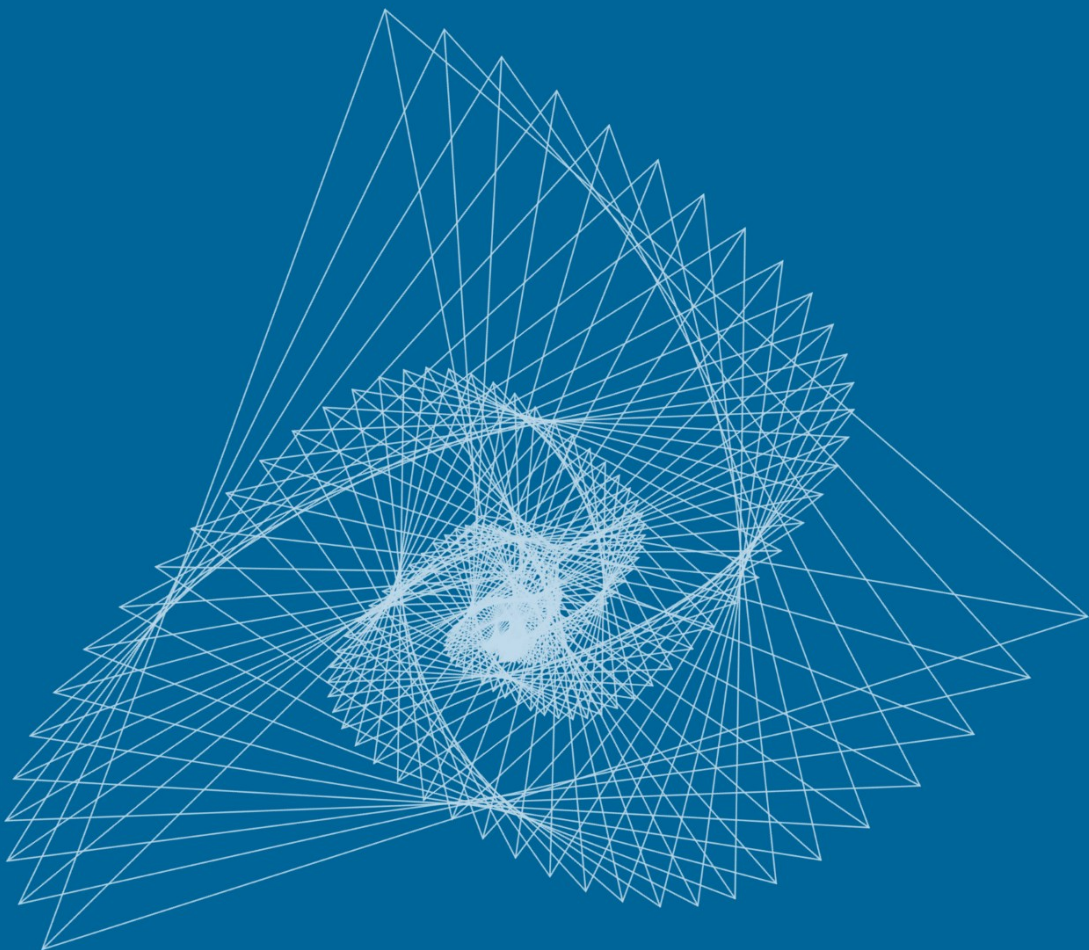
VOL. 10, NO. 1

URAL MATHEMATICAL JOURNAL

N.N. Krasovskii Institute of Mathematics and Mechanics of
the Ural Branch of Russian Academy of Sciences and
Ural Federal University named after the first President of Russia B.N.Yeltsin

ISSN: 2414-3952



2024

UMJURAN.RU
UMJ

ISSN 2414-3952
(Online)

Ural Mathematical Journal

Vol. 10, No. 1, 2024

https://umjuran.ru

Electronic Periodical Scientific Journal
Founded in 2015

The Journal is registered by the Federal Service for Supervision in the Sphere of Communication, Information Technologies and Mass Communications
Certificate of Registration of the Mass Media Эл № ФС77-61719 of 07.05.2015

Founders

N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences

Ural Federal University named after the first President of Russia B.N. Yeltsin

Contact Information

16 S. Kovalevskaya str., Ekaterinburg, Russia, 620990
Phone: +7 (343) 375-34-73 Fax: +7 (343) 374-25-81
Email: secretary@umjuran.ru
Web-site: https://umjuran.ru

TABLE OF CONTENTS

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 4–17
DOI: 10.15826/umj.2024.1.001

A CHARACTERIZATION OF MEIXNER ORTHOGONAL POLYNOMIALS VIA A CERTAIN TRANSFERT OPERATOR

Emna Abassi

Faculté des Sciences de Tunis, Université de Tunis El Manar,
Rommana 1068, Tunisia

emna.abassi@fst.utm.tn

Lotfi Khériji

Institut Préparatoire aux Etudes d'Ingénieur El Manar,
Université de Tunis El Manar,
Rommana 1068, Tunisia

kheriji@yahoo.fr

Abstract: Here we consider a certain transfert operator $\mathrm{M}_{(c,\omega)} = I_{\mathcal{P}} - c\,\tau_\omega$, $\omega \neq 0$, $c \in \mathbb{R} - \{0,1\}$, and we prove the following statement: up to an affine transformation, the only orthogonal sequence that remains orthogonal after application of this transfert operator is the Meixner polynomials of the first kind.

Keywords: Orthogonal polynomials, Regular form, Meixner polynomials, Divided-difference operator, Transfert operator, Hahn property.

1. Introduction and preliminaries

Let $\mathcal{O}$ be a linear operator acting on the space of polynomials as a lowering operator (the derivative [4, 18, 19], the q-derivative [4, 12, 14, 15], the divided-difference [1], the Dunkl [6, 8, 9, 11, 13], the q-Dunkl [5, 7, 13], other [17, 21]), a transfert operator (see [20]) or a raising operator (see [2, 3, 17]). Many researchers in this vast field cited above had the concern to characterize the $\mathcal{O}$-classical polynomial sequences that is those which fulfill the so-called Hahn property: the sequences $\{P_n\}_{n\geq 0}$ and $\{\mathcal{O}P_n\}_{n\geq 0}$ are orthogonal.

By the way, in [20], the authors characterized the $I_{(q,\omega)}$-classical orthogonal polynomials where $I_{(q,\omega)}$ is a transfert operator acting on the space of polynomials $\mathcal{P}$ and defined by [20]

$$I_{(q,\omega)} := I_{\mathcal{P}} + \omega h_q, \quad \omega \in \mathbb{C} \setminus \{0\}, \quad q \in \mathbb{C}_\omega := \big\{z \in \mathbb{C},\ z \neq 0,\ z^{n+1} \neq 1,\ 1 + \omega z^n \neq 0,\ n \in \mathbb{N}\big\},$$

with $I_{\mathcal{P}}$ being the identity operator in $\mathcal{P}$ and $(h_q f)(x) = f(qx)$, $f \in \mathcal{P}$ (homothety). Therefore, our goal is to consider the following transfert operator $\mathrm{M}_{(c,\omega)}$ acting on $\mathcal{P}$ and defined by

$$\mathrm{M}_{(c,\omega)} = I_{\mathcal{P}} - c\,\tau_\omega, \quad \omega \neq 0, \quad c \in \mathbb{R} - \{0,1\}, \tag{1.1}$$

where

$$(\tau_\omega f)(x) = f(x - \omega), \quad f \in \mathcal{P},$$

(translation) and to characterize all sequences of orthogonal polynomials $\{P_n\}_{n\geq 0}$ having the Hahn property; the resulting up an affine transformation (that is to say up a composition of a homothety and a translation; see (1.4) below), is the Meixner polynomials of the first kind (see Theorem 2 below). Indeed, in Section 2, firstly we deal with the $\mathrm{M}_{(c,\omega)}$-character by presenting some characterizations of it (see Theorem 1), secondly, we establish the system verified by the elements of second-order recurrence relation for the sequences $\{P_n\}_{n\geq 0}$ and $\{\mathrm{M}_{(c,\omega)}P_n\}_{n\geq 0}$ and thirdly we solve it to deduce the desired result (Theorem 2). Moreover, the divided-difference equation fulfilled by its canonical form and the second order linear divided-difference equation satisfied by any Meixner polynomial are highlighted.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by

$$(u)_n := \langle u, x^n\rangle, \quad n \geq 0$$

the moments of u. The form u is called regular if we can associate with it a sequence of monic polynomials $\{P_n\}_{n\geq 0}$ with $\deg P_n = n,\ n \geq 0$ ((MPS) in short) [18] such that

$$\langle u, P_m P_n\rangle = r_n \delta_{n,m}, \quad n, m \geq 0; \quad r_n \neq 0, \quad n \geq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is then called orthogonal with respect to u ((MOPS) in short). In this case, the (MOPS) $\{P_n\}_{n\geq 0}$ fulfils the standard recurrence relation ((TTRR) in short) [10, 18]

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0\,, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \end{cases} \tag{1.2}$$

where

$$\beta_n = \frac{\langle u, xP_n^2\rangle}{r_n}, \quad \gamma_{n+1} = \frac{r_{n+1}}{r_n} \neq 0, \quad n \geq 0.$$

Moreover, the regular form u will be supposed normalized that is to say $(u)_0 = 1$.

For any form u, any polynomial g and $a, \omega \in \mathbb{C}\backslash\{0\}$, $b \in \mathbb{C}$, we let $\tau_b u$, $h_a u$, gu, $Du = u'$, $D_\omega u$ be the forms defined by duality [18] namely

$$\langle \tau_b u, f\rangle = \langle u, \tau_{-b} f\rangle, \quad \langle h_a u, f\rangle = \langle u, h_a f\rangle, \quad \langle gu, f\rangle = \langle u, gf\rangle,$$
$$\langle u', f\rangle = -\langle u, f'\rangle, \quad \langle D_\omega u, f\rangle = -\langle u, D_{-\omega} f\rangle$$

where

$$(\tau_{-b}f)(x) = f(x+b), \quad (h_a f)(x) = f(ax), \quad (D_{-\omega}f)(x) = \frac{f(x) - f(x-\omega)}{\omega}, \quad f \in \mathcal{P},$$

and due to the well known formulas [1, 18] we have

$$\tau_b(fu) = (\tau_b f)(\tau_b u), \quad h_a(fu) = (h_{a^{-1}} f)\,(h_a u), \quad u \in \mathcal{P}', \quad f \in \mathcal{P}. \tag{1.3}$$

Let δ_b be the Dirac mass at b defined by

$$\langle \delta_b, f\rangle = f(b), \quad b \in \mathbb{C}, \quad f \in \mathcal{P}.$$

In addition, let $\{\widehat{P}_n\}_{n\geq 0}$ be the (MPS) defined by

$$\widehat{P}_n(x) = a^{-n}P_n(ax+b), \quad n \geq 0, \quad a \neq 0, \quad b \in \mathbb{C}.$$

If $\{P_n\}_{n\geq 0}$ is a (MOPS) associated with u, then $\{\widehat{P}_n\}_{n\geq 0}$ is a (MOPS) associated with

$$\widehat{u} = \big(h_{a^{-1}} \circ \tau_{-b}\big)u$$

and fulfilling the (TTRR) in (1.2) ($\beta_n \hookleftarrow \widehat{\beta}_n$, $\gamma_{n+1} \hookleftarrow \widehat{\gamma}_{n+1}$, $n \geq 0$) with [18]

$$\widehat{\beta}_n = \frac{\beta_n - b}{a}, \quad \widehat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0. \tag{1.4}$$

Let now $\{P_n\}_{n\geq 0}$ be a (MPS) and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$ defined by

$$\langle u_n, P_m \rangle = \delta_{n,m}, \quad n, m \geq 0.$$

Let us recall some results [18].

