Chebyshev polynomial of the first kind

$$T_n(z) = \sum_{k=0}^{[n/2]} C_n^{2k} z^{n-2k}(z^2-1)^k = \frac{1}{2}\Big((z+\sqrt{z^2-1})^n + (z-\sqrt{z^2-1})^n\Big)$$

is extremal.

P r o o f. The proof will be divided into three steps.

(1) First, we will show that $|p'(z)| \le \|T_n'\|$ for every $z \in E_1$ such that

$$|z| \le \big|a\cos(\pi/(2n)) + ib\sin(\pi/(2n))\big|.$$

Consider the trigonometric polynomial $\tau_n(t) = p_n(a\cos t + ib\sin t)$. By the chain rule,

$$|\tau_n'(t)| = \; = |p_n'(z)| \cdot |-a\sin t + ib\cos t|, \quad z = a\cos t + ib\sin t.$$

By Bernstein's inequality for trigonometric polynomials [7; 8, p. 216, Theorem 1] and (1.4),

$$|p_n'(z)| = \frac{|\tau_n'(t)|}{|-a\sin t + ib\cos t|} \le \frac{n\|p_n\|_{E_1}}{|-a\sin t + ib\cos t|} \le \frac{n\|p_n\|_{E_1}}{|1-z^2|^{1/2}}.$$

To estimate the numerator, we apply the well-known inequality [4, p. 240]

$$\|p_n\|_{E_1} \le \frac{(a+b)^n + (a-b)^n}{2}, \quad p_n \in \mathcal{P}_n^{[-1,1]}.$$

Therefore,

$$|p_n'(z)| \le \frac{n\|p_n\|_{E_1}}{|-a\sin t + ib\cos t|} \le \frac{n\big((a+b)^n + (a-b)^n\big)}{2|-a\sin t + ib\cos t|}.$$

The image of the Chebyshev polynomial of the first kind $T_n(z)$ for $z \in E_1$ is the ellipse

$$\Big\{z = \frac{(a+b)^n + (a-b)^n}{2}\cos(nt) + i\frac{(a+b)^n - (a-b)^n}{2}\sin(nt) \mid \ t \in [0, 2\pi]\Big\}.$$

Hence,

$$|T_n'(z)| = \frac{n\left|-\big((a+b)^n + (a-b)^n\big)/2 \cdot \sin(nt) + i\big((a+b)^n - (a-b)^n\big)/2 \cdot \cos(nt)\right|}{|-a\sin t + ib\cos t|}. \tag{2.5}$$

Let $z^* = a\cos(\pi/(2n)) + ib\sin(\pi/(2n))$. Note that, if $|z| = |a\cos t + ib\sin t| \le |z^*|$, then

$$|ia\sin t + b\cos t| \ge \big|ia\sin(\pi/(2n)) + b\cos(\pi/(2n))\big|.$$

Therefore, for every $z \in E_1$ such that $|z| \le |z^*|$, we have

$$|T_n'(z^*)| = \frac{n\big((a+b)^n + (a-b)^n\big)}{2\big|ia\sin(\pi/(2n)) + b\cos(\pi/(2n))\big|} \ge |p_n'(z)|.$$

Now it is clear that the following inequality holds for z such that $|z| \le |z^*|$:

$$|p_n'(z)| \le \|T_n'\|_{E_1}.$$

(2) Let us obtain the estimate for $z \in E_1$ such that $|z| > |z^*|$. The idea of the proof of this point belongs to Erdös [5, Theorem 7].

Since $a^2 \ge 1$, it is clear that

$$|z^*| = \sqrt{a^2\cos^2\frac{\pi}{2n} + b^2\sin^2\frac{\pi}{2n}} \ge \cos\frac{\pi}{2n}.$$

Further, note that $\cos(\pi/(2n))$ is the largest zero of the Chebyshev polynomial of the first kind.

Let us write the interpolation formula for $p_n'(z)$ with roots of the Chebyshev polynomial as interpolation nodes. Denote them by x_k, $k = 1, \dots, n$. We have

$$|p_n'(z)| = \left|\sum_{k=1}^{n}\frac{T_n(z)}{(z-x_k)}\frac{p_n'(x_k)}{T_n'(x_k)}\right| = \left|T_n(z)\sum_{k=1}^{n}\frac{p_n'(x_k)}{T_n'(x_k)}\frac{\overline{z}-x_k}{|z-x_k|^2}\right|.$$

Note that an angle between any two vectors $\overline{z} - x_k$ is acute for $|z| > \cos(\pi/(2n))$. Hence, if all the numbers $p_n'(x_k)/T_n'(x_k)$ are non-negative and have maximum moduli, then p_n maximizes this expression over the set $\mathcal{P}_n^{[-1,1]}$.

By Bernstein's inequality [7; 8, p. 216, Theorem 1],

$$|p_n'(x_k)| \le n/\sqrt{1-x_k^2} = |T_n'(x_k)|$$

for any $p_n \in \mathcal{P}_n^{[-1,1]}$. Therefore, the following estimate is true for $p_n \in \mathcal{P}_n^{[-1,1]}$ and $|z| > \cos(\pi/(2n))$:

$$|p_n'(z)| \le |T_n'(z)| \le \|T_n'\|_{E_1}.$$

(3) It remains to prove that

$$\|T_n'\|_{E_1} = \frac{n}{2b}[(a+b)^n - (a-b)^n].$$

It is known [6, p. 785, 22.12.2, 22.12.3] that

$$T_n'(z) = \begin{cases} 2n \displaystyle\sum_{k=1,\, k/2\notin\mathbb{Z}}^{n-2} T_{n-k}(z) + n, & (n-1)/2 \in \mathbb{Z}, \\ 2n \displaystyle\sum_{k=1,\, k/2\notin\mathbb{Z}}^{n-1} T_{n-k}(z), & (n-1)/2 \notin \mathbb{Z}. \end{cases}$$

Now it is clear that $\|T_n'\|_{E_1} = |T_n'(a)|$. Using (2.5), we see that

$$|T_n'(a)| = \|T_n'\|_{E_1} = \frac{n}{2b}\big[(a+b)^n - (a-b)^n\big].$$

□

3. Remark

The inequality considered in Section 2 is related to Markov–Bernstein–type inequalities. In 1889, A.A. Markov proved [9] that, if p_n is a polynomial of degree n, then

$$\|p_n'\|_{[-1,1]} \le n^2 \|p_n\|_{[-1,1]}.$$

Moreover, the Chebyshev polynomial of the first kind T_n is the unique extremal polynomial.

A natural generalization of this problem is to find the best constant M in the following inequality:

$$\|p_n'\|_{E_1} \le M \|p_n\|_{E_1}.$$

It is easy to see that, if $n = 1$, then T_1 is extremal, since it deviates least from zero on E_1. It can be shown that T_2 is also extremal for $n = 2$. However, A.C. Schaeffer and G. Szegö showed in 1941 [10, p. 223–225] that the solution to this problem for $n \ge 5$ is not always provided by the Chebyshev polynomial of the first kind. At present, this problem has not been solved.

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PURSUIT-EVASION DIFFERENTIAL GAMES WITH GRÖNWALL-TYPE CONSTRAINTS ON CONTROLS[1]

Bahrom T. Samatov

Namangan State Universiti,
316 Uychi Str., Namangan, 116019, Uzbekistan
samatov57@inbox.ru

Gafurjan Ibragimov

Universiti Putra Malaysia,
43400, UPM, Serdang, Selangor Darul Ehsan, Malaysia
ibragimov@upm.edu.my

Iroda V. Khodjibayeva

Namangan Engineering and Technology Institute,
7 Kosonsoy Str., Namangan, 160115, Uzbekistan
hodjira83@mail.ru

Abstract: A simple pursuit-evasion differential game of one pursuer and one evader is studied. The players' controls are subject to differential constraints in the form of the integral Grönwall inequality. The pursuit is considered completed if the state of the pursuer coincides with the state of the evader. The main goal of this work is to construct optimal strategies for the players and find the optimal pursuit time. A parallel approach strategy for Grönwall-type constraints is constructed and it is proved that it is the optimal strategy of the pursuer. In addition, the optimal strategy of the evader is constructed and the optimal pursuit time is obtained. The concept of a parallel pursuit strategy (Π-strategy for short) was introduced and used to solve the quality problem for "life-line" games by L.A. Petrosjan. This work develops and expands the works of Isaacs, Petrosjan, Pshenichnyi, and other researchers, including the authors.

Keywords: Differential game, Grönwall's inequality, Geometric constraint, Pursuit, Evasion, Optimal strategy, Domain of attainability, Life-line.

Introduction

According to the fundamental approaches in the theory of differential games developed by Pontryagin [27] and Krasovskii [22], a differential game is considered as a control problem from the point of view of either the pursuer or the evader. From this point of view, the game reduces either to the problem of pursuit (approach) or to the problem of evasion (escape). In this paper, we mainly focus on the pursuit problem.

The concept of "Differential Games" was initiated by Isaacs [20]. Differential games have been the object of research since 1960, and fundamental results were obtained by Pontryagin [27], Krasovskii [22], Bercovitz [4], Elliot and Kalton [9], Isaacs [20], Fleming [10], Friedman [11],

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Hajek [14], Ho, Bryson, and Baron [15], Petrosjan [26], Pshenichnyi [28, 29], Subbotin [38, 39], Ushakov [41], Chikrii [7], and others.

The book of Isaacs [20] contains specific game problems that were discussed in detail and proposed for further study. One of these problems is the so-called life-line problem that was initially formulated and studied for certain special cases in [20, Problem 9.5.1]. For the case when controls of both players are subject to geometric constraints, this game has been rather comprehensively studied in the works of Petrosjan [26] based on approximating measurable controls with the most efficient piecewise constant controls that realize the parallel approach strategy. Later this approach to control in differential pursuit games was termed the Π-strategy. The strategy proposed [26] in a simple pursuit game with geometric constraints became the starting point for the development of the pursuit method in games with multiple pursuers (see, e.g., [3, 5, 12, 30–34]). Differential games where both players have admissible controls satisfying integral constraints have also been considered in several works, e.g., in [3, 32, 36, 41], although this treatment has been less comprehensive than for games with geometric constraints [3, 5, 7, 12, 30]. Also, in [35], the intercept problem was studied, when objects move in the dynamic flow field.

The constructing of optimal strategies of the players and finding the value of the game are difficult and important problems of differential games. Note that in [16–19, 21, 25, 37, 40], simple-motion differential games were studied and the existence of the value of the game was proved by constructing optimal strategies of the players.

In the theory of differential games, control functions are mainly subject to geometric, integral, or mixed constraints [8, 23]. However, differential type constraints on controls also arise in some applied problems such as ecological and technical problems [1, 24].

The present paper is also devoted to a simple pursuit-evasion differential game problem. We propose Grönwall-type constraints on the players' controls [13] for the pursuit-evasion differential game. We find the optimal pursuit time and construct optimal strategies for the players.

1. Statement of the problem

There is a huge number of works where simple-motion differential games with geometric constraints on controls of the form

$$|u| \le \rho, \quad |v| \le \sigma \tag{1.1}$$

were studied. The first constraint in (1.1) means that any control function $u(t)$, $t \ge 0$, satisfies the condition

$$\|u(\cdot)\|_\infty = \operatorname*{ess\,sup}_{t\ge 0} |u(t)| \le \rho. \tag{1.2}$$

In the present paper, we propose a new set of controls of the pursuer and evader described by the following Grönwall-type constraints, respectively:

$$|u(t)|^2 \le \rho^2 + 2k\int_0^t |u(s)|^2 ds, \quad t \ge 0, \tag{1.3}$$

and

$$|v(t)|^2 \le \sigma^2 + 2k\int_0^t |v(s)|^2 ds, \quad t \ge 0, \tag{1.4}$$

where ρ and σ are given positive numbers and k is a given non-negative number.

Let the dynamics of the pursuer x and the evader y be described by the following equations:

$$\begin{aligned} \dot{x} &= u, \qquad x(0) = x_0, \\ \dot{y} &= v, \qquad y(0) = y_0, \end{aligned} \tag{1.5}$$

where $x, y, x_0, y_0, u, v \in \mathbb{R}^n$, $n \geq 1$, and $x_0 \neq y_0$.

Definition 1. *Functions $u(\cdot) = \big(u_1(\cdot), u_2(\cdot), \dots, u_n(\cdot)\big)$ and $v(\cdot) = \big(v_1(\cdot), v_2(\cdot), \dots, v_n(\cdot)\big)$ satisfying conditions (1.3) and (1.4) are called the controls of the pursuer and evader, respectively.*

Denote by $\mathbb{U}$ and $\mathbb{V}$ the sets of all controls of the pursuer and evader, respectively. Pairs $(x_0, u(\cdot))$, $u(\cdot) \in \mathbb{U}$, and $(y_0, v(\cdot))$, $v(\cdot) \in \mathbb{V}$, generate the following trajectories:

$$x(t) = x_0 + \int_0^t u(s)ds, \quad y(t) = y_0 + \int_0^t v(s)ds$$

of the pursuer and evader, respectively.

We use the following statement.

Lemma 1 (Grönwall [13]). *If*

$$|\omega(t)|^2 \leq \alpha^2 + 2k \int_0^t |\omega(s)|^2 ds,$$

then $|\omega(t)| \leq \alpha e^{kt}$, where $\omega(t)$, $t \geq 0$, is a measurable function and α and k are non-negative numbers.

By Lemma 1, if $u(\cdot) \in \mathbb{U}$ and $v(\cdot) \in \mathbb{V}$, then

$$|u(t)| \leq \rho e^{kt}, \quad |v(t)| \leq \sigma e^{kt}, \quad t \geq 0. \tag{1.6}$$

It can be easily checked that the converse is not true, that is, inequalities (1.6) do not imply inequalities (1.3) and (1.4). To define the notions of optimal strategies of the players and the optimal pursuit time, we consider two games.

1.1. The minimax payoff of the game

Denote by $B(x, r)$ the ball of radius r centered at a point x.

Definition 2. *A continuous function*

$$U(x_0, y_0, t, v), \quad U : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times B(O, \sigma e^{kt}) \to B(O, \rho e^{kt}),$$

where O stands for the origin, is called a strategy of the pursuer.

Hence, at the current time t, the pursuer is allowed to know the initial states x_0, y_0, the current time t, and the value of the evader's control $v(t)$.

Definition 3. *We say that a strategy $U = U(x_0, y_0, t, v)$ guarantees the completion of the pursuit by time $T(U)$ if, for any control of the evader $v(t)$, $t \geq 0$, we have $x(\tau) = y(\tau)$ at some time $\tau \in [0, T(U)]$, where $(x(\cdot), y(\cdot))$ is the solution of the initial value problem*

$$\begin{aligned} \dot{x} &= U(x_0, y_0, t, v(t)), \qquad x(0) = x_0, \\ \dot{y} &= v, \qquad y(0) = y_0. \end{aligned}$$

We say that $T(U)$ is a guaranteed pursuit time. Note that any number T', $T' \geq T(U)$, is also a guaranteed pursuit time corresponding to the strategy U. Denote by $T^*(U)$ the exact lower bound of the guaranteed pursuit times $T(U)$ corresponding to the strategy U.

The pursuer tries to minimize the number $T^*(U)$ by choosing their strategy U while the evader tries to maximize $T^*(U)$ by choosing their control $v(\cdot)$.

Definition 4. *A strategy U_0 is called an optimal strategy of the pursuer if $T^*(U) \geq T^*(U_0)$ for any strategy U of the pursuer. The number $T^*(U_0)$ is called the minimax payoff of the game.*

1.2. The maximin payoff of the game

Definition 5. *A continuous function*

$$V(x_0, y_0, t, x, y), \quad V : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to B(O, \sigma e^{kt}),$$

is called a strategy of the evader if the following initial value problem

$$\begin{aligned} &\dot{x} = u, && x(0) = x_0, \\ &\dot{y} = V(x_0, y_0, t, x, y), && y(0) = y_0, \end{aligned} \tag{1.7}$$

has a unique solution $(x(t), y(t))$, $t \geq 0$.

Definition 6. *We say that a strategy V guarantees the evasion on the time interval $[0, T(V))$ if, for any control $u(t)$ of the pursuer, $t \geq 0$, the condition $x(t) \neq y(t)$ holds for all $t \in [0, T(V))$, where $(x(t), y(t))$ is the solution of (1.7). The number $T(V)$ is called a guaranteed evasion time.*

Denote by $T_*(V)$ the exact upper bound of numbers $T(V)$ corresponding to the strategy V. The evader tries to maximize $T_*(V)$ by choosing their strategy V while the pursuer tries to minimize it by choosing their control $u(\cdot)$. If $T_*(V) = \infty$, we say that the evasion is possible.

Definition 7. *A strategy V_0 of the evader is called optimal if the inequality $T_*(V) \leq T_*(V_0)$ holds for any strategy V of the evader. The number $T_*(V_0)$ is called the maximin payoff of the game. If $T^*(U_0) = T_*(V_0)$, then this number is called the optimal pursuit time.*

This paper is devoted to solving the following problems under Grönwall-type constraints on the controls.

Problem 1. *Construct optimal strategies of the pursuer and evader, and find the optimal pursuit time in the game.*

Problem 2. *Solve a "life-line" differential game.*

2. The main result

In this section, we construct optimal strategies for the players and give a formula for the optimal pursuit time.

2.1. Construction of the Π_{Gr}-strategy

To construct a strategy for the pursuer, we first assume that the pursuer knows t, $x(t)$, $y(t)$, and $v(t)$ at the current time t. After constructing the strategy, we abandon the information about the current players' positions $x(t)$ and $y(t)$.

Let $x(t) \neq y(t)$, $\xi = \xi(t) = z(t)/|z(t)|$, and $z(t) = x(t) - y(t)$. Based on the classical method for deriving a Π-strategy (see, for example, [2, 20, 26, 28]), we assume that, for a constant vector $v \in \mathbb{R}^n$, the velocity $u \in \mathbb{R}^n$ is chosen so that the following relations hold:

$$u = v - \lambda\xi, \tag{2.8}$$

$$|u|^2 = |v|^2 + \delta e^{2kt}, \tag{2.9}$$

where λ is a non-negative parameter and $\delta = \rho^2 - \sigma^2$. Substituting (2.8) into (2.9), we obtain the following equation for λ:

$$\lambda^2 - 2\lambda\langle v, \xi\rangle - \delta e^{2kt} = 0,$$

where $\langle v, \xi\rangle$ denotes the inner product of vectors v and ξ in $\mathbb{R}^n$. To construct the strategy of the pursuer, we use the following root:

$$\lambda(t, v, z) = \langle v, \xi\rangle + \sqrt{\langle v, \xi\rangle^2 + \delta e^{2kt}}. \tag{2.10}$$

Note that $\lambda(t, v, z)$ is not necessarily positive for all v and z. We call the root (2.10) the resolving function (see [7],[29]) and present some of its important properties.

Property 1. *If $\delta \geq 0$, then the function $\lambda(t, v, z)$ is continuous and non-negative for all $(t, v, z) \in [0, \infty) \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$.*

Now, substituting the resolving function (2.10) into (2.8), we obtain

$$\mathbf{u}(t, v, z) = v - \lambda(t, v, z)\xi \tag{2.11}$$

that satisfies (2.9). Let $z_0 = x_0 - y_0$, and let $v(\cdot) \in \mathbb{V}$ be an arbitrary control of the evader. If the pursuer applies strategy (2.11), then, by (1.5) and (2.11), the dynamics of the vector z is described by the following initial value problem:

$$\dot{z} = \dot{x} - \dot{y} = -\lambda(t, v(t), z)\frac{z}{|z|}, \quad z(0) = z_0. \tag{2.12}$$

Obviously, for the initial value problem (2.12), the hypotheses of the Caratheodory existence theorem are satisfied if $z \neq 0$, and therefore it has a unique absolutely continuous solution $(t, z(t))$, which starts from the point $(0, z_0)$ since $z_0 \neq 0$. The following statement justifies the term of "parallel approach" for the strategy (2.11).

Lemma 2. *For every z_0, $z_0 \neq 0$, and $v(\cdot) \in \mathbb{V}$, there exists a scalar function $\Lambda(\cdot)$ such that $z(t) = z_0\Lambda(t, v(\cdot), z(\cdot))$.*

P r o o f. We obtain from (2.12) that

$$\dot{z}_i = -\frac{\lambda(t, v(t), z)}{|z|}z_i, \quad z_i(0) = z_{i0},$$

where $i = 1, 2, \ldots, n$ and z_i is a scalar coordinate of the vector $z \in \mathbb{R}^n$. Then the latter differential equation can be transformed to the form

$$z_i(t) = z_{i0}\Lambda(t, v(\cdot), z(\cdot)), \quad \Lambda(t, v(\cdot), z(\cdot)) = \exp\left\{-\int_0^t \frac{1}{|z(s)|}\lambda(s, v(s), z(s))ds\right\}.$$

and the proof of Lemma 2 is complete. □

Lemma 3. *If $\rho \geq \sigma$, then the following equation holds for every z_0, $z_0 \neq 0$ and $v(\cdot) \in \mathbb{V}$ on some time interval* $[0, t^*)$:

$$\boldsymbol{u}(t, v(t), z(t)) = \boldsymbol{u}(t, v(t), z_0). \tag{2.13}$$

P r o o f. The function $\lambda(t, v, z)$ defined by (2.10) is homogeneous in z. Therefore, $\mathbf{u}(t, v, z)$ is homogeneous in z. Hence, by Lemma 2, we obtain (2.13). This completes the proof of Lemma 3.□

By (2.13), the pursuer constructs their strategy based on the information about the current time t, the value $v(t)$, and the initial data z_0, ρ, σ, k.

Definition 8. *If $\rho \geq \sigma$, then the function*

$$\boldsymbol{u}_{Gr}(t, v) = v - \lambda_{Gr}(t, v)\xi_0, \quad \lambda_{Gr}(t, v) = \langle v, \xi_0 \rangle + \sqrt{\langle v, \xi_0 \rangle^2 + \delta e^{2kt}}, \tag{2.14}$$

where $\xi_0 = z_0/|z_0|$, is called the Π_{Gr}-strategy of the pursuer in the game.

Note that

$$|\mathbf{u}_{Gr}(t, v)|^2 = |v|^2 + \delta e^{2kt}. \tag{2.15}$$

2.2. Solution of the pursuit problem

Theorem 1. *If $\rho > \sigma$, then the Π_{Gr}-strategy guarantees the completion of the pursuit in the game on the time interval $[0, T_{Gr}]$, where*

$$T_{Gr} = \begin{cases} \dfrac{1}{k} \ln\left(1 + \dfrac{k|z_0|}{\rho - \sigma}\right), & k > 0, \\ \dfrac{|z_0|}{\rho - \sigma}, & k = 0. \end{cases}$$

P r o o f. Let $v(\cdot) \in \mathbb{V}$ be an arbitrary control of the evader, and let the pursuer use the Π_{Gr}-strategy. Use equations (1.5) and (2.14) to get the following initial value problem:

$$\dot{z} = \mathbf{u}_{Gr}(t, v(t)) - v(t) = -\lambda_{Gr}(t, v(t))\xi_0, \quad z(0) = z_0.$$

From this, we see that

$$z(t) = \Lambda_{Gr}(t, v(\cdot))z_0, \tag{2.16}$$

where

$$\Lambda_{Gr}(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \int_0^t \lambda_{Gr}(s, v(s))ds.$$

We now study the behavior of the function $\Lambda_{Gr}(t, v(\cdot))$ with respect to t. Using the definition of the function $\lambda_{Gr}(t, v)$, we obtain

$$\Lambda_{Gr}(t, v(\cdot)) \leq 1 - \frac{1}{|z_0|} \int_0^t [\sqrt{\delta e^{2ks} + \langle v(s), \xi_0 \rangle^2} - |\langle v(s), \xi_0 \rangle|]ds.$$

The function $f(t,w) = \sqrt{\delta e^{2kt} + w^2} - w$, $w \in \mathbb{R}$, is monotonely deceasing for every $t \geq 0$. Hence, by the inequality $|\langle v(t), \xi_0 \rangle| \leq |v(t)| \leq \sigma e^{kt}$, which follows from the latter inequality in (1.6), we get

$$\Lambda_{Gr}(t, v(\cdot)) \leq 1 - \frac{1}{|z_0|} \int_0^t [\sqrt{\delta e^{2ks} + \sigma^2 e^{2ks}} - \sqrt{\sigma^2 e^{2ks}}] ds = \Phi_{Gr}(t),$$

where

$$\Phi_{Gr}(t) = \begin{cases} 1 - \dfrac{\rho - \sigma}{k|z_0|} \left(e^{kt} - 1\right), & k > 0, \\ 1 - \dfrac{(\rho - \sigma)t}{|z_0|}, & k = 0. \end{cases}$$

Clearly, the function $\Phi_{Gr}(t)$ is monotonely decreasing on $[0, T_{Gr}]$ and $\Phi_{Gr}(T_{Gr}) = 0$. Consequently, there exists a time t^*, $0 \leq t^* \leq T_{Gr}$, such that $\Lambda_{Gr}(t^*, v(\cdot)) = 0$, and hence, by (2.16), $z(t^*) = 0$.

Next, we prove the admissibility of strategy (2.14) for all t, $t \geq 0$. Let $v(\cdot) \in \mathbb{V}$ be an arbitrary control of the evader. We obtain from (1.4) and (2.15) that

$$|\mathbf{u}_{Gr}(t, v(t))|^2 = |v(t)|^2 + \delta e^{2kt} \leq \sigma^2 + \delta e^{2kt} + 2k \int_0^t |v(s)|^2 ds$$

$$= \rho^2 + 2k \int_0^t \left(|v(s)|^2 + \delta e^{2ks}\right) ds = \rho^2 + 2k \int_0^t |\mathbf{u}_{Gr}(s, v(s))|^2 ds,$$

and this completes the proof. □

Theorem 2. *If $\rho > \sigma$, then, for any control of the pursuer, the evader's strategy $V(t) = -\sigma e^{kt} \xi_0$, $t \geq 0$, guarantees the inequality $x(t) \neq y(t)$ on the time interval $[0, T_{Gr})$.*

P r o o f. Let $0 \leq t < T_{Gr}$. Then

$$\begin{aligned} \langle x(t) - y(t), \xi_0 \rangle &= |y_0 - x_0| - \int_0^t \langle v(s), \xi_0 \rangle ds + \int_0^t \langle u(s), \xi_0 \rangle ds \\ &\geq |y_0 - x_0| + \sigma \int_0^t e^{ks} ds - \rho \int_0^t e^{ks} ds > 0. \end{aligned}$$

Hence, $x(t) \neq y(t)$, $0 \leq t < T_{Gr}$. This completes the proof. □

Theorems 1 and 2 allows us to conclude that T_{Gr} is the optimal pursuit time, the Π_{Gr}-strategy is an optimal strategy for the pursuer, and $V(t) = -\sigma e^{kt} \xi_0$ is an optimal strategy for the evader.

2.3. Solution of the evasion problem

We now consider the game from the evader's point of view.

Theorem 3. *If $\rho \leq \sigma$, then the evasion is possible in the game.*

P r o o f. Let $\rho \leq \sigma$ and $u(\cdot) \in \mathbb{U}$. We suggest the evader to use the strategy $V(t) = -\sigma e^{kt}\xi_0$, $t \geq 0$. Obviously, $V(\cdot) \in \mathbb{V}$. Then, for any $u(t)$, we obtain

$$|z(t)| \geq |z_0 - \int_0^t V(s)ds| - \int_0^t |u(s)|ds = |z_0| + \int_0^t \sigma e^{ks}ds - \int_0^t |u(s)|ds.$$

Using the inequality $|u(s)| \leq \rho e^{kt}$, we obtain

$$|z(t)| \geq \begin{cases} |z_0| + (\sigma - \rho)(e^{kt} - 1)/k, & k > 0, \\ |z_0| + (\sigma - \rho)t, & k = 0. \end{cases}$$

This implies that $z(t) \neq 0$, $t \geq 0$. The proof of the theorem is complete. □

2.4. Life-line differential game

The book of R. Isaacs [20] contains specific game problems, which are discussed in detail and proposed for further study. Among numerous examples considered in the book, the life-line differential game (Problem 9.5.1) occupies a special place as an example of a differential game with phase constraint. For the case when the controls of both the players are subject to geometric constraints, this game has been rather comprehensively studied in the works of L.A. Petrosjan [26] based on approximating measurable controls with the most efficient piecewise constant controls that realize the parallel approach strategy. About further development see [3, 5, 12, 30–34].

Here we mainly study the game with phase constraints for the evader on a given subset M of $\mathbf{R}^n$, which is called the life line (of the evader). (Note that, in the case $M = \emptyset$, we have a simple game.)

In the life-line differential game, the pursuer P aims to catch the evader E, i.e., to realize the equality $x(t) = y(t)$ for some $t > 0$, while E stays in the zone $\mathbf{R}^n \setminus M$. The aim of E is to reach the zone M before the pursuer catches him or to keep the relation $x(t) \neq y(t)$ for all t ($t \geq 0$). Note that M doesn't restrict the motion of P. Further, we assume that initial positions x_0 and y_0 are given such that $x_0 \neq y_0$ and $y_0 \notin M$.

Definition 9. *A strategy* $\mathbf{u}_{Gr}(v, t)$ *of the player* P *is called* winning *on the interval* $[0, T_{Gr}]$ in the lifeline game *if, for every* $v(\cdot) \in \mathbb{V}$, *there exists some time* $t^* \in [0, T_{Gr}]$ *such that*

(1) $x(t^*) = y(t^*)$;

(2) $y(t) \notin M$ *for* $t \in [0, t^*]$.

Definition 10. *A control function* $v^*(\cdot) \in \mathbb{V}$ *of the player* E *is called* winning *in the life-line game if, for every* $u(\cdot) \in \mathbb{U}$,

(1) *there exists some time* $\bar{t}$ ($\bar{t} > 0$) *such that* $y(\bar{t}) \in M$ *and* $x(t) \neq y(t)$ *for* $t \in [0, \bar{t})$; *or*

(2) $x(t) \neq y(t)$ *for all* $t \geq 0$.