Lemma 1 [18]. *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent*

(i) $\langle u, P_{m-1} \rangle \neq 0, \quad \langle u, P_n \rangle = 0, \quad n \geq m,$
(ii) $\exists \lambda_\nu \in \mathbb{C}\,, \quad 0 \leq \nu \leq m-1, \quad \lambda_{m-1} \neq 0,$

such that

$$u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu.$$

As a consequence,

– the dual sequence $\{\widehat{u}_n\}_{n\geq 0}$ of $\{\widehat{P}_n\}_{n\geq 0}$ is given by

$$\widehat{u}_n = a^n \big(h_{a^{-1}} \circ \tau_{-b}\big) u_n, \quad n \geq 0,$$

– when $\{P_n\}_{n\geq 0}$ be a (MOPS) then $u = u_0$. In this case, we have

$$u_n = r_n^{-1} P_n u_0, \quad n \geq 0$$

and reciprocally. Lastly, when u_0 is regular and Φ is a polynomial such that $\Phi u_0 = 0$, then $\Phi = 0$.

The monic Meixner polynomials $\{M_n(.;\alpha,c)\}_{n\geq 0}$ of the first kind are given by [10, 16]

$$M_n(x;\alpha,c) = (\alpha+1)_n \left(\frac{c}{c-1}\right)^n {}_2F_1\left(\begin{matrix} -n, -x \\ \alpha+1 \end{matrix} \,\middle|\, 1 - \frac{1}{c}\right), \quad n \geq 0,$$

they are orthogonal with respect to the discrete weight

$$\rho(x) = \frac{c^x (\alpha+1)_x}{x!}, \quad x \in \mathbb{N}$$

for $\alpha > -1$, $0 < c < 1$. Here, the Pochhammer symbol $(z)_n$ takes the form

$$(z)_0 = 1, \quad (z)_n = \prod_{k=1}^{n} (z+k-1), \quad n \geq 1,$$

and ${}_2F_1$ is the hypergeometric function defined by

$${}_2F_1\left(\begin{matrix} p, q \\ r \end{matrix} \,\middle|\, s\right) = \sum_{k=0}^{\infty} \frac{(p)_k (q)_k}{(r)_k} \frac{s^k}{k!}.$$

By describing exhaustively the $D_{-\omega}$-classical orthogonal polynomials in [1], the authors rediscover the (MOPS) of Meixner $\{M_n(.;\alpha,c)\}_{n\geq 0}$ orthogonal with respect to the D_{-1}-classical Meixner form $\mathcal{M}(\alpha,c)$ for $\alpha \neq -n-1$, $n \geq 0$, $c \in \mathbb{C} - \{0,1\}$ and the positive definite case occurring for $\alpha+1 > 0$, $c \in (0,\infty) - \{1\}$; they establish successively the (TTRR) elements, the divided-difference equation, the modified moments, the discrete representation and the second order linear divided-difference equation (see the following),

$$\begin{cases} \beta_n = \dfrac{c}{1-c}(\alpha+1) + \dfrac{1+c}{1-c}n, \quad \gamma_{n+1} = \dfrac{c}{(1-c)^2}(n+1)(n+\alpha+1), \quad n \geq 0, \\ D_{-1}\big((x+\alpha+1)\mathcal{M}(\alpha,c)\big) - \big((1-c^{-1})x+\alpha+1\big)\mathcal{M}(\alpha,c) = 0, \\ \big(\mathcal{M}(\alpha,c)\big)_n^{\phi} = \left(\dfrac{c}{1-c}\right)^n \dfrac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)}, \quad n \geq 0, \quad c \in \mathbb{C}-\{0,1\}, \quad \alpha+1 \in \mathbb{C}-(-\mathbb{N}), \\ \mathcal{M}(\alpha,c) = (1-c)^{\alpha+1}\displaystyle\sum_{k\geq 0}\dfrac{\Gamma(\alpha+1+k)}{\Gamma(\alpha+1)}\dfrac{c^{-k}}{k!}\delta_k, \quad 0<|c|<1, \quad \alpha \neq -n-1, \quad n \geq 0, \\ (x+\alpha+1)(D_{-1}\circ D_1 M_{n+1})(x;\alpha,c) + \big((1-c^{-1})x+\alpha+1\big)(D_1 M_{n+1})(x;\alpha,c) \\ \qquad\qquad -(n+1)(1-c^{-1})M_{n+1}(x;\alpha,c) = 0, \quad n \geq 0. \end{cases} \tag{1.5}$$

2. Main result

2.1. The $\mathrm{M}_{(c,\omega)}$-classical character

First of all, let $\omega \neq 0$ and $c \in \mathbb{R}-\{0,1\}$. By virtue of (1.1) we have

$$(\mathrm{M}_{(c,\omega)}f)(x) = f(x) - cf(x-\omega), \quad f \in \mathcal{P}. \tag{2.1}$$

Particularly,

$$(\mathrm{M}_{(c,\omega)}1)(x) = 1-c, \quad (\mathrm{M}_{(c,\omega)}\xi^n)(x) = (1-c)x^n + \text{lower degree terms}, \quad n \geq 1. \tag{2.2}$$

When $c=1$, $\mathrm{M}_{(1,\omega)}$ is not a transfert operator but a lowering one since $\mathrm{M}_{(1,\omega)} = \omega D_{-\omega}$.
From (1.1), we have

$$\mathrm{M}_{(c,\omega)} = I_{\mathcal{P}} - c\,\tau_\omega.$$

The transposed ${}^t\mathrm{M}_{(c,\omega)}$ of $\mathrm{M}_{(c,\omega)}$ is

$${}^t\mathrm{M}_{(c,\omega)} = I_{\mathcal{P}'} - c\,\tau_{-\omega} = \mathrm{M}_{(c,-\omega)},$$

leaving out a light abuse of notation without consequence.

Thus,

$$\langle \mathrm{M}_{(c,-\omega)}u, f\rangle = \langle u, \mathrm{M}_{(c,\omega)}f\rangle, \ u \in \mathcal{P}', \ f \in \mathcal{P}.$$

Particularly, by virtue of (2.2) we get

$$(M_{(c,-\omega)}u)_0 = 1-c, \quad (M_{(c,-\omega)}u)_n = (1-c)(u)_n - c\sum_{k=0}^{n-1}\binom{n}{k}(-\omega)^{n-k}(u)_k, \quad n \geq 1.$$

Lemma 2. *The following formulas hold*

$$\mathrm{M}_{(c,\omega)}(fg)(x) = f(x)(\mathrm{M}_{(1,\omega)}g)(x) + (\tau_\omega g)(x)(\mathrm{M}_{(c,\omega)}f)(x), \quad f,g \in \mathcal{P}, \tag{2.3}$$

$$\mathrm{M}_{(c,-\omega)}(fu) = (\tau_{-\omega}f)(\mathrm{M}_{(c,-\omega)}u) + (\mathrm{M}_{(1,-\omega)}f)u, \quad u \in \mathcal{P}', \quad f \in \mathcal{P}, \tag{2.4}$$

$$h_a \circ \mathrm{M}_{(c,\omega)} = \mathrm{M}_{(c,a^{-1}\omega)} \circ h_a \ \textit{in}\ \mathcal{P}, \quad h_a \circ \mathrm{M}_{(c,-\omega)} = \mathrm{M}_{(c,-a\omega)} \circ h_a \ \textit{in}\ \mathcal{P}', \quad a \in \mathbb{C}-\{0\}, \tag{2.5}$$

$$\tau_b \circ \mathrm{M}_{(c,\omega)} = \mathrm{M}_{(c,\omega)} \circ \tau_b \ \textit{in}\ \mathcal{P}, \quad \tau_b \circ \mathrm{M}_{(c,-\omega)} = \mathrm{M}_{(c,-\omega)} \circ \tau_b \ \textit{in}\ \mathcal{P}', \quad b \in \mathbb{C}. \tag{2.6}$$

P r o o f. The proof is straightforward since definitions and duality. □

Now consider a (MPS) $\{P_n\}_{n\geq 0}$. On account of (2.2), let us define the (MPS) $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ by

$$P_n^{[1]}(x;c,\omega) = \frac{(\mathrm{M}_{(c,\omega)}P_n)(x)}{1-c}, \quad \omega \neq 0, \quad c \in \mathbb{R} - \{0,1\}, \quad n \geq 0. \tag{2.7}$$

Denoting by $\{u_n^{[1]}(c,\omega)\}_{n\geq 0}$ the dual sequence of $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$, we have the result

Lemma 3. *The following formula holds*

$$\mathrm{M}_{(c,-\omega)}(u_n^{[1]}(c,\omega)) = (1-c)u_n, \quad n \geq 0. \tag{2.8}$$

P r o o f. Indeed, from the definition it follows

$$\langle u_n^{[1]}(c,\omega), P_m^{[1]}(x;c,\omega)\rangle = \delta_{n,m}, \quad n,m \geq 0,$$

so we have

$$\langle(\mathrm{M}_{(c,-\omega)}(u_n^{[1]}(c,\omega)), P_m\rangle = (1-c)\delta_{n,m}, \quad n,m \geq 0,$$

therefore,

$$\langle\mathrm{M}_{(c,-\omega)}(u_n^{[1]}(c,\omega)), P_m\rangle = 0, \quad m \geq n+1, \quad n \geq 0;$$
$$\langle\mathrm{M}_{(c,-\omega)}(u_n^{[1]}(c,\omega)), P_n\rangle = 1-c, \quad n \geq 0.$$

By virtue of Lemma 1, we get

$$\mathrm{M}_{(c,-\omega)}(u_n^{[1]}(c,\omega)) = \sum_{\nu=0}^{n} \lambda_{n,\nu} u_\nu, \quad n \geq 0.$$

But,

$$\langle\mathrm{M}_{(c,-\omega)}(u_n^{[1]}(c,\omega)), P_\mu\rangle = \lambda_{n,\mu}, \quad 0 \leq \mu \leq n,$$

with $\lambda_{n,\mu} = 0,\ 0 \leq \mu < n$ and $\lambda_{n,n} = 1-c$. The formula (2.8) is then established. □

Definition 1. *The (MPS) $\{P_n\}_{n\geq 0}$ is called $\mathrm{M}_{(c,\omega)}$-classical if $\{P_n\}_{n\geq 0}$ and $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ are orthogonal.*

Remark 1. When the (MPS) $\{P_n\}_{n\geq 0}$ is orthogonal, it satisfies the (TTRR) (1.2). When the (MPS) $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ is orthogonal, it satisfies the (TTRR) (1.2) with the notations ($\beta_n \hookleftarrow \beta_n^{[1]}$, $\gamma_{n+1} \hookleftarrow \gamma_{n+1}^{[1]}$, $n \geq 0$).