2.5. Dynamics of the attainability domain

Let conditions of Theorem 1 hold. We suppose that, at time t, $t \geq 0$, the evader E moves from a position y using the control vector

$$v(t) = \frac{w - y}{|w - y|}\sigma e^{kt}.$$

The pursuer P uses the strategy

$$\mathbf{u}_{Gr}(t, v(t)) = \frac{w - x}{|w - x|}\rho e^{kt}$$

from a position x. Then w is a point where P should meet E and

$$|w - y| = \int_t^\theta |v(s)|ds, \quad |w - x| = \int_t^\theta |\mathbf{u}_{Gr}(s, v(s))|ds \Rightarrow |w - x|/\rho = |w - y|/\sigma,$$

where θ is time when $x(\theta) = y(\theta) = w$. We define the attainability domain for the evader E in the following form:

$$A_{Gr}(x, y) = \big\{w : \ |w - x| \geq (\rho/\sigma)|w - y|\big\};$$

its boundary is know as *Apollonius' sphere.* Writing the latter in the form $|w - c_{Gr}| = R_{Gr}$, one can easily find the center $c_{Gr}(x, y)$ and the radius of *Apollonius' sphere*:

$$c_{Gr}(x, y) = (\rho^2 y - \sigma^2 x)/(\rho^2 - \sigma^2),$$

$$R_{Gr}(x, y) = \rho\sigma|x - y|/|\rho^2 - \sigma^2|.$$

The pairs $(x_0, u_{Gr}(t, v(t))$ and $(y_0, v(t))$ generate the trajectories

$$x(t) = x_0 + \int_0^t \mathbf{u}_{Gr}(s, v(s))ds, \quad y(t) = y_0 + \int_0^t v(s)ds,$$

respectively. Then, for every $(x(t), y(t))$, $t \in [0, \theta]$, we construct the sets

$$A_{Gr}(t) = A_{Gr}(x(t), y(t)) = \big\{w : \ |w - x(t)| \geq (\rho/\sigma)|w - y(t)|\big\},$$

$$A_{Gr}(0) = A_{Gr}(x_0, y_0) = \big\{w : \ |w - x_0| \geq (\rho/\sigma)|w - y_0|\big\}.$$

Theorem 4.

$$A_{Gr}(t) = x(t) + \Lambda_{Gr}(t)[A_{Gr}(0) - x_0]$$

for $t \in [0, \theta]$, *where* $\theta = \min\{t : z(t) = 0\}$.

P r o o f. Since $z(t) = \Lambda_{Gr}(t)z_0$, where $\Lambda_{Gr}(t) = \Lambda_{Gr}(t, v_t(\cdot))$ (see (2.16)), the relation $w \in A_{Gr}(t) - x(t)$ is equivalent to

$$|w| \geq (\rho/\sigma)\big|w + \Lambda_G(t)z_0\big|. \tag{2.17}$$

Obviously, it is sufficient to check (2.17) for $t \in [0, \theta)$ when $\Lambda_{Gr}(t) > 0$. Then (2.17) can be written as

$$|\Lambda_{Gr}^{-1}(t))w| \geq (\rho/\sigma)\big|\Lambda_{Gr}^{-1}(t)w + z_0\big|$$

or

$$\Lambda_{Gr}^{-1}(t)w \in A_{Gr}(0) - x_0.$$

The latter means that $w \in \Lambda_{Gr}(t)[A_{Gr}(0) - x_0]$. Thus, we have the equivalence

$$A_{Gr}(t) - x(t) = \big\{w: \ |w| \geq (\rho/\sigma)\big|w + \Lambda_{Gr}(t)z_0\big|\big\} = \Lambda_{Gr}(t)[A_{Gr}(0) - x_0],$$

hence the desired result follows. □

Theorem 5. *Monotony of Apollonius' sphere. The set $A_{Gr}(t)$ is monotone with respect to the inclusion for $t \in [0, \theta]$, i.e., if $0 \leq t_1 \leq t_2$, then $A_{Gr}(t_1) \supset A_{Gr}(t_2)$.*

P r o o f. By the properties (1.6) and (2.14)–(2.15), we have

$$|\mathbf{u}_{Gr}(t,v)|^2 = |v|^2 + \delta e^{2kt} \geq (\rho/\sigma)^2|v|^2 \Rightarrow |v - \lambda_{Gr}(t,v)\xi_0| \geq (\rho/\sigma)|v|$$

or

$$\big||z_0|v - \lambda_{Gr}(t,v)z_0\big| \geq (\rho/\sigma)|v||z_0| \Rightarrow \big|w - \lambda_{Gr}(t,v)x_0\big| \geq (\rho/\sigma)\big|w - \lambda_{Gr}(t,v)y_0\big|,$$

where $w = |z_0|v + \lambda_{Gr}(t,v)y_0$. The latter relation is equivalent to

$$|z_0|v + \lambda_{Gr}(t,v)y_0 \in \lambda_{Gr}(t,v)A_{Gr}(0).$$

From this, the convexity $A_{Gr}(0)$, and the properties of the support function (see [6])

$$F(A,\psi) = \sup_{w \in A}\langle w, \psi\rangle,$$

we get

$$\langle |z_0|v, \psi\rangle - \lambda_{Gr}(t,v)F(A_{Gr}(0) - y_0, \psi) \leq 0$$

for all ψ, $|\psi| = 1$. Consequently,

$$\big\langle v - \lambda_{Gr}(t,v)\xi_0, \psi\big\rangle - \frac{1}{|z_0|}\lambda_{Gr}(t,v)F(A_{Gr}(0) - x_0, \psi) = \frac{d}{dt}F(A_{Gr}(t), \psi) \leq 0.$$

□

2.6. Solution of the life-line game

In the life-line game, the pursuer P aims to catch the evader E, i.e., to realize the equality $x(t) = y(t)$ for some $t > 0$, while E stays in the zone $\mathbf{R}^n \setminus M$. The aim of E is to reach the zone M before the pursuer catches him or to keep the relation $x(t) \neq y(t)$ for all $t, t \geq 0$. Note that M doesn't restrict the motion of P.

Theorem 6. *If $\rho > \sigma$ and $M \bigcap A_{Gr}(x_0, y_0) = \emptyset$, then the Π_{Gr}-strategy is winning.*

P r o o f follows from Theorem 5. □

Theorem 7. *If $\rho > \sigma$ and $M \bigcap A_{Gr}(x_0, y_0) \neq \emptyset$, then there exists a control of the evader E, which is winning.*

P r o o f. Let $w \in M \bigcap A_G(x_0, y_0)$, and let E hold the control $v^*(t) = \sigma e^{kt}\nu,\ v^*(\cdot) \in \mathbb{V}$, where $\nu = (w - y_0)/|w - y_0|$. Then the time of reaching by the evader the point w is $\bar{\theta}$, and we have

$$\int_0^{\bar{\theta}} |v^*(s)|ds = |w - y_0| \Rightarrow \varphi(\bar{\theta}) := (e^{k\bar{\theta}} - 1)/k = |w - y_0|/\sigma, \tag{2.18}$$

where $\varphi(t) = (e^{kt} - 1)/k$ increases in t. We suppose that there exists a certain control function $u^*(\cdot) \in \mathbb{U}$ of the pursuer such that $x(\bar{t}) = y(\bar{t})$ and $\bar{t} < \bar{\theta}$ or $\varphi(\bar{t}) < \varphi(\bar{\theta})$. If $z(t) = x(t) - y(t)$ and $z(0) = z_0$, then, from (1.5), we get

$$z(\bar{t}) = z_0 + \int_0^{\bar{t}} (u^*(s) - v^*(t))ds = 0.$$

It follows that

$$|z_0 - \int_0^{\bar{t}} v^*(t)ds| \le \int_0^{\bar{t}} |u^*(s)|ds \le \rho\varphi(\bar{t}) \Rightarrow (\rho^2 - \sigma^2)\varphi^2(\bar{t}) + 2\sigma\langle z_0, \nu\rangle\varphi(\bar{t}) - |z_0|^2 \ge 0.$$

Hence, we get

$$\varphi(\bar{t}) \ge (\sqrt{\sigma^2\langle z_0, \nu\rangle^2 + |z_0|^2(\rho^2 - \sigma^2)} - \sigma\langle z_0, \nu\rangle)/(\rho^2 - \sigma^2). \tag{2.19}$$

Since $w \in A_G(x_0, y_0)$, we have

$$\begin{gathered}
|w - x_0| \ge (\rho/\sigma)|w - y_0| \Rightarrow |z_0 - (w - y_0)|^2 \ge (\rho/\sigma)^2|w - y_0|^2 \Rightarrow \\
|z_0|^2 - 2\langle z_0, w - y_0\rangle + |w - y_0|^2 \ge (\rho/\sigma)^2|w - y_0|^2 \Rightarrow \\
|z_0|^2 \ge \frac{|w - y_0|^2}{\sigma^2}(\rho^2 - \sigma^2) + 2|w - y_0|\langle z_0, \nu\rangle \Rightarrow \\
\sigma^2\langle z_0, \nu\rangle^2 + |z_0|^2(\rho^2 - \sigma^2) \ge \frac{|w - y_0|^2}{\sigma^2}(\rho^2 - \sigma^2)^2 + 2|w - y_0|(\rho^2 - \sigma^2)\langle z_0, \nu\rangle + \sigma^2\langle z_0, \nu\rangle^2 \Rightarrow \\
\sigma^2\langle z_0, \nu\rangle^2 + |z_0|^2(\rho^2 - \sigma^2) \ge \left[|w - y_0|(\rho^2 - \sigma^2)/\sigma + \sigma\langle z_0, \nu\rangle\right]^2 \Rightarrow \\
\sqrt{\sigma^2\langle z_0, \nu\rangle^2 + |z_0|^2(\rho^2 - \sigma^2)} \ge |w - y_0|(\rho^2 - \sigma^2)/\sigma + \sigma\langle z_0, \nu\rangle \Rightarrow \\
\left(\sqrt{\sigma^2\langle z_0, \nu\rangle^2 + |z_0|^2(\rho^2 - \sigma^2)} - \sigma\langle z_0, \nu\rangle\right)/(\rho^2 - \sigma^2) \ge |w - y_0|/\sigma = \varphi(\bar{\theta}).
\end{gathered}$$

Then, from (2.18)–(2.19), we get $\varphi(\bar{t}) \ge \varphi(\bar{\theta})$ or $\bar{t} \ge \bar{\theta}$, which contradict our assumption. $\square$

Theorem 8. *If $\sigma \ge \rho$, then there exists a control of the evader E, which is winning in the life-line game.*

P r o o f follows from Theorem 3. $\square$

3. Conclusion

In the present paper, we have studied a simple pursuit-evasion differential game of one pursuer and one evader. We have proposed Grönwall-type constraints on the players' controls and constructed the Π_{Gr}-strategy for the pursuer. We have shown that the Π_{Gr}-strategy is an optimal strategy for the pursuer. Also, we have constructed an optimal strategy for the evader and found the optimal pursuit time. The results obtained show that the optimal strategies U and V of the players satisfy the conditions $|U| = \rho e^{kt}$ and $|V| = \sigma e^{kt}$, respectively. For the completeness of the results, we have also studied an evasion life-line game.

There is a large scope for further investigations. For example, differential games of many players with Grönwall-type constraints on the players' controls can be studied.

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THE LOCAL DENSITY AND THE LOCAL WEAK DENSITY IN THE SPACE OF PERMUTATION DEGREE AND IN HATTORI SPACE

Tursun K. Yuldashev[†], **Farkhod G. Mukhamadiev**[††]

National University of Uzbekistan,
700174, Tashkent, Uzbekistan

[†]tursun.k.yuldashev@gmail.com, [††]farhod8717@mail.ru

Abstract: In this paper, the local density (ld) and the local weak density (lwd) in the space of permutation degree as well as the cardinal and topological properties of Hattori spaces are studied. In other words, we study the properties of the functor of permutation degree SP^n and the subfunctor of permutation degree SP^n_G, P is the cardinal number of topological spaces. Let X be an infinite T_1-space. We prove that the following propositions hold.
(1) Let $Y^n \subset X^n$; (A) if $d\,(Y^n) = d\,(X^n)$, then $d\,(SP^nY) = d\,(SP^nX)$; (B) if $lwd\,(Y^n) = lwd\,(X^n)$, then $lwd\,(SP^nY) = lwd\,(SP^nX)$.
(2) Let $Y \subset X$; (A) if $ld\,(Y) = ld\,(X)$, then $ld\,(SP^nY) = ld\,(SP^nX)$; (B) if $wd\,(Y) = wd\,(X)$, then $wd\,(SP^nY)$ $= wd\,(SP^nX)$.
(3) Let n be a positive integer, and let G be a subgroup of the permutation group S_n. If X is a locally compact T_1-space, then SP^nX, SP^n_GX, and $\exp_n X$ are k-spaces.
(4) Let n be a positive integer, and let G be a subgroup of the permutation group S_n. If X is an infinite T_1-space, then $n\,\pi\,w\,(X) = n\,\pi\,w\,(SP^nX) = n\,\pi\,w\,\left(SP^n_GX\right) = n\,\pi\,w\,(\exp_n X)$.

We also have studied that the functors SP^n, SP^n_G, and $\exp_n$ preserve any k-space. The functors SP^2 and SP^3_G do not preserve Hattori spaces on the real line. Besides, it is proved that the density of an infinite T_1-space X coincides with the densities of the spaces X^n, SP^nX, and $\exp_n X$. It is also shown that the weak density of an infinite T_1-space X coincides with the weak densities of the spaces X^n, SP^nX, and $\exp_n X$.

Keywords: Local density, Local weak density, Space of permutation degree, Hattori space, Covariant functors.

1. Introduction

In mathematical research in the modern world, a special place is occupied by the study of the topological properties of objects in various topological spaces. Research in general topology is topical, where the properties of topological spaces and their continuous mappings, operations on topological spaces and their mappings, as well as the classification of topological spaces are studied. This section of general topology uses concepts such as neighborhood, closure, compactness, density, separability, cardinal number, π-base of sets, sum, intersection, Tikhonov product, and others. An overview of the main stages in the development of set-theoretic topology is given in [1]. Some cardinal properties of topological spaces related to weak density were studied in [4]. In [5], some cardinal properties of Hattori spaces and their hyperspaces were studied. In [2, 6], some properties of topological spaces related to local density and local weak density in various topological spaces were studied.

Along with the concepts of local τ-density and local weak τ-density in various topological spaces, we are interested in such concepts as hereditary Souslin number, hereditary density, hereditary π-weight, hereditary Shanin number, hereditary pre-Shanin number, hereditary caliber, hereditary precaliber, hereditary weak density, hereditary Lindelöf number, and hereditary extent of topological spaces.

Denote by P the cardinal number of topological spaces. Let SP^n be a functor of permutation degree, and let SP^n_G be a subfunctor of the functor of permutation degree.

At the Prague topological symposium in 1981, V.V. Fedorchuk posed the following general problem in the theory of covariant functors [8] and thus created a new direction of research in this area of topology.

Problems. Let P be a geometrical or topological property, and let F be a covariant functor. If X has the property P, does $F(X)$ have the same property P? The opposite problem: for which functors $F(X)$ the space X has the property P if $F(X)$ has this property?

In [2], it was proved that the property of local density and the property of local weak density coincide for stratifiable spaces. These cardinal numbers are preserved under open mappings and are inverse invariant of a class of closed irreducible mappings.

In our present work, we prove that the following propositions are true for an infinite T_1-space X:

(1) if $ld\,(Y) = ld\,(X)$ for $Y \subset X$, then $ld\,(SP^n Y) = ld\,(SP^n X)$;

(2) if $lwd\,(Y) = lwd\,(X)$ for $Y \subset X$, then $lwd\,(SP^n Y) = lwd\,(SP^n X)$;

(3) if X is a locally compact T_1-space, n is a positive integer, and G is a subgroup of the permutation group S_n, then $SP^n X$, $SP^n_G X$, and $\exp_n X$ are k-spaces;

(4) if X is an infinite T_1-space, n is a positive integer, and G is a subgroup of the permutation group S_n, then

$$n\,\pi\,w\,(X) = n\,\pi\,w\,(SP^n X) = n\,\pi\,w\,(SP^n_G X) = n\,\pi\,w\,(\exp_n X)\,.$$

We also prove that the functors SP^2 and SP^3_G do not preserve Hattori spaces on the real line. In addition, we prove that the density of an infinite T_1-space X coincides with the densities of the spaces X^n, $SP^n X$, and $\exp_n X$. We show that the weak density of an infinite T_1-space X coincides with the weak densities of the spaces X^n, $SP^n X$, and $\exp_n X$.