Theorem 1. *For any (MOPS) $\{P_n\}_{n\geq 0}$, the following assertions are equivalent.*

a) *The sequence $\{P_n\}_{n\geq 0}$ is $\mathrm{M}_{(c,\omega)}$-classical.*
b) *There exist a polynomial ϕ monic, $\deg\phi \leq 1$ and a constant $K \neq 0$ such that*

$$\mathrm{M}_{(c,-\omega)}(\phi u_0) - K^{-1}(1-c)u_0 = 0, \tag{2.9}$$
$$1 - c - K\phi'(0)\,\omega\, n \neq 0, \quad n \geq 0. \tag{2.10}$$

c) *There exist a polynomial ϕ monic,* $\deg\phi\le 1$, *a constant* $K\ne 0$ *and a sequence of complex numbers* $\{\lambda_n\}_{n\ge 0}$, $\lambda_n\ne 0$, $n\ge 0$, *such that*

$$\begin{aligned}&\big(K\phi(x)-1+c\big)(\mathrm{M}_{(c,-\omega)}\circ\mathrm{M}_{(c,\omega)}P_n)(x)\\ &+(c-1)(K\phi(x)-1)(\mathrm{M}_{(c,\omega)}P_n)(x)=\lambda_n P_n(x),\quad n\ge 0.\end{aligned}\tag{2.11}$$

P r o o f. a) $\Rightarrow$ b), a) $\Rightarrow$ c).

From (2.8) and the regularity of u_0 and $u_0^{[1]}(c,\omega)$, we have

$$\mathrm{M}_{(c,-\omega)}\big(P_n^{[1]}(.;c,\omega)u_0^{[1]}(c,\omega)\big)=\zeta_n\,P_nu_0,\quad n\ge 0,$$

with

$$\zeta_n=(1-c)\,\frac{\langle u_0^{[1]}(c,\omega),(P_n^{[1]}(.;c,\omega))^2\rangle}{\langle u_0,P_n^2\rangle},\quad n\ge 0.$$

By (2.4), we get

$$(\tau_{-\omega}P_n^{[1]}(.;c,\omega))\mathrm{M}_{(c,-\omega)}(u_0^{[1]}(c,\omega))+(\mathrm{M}_{(1,-\omega)}P_n^{[1]}(.;c,\omega))u_0^{[1]}(c,\omega)=\zeta_n\,P_nu_0,\quad n\ge 0.$$

In accordance with the definition of $\mathrm{M}_{(c,-\omega)}$, one may write

$$\mathrm{M}_{(c,-\omega)}(u_0^{[1]}(c,\omega))=u_0^{[1]}(c,\omega)-c(\tau_{-\omega}u_0^{[1]}(c,\omega)),$$

which yields

$$P_n^{[1]}(.;c,\omega)u_0^{[1]}(c,\omega)-c(\tau_{-\omega}P_n^{[1]}(.;c,\omega))(\tau_{-\omega}u_0^{[1]}(c,\omega))=\zeta_n\,P_nu_0,\quad n\ge 0.\tag{2.12}$$

Taking $n=0$ in (2.12) leads to

$$u_0^{[1]}(c,\omega)-c(\tau_{-\omega}u_0^{[1]}(c,\omega))=(1-c)u_0.\tag{2.13}$$

Injecting (2.13) in (2.12) gives

$$\big\{P_n^{[1]}(.;c,\omega)-(\tau_{-\omega}P_n^{[1]}(.;c,\omega))\big\}u_0^{[1]}(c,\omega)=\big\{\zeta_n\,P_n-(1-c)(\tau_{-\omega}P_n^{[1]}(.;c,\omega))\big\}u_0,\quad n\ge 0.\tag{2.14}$$

Now, taking $n=1$ in (2.14), we obtain

$$u_0^{[1]}(c,\omega)=K\phi(x)u_0,\tag{2.15}$$

where K be a normalization constant since ϕ monic and

$$K\phi(x)=\frac{1-c}{\omega}\Big\{(1-\frac{\gamma_1^{[1]}}{\gamma_1})x+\omega+\frac{\gamma_1^{[1]}}{\gamma_1}\beta_0-\beta_0^{[1]}\Big\}.$$

Applying the operator $\tau_{-\omega}$ to (2.15), we get

$$(\tau_{-\omega}u_0^{[1]}(c,\omega))=K(\tau_{-\omega}\phi)(x)(\tau_{-\omega}u_0).\tag{2.16}$$

Replacing (2.16) and (2.15) in (2.13) leads to the desired result (2.9). By virtue of (2.15), the formula in (2.14) becomes

$$\Big\{K\phi\Big(P_n^{[1]}(.;c,\omega)-(\tau_{-\omega}P_n^{[1]}(.;c,\omega))\Big)+(1-c)(\tau_{-\omega}P_n^{[1]}(.;c,\omega))-\zeta_n\,P_n\Big\}u_0=0,\quad n\ge 0.$$

Therefore,

$$K\phi\Big(P_n^{[1]}(.;c,\omega)-(\tau_{-\omega}P_n^{[1]}(.;c,\omega))\Big)+(1-c)(\tau_{-\omega}P_n^{[1]}(.;c,\omega))-\zeta_n\,P_n=0,\quad n\geq 0,$$

thanks to the regularity of u_0. Moreover, from (2.1) with the change $\omega\leftarrow -\omega$, we may write

$$(\tau_{-\omega}P_n^{[1]}(.;c,\omega))=c^{-1}\Big(P_n^{[1]}(.;c,\omega)-(\mathrm{M}_{(c,-\omega)}P_n^{[1]}(.;c,\omega))\Big),\quad n\geq 0.$$

Consequently, the last equation becomes

$$\begin{aligned}\big(K\phi(x)-1+c\big)(\mathrm{M}_{(c,-\omega)}\circ\mathrm{M}_{(c,\omega)}P_n)(x)+(c-1)(K\phi(x)-1)(\mathrm{M}_{(c,\omega)}P_n)(x)\\=c(1-c)\zeta_nP_n(x),\quad n\geq 0.\end{aligned}\tag{2.17}$$

Writing into (2.17)

$$\begin{cases}\phi(x)=\phi'(0)x+\phi(0),\\ (\mathrm{M}_{(c,\omega)}P_n)(x)=P_n(x)-cP_n(x-\omega),\\ (\mathrm{M}_{(c,-\omega)}\circ\mathrm{M}_{(c,\omega)}P_n)(x)=(1+c^2)P_n(x)-c\big(P_n(x-\omega)+P_n(x+\omega)\big),\\ P_n(x)=\displaystyle\sum_{k=0}^{n}a_{n,k}x^k,\quad a_{n,n}=1,\quad n\geq 0,\end{cases}$$

and by comparing the degrees we obtain

$$1-c-K\phi'(0)\,\omega\,n=\zeta_n\neq 0,\quad n\geq 0.$$

Hence (2.10) and a) $\Rightarrow$ b).
Finally, (2.17) is (2.11) with $\lambda_n=c(1-c)\zeta_n\neq 0$, $n\geq 0$. We have also proved that a) $\Rightarrow$ c).

b) $\Rightarrow$ a) Let us suppose that there exist a polynomial ϕ monic, $\deg\phi\leq 1$ and a constant $K\neq 0$ such that (2.9)–(2.10) are valid. From (2.9), we have

$$0=\langle\mathrm{M}_{(c,-\omega)}(\phi u_0)-K^{-1}(1-c)u_0,1\rangle=(1-c)\big(\langle u_0,\phi\rangle-K^{-1}\big).$$

Thus,

$$K^{-1}=\langle u_0,\phi\rangle=\phi'(0)\beta_0+\phi(0)=\phi(\beta_0).$$

Necessarily, $\phi(\beta_0)\neq 0$. Let $v=K\phi u_0$. We are going to prove that the (MPS) $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ is orthogonal with respect to v. We have successively

$$\langle v,P_0^{[1]}(.;c,\omega)\rangle=K\langle u_0,\phi\rangle=1,\tag{2.18}$$

for all $n\geq 1$,

$$\begin{aligned}\langle v,P_n^{[1]}(.;c,\omega)\rangle&=\frac{K}{1-c}\langle\phi u_0,\mathrm{M}_{(c,\omega)}P_n\rangle=\frac{K}{1-c}\langle\mathrm{M}_{(c,-\omega)}(\phi u_0),P_n\rangle\\&\underset{(2.9)}{=}\frac{K}{1-c}\langle K^{-1}(1-c)u_0,P_n\rangle=0,\end{aligned}$$

and for $m \geq 1,\ n \geq 0$,

$$\langle v, x^m P_n^{[1]}(.;c,\omega)\rangle = \frac{K}{1-c}\,\langle \phi u_0, x^m (P_n(x) - cP_n(x-\omega))\rangle$$
$$= \frac{K}{1-c}\,\langle \phi u_0, x^m P_n(x)\rangle - \frac{Kc}{1-c}\,\langle \phi u_0, \tau_\omega\big((\xi+\omega)^m P_n(\xi)\big)(x)\rangle$$
$$= \frac{K}{1-c}\,\langle \phi u_0, x^m P_n(x)\rangle - \frac{K}{1-c}\,\langle c\tau_{-\omega}(\phi u_0), (x+\omega)^m P_n(x)\rangle$$
$$\underset{c\tau_{-\omega}(\phi u_0)=(\phi-K^{-1}(1-c))u_0}{=} \frac{K}{1-c}\,\langle \phi u_0, (x^m - (x+\omega)^m)P_n(x)\rangle + \langle u_0, (x+\omega)^m P_n(x)\rangle,$$

or equivalently, for $m \geq 1,\ n \geq 0$,

$$\langle v, x^m P_n^{[1]}(.;c,\omega)\rangle = -\frac{K\phi'(0)}{1-c}\sum_{k=1}^{m}\binom{m}{k-1}\omega^{m-k+1}\langle u_0, x^k P_n(x)\rangle$$
$$-\frac{K\phi(0)}{1-c}\sum_{k=0}^{m-1}\binom{m}{k}\omega^{m-k}\langle u_0, x^k P_n(x)\rangle + \sum_{k=0}^{m}\binom{m}{k}\omega^{m-k}\langle u_0, x^k P_n(x)\rangle$$

from which thanks to the orthogonality of $\{P_n\}_{n\geq 0}$ and (2.10) we get

$$\begin{cases} \langle v, x^m P_n^{[1]}(.;c,\omega)\rangle = 0, \quad 1 \leq m \leq n-1, \quad n \geq 2, \\ \langle v, x^n P_n^{[1]}(.;c,\omega)\rangle = \left(1 - \dfrac{K\phi'(0)}{1-c}\,n\,\omega\right)\langle u_0, P_n^2\rangle \neq 0, \quad n \geq 1. \end{cases} \tag{2.19}$$

By the identities in (2.18)–(2.19), we see that $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ is orthogonal with respect to v. We then obtain the desired result.