2. Auxiliary material

Recall some notation, definitions, and statements that are widely used in this paper. The permutation group of X is the group of all permutations (one-to-one and onto mappings $X \to X$). The permutation group of a set X is usually denoted by $S(X)$. If $X = \{1, 2, 3, \ldots, n\}$, then $S(X)$ is denoted by S_n.

Let X^n be the nth power of a compact set X. The permutation group S_n of all permutations acts on the nth power X^n as the permutation of coordinates. The set of all orbits of this action with quotient topology is denoted by $SP^n X$. Thus, points of the space $SP^n X$ are finite subsets (equivalence classes) of the product X^n. Thus, two points $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in X^n$ are equivalent if there is a permutation $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$. The space $SP^n X$ is called the n-permutation degree of a space X. An equivalent relation by which we obtain the space $SP^n X$ is called the symmetric equivalence relation. The nth permutation degree is always a quotient of X^n. Thus, the quotient mapping is denoted as $\pi^s_n\colon X^n \to SP^n X$, where $\pi^s_n\big((x_1, x_2, \ldots, x_n)\big) = \big[x = (x_1, x_2, \ldots, x_n)\big]$ is an orbit of the point $x = (x_1, x_2, \ldots, x_n) \in X^n$.

The concept of permutation degree has generalizations. Let $SP^n_G X$ be any subgroup of the group S_n. Then it also acts on X^n as the group of permutations of coordinates. Consequently, it generates a G-symmetric equivalence relation on X^n. This quotient space of the product X^n under the G-symmetric equivalence relation is called the G-permutation degree of the space X and is denoted by $SP^n_G X$. The operation SP^n_G is also a covariant functor in the category of compact sets and is said to be a functor of G-permutation degree. If $G = S_n$, then $SP^n_G = SP^n$. If the group $SP^n_G X$ consists of only one element, then $SP^n_G X = X^n$.

Let X be a T_1-space. The collection of all nonempty closed subsets of X is denoted by $\exp X$. The family B of all sets of the form

$$O\langle U_1, \ldots, U_n\rangle = \Big\{F : F \in \exp X,\ F \subset \bigcup_{i=1}^{n} U_i,\ F \cap U_i \neq \emptyset,\ i = 1, 2, \ldots, n\Big\}$$

generates a topology on the set $\exp X$, where $U_1, \ldots, U_n$ is a family of open sets of X. This topology is called the Vietoris topology. The set $\exp X$ with the Vietoris topology is called the exponential space or the hyperspace of X [9]. Let X be a T_1-space. Denote by $\exp_n X$ the set of all closed subsets of X such that $\exp_n X = \{F \in \exp X : |F| \leq n\}$.

We use the following notation:

$$\exp_\omega X = \cup\{\exp_n X : n = 1, 2, \ldots\}, \quad \exp_c X = \{F \in \exp X : F \text{ is compact in } X\}.$$

It is clear that $\exp_n X \subset \exp_\omega X \subset \exp_c X \subset \exp X$ for any topological space X. Moreover, if $G_1 \subset G_2$ for subgroups G_1 and G_2 of the permutation group

$$\pi_n^s\big((x_1, x_2, \ldots, x_n)\big) = \big[x = (x_1, x_2, \ldots, x_n)\big] \in X^n,$$

then we have the following chain of factorizations of functors [9]:

$$X^n \to SP^n_{G_1} X \to SP^n_{G_2} X \to SP^n X \to \exp_n X.$$

A subset D of a topological space X is called a dense set in X if $[D] = X$. Define the density $d(X)$ of X by $d(X) = \min\{|D| : D \text{ is a dense subset of } X\}$ [7].

We say that the local density of a topological space X is τ at a point x if τ is the smallest cardinal number such that x has a neighbourhood of density τ in X. The local density at a point x is denoted by $ld(x)$. The local density of a topological space X is defined as the supremum of all numbers $ld(x)$ for $x \in X$: $ld(X) = \sup\{ld(x) : x \in X\}$ [2, 6]. It is known that $ld(X) \leq d(X)$ for any topological space.

Example. Let $\mathbb{R}$ be the real line with discrete topology. In the discrete topological space $(\mathbb{R}, \tau_d)$, every point $x \in \mathbb{R}$ has the one-point neighbourhood $\{x\}$. It follows that $ld(\mathbb{R}, \tau_d) = 1$. On the other hand, the boundary set of any set is empty in a discrete space, and hence the only dense set is the space itself. This means that $d(\mathbb{R}, \tau_d) = |\mathbb{R}| = c$. Then $1 = ld(\mathbb{R}, \tau_d) < d(\mathbb{R}, \tau_d) = c$.

We say that the weak density of a topological space is $\tau \geq \aleph_0$ if τ is the smallest cardinal number such that there exists a π-base coinciding with τ centered systems of open sets, i.e., there is a π-base $B = \bigcup\{B_\alpha : \alpha \in A\}$, where B_α is a centered system of open sets for every $\alpha \in A$, $|A| = \tau$.

The weak density of a topological space X is denoted by $wd(X)$. If $d(X) = \tau \geq \aleph_0$, then $wd(X) \leq \tau$. Similarly, if Y is dense in a topological space X, then $wd(Y) = wd(X)$ [3]. The following theorem and proposition were proved in [3].

Theorem 1. *Let $\{X_\alpha : \alpha \in A\}$ be a family of topological spaces such that $wd(X_\alpha) \leq \tau \geq \aleph_0$ for every $\alpha \in A$, where $|A| \leq 2^\tau$. Then $wd\left(\prod_{\alpha\in A} X_\alpha\right) \leq \tau$.*

Proposition 1. *Assume that X and Y are topological spaces and there exists a continuous "onto" mapping $f : X \to Y$. Then $wd(Y) \leq wd(X)$.*

A topological space X is called a locally weak τ-dense space at a point $x \in X$ if τ is the smallest cardinal number such that x has a neighbourhood of weak density τ in X. The local weak density at a point x is denoted by $lwd(x)$. The local weak density of a topological space X is defined as the supremum of all numbers $lwd(x)$ for $x \in X$: $lwd(X) = \sup\{lwd(x) : x \in X\}$ [2, 6]. If X is a space of local density τ and $f : X \to Y$ is an open continuous "onto" mapping, then Y is a space of local density τ [12]. The quotient mapping $\pi_n^s : X^n \to SP^n X$ is a clopen continuous onto mapping [13].

The following two statements are from [11].

Proposition 2. *If X is a topological space, then $\exp_n X$ is dense in $\exp X$.*

Proposition 3. *X is separable if and only if $\exp X$ is separable.*

These propositions imply that, for any infinite T_1-space X, we have

$$lwd\,(X) = lwd\,(X^n) = lwd\,(SP^n X).$$

The following theorem was proved in [4].

Theorem 2. *Let X be an infinite T_1-space. Then $wd\,(X) = wd\,(\exp_n X) = wd\,(\exp X)$.*

To substantiate our results, we also use the following notation and definitions from [7].

An uncountable cardinal number τ is a caliber of a topological space if every family of cardinality τ consisting of nonempty open sets contains subfamily of the same cardinality with nonempty intersection. The caliber of a topological space X is denoted by $k\,(X)$.

The cardinal number $\min\{\tau\colon \tau^+ \text{ is a caliber of } X\}$ is called the Shanin number of X and is denoted by $sh\,(X)$.

A cardinal number $\tau > \aleph_0$ is called a precaliber of a space X if every family of cardinality τ consisting of nonempty open subsets of X contains a subfamily of cardinality τ with finite intersection. Define

$$pk\,(X) = \{\tau^+ : \tau \text{ is a precaliber of } X\}.$$

The cardinal number $psh\,(X) = \min\{\tau^+ : \tau \text{ is a precaliber of } X\}$ is called the pre-Shanin number. We always have $c\,(X) \leq psh\,(X) \leq sh\,(X) \leq d\,(X)$.

The Lindelöf number $l(X)$ of X is defined as $l(X) = \min\{\tau :$ every open cover of X has a refinement of cardinality $\leq \tau\} + \aleph_0$. If $l(X) = \aleph_0$, i.e., every open cover has a countable refinement, we say that X is a Lindelöf space.

The notion of cellularity (Souslin number) $c\,(X)$ of X is defined as $c\,(X)$ =min$\{\tau :$ every family of pairwise disjoint nonempty open subsets of X has cardinality $\leq \tau\} + \aleph_0$. If $c(X) = \aleph_0$, we say that X has the countable chain condition (Souslin property).

The spread $s(X)$ and the extent $e(X)$ are defined as follows: $s(X)$ =sup$\{|D| : D$ is a discrete subset of $X\} + \aleph_0$ and $e(X)$ =sup$\{|D| : D$ is a discrete closed subset of $X\} + \aleph_0$, respectively.

For a metrizable space X, we have $l\,(X) = d\,(X) = c\,(X) = s\,(X) = e\,(X)$.

For a cardinal function φ, we define the corresponding hereditary cardinal function $h\varphi = \sup\{\varphi(Y) : Y \subset X\}$. For example, we have the hereditary Souslin number $hc\,(X)$, the hereditary density $hd\,(X)$, the hereditary π-weight $h\pi w\,(X)$, and the hereditary Shanin number $hsh\,(X)$. Similar symbols we use to denote the hereditary pre-Shanin number, the hereditary caliber, the hereditary precaliber, the hereditary weak density, the hereditary Lindelöf number, and the hereditary extent of the space X, respectively: $hpsh\,(X)$, $hk\,(X)$, $hpk\,(X)$, $hwd\,(X)$, $hl\,(X)$, and $he\,(X)$.

It is easy to see that the hereditary Souslin number $hc\,(X)$ of a space X coincides with its spread $s\,(X)$.

Definition 1. *A topological space is a k-space if it is a quotient image of some topological space Y.*

Recall that a topological space is locally compact if, for every $x \in X$, there exists a neighbourhood U of x such that $[U]$ is a compact subspace of X.

In 2010, Hattori defined [10] the following topology on $\mathbb{R}$. Let $\mathbb{R}$ be the real line and $\mathbb{A} \subseteq \mathbb{R}$. The topology $\tau(\mathbb{A})$ on $\mathbb{R}$ is defined as follows:

(1) for each $x \in \mathbb{A}$, $\{(x-\varepsilon,\, x+\varepsilon) : \varepsilon > 0\}$ is the neighbourhood base at x;

(2) for each $x \in \mathbb{R}\backslash\mathbb{A}$, $\{[x,\, x+\varepsilon) : \varepsilon > 0\}$ is the neighbourhood base at x.

The space $(\mathbb{R}, \tau(\mathbb{A}))$ is called [5] a Hattori space. Let $\tau_{\mathbb{E}}$ be the Euclidean topology on $\mathbb{R}$. Note that, for any $\mathbb{A}, \mathbb{B} \subseteq \mathbb{R}$, we have $\mathbb{A} \supseteq \mathbb{B}$ if $\tau(\mathbb{A}) \subseteq \tau(\mathbb{B})$, in particular, $\tau(\mathbb{R}) = \tau_{\mathbb{E}} \subseteq \tau(\mathbb{A})$ and $\tau(\mathbb{B}) \subseteq \tau(\emptyset) = \tau_S$. We set $P_{\text{top}}(\mathbb{R}) = \{\tau(\mathbb{A}) : \mathbb{A} \subseteq \mathbb{R}\}$ and define a partial order $\leq$ on $P_{\text{top}}(\mathbb{R})$ by the inclusion: $\tau(\mathbb{A}) \leq \tau(\mathbb{B})$ if $\tau(\mathbb{A}) \subseteq \tau(\mathbb{B})$.

3. Main results

Theorem 3. *Let X be an infinite T_1-space, and let Y^n be dense in X^n. Then SP^nY is also dense in SP^nX.*

P r o o f. Let Y^n be a dense subset of X^n, and let SP^nU be an arbitrary open set from SP^nX. Since the mapping $\pi_n^s : X^n \to SP^nX$ is continuous, the set $(\pi_n^s)^{-1}(SP^nU) \subset X^n$ is open. Thus, taking into account the density of Y^n in X^n, we conclude that $(\pi_n^s)^{-1}(SP^nU) \cap Y^n \neq \emptyset$. Therefore, there exists $y \in Y^n$ such that $y \in (\pi_n^s)^{-1}(SP^nU)$. Then $\pi_n^s(y) \in SP^nU$ (and $\pi_n^s(y) \in SP^nY$). Hence, we have $SP^nU \cap SP^nY \neq \emptyset$ for every open set SP^nU. This means that the set SP^nY is dense in SP^nX. Theorem 3 is proved. □

Corollary 1. *If X is an infinite T_1-space and Y^n is a subset of X^n such that $d(Y^n) = d(X^n)$, then $d(SP^nY) = d(SP^nX)$.*

Proposition 4. *Assume that X is an infinite T_1-space, n is a positive number, and G_1 and G_2 are subgroups of the permutation group S_n such that $G_1 \subset G_2$. Then*

$$d(X) = d(X^n) = d\left(SP^n_{G_1}X\right) = d\left(SP^n_{G_2}X\right) = d(SP^nX) = d\left(\exp_n X\right).$$

P r o o f. Let X be an infinite T_1-space. Taking into account that

$$X^n \to SP^n_{G_1}X \to SP^n_{G_2}X \to SP^nX \to \exp_n X$$

and the fact that continuous mappings do not increase the density of topological spaces, we directly obtain the inequalities

$$d(X) \geq d(X^n) \geq d\left(SP^n_{G_1}X\right) \geq d\left(SP^n_{G_2}X\right) \geq d(SP^nX) \geq d(\exp_n X).$$

By Propositions 2 and 3, we get $d(X) = d\left(\exp_n X\right)$, and hence

$$d(X) = d(X^n) = d\left(SP^n_{G_1}X\right) = d\left(SP^n_{G_2}X\right) = d(SP^nX) = d(\exp_n X).$$

Proposition 4 is proved. □

Theorem 4. *Let X be an infinite T_1-space, and let Y^n be a locally dense set in X^n. Then SP^nY is also locally dense in SP^nX.*

P r o o f. The set Y^n is locally dense in X^n. By definition, for any point $y \in Y^n$, there exists a neighbourhood $Oy \subset X^n$ such that Oy is dense in X^n. Then Theorem 3 implies that $SP^n(Oy)$ is also dense in SP^nX. On the other hand, the quotient mapping $\pi_n^s : X^n \to SP^nX$ is an open mapping. Therefore, $SP^n(Oy)$ is a neighbourhood of the point $\pi_n^s(y) \in SP^nY$. Then SP^nY is locally dense in SP^nX. Theorem 4 is proved. □

Corollary 2. *If X is an infinite T_1-space and $Y \subset X$ is such that $ld\,(Y) = ld\,(X)$, then*

$$ld\,(SP^nY) = ld\,(SP^nX)\,.$$

Theorem 5. *Let X be an infinite T_1-space. Then $wd\,(X) = wd\,(SP^nX)$.*

P r o o f. First, we will show that $wd\,(SP^nX) \leq wd\,(X)$. Suppose that $wd\,(X) = \tau \geq \aleph_0$. Then $wd\,(X^n) = \tau$ by Theorem 1. The space SP^nX is a continuous image of the space X^n. Proposition 1 implies that $wd\,(SP^nX) \leq \tau$.