c) $\Rightarrow$ b) Comparing the degrees in (2.11), we can deduce (2.10). Making $n = 0$ into (2.11), we obtain

$$\lambda_0 = c(1-c)^2. \tag{2.20}$$

Moreover, from definitions, (2.11) may be written as

$$\phi\big((\mathrm{M}_{(c,\omega)}P_n) - (\tau_{-\omega}\circ \mathrm{M}_{(c,\omega)}P_n)\big) + K^{-1}(1-c)(\tau_{-\omega}\circ \mathrm{M}_{(c,\omega)}P_n) = c^{-1}K^{-1}\lambda_n P_n, \quad n \geq 0,$$

then,

$$\langle u_0, \phi\big((\mathrm{M}_{(c,\omega)}P_n) - (\tau_{-\omega}\circ \mathrm{M}_{(c,\omega)}P_n)\big) + K^{-1}(1-c)(\tau_{-\omega}\circ \mathrm{M}_{(c,\omega)}P_n)\rangle = c^{-1}K^{-1}\lambda_n\langle u_0, P_n\rangle, \quad n \geq 0.$$

Equivalently,

$$\langle \mathrm{M}_{(c,-\omega)}(\phi u_0) - (\mathrm{M}_{(c,-\omega)}\circ\tau_\omega)(\phi u_0) + K^{-1}(1-c)(\mathrm{M}_{(c,-\omega)}\circ\tau_\omega u_0), P_n\rangle = c^{-1}K^{-1}\lambda_n\langle u_0, P_n\rangle, \quad n \geq 0.$$

By virtue of Lemma 1 and (2.20), we get

$$\mathrm{M}_{(c,-\omega)}(\phi u_0) - (\mathrm{M}_{(c,-\omega)}\circ\tau_\omega)(\phi u_0) + K^{-1}(1-c)(\mathrm{M}_{(c,-\omega)}\circ\tau_\omega u_0) - K^{-1}(1-c)^2 u_0 = 0.$$

A similar expression is

$$\begin{aligned} \mathrm{M}_{(c,-\omega)}(\phi u_0) - K^{-1}(1-c)u_0 = (\mathrm{M}_{(c,-\omega)}\circ\tau_\omega)(\phi u_0) \\ -K^{-1}(1-c)(\mathrm{M}_{(c,-\omega)}\circ\tau_\omega u_0) - K^{-1}(1-c)cu_0. \end{aligned} \tag{2.21}$$

But, by (2.6) and definition of the operator $(\mathrm{M}_{(c,-\omega)}$, we have for the right side of (2.21),

$$\begin{aligned}(\mathrm{M}_{(c,-\omega)} \circ \tau_\omega)(\phi u_0) - K^{-1}(1-c)(\mathrm{M}_{(c,-\omega)} \circ \tau_\omega u_0) - K^{-1}(1-c)cu_0 \\ = \tau_\omega\big(\mathrm{M}_{(c,-\omega)}(\phi u_0)\big) - K^{-1}(1-c)\tau_\omega\big((\mathrm{M}_{(c,-\omega)}u_0) + c\tau_{-\omega}u_0\big) \\ = \tau_\omega\big(\mathrm{M}_{(c,-\omega)}(\phi u_0) - K^{-1}(1-c)u_0\big).\end{aligned}$$

Therefore, (2.21) becomes

$$\mathrm{M}_{(1,\omega)}\big(\mathrm{M}_{(c,-\omega)}(\phi u_0) - K^{-1}(1-c)u_0\big) = 0.$$

From the fact that the operator $\mathrm{M}_{(1,\omega)}$ is injective in $\mathcal{P}'$ we get (2.9). □

Lemma 4. *If u_0 satisfies (2.9), then $\widehat{u}_0 = \big(h_{a^{-1}} \circ \tau_{-b}\big)u_0$ fulfills the equation*

$$\mathrm{M}_{(c,-\omega a^{-1})}\big(a^{-\deg\phi}\phi(ax+b)\widehat{u}_0\big) - a^{-\deg\phi}K^{-1}(1-c)\widehat{u}_0 = 0.$$

P r o o f. We need the following formulas which are easy to prove from (1.3)

$$g(\tau_b u) = \tau_b\big((\tau_{-b}g)u\big); \quad g(h_a u) = h_a\big((h_a g)u\big), \quad g \in \mathcal{P}, \quad u \in \mathcal{P}'. \tag{2.22}$$

Now, with $u_0 = \big(\tau_b \circ (h_a\big)\widehat{u}_0$, we have

$$-K^{-1}(1-c)u_0 = \big(\tau_b \circ (h_a\big)\big(-K^{-1}(1-c)\widehat{u}_0\big).$$

Further,

$$\begin{aligned}\mathrm{M}_{(c,-\omega)}(\phi u_0) = \mathrm{M}_{(c,-\omega)}\big(\phi(\tau_b(h_a\widehat{u}_0))\big) \underset{(2.22)}{=} \mathrm{M}_{(c,-\omega)}\big(\tau_b((\tau_{-b}\phi)(h_a\widehat{u}_0))\big) \\ \underset{(2.6)}{=} (\tau_b \circ \mathrm{M}_{(c,-\omega)})\big((\tau_{-b}\phi)(h_a\widehat{u}_0)\big) \underset{(2.22)}{=} (\tau_b \circ \mathrm{M}_{(c,-\omega)})\big(h_a((h_a \circ \tau_{-b}\phi)\widehat{u}_0)\big) \\ \underset{(2.5)}{=} \big(\tau_b \circ h_a \circ \mathrm{M}_{(c,-\omega a^{-1})}\big)\big((h_a \circ \tau_{-b}\phi)\widehat{u}_0\big).\end{aligned}$$

Consequently, equation (2.9) becomes

$$\tau_b \circ h_a\Big(\mathrm{M}_{(c,-\omega a^{-1})}\big(\phi(ax+b)\big)\widehat{u}_0\big) - K^{-1}(1-c)\widehat{u}_0\Big) = 0.$$

This leads to the desired equality. □

2.2. Determination of all $\mathrm{M}_{(c,\omega)}$-classical (MOPS)s

Lemma 5. *Let $\{P_n\}_{n\geq 0}$ be a $\mathrm{M}_{(c,\omega)}$-classical (MOPS). The following equality holds*

$$\frac{c}{1-c}\,\omega\, P_{n+1}(x-\omega) = (\beta_{n+1} - \beta^{[1]}_{n+1})P^{[1]}_{n+1}(x;c,\omega) + (\gamma_{n+1} - \gamma^{[1]}_{n+1})P^{[1]}_n(x;c,\omega), \quad n \geq 0. \tag{2.23}$$

P r o o f. From the (TTRR) (1.2) we have

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0. \tag{2.24}$$

Applying the transfert operator to (2.24), using (2.3) and (2.7) we obtain

$$\begin{aligned}(1-c)P^{[1]}_{n+2}(x;c,\omega) = (1-c)(x-\beta_{n+1})P^{[1]}_{n+1}(x;c,\omega) + c\,\omega P_{n+1}(x-\omega) \\ -\gamma_{n+1}(1-c)P^{[1]}_n(x;c,\omega), \quad n \geq 0.\end{aligned} \tag{2.25}$$

But from the (TTRR) of $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$, one may write

$$xP_n^{[1]}(.;c,\omega) = P_{n+2}^{[1]}(.;c,\omega) + \beta_{n+1}^{[1]}P_{n+1}^{[1]}(.;c,\omega) + \gamma_{n+1}^{[1]}P_n^{[1]}(.;c,\omega), \quad n \geq 0. \tag{2.26}$$

Now, injecting (2.26) in (2.25) leads to the desired result (2.23). □

Proposition 1. *The coefficients β_n, γ_{n+1}, $\beta_n^{[1]}$, $\gamma_{n+1}^{[1]}$ satisfy the following system*

$$\beta_n - \beta_n^{[1]} = \omega\,\frac{c}{1-c}, \quad n \geq 0, \tag{2.27}$$

$$\gamma_{n+1} - \gamma_{n+1}^{[1]} = -\omega^2\,\frac{c}{(1-c)^2}\,(n+1), \quad n \geq 0, \tag{2.28}$$

$$\beta_{n+1} - \beta_n = \omega\,\frac{1+c}{1-c}, \quad n \geq 0, \tag{2.29}$$

$$\gamma_n^{[1]} = \frac{n}{n+1}\,\gamma_{n+1}, \quad n \geq 1. \tag{2.30}$$

P r o o f. Firstly, the higher degree test in (2.23) yields

$$\beta_{n+1} - \beta_{n+1}^{[1]} = \omega\,\frac{c}{1-c},\ n \geq 0. \tag{2.31}$$

Secondly, $n = 0$ in (2.23) gives

$$\gamma_1 - \gamma_1^{[1]} = -\omega\,\frac{c}{1-c}\,(\omega + \beta_0 - \beta_0^{[1]}). \tag{2.32}$$

Thirdly, applying the transfert operator $\mathrm{M}_{(c,\omega)}$ to

$$P_1(x) = x - \beta_0$$

and by virtue of (2.7) and (2.31)–(2.32) we get (2.27) and

$$\gamma_1 - \gamma_1^{[1]} = -\omega^2\,\frac{c}{(1-c)^2}. \tag{2.33}$$

Thanks to (2.27), the formula in (2.23) becomes

$$c\,\omega\,P_{n+1}(x-\omega) = c\,\omega\,P_{n+1}^{[1]}(x;c,\omega) + (1-c)(\gamma_{n+1} - \gamma_{n+1}^{[1]})P_n^{[1]}(x;c,\omega), \quad n \geq 0. \tag{2.34}$$

Moreover, multiplication of (2.24) by $c\omega$ with the change $x \leftarrow x - \omega$ yields

$$c\,\omega P_{n+2}(x-\omega) = (x - \omega - \beta_{n+1})c\,\omega P_{n+1}(x-\omega) - \gamma_{n+1}c\,\omega P_n(x-\omega), \quad n \geq 0. \tag{2.35}$$

Replacing (2.34) for the index n, $n+1$, $n+2$ in (2.35), using (2.26) for the index n, $n+1$, the formula in (2.27) and the fact that $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ is a basis , we obtain successively

$$(\gamma_{n+2}^{[1]} - \gamma_{n+2}) - (\gamma_{n+1}^{[1]} - \gamma_{n+1}) = \omega^2\frac{c}{(1-c)^2}, \quad n \geq 0, \tag{2.36}$$

$$(\gamma_{n+1}^{[1]} - \gamma_{n+1})\Big\{(1-c)(\beta_n - \beta_{n+1}) + (1+c)\omega\Big\} = 0, \tag{2.37}$$

$$(\gamma_{n+1}^{[1]} - \gamma_{n+1})\gamma_n^{[1]} = (\gamma_n^{[1]} - \gamma_n)\gamma_{n+1}, \quad n \geq 1. \tag{2.38}$$

Summing on (2.36) and taking into account (2.33) lead to (2.28) and (2.37) yields (2.29). Lastly, (2.30) is a direct consequence of (2.38) and (2.28).