Now we will prove that $wd\,(SP^nX) \geq wd\,(X^n)$. To this end, assume that $wd\,(SP^nX) = \tau \geq \aleph_0$. This means that there exists $SP^nB = \cup\{SP^nB_\alpha : \alpha \in A,\ |A| = \tau\}$ and this is a π-base in SP^nX, where $SP^nB_\alpha = \{SP^nU_s^\alpha : s \in A_\alpha\}$ is a centered system of nonempty open sets for every $\alpha \in A$.

We set

$$B_\alpha = \big\{ (\pi_n^s)^{-1}\,(SP^nU_s^\alpha) : s \in A_\alpha \big\}, \quad B = \cup\big\{B_\alpha : \alpha \in A\big\}.$$

Let us show that B_α is a centered system of nonempty open sets in X^n for every $\alpha \in A$. For every finite subfamily $\big\{SP^nU_{s_i}^\alpha\big\}_{i=1}^k$ of SP^nB_α, we have $\cap_{i=1}^k SP^nU_{s_i}^\alpha \neq \emptyset$. Then

$$\emptyset \neq (\pi_n^s)^{-1}\left(\cap_{i=1}^k SP^nU_{s_i}^\alpha\right) = \cap_{i=1}^k\big((\pi_n^s)^{-1}\,(SP^nU_{s_i}^\alpha)\big).$$

This shows that $B_\alpha = \big\{ (\pi_n^s)^{-1}\,(SP^nU_s^\alpha) : s \in A_\alpha \big\}$ is also a centered system of nonempty open sets in X^n. Now, we show that B is a π-base in X^n. Since

$$SP^nB = \cup\big\{SP^nB_\alpha : \alpha \in A,\ |A| = \tau\big\}$$

is a π-base of SP^nX, for every open subset SP^nU of SP^nX, there exists $SP^nU_s^\alpha \in SP^nB_\alpha \subset SP^nB$ such that $SP^nU_s^\alpha \subset SP^nU$. Since the quotient mapping $\pi_n^s : X^n \to SP^nX$ is open and onto, we have

$$(\pi_n^s)^{-1}\,(SP^nU_s^\alpha) \subset (\pi_n^s)^{-1}\,(SP^nU)\,.$$

This means that B is a π-base in X^n. Therefore, we have $wd\,(X^n) \leq \tau$. Theorem 5 is proved. □

Corollary 3. *If X is an infinite T_1-space and $Y \subset X$ is such that $wd\,(Y) = wd\,(X)$, then*

$$wd\,(SP^nY) = wd\,(SP^nX)\,.$$

Theorem 6. *Let X be an infinite T_1-space, and let Y^n be locally weakly dense in X^n. Then SP^nY is locally weakly dense in SP^nX.*

P r o o f. Suppose that X is an infinite T_1-space and $Y^n \subset X^n$ is locally weakly dense. Then, for every point $y \in Y^n$, there exists a neighbourhood Oy such that Oy is weakly dense in X^n. According to Theorem 5, $SP^n(Oy) = \{\pi_n^s(y') : y' \in Oy\}$ is also weakly dense in SP^nX. This means that, for every point $\pi_n^s(y) \in SP^nY$, there exists $SP^n(Oy)$ such that it is weakly dense in SP^nX. This shows that SP^nY is locally weakly dense in SP^nX. Theorem 6 is proved. □

Corollary 4. *If X is an infinite T_1-space and $Y^n \subset X^n$ is such that $lwd\,(Y^n) = lwd\,(X^n)$, then $lwd\,(SP^nY) = lwd\,(SP^nX)$.*

Proposition 5. *Assume that X is an infinite T_1-space, n is a positive number, and G_1 and G_2 are subgroups of the permutation group S_n such that $G_1 \subset G_2$. Then*

$$wd\,(X) = wd\,(X^n) = wd\,\left(SP_{G_1}^nX\right) = wd\,\left(SP_{G_2}^nX\right) = wd\,(SP^nX) = wd\,(\exp_n X).$$

P r o o f. Let X be an infinite T_1-space. Taking into account that

$$X^n \to SP^n_{G_1}X \to SP^n_{G_2}X \to SP^nX \to \exp_n X$$

and the fact that continuous mappings do not increase the weak density of topological spaces, we directly obtain the inequalities

$$wd\,(X) \geq wd\,(X^n) \geq wd\,\left(SP^n_{G_1}X\right) \geq wd\,\left(SP^n_{G_2}X\right) \geq wd\,(SP^nX) \geq wd\,(\exp_n X).$$

According to Theorem 2, $wd\,(X) = wd\,(\exp_n X)$. Hence, we get

$$wd(X) = wd\,(X^n) = wd\,\left(SP^n_{G_1}X\right) = wd\,\left(SP^n_{G_2}X\right) = wd\,(SP^nX) = wd\,(\exp_n X).$$

Proposition 5 is proved. □

Proposition 6. *Assume that X is a locally compact T_1-space, n is a positive integer, and G is a subgroup of the permutation group S_n. Then SP^nX, SP^n_GX, and $\exp_n X$ are k-spaces.*

P r o o f. Let X be a locally compact T_1-space. Then X^n is a locally compact space for each $n \in \mathbb{N}$. The spaces SP^nX, SP^n_GX, and $\exp_n X$ become quotient images of the space X^n. Therefore, SP^nX, SP^n_GX, and $\exp_n X$ are k-spaces. Proposition 6 is proved. □

Corollary 5. *The functors SP^n, SP^n_G, and $\exp_n$ preserve any k-space.*

Proposition 7. *Assume that X is an infinite T_1-space, n is a positive integer, and G is a subgroup of the permutation group S_n. Then $n\,\pi w\,(SP^nX) = n\,\pi w\,(X)$.*

P r o o f. It was proved in Proposition 4 that $d\,(SP^nX) = d\,(X)\,,\ n \in \mathbb{N}$. It is known that any dense set $M \subset X$ can be a π-net of this space. Hence, we have $n\,\pi w\,(SP^nX) = n\,\pi\,w\,(X)$. Proposition 7 is proved. □

Corollary 6. *Assume that X is an infinite T_1-space, n is a positive integer, and G is a subgroup of the permutation group S_n. Then*

$$\begin{aligned} n\,\pi\,w(X) = n\,\pi\,w(SP^nX) = n\,\pi\,w(SP^n_GX) = n\,\pi\,w(SP^n_{G_1}X) = \\ = n\,\pi\,w(SP^n_{G_2}X) = n\,\pi\,w(\exp_n X) = n\,\pi\,w(\exp_\omega X) = n\,\pi\,w(\exp X). \end{aligned}$$

Theorem 7. *Let $\mathbb{A}$ be a subset of $\mathbb{R}$ such that $int\,(\mathbb{R}\backslash\mathbb{A}) \neq \emptyset$. Then the following nonequalities hold for the Hattori space $(\mathbb{R},\,\tau\,(\mathbb{A}))$ and the functor of permutation degree SP^2:*

(1) $s\,(\mathbb{R},\,\tau(\mathbb{A})) \neq s\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(2) $hd\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hd\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(3) $h\pi w\,(\mathbb{R},\,\tau(\mathbb{A})) \neq h\pi\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(4) $hsh\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hsh\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(5) $hc\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hc\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(6) $hk\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hk\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(7) $hpk\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hpk\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(8) $hpsh\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hpsh\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(9) $hwd\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hwd\,\big(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\big)$;

(10) $hl\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hl\,\big(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\big)$;

(11) $he\,(\mathbb{R},\,\tau(\mathbb{A})) \neq he\,\big(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\big)$.

P r o o f. It is known that the space SP^2X contains the squared Hattori space X^2. However, X^2 contains a discrete set of cardinality c. The other nonequalities can be easily checked. Theorem 7 is proved. □

Corollary 7. *The functor SP^2 does not preserve Hattori spaces on the real line.*

Corollary 8. *Let $\mathbb{A}$ be a subset of $\mathbb{R}$ such that $int\,(\mathbb{R}\backslash\mathbb{A}) \neq \emptyset$, and let G be an arbitrary subgroup of the group S_3. Then the following nonequalities hold for the Hattori space $(\mathbb{R},\,\tau\,(\mathbb{A}))$ and the functor of permutation degree SP^3_G:*

(1) $s\;(\mathbb{R},\,\tau\,(\mathbb{A})) \neq s\;\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(2) $hd\;(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hd\;\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(3) $h\,\pi\,w\;(\mathbb{R},\,\tau\,(\mathbb{A})) \neq h\,\pi\,w\;\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(4) $hsh\;(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hsh\;\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(5) $hc\;(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hc\;\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(6) $hk\;(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hk\;\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(7) $hpk\;(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hpk\;\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(8) $hpsh\;(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hpsh\;\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(9) $hwd\;(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hwd\;\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(10) $hl\;(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hl\;\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(11) $he\;(\mathbb{R},\,\tau\,(\mathbb{A})) \neq he\;\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$.

Corollary 9. *The functor SP^3_G does not preserve Hattori spaces on the real line.*

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THE VARIETY GENERATED BY AN AI-SEMIRING OF ORDER THREE

Xianzhong Zhao[1], Miaomiao Ren[1,†], Siniša Crvenković[2], Yong Shao[1], Petar Dapić[2]

[1]Northwest University, Xi'an, 710069, P.R. China

[2]University of Novi Sad, Novi Sad, 21102, Serbia

[†]miaomiaoren@yeah.net

Abstract: Up to isomorphism, there are 61 ai-semirings of order three. The finite basis problem for these semirings is investigated. This problem for 45 semirings of them is answered by some results in the literature. The remaining semirings are studied using equational logic. It is shown that with the possible exception of the semiring S_7, all ai-semirings of order three are finitely based.

Keywords: Ai-semiring, Identity, Finitely based variety.

Introduction and preliminaries

By a *variety* we mean a class of algebras of the same type that is closed under subalgebras, homomorphic images and direct products. It is well-known (Birkhoff's theorem) that a class of algebras of the same type is a variety if and only if it is an equational class. One of the fundamental problems about a variety is the so called finite basis problem, that is, whether it can be defined by finitely many identities. If the answer is positive, then it is called *finitely based*. Otherwise, it is called *nonfinitely based*. An algebra A is said to be finitely based (resp., nonfinitely based) if the variety generated by A is finitely based (resp., nonfinitely based).

In 1951 Lyndon [9] showed that all two-element algebras are finitely based and formulated the problem whether every finite algebra is finitely based. This problem has been answered negatively, since a certain seven-element groupoid [10] was shown to be nonfinitely based. Some classical algebras are finite based. For example, so are every finite group [15], every finite associative ring [6, 8], every finite lattice [11] and every commutative semigroup [18]. However, not every finite semigroup and not every finite semiring are finitely based. The first example of an nonfinitely based finite semigroup (resp., semiring) has been given by Perkins [18] (resp., Dolinka [1]).

To seek a ultimate solution to the finite basis problem for finite algebras, Tarski [24] proposed the following problem: Is there an algorithm to decide whether a finite algebra is finitely based? McKenzie [12] negatively answered this problem for finite groupoids. However, this problem is still open when restricted to finite semigroups and finite semirings.

By a *semiring* we mean an algebra $(S, +, \cdot)$ such that

- the additive reduct $(S, +)$ is a commutative semigroup;
- the multiplicative reduct $(S, \cdot)$ is a semigroup;
- $(S, +, \cdot)$ satisfies the identities $x(y + z) \approx xy + xz$ and $(y + z)x \approx yx + zx$.

One can easily find many examples of semirings in almost all branches of mathematics. Semirings can be regarded as a common generalization of both rings and distributive lattices. They have been widely applicated in theoretical computer science and information science. We shall say that a semiring is an *additively idempotent semiring* (ai-semiring for short) if its additive reduct is a semilattice, i.e., a commutative idempotent semigroup. The variety of all ai-semirings is denoted by $\mathbf{AI}$. Let $P_f(X^+)$ denote the set of all finite non-empty subsets of the free semigroup X^+ on a countably infinite set X of variables. If we define an addition and a multiplication on $P_f(X^+)$ by

$$A + B = A \cup B, \quad A \circ B = \{ab \mid a \in A, b \in B\},$$

then $(P_f(X^+), +, \circ)$ is free in $\mathbf{AI}$ with respect to the mapping $\varphi\colon X \to P_f(X^+), x \mapsto \{x\}$ (see [7, Theorem 2.5]). An ai-semiring identity ($\mathbf{AI}$-identity for short) over X is an expression of the form $u \approx v$, where $u, v \in P_f(X^+)$. For convenience, we write $u_1 + u_2 + \cdots + u_k \approx v_1 + v_2 + \cdots + v_\ell$ for the ai-semiring identity $\{u_i \mid 1 \leq i \leq k\} \approx \{v_j \mid 1 \leq j \leq \ell\}$.

In the last decades, several authors studied the finite basis problem for various semiring varieties. There is a rich literature on this subject (see [1–5, 16, 17, 19–23, 25, 27, 28]). Dolinka [1] found the first example of a finite nonfinitely based ai-semiring. In [2] he provided a sufficient condition under which an ai-semiring is inherently nonfinitely based, i.e., the variety $\mathbf{V}$ generated by this semiring is locally finite and every locally finite variety $\mathbf{W}$ for which $\mathbf{V} \subseteq \mathbf{W}$ is nonfinitely based. As an application, it was shown in [3, 4] that some ai-semirings are inherently nonfinitely based[1]. McKenzie and Romanowska [13] showed that all ai-semirings satisfying $x^2 \approx x$ and $xy \approx yx$ are finitely based. Zhao et al. [27, 28] considered the finite basis problem for ai-semirings satisfying $x^2 \approx x$ that are related to Green's relations. Based on the work of [13, 27, 28], Ghosh et al. [5] and Pastijn [16] proved that all ai-semirings satisfying $x^2 \approx x$ are finitely based. Ren et al. [21] showed that this result holds for all ai-semirings satisfying $x^3 \approx x$. However, not every ai-semirings satisfying $x^n \approx x$ ($n \geq 4$) is finitely based (see [22]). Recently, Ren et al. [20] answered the finite basis problem for ai-semirings satisfying $x^n \approx x$ and $xy \approx yx$ in which $n-1$ is square-free. From these references one can find that semirings of small order have played an important role. This motivates some authors to investigate the finite basis problem for ai-semirings of small order. In this direction, Shao and Ren [23] considered the variety generated by all ai-semirings of order two. Vechtomov and Petrov [25] studied the variety generated by all semirings of order two whose multiplicative reduct is a semilattice. Moreover, McNulty and Willard [14] initiated the study of the finite basis problem for algebras of order three. The present paper follows this line of investigation. We shall systematically study the finite basis problem for ai-semirings of order three. For this, the following information about ai-semirings of order two in [23] are necessary.

Up to isomorphism, there are exactly 6 ai-semirings of order two, which are listed as L_2, R_2, M_2, D_2, N_2 and T_2 in Table 1. We assume that the underlying set of each of these semirings is $\{0, 1\}$. Their Cayley tables for addition and multiplication are listed in the 2nd and respectively the 3rd columns of Table 1 while the 4th column contains their equational bases.

To present the solution of the word problem for ai-semirings of order two, we need to introduce the following notations. Let ω be an element of X^+ and x an element of ω. Then

- ⋄ $c(\omega)$ denotes the *content* of ω, i.e., the set of all variables occurring in ω.
- ⋄ $h(\omega)$ denotes the *head* of ω, i.e., the first variable occurring in u.
- ⋄ $t(\omega)$ denotes the *tail* of ω, i.e., the last variable occurring in u.
- ⋄ $\ell(\omega)$ denotes the *length* of ω, i.e., is the number of variables occurring in u, where each letter is counted as many times as it occurs in u.
- ⋄ $m(x,\ \omega)$ denotes the *multiplicity* of x in ω, i.e., the number of occurrences of x in w.