Now, we are able to solve the system (2.27)–(2.30).
Summing on (2.29) leads to

$$\beta_n = \beta_0 + \omega\,\frac{1+c}{1-c}\,n, \quad n \geq 0. \tag{2.39}$$

Injecting (2.39) in (2.27) yields

$$\beta_n^{[1]} = \beta_0 - \omega\,\frac{c}{1-c} + \omega\,\frac{1+c}{1-c}\,n, \quad n \geq 0. \tag{2.40}$$

Also, injecting (2.30) in (2.28) gives

$$\frac{\gamma_{n+2}}{n+2} - \frac{\gamma_{n+1}}{n+1} = \omega^2 \frac{c}{(1-c)^2}, \quad n \geq 0.$$

Summing the previous equality leads to

$$\gamma_{n+1} = (n+1)\left(\gamma_1 + \omega^2 \frac{c}{(1-c)^2}\,n\right), \quad n \geq 0. \tag{2.41}$$

After replacing (2.41) in (2.30) we deduce the following

$$\gamma_{n+1}^{[1]} = (n+1)\left(\gamma_1 + \omega^2 \frac{c}{(1-c)^2}\,(n+1)\right), \quad n \geq 0. \tag{2.42}$$

□

Corollary 1. *Let $\{P_n\}_{n\geq 0}$ be a $\mathrm{M}_{(c,\omega)}$-classical (MOPS). The following statements hold.*

1) *The recurrence elements of $\{P_n\}_{n\geq 0}$ are*

$$\begin{cases} \beta_n = \omega\left(\dfrac{\beta_0}{\omega} + \dfrac{1+c}{1-c}\,n\right), \quad n \geq 0, \\ \gamma_{n+1} = \omega^2 \dfrac{c}{(1-c)^2}(n+1)\left(n + \dfrac{(1-c)^2}{c}\,\dfrac{\gamma_1}{\omega^2}\right), \quad n \geq 0. \end{cases} \tag{2.43}$$

2) *The recurrence elements of $\{P_n^{[1]}(.;c,\omega)\}_{n\geq 0}$ are*

$$\begin{cases} \beta_n^{[1]} = \omega\left(\dfrac{\beta_0}{\omega} - \dfrac{c}{1-c} + \dfrac{1+c}{1-c}\,n\right), \quad n \geq 0, \\ \gamma_{n+1}^{[1]} = \omega^2 \dfrac{c}{(1-c)^2}(n+1)\left(n + 1 + \dfrac{(1-c)^2}{c}\,\dfrac{\gamma_1}{\omega^2}\right), \quad n \geq 0. \end{cases} \tag{2.44}$$

P r o o f. The formula (2.43) is a consequence of (2.39) and (2.41). Also, (2.44) is a direct result from (2.40) and (2.42).

Theorem 2. *Up to an affine transformation, the only $\mathrm{M}_{(c,1)}$-classical (MOPS) is the Meixner's one of the first kind.*

P r o o f. The classification of the canonical situations depends on the fact that $\beta_0 \neq 0$ or $\beta_0 = 0$.

$\beta_0 \neq 0$. For (2.43)–(2.44), put

$$\omega\,\beta_0 = (1-c)\gamma_1$$

and

$$\frac{(1-c)^2}{c}\,\frac{\gamma_1}{\omega^2} = \alpha + 1.$$

Then,

$$\frac{\beta_0}{\omega} = \frac{c}{1-c}(\alpha+1).$$

Now, for (2.43), choosing $a = \omega,\ b = 0$ in (1.4) and thanks to (2.5)–(2.6) this yields

$$\begin{cases} \widehat{\beta}_n = \dfrac{c}{1-c}(\alpha+1) + \dfrac{1+c}{1-c}n, \quad n \geq 0, \\ \widehat{\gamma}_{n+1} = \dfrac{c}{(1-c)^2}(n+1)(n+\alpha+1), \quad n \geq 0. \end{cases}$$

Therefore (see (1.5)),

$$\widehat{P}_n = M_n(.;\alpha,c), \quad n \geq 0,$$

with $\alpha \neq -n-1,\ n \geq 0$. Next, for (2.44), choosing

$$a = \omega, \quad b = -\frac{2\omega\, c}{1-c}$$

in (1.4) and thanks to (2.5)–(2.6) this yields

$$\begin{cases} \widehat{\beta}^{[1]}_n = \dfrac{c}{1-c}(\alpha+2) + \dfrac{1+c}{1-c}n, \quad n \geq 0, \\ \widehat{\gamma}^{[1]}_{n+1} = \dfrac{c}{(1-c)^2}(n+1)(n+\alpha+2), \quad n \geq 0. \end{cases}$$

Thus,

$$\widehat{P}^{[1]}_n = M_n(.;\alpha+1,c), \quad n \geq 0,$$

with $\alpha \neq -n-2,\ n \geq 0$.

$\beta_0 = 0$. In this case, (2.43)–(2.44) become successively,

$$\begin{cases} \beta_n = \omega\dfrac{1+c}{1-c}n, \quad n \geq 0, \\ \gamma_{n+1} = \omega^2\dfrac{c}{(1-c)^2}(n+1)\left(n + \dfrac{(1-c)^2}{c}\dfrac{\gamma_1}{\omega^2}\right), \quad n \geq 0, \end{cases} \tag{2.45}$$

$$\begin{cases} \beta^{[1]}_n = \omega\left(-\dfrac{c}{1-c} + \dfrac{1+c}{1-c}n\right), \quad n \geq 0, \\ \gamma^{[1]}_{n+1} = \omega^2\dfrac{c}{(1-c)^2}(n+1)\left(n+1 + \dfrac{(1-c)^2}{c}\dfrac{\gamma_1}{\omega^2}\right), \quad n \geq 0. \end{cases} \tag{2.46}$$

For (2.45), putting

$$\frac{(1-c)^2}{c}\frac{\gamma_1}{\omega^2} = \alpha+1,$$

and choosing in (1.4)

$$a = \omega, \quad b = -\frac{\omega\, c}{1-c}(\alpha+1),$$

we obtain

$$\begin{cases} \widehat{\beta}_n = \dfrac{c}{1-c}(\alpha+1) + \dfrac{1+c}{1-c}n, \quad n \geq 0, \\ \widehat{\gamma}_{n+1} = \dfrac{c}{(1-c)^2}(n+1)(n+\alpha+1), \quad n \geq 0. \end{cases}$$

Consequently,

$$\widehat{P}_n = M_n(.;\alpha,c), \quad n \geq 0,$$

with $\alpha \neq -n-1,\ n \geq 0$. For (2.46), putting

$$\frac{(1-c)^2}{c}\,\frac{\gamma_1}{\omega^2} = \alpha + 1$$

and choosing in (1.4)

$$a = \omega, \quad b = -\frac{\omega\, c}{1-c}(\alpha+3),$$

we get

$$\begin{cases} \widehat{\beta}_n^{[1]} = \dfrac{c}{1-c}(\alpha+2) + \dfrac{1+c}{1-c}\,n, \quad n \geq 0, \\ \widehat{\gamma}_{n+1}^{[1]} = \dfrac{c}{(1-c)^2}\,(n+1)(n+\alpha+2), \quad n \geq 0. \end{cases}$$

Equivalently,

$$\widehat{P}_n^{[1]} = M_n(.;\alpha+1,c), \quad n \geq 0,$$

with $\alpha \neq -n-2,\ n \geq 0$.

The theorem is then proved. □

Remark 2. On account of Theorem 1, Theorem 2 and after some easy calculations we get for the divided-difference equation (2.9) fulfilled by the Meixner form $\mathcal{M}(\alpha,c)$,

$$\mathrm{M}_{(c,-1)}\Big(\Big(x - \frac{1+c}{1-c}\,(\alpha+1)\Big)\mathcal{M}(\alpha,c)\Big) + (\alpha+1)\mathcal{M}(\alpha,c) = 0,$$

and also for the second order linear divided-difference equation (2.11) satisfied by any Meixner polynomial $M_n(.;\alpha,c)$, for all $n \geq 0$,

$$\Big(-\frac{1-c}{\alpha+1}x + 2c\Big)(\mathrm{M}_{(c,-1)} \circ \mathrm{M}_{(c,1)} M_n)(x;\alpha,c) + (1-c)\Big(\frac{1-c}{\alpha+1}x - c\Big)(\mathrm{M}_{(c,1)} M_n)(x;\alpha,c)$$
$$= c(1-c)^2\,\frac{n+\alpha+1}{\alpha+1} M_n(x;\alpha,c).$$

Acknowledgements

The authors thank the anonymous referees for their careful reading of the manuscript and corrections.

REFERENCES

1. Abdelkarim F., Maroni P. The D_ω-classical orthogonal polynomials. *Results. Math.*, 1997. Vol. 32, No. 1–2. P. 1–28. DOI: 10.1007/BF03322520
2. Aloui B. Characterization of Laguerre polynomials as orthogonal polynomials connected by the Laguerre degree raising operator. *Ramanujan J.*, 2018. Vol. 45, No. 2. P. 475–481. DOI: 10.1007/s11139-017-9901-x
3. Aloui B. Chebyshev polynomials of the second kind via raising operator preserving the orthogonality. *Period. Math. Hung.*, 2018. Vol. 76, No. 1. P. 126–132. DOI: 10.1007/s10998-017-0219-7
4. Aloui B., Chammam W. Classical orthogonal polynomials and some new properties for their centroids of zeroes. *Anal. Math.*, 2020. Vol. 46, No. 1. P. 13–23. DOI: 10.1007/s10476-020-0012-3
5. Aloui B., Souissi J. Characterization of q-Dunkl-classical symmetric orthogonal q-polynomials. *Ramanujan J.*, 2021. Vol. 57, No. 4. P. 1355–1365. DOI: 10.1007/s11139-021-00425-8

6. Ben Salah I., Ghressi A., Khériji L. A characterization of symmetric T_μ-classical monic orthogonal polynomials by a structure relation. *Integral Transforms Spec. Funct.*, 2014. Vol. 25, No. 6. P. 423–432. DOI: 10.1080/10652469.2013.870339
7. Bouanani A., Khériji L., Ihsen Tounsi M. Characterization of q-Dunkl Appell symmetric orthogonal q-polynomials. *Expo. Math.*, 2010. Vol. 28, No. 4. P. 325–336. DOI: 10.1016/j.exmath.2010.03.003
8. Bouras B., Habbachi Y., Marcellán F. Rodrigues formula and recurrence coefficients for non-symmetric Dunkl-classical orthogonal polynomials. *Ramanujan J.*, 2021. Vol. 56, No. 2. P. 451–466. DOI: 10.1007/s11139-021-00419-6
9. Chaggara H. Operational rules and a generalized Hermite polynomials. *J. Math. Anal. Appl.*, 2007. Vol. 332, No. 1. P. 11–21. DOI: 10.1016/j.jmaa.2006.09.068
10. Chihara T. S. *An Introduction to Orthogonal Polynomials.* New York: Gordon and Breach, 1978. 249 p.
11. Ghressi A., Khériji L. A new characterization of the generalized Hermite linear form. *Bull. Belg. Math. Soc. Simon Stevin.*, 2008. Vol. 15, No. 3. P. 561–567. DOI: 10.36045/bbms/1222783100
12. Ghressi A., Khériji L. Integral and discrete representations of certain Brenke type orthogonal q-polynomials. *Ramanujan J.*, 2013. Vol. 30, No. 2. P. 163–171. DOI: 10.1007/s11139-012-9421-7
13. Habbachi Y. A new characterization of symmetric Dunkl and q-Dunkl orthogonal polynomials. Ural Math. J., 2023. Vol. 9, No. 2. P. 109–120. DOI: 10.15826/umj.2023.2.009
14. Khériji L., Maroni P. The H_q-classical orthogonal polynomials. *Acta. Appl. Math.*, 2002. Vol. 71, No. 1. P. 49–115. DOI: 10.1023/A:1014597619994
15. Khériji L., Maroni P. On the natural q-analogues of the classical orthogonal polynomials. *Eurasian Math. J.*, 2013. Vol. 4, No. 2. P. 82–103.
16. Koekoek R., Swarttouw R. F. *The Askey-Scheme of Hypergeometric Orthogonal Polynomials and its q-Analogue.* Rep. Fac. Tech. Math. Inf., Delft University of Technology, The Netherlands. Report 98–17. Delft: TU Delft, 1998. 168 p.
17. Koornwinder T. H. Lowering and raising operators for some special orthogonal polynomials. In: *Jack, Hall-Littlewood and Macdonald Polynomials. Contemp. Math., vol. 417.* Providence, RI: AMS, 2006. P. 227–238. DOI: 10.1090/conm/417/07924
18. Maroni P. Une théorie algébrique des polynômes orthogonaux. Applications aux polynômes orthogonaux semi-classiques. In: *Orthogonal Polynomials and their Applications, C. Brezinski et al. (eds.), IMACS Ann. Comput. Appl. Math., vol. 9*, 1991. P. 95–130. (in French)
19. Maroni P. Variations around classical orthogonal polynomials. Connected problems. *J. Comput. Appl. Math.*, 1993, Vol. 48, No. 1–2. P. 133–155. DOI: 10.1016/0377-0427(93)90319-7
20. Maroni P., Mejri M. The $I_{(q,\omega)}$-classical orthogonal polynomials. *Appl. Numer. Math.*, 2002. Vol. 43, No. 4. P. 423–458. DOI: 10.1016/S0168-9274(01)00180-5
21. Srivastava H. M., Ben Cheikh Y. Orthogonality of some polynomial sets via quasi-monomiality. *Appl. Math. Comput.*, 2003. Vol. 141, No. 2–3. P. 415–425. DOI: 10.1016/S0096-3003(02)00961-X