[1]The semiring varieties in Dolinka's papers are types of $(2, 2, 0)$.

Table 1. The 2-element ai-semirings

Semiring	Addition	Multiplication	Equational basis
L_2	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}0&0\\1&1\end{matrix}$	$xy \approx x$
R_2	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}0&1\\0&1\end{matrix}$	$xy \approx y$
M_2	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}0&1\\1&1\end{matrix}$	$x + y \approx xy$
D_2	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}0&0\\0&1\end{matrix}$	$x^2 \approx x,\ xy \approx yx,\ x + xy \approx x$
N_2	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}0&0\\0&0\end{matrix}$	$xy \approx zt,\ x + x^2 \approx x$
T_2	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}1&1\\1&1\end{matrix}$	$xy \approx zt,\ x + x^2 \approx x^2$

The following result follows from [23, Lemma 1.1]. We shall directly apply it without further notice.

Lemma 1. *Let $u \approx v$ be an nontrivial* **AI***-identity, where $u = u_1 + \cdots + u_k, v = v_1 + \cdots + v_\ell, u_i, v_j \in X^+, 1 \leq i \leq k, 1 \leq j \leq \ell$. Then*

(i) L_2 satisfies $u \approx v$ if and only if $\{h(u_i) \mid 1 \leq i \leq k\} = \{h(v_j) \mid 1 \leq j \leq \ell\}$;

(ii) R_2 satisfies $u \approx v$ if and only if $\{t(u_i) \mid 1 \leq i \leq k\} = \{t(v_j) \mid 1 \leq j \leq \ell\}$;

(iii) M_2 satisfies $u \approx v$ if and only if $\bigcup\{c(u_i) \mid 1 \leq i \leq k\} = \bigcup\{c(v_j) \mid 1 \leq j \leq \ell\}$;

(iv) D_2 satisfies $u \approx v$ if and only if $(\forall u_i \in u)(\exists v_j \in v)c(v_j) \subseteq c(u_i)$ and $(\forall v_k \in v)(\exists u_\ell \in u)c(u_\ell) \subseteq c(v_k)$;

(v) N_2 satisfies $u \approx v$ if and only if $\{u_i \in u \mid \ell(u_i) = 1\} = \{v_j \in v \mid \ell(v_j) = 1\}$;

(vi) T_2 satisfies $u \approx v$ if and only if $\{u_i \in u \mid \ell(u_i) \geq 2\} \neq \emptyset, \{v_j \in v \mid \ell(v_j) \geq 2\} \neq \emptyset$.

Up to isomorphism, there are 61 ai-semirings of order three[2], which are listed as S_i, $1 \leq i \leq 61$ in Table 2. We assume that the carrier set of each of these semirings is $\{1, 2, 3\}$. Their Cayley tables for addition and multiplication are listed in Table 2. It is easy to check that there are 24 ai-semirings of order three satisfying $x^3 \approx x$. By the main results of [21] we have that these semirings are all finitely based. So we only need to study the finite basis problem for the remaining 37 semirings. In fact, some of these semirings are members of the variety which are generated by all ai-semirings of order two. By the the main result of [23] it follows that they are all finitely based. Thus we have

Proposition 1. *The following ai-semirings are finitely based:* S_1, S_3, S_5, S_8, S_9, S_{10}, S_{11}, S_{12}, S_{13}, S_{14}, S_{15}, S_{16}, S_{17}, S_{18}, S_{19}, S_{20}, S_{21}, S_{22}, S_{23}, S_{24}, S_{25}, S_{26}, S_{27}, S_{28}, S_{29}, S_{30}, S_{31}, S_{32}, S_{33}, S_{34}, S_{35}, S_{36}, S_{37}, S_{38}, S_{39}, S_{40}, S_{41}, S_{42}, S_{43}, S_{48}, S_{49}, S_{50}, S_{51}, S_{52} *and* S_{61}.

For an ai-semiring S, S^* denotes the (multiplicative) left-right dual of S. It is easy to see that if S is finitely based, so is S^*. Thus, in the remaining we only need to study the finite basis problem for S_2, S_4, S_7, S_{44}, S_{46}, S_{47}, S_{53}, S_{55}, S_{57}, S_{58}, S_{59} and S_{60}. The following theorem is our main result.

Theorem 1. *With the possible exception of S_7, all ai-semirings of order three are finitely based.*

[2]We wrote a program and obtained this result.

Table 2. The 3-element ai-semirings

Semiring	+			·			Semiring	+			·		
	1	1	1	1	1	1		1	1	1	1	1	1
S_1	1	2	1	1	1	1	S_2	1	2	1	1	1	1
	1	1	3	1	1	1		1	1	3	1	1	2
	1	1	1	1	1	1		1	1	1	1	1	1
S_3	1	2	1	1	1	1	S_4	1	2	1	1	1	1
	1	1	3	1	1	3		1	1	3	1	2	3
	1	1	1	1	1	1		1	1	1	1	1	1
S_5	1	2	1	1	1	1	S_6	1	2	1	1	1	2
	1	1	3	3	3	3		1	1	3	1	1	3
	1	1	1	1	1	1		1	1	1	1	1	1
S_7	1	2	1	1	1	2	S_8	1	2	1	1	2	1
	1	1	3	1	2	3		1	1	3	1	1	3
	1	1	1	1	1	1		1	1	1	1	1	1
S_9	1	2	1	1	2	1	S_{10}	1	2	1	1	2	3
	1	1	3	3	3	3		1	1	3	1	3	2
	1	1	1	1	1	1		1	1	1	1	1	3
S_{11}	1	2	1	2	2	2	S_{12}	1	2	1	1	1	3
	1	1	3	3	3	3		1	1	3	1	1	3
	1	1	1	1	1	3		1	1	1	1	1	3
S_{13}	1	2	1	1	1	3	S_{14}	1	2	1	1	2	3
	1	1	3	3	3	3		1	1	3	1	1	3
	1	1	1	1	1	1		1	1	1	1	2	3
S_{15}	1	2	1	1	2	1	S_{16}	1	2	1	1	2	3
	1	1	3	3	3	3		1	1	3	1	2	3
	1	1	1	2	2	2		1	1	3	1	1	1
S_{17}	1	2	1	2	2	2	S_{18}	1	2	3	1	1	1
	1	1	3	2	2	2		3	3	3	1	1	1
	1	1	3	1	1	1		1	1	3	1	1	1
S_{19}	1	2	3	1	1	1	S_{20}	1	2	3	1	1	1
	3	3	3	1	1	3		3	3	3	3	3	3
	1	1	3	1	1	1		1	1	3	1	1	1
S_{21}	1	2	3	1	2	1	S_{22}	1	2	3	1	2	1
	3	3	3	1	1	1		3	3	3	1	1	3
	1	1	3	1	1	1		1	1	3	1	1	1
S_{23}	1	2	3	1	2	1	S_{24}	1	2	3	2	2	2
	3	3	3	3	3	3		3	3	3	1	1	1
	1	1	3	1	1	1		1	1	3	1	1	1
S_{25}	1	2	3	2	2	2	S_{26}	1	2	3	2	2	2
	3	3	3	1	1	3		3	3	3	3	3	3
	1	1	3	1	1	3		1	1	3	1	1	3
S_{27}	1	2	3	1	1	3	S_{28}	1	2	3	1	1	3
	3	3	3	1	1	3		3	3	3	3	3	3
	1	1	3	1	1	3		1	1	3	1	1	3
S_{29}	1	2	3	1	2	3	S_{30}	1	2	3	1	2	3
	3	3	3	1	1	3		3	3	3	3	3	3

S_{31}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&1&3\\2&2&3\\3&3&3\end{matrix}$	S_{32}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&2&1\\1&2&1\\1&2&1\end{matrix}$
S_{33}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&2&1\\1&2&1\\1&2&3\end{matrix}$	S_{34}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&2&1\\2&2&2\\1&2&1\end{matrix}$
S_{35}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&2&1\\2&2&2\\1&2&3\end{matrix}$	S_{36}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&2&1\\2&2&2\\3&2&3\end{matrix}$
S_{37}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&2&3\\1&2&3\\1&2&3\end{matrix}$	S_{38}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&2&3\\1&2&3\\3&3&3\end{matrix}$
S_{39}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&2&3\\2&2&2\\1&2&3\end{matrix}$	S_{40}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&2&3\\2&2&2\\3&2&3\end{matrix}$
S_{41}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&2&3\\2&2&2\\3&3&3\end{matrix}$	S_{42}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&2&3\\2&2&3\\3&2&3\end{matrix}$
S_{43}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}1&2&3\\2&2&3\\3&3&3\end{matrix}$	S_{44}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}2&2&1\\2&2&2\\1&2&3\end{matrix}$
S_{45}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}2&2&1\\2&2&2\\2&2&3\end{matrix}$	S_{46}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}2&2&2\\2&2&2\\1&2&3\end{matrix}$
S_{47}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}2&2&2\\2&2&2\\2&2&1\end{matrix}$	S_{48}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}2&2&2\\2&2&2\\2&2&2\end{matrix}$
S_{49}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}2&2&2\\2&2&2\\2&2&3\end{matrix}$	S_{50}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}2&2&2\\2&2&2\\3&3&3\end{matrix}$
S_{51}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}2&2&3\\2&2&3\\2&2&3\end{matrix}$	S_{52}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}2&2&3\\2&2&3\\3&3&3\end{matrix}$
S_{53}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}3&1&3\\1&2&3\\3&3&3\end{matrix}$	S_{54}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}3&1&3\\3&2&3\\3&3&3\end{matrix}$
S_{55}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}3&2&3\\2&2&2\\3&2&3\end{matrix}$	S_{56}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}3&2&3\\3&2&3\\3&2&3\end{matrix}$
S_{57}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}3&3&3\\1&2&3\\3&3&3\end{matrix}$	S_{58}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}3&3&3\\2&2&2\\3&3&3\end{matrix}$
S_{59}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}3&3&3\\3&1&3\\3&3&3\end{matrix}$	S_{60}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}3&3&3\\3&2&3\\3&3&3\end{matrix}$
S_{61}	$\begin{matrix}1&1&3\\1&2&3\\3&3&3\end{matrix}$	$\begin{matrix}3&3&3\\3&3&3\\3&3&3\end{matrix}$			

1. The proof of Theorem 1

In this section we shall provide the proof of Theorem 1. Let $\mathbf{HSP}(S)$ denote the variety generated by an ai-semiring S and $\underline{k}$ the set $\{1, 2, \ldots, k\}$ for a positive integer k. We start with a technique that will be used repeatedly in the sequel. Suppose that Σ is a set of identities which include the identities that determine $\mathbf{AI}$ and that $u \approx v$ is an $\mathbf{AI}$-identity, where $u = u_1 + \cdots + u_k, v = v_1 + \cdots + v_\ell, u_i, v_j \in X^+, i \in \underline{k}, j \in \underline{\ell}$. Then it is easy to see that the ai-semring variety defined by $u \approx v$ is equal to the ai-semiring variety defined by the simpler identities $u \approx u + v_j, v \approx v + u_i, i \in \underline{k}, j \in \underline{\ell}$. Thus, to show that $u \approx v$ is derivable from Σ, we only need to show that $u \approx u + v_j, v \approx v + u_i, i \in \underline{k}, j \in \underline{\ell}$ can be derived from Σ.

Proposition 2. $\mathbf{HSP}(S_2)$ *is the ai-semiring variety determined by the identities*

$$x_1x_2x_3 \approx y_1y_2y_3, \tag{1.1}$$

$$x + x^2 \approx x^3, \tag{1.2}$$

$$x^2 + y^2 \approx xy, \tag{1.3}$$

$$x^3 + y \approx x^3. \tag{1.4}$$

P r o o f. An $\mathbf{AI}$-term is said to be in canonical form if it is equal to one of the following terms: $x_1 + \cdots + x_m$, $x_1^2 + \cdots + x_m^2$, $x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$ and x^3, where $x_1, \ldots, x_m$ are distinct variables, $y_1, \ldots, y_n$ are distinct variables, and $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$. Suppose that $u = u_1 + u_2 + \cdots + u_k$ is an $\mathbf{AI}$-term, where $u_i \in X^+, i \in \underline{k}$. We shall show that there exists an $\mathbf{AI}$-term u' in canonical form such that the identities (1.1)–(1.4) and the identities determining $\mathbf{AI}$ imply the identity $u \approx u'$. The following cases are needed.

- $\ell(u_i) = 1$ for all $i \in \underline{k}$. Then $u = x_1 + \cdots + x_m$.
- $\ell(u_i) = 2$ for all $i \in \underline{k}$. Then the identity (1.3) implies $u \approx x_1^2 + \cdots + x_m^2$.
- $\ell(u_i) \leq 2$ for all $i \in \underline{k}$, $\ell(u_{i_1}) = 1$ for some $i_1 \in \underline{k}$ and $\ell(u_{i_2}) = 2$ for some $i_2 \in \underline{k}$. If $c(u_i) \bigcap c(u_j) \neq \emptyset$ for some u_i and u_j with $\ell(u_i) = 1$ and $\ell(u_j) = 2$, then the identities (1.2)–(1.4) implies $u \approx x^3$. Otherwise, we have that the identity (1.3) implies $u \approx x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$.
- $\ell(u_i) \geq 3$ for some $i \in \underline{k}$. Then the identities (1.1) and (1.4) imply $u \approx x^3$.

It is routine to check that S_2 satisfies the identities (1.1)–(1.4). In the remainder we shall show that every identity which is satisfied in S_2 can be derived from the identities (1.1)–(1.4) and the identities determining $\mathbf{AI}$. By the above arguments it is enough to show that if S_2 satisfies an identity $u \approx v$, where u and v are $\mathbf{AI}$-terms in canonical forms, then the identities (1.1)–(1.4) and the identities determining $\mathbf{AI}$ imply $u \approx v$. Notice that T_2 can be embedded into S_2. We only need to consider the following cases:

- $u = x_1 + \cdots + x_m, v = y_1 + \cdots + y_n$. It is easy to see that $u \approx v$ is trivial.
- $u = x_1^2 + \cdots + x_m^2, v = y_1^2 + \cdots + y_n^2$. It is easy to see that $u \approx v$ is trivial.
- $u = x_1^2 + \cdots + x_m^2, v = y_1 + \cdots + y_k + z_1^2 + \cdots + z_\ell^2$. Let $\varphi : P_f(X^+) \to S_2$ be a semiring homomorphism such that $\varphi(x) = 3$ for every variable x in X. Then $\varphi(u) = 2$ and $\varphi(v) = 1$, a contradiction. Thus $u \approx v$ is not satisfied in S_2.
- $u = x_1^2 + \cdots + x_m^2, v = t^3$. This case is similar to the preceding one.

- $u = y_1 + \cdots + y_k + z_1^2 + \cdots + z_\ell^2, v = y'_1 + \cdots + y'_m + {z'}_1^2 + \cdots + {z'}_n^2$. It is easy to see that $u \approx v$ is trivial.

- $u = y_1 + \cdots + y_k + z_1^2 + \cdots + z_\ell^2, v = x^3$. Let $\varphi : P_f(X^+) \to S_2$ be a semiring homomorphism such that $\varphi(y_i) = 2$, $\varphi(z_j) = 3$ and $\varphi(t) = 1$ for all $i \in \underline{k}$, $j \in \underline{\ell}$. Then $\varphi(u) = 2$ and $\varphi(v) = 1$, a contradiction. Thus $u \approx v$ is not satisfied in S_2.

- $u = x_1^3, v = x_2^3$. Then the identity (1) implies $u \approx v$.

This completes the proof. □

Proposition 3. $\mathbf{HSP}(S_4)$ *is the ai-semiring variety determined by the identities*

$$xy \approx x^2y, \tag{1.5}$$
$$xyz \approx yxz, \tag{1.6}$$
$$x + y^2 \approx xy^2, \tag{1.7}$$
$$x + yz \approx yx + yz. \tag{1.8}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to one of the following terms: $x_1 + \cdots + x_m$, $x_1^2 \cdots x_m^2$ and $x_1^2 \cdots x_m^2(y_1 + \cdots + y_n)$, where $x_1, \ldots, x_m$ are distinct variables, $y_1, \ldots, y_n$ are distinct variables and $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$. Suppose that $u = u_1 + u_2 + \cdots + u_k$ is an **AI**-term, where $u_i \in X^+$, $i \in \underline{k}$. We shall show that there exists an **AI**-term u' in canonical form such that the identities (1.5)–(1.8) and the identities determining **AI** imply the identity $u \approx u'$. The following cases are needed.

- $\ell(u_i) = 1$ for all $i \in \underline{k}$. Then $u = x_1 + \cdots + x_m$.

- $m(t(u_i), u_i) \geq 2$ for some $j \in \underline{k}$. Then the identities (1.5)–(1.7) imply $u \approx x_1^2 \cdots x_m^2$.

- $\ell(u_i) \geq 2$ for some $i \in \underline{k}$, $m(t(u_j), u_j) = 1$ for every $j \in \underline{k}$. Then the identities (1.5), (1.6) and (1.8) imply $u \approx x_1^2 \cdots x_m^2(y_1 + \cdots + y_n)$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$.

It is routine to check that S_4 satisfies the identities (1.5)–(1.8). By the above arguments it is enough to show that if S_4 satisfies an identity $u \approx v$, where u and v are **AI**-terms in canonical forms, then the identities (1.5)–(1.8) and the identities determining **AI** imply $u \approx v$. Since T_2 can be embedded into S_4, we only need to consider the following cases:

- $u = x_1 + \cdots + x_m, v = y_1 + \cdots + y_n$. It is easy to see that $u \approx v$ is trivial.

- $u = x_1^2 \cdots x_m^2, v = y_1^2 \cdots y_n^2$. It is easy to see that $u \approx v$ is trivial.

- $u = x_1^2 \cdots x_m^2, v = y_1^2 \cdots y_k^2(z_1 + \cdots + z_\ell)$. Let $\varphi : P_f(X^+) \to S_4$ be a semiring homomorphism such that $\varphi(y_i) = 3$, $\varphi(z_j) = 2$, $i \in \underline{k}$, $j \in \underline{\ell}$, $\varphi(x) = 1$ for every remaining variable x. Then $\varphi(u) = 1$ or 3, $\varphi(v) = 2$, a contradiction. Thus $u \approx v$ is not satisfied in S_4.

- $u = x_1^2 \cdots x_m^2(y_1 + \cdots + y_n)$, $v = z_1^2 \cdots z_k^2(t_1 + \cdots + t_\ell)$. It is easy to see that $u \approx v$ is trivial.

This completes the proof. □

Proposition 4. $\mathbf{HSP}(S_{44})$ *is the ai-semiring variety determined by the identities*

$$x^3 \approx x^2, \tag{1.9}$$
$$xy \approx yx, \tag{1.10}$$
$$x + xy \approx x, \tag{1.11}$$
$$x^2y + xy^2 \approx xy. \tag{1.12}$$

P r o o f. An **AI**-term $u = u_1 + \cdots + u_n$ is said to be in canonical form if every u_i is equal to one of the following terms: $x, x_1^2 \cdots x_m^2$ and $x_1^2 \cdots x_m^2 y$, where $x_1, \ldots, x_m$ are distinct variables and $y \neq x_i$ for every $i \in \underline{m}$. Let $p = x_1 \cdots x_n$ be an element of X^+ such that $n \geq 2$. By induction on n we have that the identity (1.10) and (1.12) imply $p \approx \sum_{1 \leq i \leq n} x_1^2 \cdots x_{i-1}^2 x_{i+1}^2 \cdots x_n^2 x_i$. It follows that for any **AI**-term u, there exists an **AI**-term u' in canonical form such that (1.9)–(1.12) imply $u \approx u'$.

It is easy to check that S_{44} satisfies the identities (1.9)–(1.12). To show that **HSP**(S_{44}) is determined by (1.9)–(1.12), by the above arguments it suffices to show that if S_{44} satisfies $u+p \approx u$, where $u+p$ and u are **AI**-terms in canonical forms, then the identities (1.9)–(1.12) and the identities determining **AI** imply $u + p \approx u$. We shall consider the following three cases.

- $p = x$. Since N_2 can be embedded into S_{44}, there exists some u_i in u such that $u_i = x$. It follows that $u + p \approx u$ is trivial.

- $p = x_1^2 \cdots x_m^2$. Since D_2 can be embedded into S_{44}, there exists some u_i in u such that $c(u_i) \subseteq c(p)$ and so (1.9) and (1.10) imply $u_i p \approx p$. Further, we have
$$u \approx u + u_i \overset{(1.11)}{\approx} u + u_i + u_i p \approx u + u_i + p \approx u + p.$$

- $p = x_1^2 \cdots x_m^2 y$. Since D_2 can be embedded into S_{44}, we have that $\{u_i \in u \mid c(u_i) \subseteq c(p)\}$ is non-empty. Suppose that for any u_i in $\{u_i \in u \mid c(u_i) \subseteq c(p)\}$, there exists x in $c(u_i)$ such that $m(x,p) < m(x,u_i)$. That is to say, $m(y,u_i) = 2$ for every u_i in $\{u_i \mid c(u_i) \subseteq c(p)\}$. Let $\varphi : P_f(X^+) \to S_{44}$ be a semiring homomorphism such that $\varphi(z) = 2$ for every $z \in X \setminus c(p)$, $\varphi(x_i) = 3$ for every $i \in \underline{m}$ and $\varphi(y) = 1$. Then $\varphi(u) = 2$ and $\varphi(u+p) = 1$, a contradiction. Thus there exists u_i in $\{u_i \in u \mid c(u_i) \subseteq c(p)\}$ such that $m(x,u_i) \leq m(x,p)$ for every x in $c(u_i)$. If $y \in c(u_i)$, then $m(y,u_i) = 1$ and so (1.9) and (1.10) imply $u_i x_1^2 \cdots x_m^2 \approx p$. Further, we have
$$u \approx u + u_i \overset{(1.11)}{\approx} u + u_i + u_i x_1^2 \cdots x_m^2 \approx u + p.$$
If $y \notin c(u_i)$, then (1.9) and (1.10) imply $u_i p \approx p$. We therefore have
$$u \approx u + u_i \overset{(1.11)}{\approx} u + u_i + u_i p \approx u + p.$$

This completes the proof. □

Proposition 5. **HSP**(S_{46}) *is the ai-semiring variety determined by the identities*

$$x^2 y \approx xy, \tag{1.13}$$
$$x^2 y^2 \approx y^2 x^2, \tag{1.14}$$
$$xyz \approx yxz, \tag{1.15}$$
$$x + xy \approx x, \tag{1.16}$$
$$x + yx \approx x. \tag{1.17}$$

P r o o f. An **AI**-term $u = u_1 + \cdots + u_n$ is said to be in canonical form if every u_i is equal to one of the following terms: $x, x_1^2 \cdots x_m^2$ and $x_1^2 \cdots x_m^2 y$, where $y \neq x_i$ for all $i \in \underline{m}$. Let p be an element of X^+ such that $\ell(p) \geq 2$. If $m(t(p),p) = 1$, then the identities (1.13)–(1.15) imply $p \approx x_1^2 \cdots x_m^2 y$. If $m(t(p),p) \geq 2$, then the identities (1.13)–(1.15) imply $p \approx x_1^2 \cdots x_m^2$. It follows that for any **AI**-term u, there exists an **AI**-term u' in canonical form such that (1.13)–(1.15) imply $u \approx u'$.

It is routine to check that S_{46} satisfies the identities (1.13)–(1.17). To show that $\mathbf{HSP}(S_{46})$ is the ai-semiring variety determined by (1.13)–(1.17), by the above arguments it suffices to show that if S_{46} satisfies $u+p \approx u$, where $u+p$ and u are **AI**-terms in canonical form, then (1.13)–(1.17) imply $u+p \approx u$. The following three cases are necessary.

- $p = x$. Since D_2 can be embedded into S_{46}, there exists some u_i in u such that $c(u_i) = \{x\}$. Suppose that $u_i = x^2$ for every u_i in u with $c(u_i) = \{x\}$. Let $\varphi : X \to S_{46}$ be a semiring homomorphism such that $\varphi(x) = 1$ and $\varphi(y) = 2$ for every $y \neq x$. Then $\varphi(u) = 2$ and $\varphi(u+p) = 1$, a contradiction. Thus there exists u_i in u such that $u_i = x$ and so $u+p \approx u$ is trivial.

- $p = x_1^2 \cdots x_m^2$. Since D_2 can be embedded into S_{46}, there exists some u_i in u such that $c(u_i) \subseteq c(p)$ and so (1.13)–(1.15) imply $p \approx u_i p$. We now have

$$u+p \approx u+u_i+p \approx u+u_i+u_ip \overset{(1.16)}{\approx} u+u_i \approx u.$$

- $p = x_1^2 \cdots x_m^2 y$. Since D_2 can be embedded into S_{46}, it follows that $\{u_i \in u \mid c(u_i) \subseteq c(p)\}$ is non-empty. Suppose that $m(y, u_i) = 2$ for every u_i in $\{u_i \in u \mid c(u_i) \subseteq c(p)\}$. Let $\varphi : P_f(X^+) \to S_{46}$ be a semiring homomorphism such that $\varphi(z) = 2$ for every $z \notin c(p)$, $\varphi(x_i) = 3$ for every $i \in \underline{m}$ and $\varphi(y) = 1$. Then $\varphi(u) = 2$ and $\varphi(u+p) = 1$, a contradiction. Thus we only need to consider the following cases:

  ◇ $(\exists u_i \in \{u_i \in u \mid c(u_i) \subseteq c(p)\})\ y \notin c(u_i)$. Then

  $$u+p \approx u+u_i+p \overset{(1.13)-(1.15)}{\approx} u+u_i+u_ip \overset{(1.16)}{\approx} u+u_i \approx u.$$

  ◇ $(\exists u_i \in \{u_i \in u \mid c(u_i) \subseteq c(p)\})\ y \in c(u_i)$, $t(u_i) = y$ and $m(y, u_i) = 1$. Then

  $$u+p \approx u+u_i+p \overset{(1.13)-(1.15)}{\approx} u+u_i+x_1^2 \cdots x_m^2 u_i \overset{(1.17)}{\approx} u+u_i \approx u.$$

This completes the proof. □

Proposition 6. $\mathbf{HSP}(S_{47})$ *is the ai-semiring variety determined by the identities*

$$xy \approx yx, \tag{1.18}$$
$$x + xy \approx x, \tag{1.19}$$
$$x^2 + xy \approx x^2, \tag{1.20}$$
$$x + x_1x_2x_3 \approx x. \tag{1.21}$$

P r o o f. It is easy to verify that S_{47} satisfies the identities (1.18)–(1.21). In the remainder it suffices to show that every identity which is satisfied in S_{47} is derivable from (1.18)–(1.21). Let $u+p \approx u$ be such an identity, where $u = u_1 + \cdots + u_m, u_i, p \in X^+, i \in \underline{m}$. We consider the following three cases.

- $\ell(p) = 1$. Since N_2 can be embedded into S_{47}, there exists u_i in u such that $u_i = p$. Thus $u+p \approx u$ is trivial.

- $\ell(p) = 2$. Suppose that for any u_i in u, $c(u_i) \nsubseteq c(p)$. Let $\varphi\colon P_f(X^+) \to S_{47}$ be a semiring homomorphism such that $\varphi(z) = 2$ for every $z \in X \setminus c(p)$ and $\varphi(x) = 3$ for every $x \in c(p)$. Then $\varphi(u) = 2$ and $\varphi(u + p) = 1$, a contradiction. Thus there exists u_i in u such that $c(u_i) \subseteq c(p)$. Assume that $\ell(u_i) \geq 3$ for every u_i in $\{u_i \in u \mid c(u_i) \subseteq c(p)\}$. Then $\varphi(u) = 2$ and $\varphi(u+p) = 1$, a contradiction. This implies that there exists u_i in u such that $c(u_i) \subseteq c(p)$ and $\ell(u_i) \leq 2$. Further, the identities (1.18)–(1.20) imply

$$u + p \approx u + u_i + p \approx u + u_i \approx u.$$

- $\ell(p) \geq 3$. Then $u + p \approx u$ can be derived from (1.21).

This completes the proof. □

Proposition 7. **HSP**(S_{53}) *is the ai-semiring variety determined by the identities*

$$xy \approx yx, \tag{1.22}$$
$$xy + y^2 \approx x + y^2, \tag{1.23}$$
$$x + xy \approx xy, \tag{1.24}$$
$$xy + yz + xz \approx xyz. \tag{1.25}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to one of the following terms: $x_1 + \cdots + x_m$, $x_1y_1 + \cdots + x_my_m$, and $x_1 + \cdots + x_m + y_1z_1 + \cdots + y_nz_n$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j, z_j \mid j \in \underline{n}\} = \emptyset$, x^2 and xy can not occur simultaneously. Suppose that $u = u_1 + u_2 + \cdots + u_k$ is an **AI**-term, where $u_i \in X^+, i \in \underline{k}$. It is easy to show that there exists an **AI**-term u' in canonical form such that the identities (1.22)–(1.25) imply the identity $u \approx u'$.

It is routine to check that S_{53} satisfies the identities (1.22)–(1.25). In the remainder we shall show that every identity which is satisfied in S_{53} can be derived from the identities (1.22)–(1.25). By the above arguments it suffices to show that if S_{53} satisfies an identity $u \approx v$, where u and v are **AI**-terms in canonical form, then the identities (1.22)–(1.25) and the identities determining **AI** imply $u \approx v$. Notice that T_2 can be embedded into S_{53}. We only need to consider the following cases:

- $u = x_1 + \cdots + x_m$, $v = y_1 + \cdots + y_n$. It follows immediately that $u \approx v$ is trivial.

- $u = x_1y_1 + \cdots + x_my_m$, $v = z_1s_1 + \cdots + z_ns_n$. For any $i \in \underline{m}$, suppose that $\{x_1, y_1\}$ is not equal to $\{z_j, s_j\}$ for every $j \in \underline{n}$. Consider the following two subcases:

  ◇ $x_i = y_i$. Let $\varphi : P_f(X^+) \to S_{53}$ be a semiring homomorphism such that $\varphi(x_i) = 1$ and $\varphi(z) = 2$ for every $z \in X \setminus \{x_i\}$. Then $\varphi(u) = 3$ and $\varphi(v) = 1$, a contradiction.

  ◇ $x_i \neq y_i$. Since M_2 can be embedded into S_{53}, we can deduce that S_{53} satisfies one of the following identities: $x_iy_i \approx x_i^2 + y_i^2$, $x_iy_i \approx x_i^2 + y_i$, $x_iy_i \approx x_i + y_i^2$ and $x_iy_i \approx x_i + y_i$, a contradiction.

  Thus $\{x_i, y_i\}$ is equal to $\{z_j, s_j\}$ for some $j \in \underline{n}$. Similarly, for any $j \in \underline{n}$, $\{z_j, s_j\}$ is equal to $\{x_i, y_i\}$ for some $i \in \underline{m}$. Hence $u \approx v$ is trivial.