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 18–27

DOI: 10.15826/umj.2024.1.002

AN EXPLICIT ESTIMATE FOR APPROXIMATE SOLUTIONS OF ODES BASED ON THE TAYLOR FORMULA

Abdulla Kh. Abdullayev

V.I. Romanovskiy Institute of Mathematics of Uzbekistan Academy of Sciences,
9 University Str., 100174 Tashkent, Uzbekistan

abduganiax@mail.ru

Abdulla A. Azamov

V.I. Romanovskiy Institute of Mathematics of Uzbekistan Academy of Sciences,
9 University Str., 100174 Tashkent, Uzbekistan

abdulla.azamov@gmail.com

Marks B. Ruziboev

Kimyo International University,
156 Shota Rustaveli Str., 100121 Tashkent, Uzbekistan

marx.ruziboev@gmail.com

Abstract: In this paper, we consider a third-order explicit scheme based on Taylor's formula to obtain an approximate solution for the Cauchy problem of systems of ODEs. We prove an estimate for the accuracy of the approximate solution with an explicit constant that depends only on the right-hand side of the equation and the domain of the solution.

Keywords: Dynamical systems, Cauchy problem, Approximate solution, Taylor formula, Accuracy of approximate solution, Level of accuracy, Error term.

1. Introduction

It is needless to note the importance of estimating accuracy for approximate solutions of ODEs. Here we consider the problem

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{1.1}$$

where $x \in D \subset \mathbb{R}^d$ and D is a convex domain. In what follows, we assume that the function $f : D \to \mathbb{R}^d$ is three times differentiable with continuous derivatives in D. In practice, establishing the highest possible accuracy of an approximate solution is one of the key problems. Thus, the efficiency of an approximate solution is determined by its accuracy. Let $x_*(t)$ be a solution of (1.1) in the interval $0 \leq t \leq T$ for some $T > 0$, and let $\widehat{x}(t)$ be its approximate solution (obtained by some scheme) on the same interval. The accuracy of the scheme is expressed by an inequality of the form

$$\sup_{0 \leq t \leq T} |x_*(t) - \widehat{x}(t)| \leq Ce^{LT} h^s, \tag{1.2}$$

where L is the Lipschitz constant of the function f, h is the mesh size, and s is the order of accuracy of the method. Approximate solution schemes can be implicit and explicit. In this paper, we are concerned with explicit methods. Frequently used one-step approximate solution schemes can be divided into two groups:

(1) Schemes based on Taylor's expansion of the solution;
(2) Runge–Kutta-type methods.

Schemes based on Taylor's expansion are easy to implement, but experts prefer Runge–Kutta-type methods. This preference is caused by the fact that the error estimates for schemes based on Taylor's formula contain derivatives of the function f, which can be challenging to estimate. However, with the development of computer algebra, computations of derivatives of a rather wide range of functions can be automated [7, 12, 13]. Therefore, schemes based on Taylor's formula can be implemented without any extra hurdles, due to the simplicity of implementation.

On the other hand, since the 1960s, computer simulations have been used extensively to study dynamical systems described by nonlinear systems of ordinary differential equations (ODEs). Recently, with the rise of computational power and computers being widely available, computer-assisted proofs have come into play. These proofs, however, require verification of the accuracy of the scheme, i.e., proof of inequality (1.2) with explicit constants.

Estimates for the accuracy of approximate solutions to ODEs are studied extensively in literature [2, 4, 6]. For example, the monographs [10, 14, 16] contain estimates for Runge–Kutta methods. Part III of the well-known monograph [3] is devoted to approximate solutions of ODEs. Schemes based on Taylor's formula are briefly discussed in the first section from a methodological viewpoint. But no estimates are provided. It is surprising that, among the vast amount of literature devoted to the approximate solutions of ODEs, we did not find estimates with explicit constants. Most estimates give the order of approximation, which is insufficient if we want to use approximate solutions in the proofs. Except [10], which, citing [5], gives an inequality for the Runge–Kutta method. In [15, Sect. 2, Part II], the authors mention a scheme based on Taylor's formula, but do not consider the problem of accuracy estimates. Some authors claim that if one takes the first n terms of Taylor's expansion, then the error term will be of the form

$$\max_{0\le n\le N} |x_n - y(x_n)| = O(h^n).$$

But they neither provide proof nor speak about constants involved in $O(h^n)$.

In [7], which is one of the most comprehensive monographs on approximate solutions of ODEs, the authors claim that the difference between exact and approximate solutions is estimated by the remainder term of Taylor's expansion and in just one step it will be $O(h^{p+1})$ [8, Sect. 318, p. 180]; no further details are given. In [17, 18], the problem of estimating the error is investigated for methods of approximation of the integral

$$x(t+h) = x(t) + \int_t^{t+h} f[x(t)]dt.$$

Paper [8] provides explicit estimates for the approximate computation of this integral. This is equivalent to considering the first term of Taylor's expansion, which provides the first-order Taylor approximation scheme. The author considers multi-step approximate solution schemes but does not give inequalities of the form (1.2).

It turns out that schemes based on Taylor's formula are more convenient than schemes based on Runge–Kutta methods for obtaining explicit constants in (1.2). For example, in [1], an estimate of type (1.2) is obtained for a second-order scheme based on Taylor's formula. Also, they are easier to implement in numerical approximations than Runge–Kutta-type methods. Keeping in mind an application of the approximate solution schemes in computer-aided proofs, in the present paper, we consider a third-order scheme based on Taylor's formula for Cauchy problem (1.1) and give an explicit constant C, for which inequality (1.2) holds with $s = 3$.

2. Approximate solution schemes based on Taylor's formula

For a continuously differentiable function $f(x)$, the initial value problem (1.1) has a unique solution; however, it is challenging to estimate the interval where the solution exists [11]. On the other hand, for approximation schemes, we need to know the existence of the solution. Therefore, our first standing assumption is the following.

Assumption A. *Fix $T > 0$. The solution $x_*(t)$ to the Cauchy problem (1.1) is defined on the interval $[0, T]$.*

Usually, when considering approximate solutions, one tries to find values of the approximate solution $x_*(t)$ on a mesh $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$. Here, to simplify the exposition, without loss of generality, we consider the uniform mesh $t_n = nh$, where $h = T/N$, $N \in \mathbb{N}^+$.

We start with the Taylor expansion of the exact solution with accuracy $O(h^3)$:

$$x_*(t+nh) = x_*(t) + h\dot{x}_*(t) + \frac{h^2}{2}\ddot{x}_*(t) + \frac{h^3}{6}\dddot{x}_*(t) + R_4(t,h).$$

By definition, we have

$$\begin{aligned}\dot{x}_*(t) &= f[x_*(t)],\\ \ddot{x}_*(t) &= f'[x_*(t)]f[x_*(t)],\\ \dddot{x}_*(t) &= f''[x_*(t)]f[x_*(t)]f[x_*(t)] + f'[x_*(t)]f'[x_*(t)]f[x_*(t)].\end{aligned}$$

We also need the fourth derivative $x^{IV}(t)$. To shorten the notation, we interpret derivatives of the function f as operators acting on $x_*(t)$ and write

$$\begin{aligned}\dot{x}_*(t) &= f[x_*(t)],\\ \ddot{x}_*(t) &= (f'f)[x_*(t)],\\ \dddot{x}_*(t) &= (f''ff + f'f'f)[x_*(t)].\end{aligned}$$

In particular, on the uniform mesh,

$$\begin{aligned}x_*[(n+1)h] &= x_*(nh) + hf[x_*(nh)] + \frac{h^2}{2}(f'f)[x_*(nh)]\\ &+ \frac{h^3}{6}(f''ff + f'f'f)[x_*(nh)] + R_4(nh,h), \quad n = 0,1,2,\dots,N-1.\end{aligned}$$

By neglecting the remainder term, we obtain the following recurrent formula for the approximate solution:

$$x_{n+1} = hf(x_n) + \frac{h^2}{2}(f'f)(x_n) + \frac{h^3}{6}(f''ff + f'f'f)(x_n). \tag{2.1}$$

It is expected that elements of the sequence x_n $(n = 1,2,\dots,N)$ defined by (2.1) are close to the values of the exact solution at the points $x_*(h), x_*(2h), \dots, x_*(Nh)$. An intuitive way to measure this closeness is to compute the value

$$\rho = \max_{1\le n\le N} |x_*(nh) - x_n|. \tag{2.2}$$

However, the quantity (2.2) does not provide any information about the behavior of the solution on the interval $((n-1)h, nh)$. The main aim of this paper is to derive such estimates. If we want to apply numerical solutions in computer-aided proofs, then we cannot ignore the behavior of the system on the interval $(nh,(n+1)h)$. Certainly, for $t \in (nh,(n+1)h)$, we can estimate $x_*(t)$ as follows. For $t \in [0,T]$, let $n(t) = [t/h]$. Then $|t - n(t)h| < h$ and

$$|x_*(t) - x_*(n(t)h)| \leq \int\limits_{n(t)h}^{h} |f[x(s)]|ds \leq M_0 h,$$

where M_0 is the maximum value of $|f(x)|$ on some compact subset of D. The above inequality together with (2.2) imply that

$$|x_*(t) - x_n| \leq M_0 h + |x_*[n(t)h] - x_n| \leq M_0 h + \rho,$$

which shows that the difference between the exact and approximate solutions is, at best, of order h. However, this estimate is very rough and insufficient for our purposes. To get a better estimate, we use generalized Euler polygons.

From now on, we fix N, h, and $T > 0$ such that $Nh = T$ and define

$$\sigma_t = nh \quad \text{if} \quad t \in [nh,(n+1)h) \quad \text{for} \quad n = 0,\dots,N-1.$$

We start with the definition of an approximate solution.