- $u = x_1y_1 + \cdots + x_my_m, v = z_1 + \cdots + z_k + s_1t_1 + \cdots + s_\ell t_\ell$. We consider the following two subcases:

  ◇ $\{x_i, y_i\} \subseteq \{z_i \mid i \in \underline{k}\}$ for some $i \in \underline{m}$. Let $\psi : P_f(X^+) \to S_{53}$ be a semiring homomorphism such that $\psi(z_i) = 1$ for every $i \in \underline{k}$ and $\psi(x) = 2$ for every $z \in X \setminus \{z_i \mid i \in \underline{k}\}$. Then $\psi(u) = 3$ and $\psi(v) = 1$, a contradiction.

- ◇ $\{x_i, y_i\} \nsubseteq \{z_i \mid i \in \underline{k}\}$ for every $i \in \underline{m}$. Notice that M_2 can be embedded into S_{53}. Let $\theta : P_f(X^+) \to S_{53}$ be a semiring homomorphism such that $\theta(x_i) = \theta(y_i) = 1$ if $\{x_i, y_i\} \bigcap \{z_i \mid i \in \underline{k}\} \neq \emptyset$, and $\theta(y) = 2$ for every remaining variable y. Then $\theta(u) = 3$ and $\theta(v) = 1$, a contradiction.

This shows that $u \approx v$ is not satisfied in S_{53}.

- $u = x_1 + \cdots + x_m + y_1 z_1 + \cdots + y_n z_n, v = x'_1 + \cdots + x'_k + y'_1 z'_1 + \cdots + y'_\ell z'_\ell$. Suppose that $x_1 = y'_i$ for some i.
  - ◇ $y'_i = z'_i$. Choose every variable in $X \setminus \{x_1\}$ to 2. Then S_{53} satisfies $x_1 \approx x_1^2$, a contradiction.
  - ◇ $y'_i \neq z'_i$. Choose every variable in $X \setminus \{y'_i, z'_i\}$ to 2. Then S_{53} satisfies $y'_i + z'_i \approx y'_i z'_i$ or $y'_i + {z'_i}^2 \approx y'_i z'_i$, a contradiction.

  This implies that $x_1 + \cdots + x_m \approx x'_1 + \cdots + x'_k$ is trivial and so S_{53} satisfies $y_1 z_1 + \cdots + y_n z_n \approx y'_1 z'_1 + \cdots + y'_\ell z'_\ell$. By the preceding case it follows that $y_1 z_1 + \cdots + y_n z_n \approx y'_1 z'_1 + \cdots + y'_\ell z'_\ell$ is trivial. Hence $u \approx v$ is trivial.

This completes the proof. □

Proposition 8. **HSP**(S_{55}) *is the ai-semiring variety determined by the identities*

$$xy \approx yx, \tag{1.26}$$
$$xy \approx x^2 y, \tag{1.27}$$
$$xy \approx xy + xyz, \tag{1.28}$$
$$x^2 \approx x^2 + x. \tag{1.29}$$

P r o o f. It is routine to check that S_{55} satisfies (1.26)–(1.29). In the remainder it suffices to show that if S_{55} satisfies $u \approx u + q$, where $u = u_1 + u_2 + \cdots + u_m, u_i, q \in X^+, i \in \underline{m}$, then (1.26)–(1.29) and the identities determining **AI** imply the identity $u \approx u + q$. Choose $Z = (\bigcup_{i \in \underline{m}} c(u_i)) \setminus c(q)$. By [21, Lemma 2.11] we have that T_2 satisfies $D_Z(u) \approx D_Z(u) + q$, where $D_Z(u)$ denotes the sum of terms u_i for which $c(u_i) \subseteq c(q)$. We may assume that $D_Z(u) = u_1 + u_2 + \cdots + u_k$. The following two cases are necessary.

- $\ell(q) = 1$. Then (1.27) and (1.29) implies $u \approx u + q$.
- $\ell(q) \geq 2$. Then there exists u_i with $c(u_i) \subseteq c(q)$ such that $\ell(u_i) \geq 2$. Further, by (1.26)–(1.28) we have
$$u \approx u + u_i \overset{(1.28)}{\approx} u + u_i + u_i q \overset{(1.26),(1.27)}{\approx} u + u_i + q \approx u + q.$$

This completes the proof. □

Proposition 9. **HSP**(S_{57}) *is the ai-semiring variety determined by the identities*

$$xyz \approx yxz, \tag{1.30}$$
$$x^2 y \approx xy, \tag{1.31}$$
$$x + yz \approx yx + yz, \tag{1.32}$$
$$x^2 + xy \approx xy. \tag{1.33}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to $x_1 + \cdots + x_m$, x^2 or $x_1 \cdots x_m(y_1 + \cdots + y_n)$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$. Let u be an arbitrary **AI**-term. It is easy to see that there exists an **AI**-term u' in canonical form such that (1.30)–(1.33) imply $u \approx u'$.

It is routine to check that S_{57} satisfies (1.30)–(1.33). In the remainder it is enough to show that if S_{57} satisfies $u \approx v$ where u and v are **AI**-terms in canonical form, then (1.30)–(1.33) and the identities determining **AI** imply $u \approx v$. Notice that both M_2 and T_2 can be embedded into S_{57}. We consider the following nontrivial case that $u = x_1 \cdots x_m(y_1 + \cdots + y_n)$, $v = z_1 \cdots z_k(t_1 + \cdots + t_\ell)$. For a fixed x_i, suppose that it is not equal to z_j for every $j \in \underline{k}$. Since M_2 can be embedded into S_{57}, it follows that x_i must be equal to some t_j. Choose x_i to 1 and every other variable to 2. We have that $3 = 1$, a contradiction. Thus $x_1 \cdots x_m \approx z_1 \cdots z_k$ is trivial. Choose x_i to 2 for every $i \in \underline{m}$. Then S_{57} satisfies $y_1 + \cdots + y_n \approx t_1 + \cdots + t_\ell$. Thus $y_1 + \cdots + y_n \approx t_1 + \cdots + t_\ell$ is trivial and so is $u \approx v$. □

Proposition 10. **HSP**(S_{58}) *is the ai-semiring variety determined by the identities*

$$xy \approx x^2, \tag{1.34}$$

$$x^2 \approx x + x^2. \tag{1.35}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to $x_1 + \cdots + x_m$, $y_1^2 + \cdots + y_n^2$ or $x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$. Let u be an arbitrary **AI**-term. It is easy to see that there exists an **AI**-term u' in canonical form such that (1.34) and (1.35) imply $u \approx u'$.

It is routine to check that S_{58} satisfies (1.34) and (1.35). In the remainder it is enough to show that if S_{58} satisfies $u \approx v$, where u and v are **AI**-terms in canonical form, then (1.34), (1.35) and the identities determining **AI** imply $u \approx v$. Notice that T_2 can be embedded into S_{58}. The following two cases are necessary.

- $u = x_1 + \cdots + x_m$, $v = y_1 + \cdots + y_n$. Then $u \approx v$ is trivial.

- $u = x_1^2 + \cdots + x_m^2$, $v = y_1^2 + \cdots + y_n^2$. Since L_2 can be embedded into S_{58}, it follows that $u \approx v$ is trivial.

- $u = x_1^2 + \cdots + x_k^2$, $v = y_1 + \cdots y_m + z_1^2 + \cdots + z_n^2$. Let $\psi : P_f(X^+) \to S_{58}$ be a semiring homomorphism such that $\psi(y_i) = 1$ for every $i \in \underline{m}$ and $\psi(x) = 2$ for every remaining variable x. Since L_2 can be embedded into S_{58}, it follows that $\psi(u) = 3$ and $\psi(v) = 1$, a contradiction.

- $u = x_1 + \cdots x_m + y_1^2 + \cdots + y_n^2$, $v = z_1 + \cdots z_k + t_1^2 + \cdots + t_\ell^2$. Since L_2 can be embedded into S_{58}, we have

  $$\{x_i \mid i \in \underline{m}\} \bigcup \{y_i \mid i \in \underline{n}\} = \{z_j \mid j \in \underline{k}\} \bigcup \{t_j \mid j \in \underline{\ell}\}.$$

  For a fixed x_i, suppose that it is not equal to z_j for every $j \in \underline{k}$. Then x_i must be equal to some t_j. Choose x_i to 1 and every remaining variable to 2. We have that $1 = 3$, a contradiction. Thus x_i is equal to some z_i and so $\{x_i \mid i \in \underline{m}\} = \{z_j \mid j \in \underline{k}\}$. Hence $u \approx v$ is trivial.

This completes the proof. □

Proposition 11. **HSP**(S_{59}) *is the ai-semiring variety determined by the identities*

$$x_1x_2x_3 \approx y_1y_2y_3, \tag{1.36}$$
$$x^3 + y \approx x^3, \tag{1.37}$$
$$x^2 + y^2 \approx xy, \tag{1.38}$$
$$x + x^2 \approx x^2. \tag{1.39}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to x^3, $x_1 + \cdots + x_m$, $y_1^2 + \cdots + y_n^2$ or $x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$, where $\{x_i \mid i \in \underline{m}\} \bigcap \{y_j \mid j \in \underline{n}\} = \emptyset$. Let u be an arbitrary **AI**-term. It is easy to see that there exists an **AI**-term u' in canonical form such that (1.36)-(1.39) imply $u \approx u'$.

It is routine to check that S_{59} satisfies (1.36)-(1.39). In the remainder it is enough to show that if S_{59} satisfies $u \approx v$, where u and v are **AI**-terms in canonical form, then (1.36)-(1.39) and the identities determining **AI** imply $u \approx v$. Notice that T_2 can be embedded into S_{59}. The following cases are necessary.

- $u = x_1 + \cdots + x_m$, $v = y_1 + \cdots + y_n$. It is easy to see that $u \approx v$ is trivial.
- $u = t_1^3$, $v = t_2^3$. Then (1.36) implies $u \approx v$.
- $u = t_1^3$, $v = y_1^2 + \cdots + y_n^2$. Choose every variable to 2. Then $3 = 1$, a contradiction. Thus S_{59} does not satisfy $u \approx v$.
- $u = t_1^3$, $v = x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$. Choose every variable to 2. Then $3 = 1$, a contradiction. Thus S_{59} does not satisfy $u \approx v$.
- $u = x_1^2 + \cdots + x_m^2$, $v = y_1^2 + \cdots + y_n^2$. It is easy to see that $u \approx v$ is trivial.
- $u = x_1^2 + \cdots + x_m^2$, $v = y_1 + \cdots + y_k + z_1^2 + \cdots + z_\ell^2$. Consider the following two subcases.
  - ◇ $\{y_i \mid i \in \underline{k}\} \nsubseteq \{x_i \mid i \in \underline{m}\}$. Choose y_i to 3, where $y_i \notin \{x_i \mid i \in \underline{m}\}$. Choose every other variable to 2. Then 1=3, a contradiction.
  - ◇ $\{y_i \mid i \in \underline{k}\} \subseteq \{x_i \mid i \in \underline{m}\}$. Choose y_i to 1 for every $i \in \underline{m}$ and every other variable to 2. Then 3=1, a contradiction.

  Thus S_{59} does not satisfy $u \approx v$.
- $u = x_1 + \cdots + x_m + y_1^2 + \cdots + y_n^2$, $v = z_1 + \cdots + z_k + s_1^2 + \cdots + s_\ell^2$. Fix x_i. Suppose that x_i is not equal to z_j for every $j \in \underline{k}$.
  - ◇ $x_i \in \{s_j \mid j \in \underline{\ell}\}$. Choose x_i to 1 and every other variable to 2. Then 1=3, a contradiction.
  - ◇ $x_i \notin \{s_j \mid j \in \underline{\ell}\}$. Choose x_i to 3 and every other variable to 2. Then 3=1, a contradiction.

  Thus x_i is equal to z_j for some $j \in \underline{k}$ and so $x_1 + \cdots + x_m \approx z_1 + \cdots + z_k$ is trivial. Further, $y_1^2 + \cdots + y_n^2 \approx s_1^2 + \cdots + s_\ell^2$ holds in S_{59}. By the case 5 we have that this identity is trivial and so is $u \approx v$.

This complete the proof. □

Proposition 12. **HSP**(S_{60}) *is the ai-semiring variety determined by the identities*

$$x^3 \approx x^2, \tag{1.40}$$
$$x^2 + y^2 \approx xy, \tag{1.41}$$
$$x + x^2 \approx x^2. \tag{1.42}$$

P r o o f. An **AI**-term is said to be in canonical form if it is equal to $x_1+\cdots+x_m$, $y_1^2+\cdots+y_n^2$ or $x_1+\cdots+x_m+y_1^2+\cdots+y_n^2$, where $\{x_i \mid i \in \underline{m}\}\bigcap\{y_j \mid j \in \underline{n}\} = \emptyset$. Let u be an arbitrary **AI**-term. It is easy to see that there exists an **AI**-term u' in canonical form such that (1.40)-(1.42) imply $u \approx u'$.

It is routine to check that S_{60} satisfies (1.40)-(1.42). In the remainder it suffices to show that if S_{60} satisfies $u \approx v$, where u and v are terms in canonical form, then (1.40)-(1.42) and the identities determining **AI** imply $u \approx v$. Notice that T_2 can be embedded into S_{60}. The following cases are necessary.

- $u = x_1+\cdots+x_m$, $v = y_1+\cdots+y_n$. It is easy to see that $u \approx v$ is trivial.
- $u = x_1^2+\cdots+x_m^2$, $v = y_1^2+\cdots+y_n^2$. Since M_2 can be embedded into S_{60}, it follows that $u \approx v$ is trivial.
- $u = x_1^2+\cdots+x_m^2, v = y_1+\cdots+y_k+z_1^2+\cdots+z_\ell^2$. Then $\{y_i \mid i \in \underline{k}\} \subseteq \{x_i \mid i \in \underline{m}\}$. Choose y_i to 1 for every $i \in \underline{k}$ and every other variable to 2. Then 3=1, a contradiction. Thus S_{60} does not satisfy $u \approx v$.
- $u = x_1+\cdots+x_m+y_1^2+\cdots+y_n^2$, $v = z_1+\cdots+z_k+s_1^2+\cdots+s_\ell^2$. Then

  $$\{x_i \mid i \in \underline{m}\}\bigcup\{y_i \mid i \in \underline{n}\} = \{z_j \mid j \in \underline{k}\}\bigcup\{s_j \mid j \in \underline{\ell}\}.$$

  Fix x_i. Suppose that x_i is not equal to z_j for all $j \in \underline{k}$. Choose x_i to 1 for all $i \in \underline{k}$ and every other variable to 2. Then 1=3, a contradiction. Thus $x_i = z_j$ for some $j \in \underline{k}$. This implies that $u \approx v$ is trivial.

This complete the proof. □

By Propositions 1–12 we immediately complete the proof of Theorem 1.

2. Conclusion

We have answered the finite basis problem for all ai-semirings of order three except S_7. This will lay a solid foundation for our subsequent work about ai-semiring varieties. Moreover, we conjecture that the semiring S_7 is nonfinitely based. In contrast to the rich results in the theory of semigroup varieties [26], there are still many problems to be solved in the theory of semiring varieties. In particular, it is of the interest to study the variety generated by all ai-semirings of order three.

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Contact Information

16 S. Kovalevskaya str., Ekaterinburg, Russia, 620990

Phone: +7 (343) 375-34-73

Fax: +7 (343) 374-25-81

Email: secretary@umjuran.ru

Web-site: https://umjuran.ru

N.N. Krasovskii Institute of Mathematics and Mechanics
of the Ural Branch of Russian Academy of Sciences

Ural Federal University named after the first President of Russia B.N. Yeltsin

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