Definition 1. *A continuous function $\widehat{x}(t) : \mathbb{R} \to [0:T]$ satisfying the equation*

$$\widehat{x}(t) = x_0 + \int\limits_0^t \Big(f[\widehat{x}(\sigma_s)] + (s-\sigma_s)(f'f)[\widehat{x}(\sigma_s)] + \frac{(s-\sigma_s)^2}{2}(f''ff + f'f'f)[\widehat{x}(\sigma_s)] \Big) ds \tag{2.3}$$

is called an approximate solution of (1.1)*.*

Although (2.3) looks like an integral equation, it is a recurrent formula, and we can construct $\widehat{x}(t)$ explicitly step by step. Therefore, (2.3) defines a function $\widehat{x}(t)$ on $[0,T]$ as an approximate solution; i.e., our notion of approximate solution is well defined.

Lemma 1. *The equality $\widehat{x}(nh) = x_n$ holds for all $n > 0$.*

The lemma is proved easily by induction on the intervals $[0, nh]$. We use this lemma to reduce the problem to deriving estimates of the form (1.2) for the difference $|x_*(t) - \widehat{x}(t)|$.

The remainder term of the Taylor expansion is given by the formula (see for example, [9])

$$R_4(t,h) = \int\limits_0^1 \frac{(1-s)^3}{3!} x_*^{(IV)}(t+sh)h^4 ds,$$

where

$$x_*^{(IV)}(s) = \big(f'''fff + 3f''f'ff + f'f''ff + f'f'f'f\big)[x_*(t)].$$

By the formula for R_4, the error term is estimated by the maximum values of the derivatives of f. To estimate the derivatives effectively along the solution and approximate solution, we need them to stay in some compact set. Therefore, we assume the following.

Assumption B. *Let K be a convex and compact domain in $\mathbb{R}^d$. We assume that values of the exact solution $x_*(t)$ and the approximate solution $\widehat{x}(t)$ remain in K for all $t \in [0,T]$.*

Since, K is compact by definition, Assumptions A and B allow us to define

$$\begin{aligned} M_0 &= \max_{x\in K} |f(x)|, \quad M_1 = \max_{x\in K} \|f'(x)\|, \\ M_2 &= \max_{x\in K} \|f''(x)\|, \quad M_3 = \max_{x\in K} \|f'''(x)\|. \end{aligned} \tag{2.4}$$

Using these quantities, we obtain the following estimate for the remainder term of the Taylor formula:

$$|R_4(nh,h)| \le \frac{h^4}{24} \max_{0\le t\le T} |x_*^{IV}(s)|ds \le \frac{h^4}{24}\left(M_3M_0^3 + 4M_2M_1M_0^2 + M_1^3M_0\right).$$

In the literature, when estimating the accuracy of approximate solutions, many authors claim that the error is bounded by $|R_4(nh,h)|$. However, this is not true, since $R_4(nh,h)$ is the difference between $x_*(t)$ and its Taylor expansion, which does not directly imply any conclusions for the difference $x_*(t) - \widehat{x}(t)$. In the present paper, we show that it is possible to obtain an explicit estimate for the latter. The main result of the this paper is the following theorem.

Theorem 1. *Under Assumptions A and B, the following inequality holds*:

$$|x_*(t) - \widehat{x}(t)| \le \frac{e^{M_1T} - 1}{6M_1}(L_0 + L_1h + L_2h^2)h^3,$$

where

$$\begin{aligned} L_0 &= 5M_0^2M_1M_2 + M_0M_1^3 + M_0^3M_3, \\ L_1 &= \frac{1}{4}(M_0^3M_2^2 + 4M_0^3M_1M_3 + 9M_0^2M_1^2M_2), \\ L_2 &= \frac{1}{2}(M_0^4M_2M_3 + M_0^3M_1^2M_3 + 2M_0^3M_1M_2^2 + 2M_0^2M_1^3M_2). \end{aligned}$$

Note that our estimate for the accuracy of the method is explicit and can be computed effectively in terms of the right-hand side of the initial value problem (1.1).

3. Proof of Theorem 1

In this section, we prove the main theorem. The key ingredient of the proof is a discretization of the time t using a piecewise constant function σ_s.

In what follows, we repeatedly use the following formula for the derivative of the approximate solution. By (2.3), for $t \ne nh$ $(n = 1, 2, \dots, N-1)$,

$$\dot{\widehat{x}}(t) = f[\widehat{x}(\sigma_t)] + (t-\sigma_t)(f'f)[\widehat{x}(\sigma_t)] + \frac{(t-\sigma_t)^2}{2}(f''f^2 + (f')^2f)[\widehat{x}(\sigma_t)]. \tag{3.1}$$

Further, taking into account that $0 \le t - \sigma_t \le h$, we have

$$|\dot{\widehat{x}}(\sigma_t)| \le M_0\Big(1 + hM_1 + \frac{h^2}{2}(M_2M_0 + M_1^2)\Big). \tag{3.2}$$

P r o o f. For the exact solution, we have the equality

$$x_*(t) = x_0 + \int_0^t f[x_*(s)]ds.$$

Using this equality and equation (2.3) and adding and subtracting the term

$$\int_0^t f[\widehat{x}(s)]ds,$$

we obtain

$$|x_*(t) - \widehat{x}(t)| \le \int_0^t I(s)ds + \int_0^t \big|f[x_*(s)] - f[\widehat{x}(s)]\big|ds,$$

where

$$I(s) = f[\widehat{x}(s)] - f[\widehat{x}(\sigma_s)] - (s-\sigma_s)(f'f)[\widehat{x}(\sigma_s)] - \frac{(s-\sigma_s)^2}{2}(f''f^2 + f'^2 f)[\widehat{x}(\sigma_s)].$$

We are going to derive an upper estimate of the form Ch^3 for $I(s)$. By the fundamental rule of the calculus, we have

$$f[\widehat{x}(s)] - f[\widehat{x}(\sigma_s)] = \int_{\sigma_s}^s \frac{df[\widehat{x}(r)]}{dr}dr = \int_{\sigma_s}^s f'[\widehat{x}(r)]\dot{\widehat{x}}(r)dr.$$

Substituting the derivative of the approximate solution $\dot{\widehat{x}}(r)$ given in (3.2) into the right-hand side, we obtain

$$f[\widehat{x}(s)]-f[\widehat{x}(\sigma_s)]=\int_{\sigma_s}^s f'[\widehat{x}(r)]\Big\{f[\widehat{x}(\sigma_r)]+(r-\sigma_r)(f'f)[\widehat{x}(\sigma_r)] + \frac{(r-\sigma_r)^2}{2}(f''f^2+f'^2f)[\widehat{x}(\sigma_r)]\Big\}dr.$$

Note that $\sigma_r = \sigma_s$ in this equation since $r \in [\sigma_s, s]$ and σ is piecewise constant. Denote the latter term on the right-hand side by C_1:

$$C_1 = \int_{\sigma_s}^s \frac{(r-\sigma_s)^2}{2}(f''f^2 + f'^2 f)[\widehat{x}(\sigma_r)]f'[\widehat{x}(r)]dr.$$

Using (2.4), we obtain the estimate

$$|C_1| \le (M_2M_1M_0^2 + M_1^3M_0)\int_{\sigma_s}^s \frac{(r-\sigma_s)^2}{2}dr \le D_1h^3, \tag{3.3}$$

where $D_1 = (M_2M_1M_0^2 + M_1^3M_0)/6$.

Therefore, we obtain

$$\begin{aligned} I(s) = \int_{\sigma_s}^s \big\{f'[\widehat{x}(r)] - f'[\widehat{x}(\sigma_s)]\big\}f[\widehat{x}(\sigma_s))]dr \\ + \int_{\sigma_s}^s f'[\widehat{x}(r)](f'f)[\widehat{x}(\sigma_s)]dr - \frac{(s-\sigma_s)^2}{2}(f''f^2 + f'^2 f)[\widehat{x}(\sigma_s)] + C_1. \end{aligned} \tag{3.4}$$

Denote the first term of this expression by $J(s)$:

$$J(s) = \int_{\sigma_s}^s \big\{f'[\widehat{x}(r)] - f'[\widehat{x}(\sigma_s)]\big\}f[\widehat{x}(\sigma_s)]dr.$$

We estimate $J(s)$, using the fundamental rule of calculus and the derivative of the approximate solution given by (3.2):

$$J(s) = \int_{\sigma_s}^{s}\int_{\sigma_s}^{r} \frac{df'[\widehat{x}(u)]}{du} du f[\widehat{x}(\sigma_s)]dr = \int_{\sigma_s}^{s}\int_{\sigma_s}^{r} f''[\widehat{x}(u)]\Big\{ f[\widehat{x}(\sigma_u)] + (u-\sigma_u)(f'f)[\widehat{x}(\sigma_u)] + \frac{(u-\sigma_u)^2}{2}(f''f^2+(f')^2 f)[\widehat{x}(\sigma_u)]\Big\} f[\widehat{x}(\sigma_s)]dudr. \tag{3.5}$$

Define

$$C_2 = \int_{\sigma_s}^{s}\int_{\sigma_s}^{r} \Big\{(u-\sigma_u) f''[\widehat{x}(u)](f'f)[\widehat{x}(\sigma_u)]f[\widehat{x}(\sigma_s)] + \frac{(u-\sigma_u)^2}{2} f''[\widehat{x}(u)](f''f^2+(f')^2 f)[\widehat{x}(\sigma_u)]f[\widehat{x}(\sigma_s)]\Big\}dudr. \tag{3.6}$$

Thus, taking into account that $\sigma_u=\sigma_s$ and using (2.4), we obtain the inequality

$$|C_2| \le \int_{\sigma_s}^{s}\int_{\sigma_s}^{r} \left| M_2M_1M_0^2(u-\sigma_s) + (M_2^2M_0^3 + M_2M_1^2M_0^2)\frac{(u-\sigma_s)^2}{2}\right| dudr \le D_2h^3 + D_3h^4, \tag{3.7}$$

where

$$D_2 = \frac{1}{6}M_1M_2M_0^2, \quad D_3 = \frac{1}{24}(M_2^2M_0^3 + M_2M_1^2M_0^2).$$

Consequently, substituting (3.6) into (3.5) and then (3.5) into (3.4) and denoting C_1+C_2 by C_3, we obtain

$$I(s) = \int_{\sigma_s}^{s}\int_{\sigma_s}^{r} f''[\widehat{x}(u)]f[\widehat{x}(u)]f[\widehat{x}(\sigma_s)]dudr + \int_{\sigma_s}^{s} f'[\widehat{x}(r)](r-\sigma_r)(f'f)[\widehat{x}(\sigma_r)]dr - \frac{(s-\sigma_s)^2}{2}(f''f^2 + f'^2 f)[\widehat{x}(\sigma_s)] + C_3 = A(s) + B(s) + C_3, \tag{3.8}$$

where we used the following notation:

$$A(s) = \int_{\sigma_s}^{s}\int_{\sigma_s}^{r} \{(f''f)[\widehat{x}(u)] - (f'f)[\widehat{x}(\sigma_s)]\} f[\widehat{x}(\sigma_s)]dudr,$$

$$B(s) = \int_{\sigma_s}^{s} (r-\sigma_s)\{f'[\widehat{x}(r)] - f'[\widehat{x}(\sigma_s)]\}(f'f)[\widehat{x}(\sigma_s)]dr.$$

Combining (3.3) and (3.7), we obtain

$$|C_3| \le D_1h^3 + D_2h^3 + D_3h^4. \tag{3.9}$$

It remains to estimate $A(s)$ and $B(s)$. We have

$$A(s) = \int_{\sigma_s}^{s}\int_{\sigma_s}^{r}\left(\int_{\sigma_s}^{u} \frac{d}{dv}\left(f''[\widehat{x}(v)]f[\widehat{x}(v)]\right) dv\right) f[\widehat{x}(\sigma_s)]dudr.$$

which implies

$$A(s)=\int_{\sigma_s}^{s}\int_{\sigma_s}^{r}\Big(\int_{\sigma_s}^{u}\{f'''[\widehat{x}(v)]\dot{\widehat{x}}(v)f[\widehat{x}(v)]+f''[\widehat{x}(v)]f'[\widehat{x}(v)]\dot{\widehat{x}}(v)\}dv\Big)f[\widehat{x}(\sigma_s)]dudr. \tag{3.10}$$

Therefore, taking the absolute value of the expression under the outer integral (integration with respect to r), using estimates (2.4) and (3.2), and taking into account that $t-\sigma_t\le h$ and $\sigma_u=\sigma_s=\sigma_r$, we obtain

$$\begin{aligned}|A(s)|&\le\int_{\sigma_s}^{s}\int_{\sigma_s}^{r}\int_{\sigma_s}^{u}\Big(M_0+hM_1M_0+\frac{h^2}{2}(M_2M_0^2+M_1^2M_0)\Big)(M_3M_0^2+M_0M_1M_2)dvdudr\\&\le(M_3M_0^2+M_0M_1M_2)\Big(M_0+hM_1M_0+\frac{h^2}{2}(M_2M_0^2+M_1^2M_0)\Big)\frac{h^3}{6}.\end{aligned}$$

Similarly,

$$\begin{aligned}B(s)&=\int_{\sigma_s}^{s}(r-\sigma_s)\Big\{\int_{\sigma_s}^{r}\frac{d}{du}f'[\widehat{x}(u)]du\Big\}(f'f)[\widehat{x}(\sigma_s)]dr\\&=\int_{\sigma_s}^{s}(r-\sigma_s)\int_{\sigma_s}^{r}f''[\widehat{x}(u)]\dot{\widehat{x}}(u)du\}(f'f)[\widehat{x}(\sigma_s)]dr.\end{aligned}$$

Again, using (2.4) and (3.2), and taking into account that $t-\sigma_t\le h$ and $\sigma_s=\sigma_r$ in the above equation, we obtain

$$\begin{aligned}|B(s)|&\le\frac{(r-\sigma_s)^3}{3}M_0M_1M_2\Big(M_0+hM_1M_0+\frac{h^2}{2}(M_2M_0^2+M_1^2M_0)\Big)\\&\le M_0M_1M_2\Big(M_0+hM_1M_0+\frac{h^2}{2}(M_2M_0^2+M_1^2M_0)\Big)\frac{h^3}{3}.\end{aligned} \tag{3.11}$$

Finally, substituting (3.9), (3.10), and (3.11) into (3.8), we obtain the inequality

$$\begin{aligned}|I(s)|\le&\frac{1}{6}(M_2M_1M_0^2+M_1^3M_0))h^3+\frac{1}{6}M_1M_2M_0^2h^3+\frac{1}{24}(M_2^2M_0^3+M_2M_1^2M_0^2)h^4\\&+(M_3M_0^2+M_0M_1M_2)\left(M_0+hM_1M_0+\frac{h^2}{2}(M_2M_0^2+M_1^2M_0)\right)\frac{h^3}{6}\\&+M_0M_1M_2\left(M_0+hM_1M_0+\frac{h^2}{2}(M_2M_0^2+M_1^2M_0)\right)\frac{h^3}{3}\\&=\frac{1}{6}(L_0+L_1h+L_2h^2)h^3,\end{aligned}$$

with the required constants L_0, L_1, and L_2.

Now, we use the compactness of the domain K and smoothness of f to obtain

$$|f[x_*(s)]-f[\widehat{x}(\sigma_s)]|\le M_1|x_*(s)-\widehat{x}(s)|. \tag{3.12}$$

Using inequalities (3.1), (3.2), and (3.12), we get the inequality

$$\begin{aligned}|x_*(t)-\widehat{x}(t)|&\le\int_0^t|I(s)|ds+\int_0^t|f[x_*(s)]-f[\widehat{x}(\sigma_s)]|ds\\&\le\frac{1}{6}(L_0+L_1h+L_2h^2)h^3t+\int_0^t M_1|x_*(s)-\widehat{x}(s)|.\end{aligned} \tag{3.13}$$

In (3.13), we apply Grönwall's inequality. For our purposes, the following version is the most convenient. Let $u : R \to R$ be a continuous function such that $u(t) \geq 0$ for $t \geq 0$ and

$$u(t) \leq Ct + M \int_0^t u(s)ds$$

for some $C, M > 0$. Then, the following inequality holds:

$$u(t) \leq C\frac{e^{Mt} - 1}{M}.$$

Applying Gröwnall's inequality to estimate (3.13), we obtain

$$|x_*(t) - \widehat{x}(t)| \leq \frac{1}{6}(L_0 + L_1 h + L_2 h^2)h^3 \frac{e^{M_1 t} - 1}{M_1}.$$

This completes the proof. □

4. Conclusion

1. If a nonautonomous system is considered in a d-dimensional space, then we can interpret it as an autonomous system in the $(d+1)$-dimensional space. In particular, we can consider the following Cauchy problem:

$$\dot{x} = f(t, x), \quad x(0) = x_0 \Leftrightarrow \begin{cases} \dfrac{dx}{dt} = f(\xi, x), & x(0) = x_0, \\ \dfrac{d\xi}{dt} = 1, & \xi(0) = 0. \end{cases}$$

   Therefore, the difference between the exact and approximate solutions can be estimated by the same expression with the constant $\sqrt{M_0^2 + 1}$ insted of M_0.

2. In the proof of the estimate for the difference between the exact and approximate solutions, we obtained

$$\max_{0 \leq t \leq T} |x_*(t) - \widehat{x}(t)| \leq C\frac{e^{M_1 t} - 1}{M_1} h^3$$

   with the coefficient that is considerably larger than expected, where the constant C is a fifth-order polynomial of the constants M_0, M_1, M_2, and M_3. If these constants are not very large, i.e., on the order of 1, then the coefficient C does not affect the choice of the mesh size. In this case, the mesh size would mostly depend on $(e^{M_1 t} - 1)/M_1$. On the other hand, if the constants M_0, M_1, M_2, and M_3 are on about 10, then the coefficient at h^3 is on the order of 10^5, which would affect the choice of h essentially, in certain cases, it may even invalidate the approximate solution scheme.

3. Another interesting question is whether it is possible to simplify the proof of the main theorem. The authors think that the proofs cannot be simplified considerably.

4. It is possible to prove a similar theorem with the order of accuracy h^4; i.e., we can consider the fourth-order scheme

$$x_{n+1} = x_n + hf(x_n) + \frac{h^2}{2}(f'f)(x_n) + \frac{h^3}{6}\big(f''ff + f'f'f\big)(x_n) + \\ + \frac{h^4}{24}\big(f'''fff + 3f''f'ff + f'f''ff + f'f'f'f\big)(x_n)$$

   and prove an analogous theorem.

REFERENCES

1. Azamov A. A., Abdullaev A. X., Tilavov A. M. On the derivation of an inequality for estimating the accuracy of an approximate solution to the initial value problem. *Bull. Inst. Math.*, 2022. Vol. 5, No. 5. P. 105–111.
2. Babenko K. I. *Foundations of Numerical Analysis* [Osnovy chislennogo analiza]. 2nd edition. Moscow, Izhevck: Regulyarnaya i haoticheskaya dinamika, 2002. 848 p. (in Russian)
3. Bakhvalov N. S. *Numerical Methods (Analysis, Algebra, Ordinary Differential Equations)* [CHislennye metody (analiz, algebra, obyknovennye differencial'nye uravneniya)]. Moscow: Nauka, 1975. 632 p.
4. Berezin I. S., Zhidkov P. N. *Computing Methods*. Vol. 2. Pergamon Press, 1965. 464 p.
5. Bieberbach L. On the remainder of the Runge–Kutta formula in the theory of ordinary differential equations. *J. Appl. Math. Phys. (ZAMP)*, 1951. Vol. 2. P. 233–248. DOI: 10.1007/BF02579687 (in German)
6. Brenan K. E., Campbell S. L., Petzold L. R. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations.* Philadelphia: SIAM, 1995. XII+251 p. DOI: 10.1137/1.9781611971224
7. Buchberger B., Collins G. E., Loos R., Albrecht R. *Computer Algebra: Symbolic and Algebraic Computation.* Vienna: Springer, 1983. VII+283 p. DOI: 10.1007/978-3-7091-7551-4
8. Butcher J. C. *Numerical Methods for Ordinary Differential Equations*. 3rd edition. United Kingdom: John Wiley & Sons Ltd., 2016. 544 p.
9. Cartan H. *Formes Différentielles*. Hermann Paris, 1967. 190 p. (in French)
10. Hairer E., Norsett S., Wanner G. *Solving Ordinary Differential Equations I. Nonstiff Problems.* 2nd edition. Berlin, Heidelberg: Springer, 2008. XV+528 p. DOI: 10.1007/978-3-540-78862-1
11. Hartman P. *Ordinary Differential Equation*. SIAM, 2002. XVIII+624 p. DOI: 10.1137/1.9780898719222
12. Gathen J., Gerhard J. *Modern Computer Algebra*. 3rd edition. Cambridge University Press, 2013. 796 p. DOI: 10.1017/CBO9781139856065
13. Geddes K. O., Czapor S. R., Labahn G. *Algorithms for Computer Algebra*. NY: Springer, 2007. XXII+586 p. DOI: 10.1007/b102438
14. Kendall A. E. *An Introduction to Numerical Analysis*. 2nd edition. John Wiley & Sons, Inc., 1989. 693 p.
15. Krilov V. I., Bobkov V. V., Monastirniy P. I. *Computational Methods* [Vychislitel'nye metody]. Vol. II. Moscow: Nauka, 1977. 399 p. (in Russian)
16. Lambert J. D. *Numerical Methods for Ordinary Differential Systems: The Initial Value Problem*. Wiley, 1992. 304 p.
17. Milne W. E. *Numerical Calculus*. Princeton University Press, 1949. 404 p.
18. Milne W. E. The Remainder in linear methods of approximation. *J. Res. Nat. Bur. Stand.*, 1949. Vol. 43. Art. no. RP2401. P. 501–511.

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 28–43

DOI: 10.15826/umj.2024.1.003

ON A GROUP EXTENSION INVOLVING THE SPORADIC JANKO GROUP J_2

Ayoub B. M. Basheer

School of Mathematical and Computer Sciences, University of Limpopo (Turfloop), P. Bag X1106, Sovenga 0727, South Africa

Mathematics Program, Faculty of Education and Arts, Sohar University, Sohar, Oman

ayoubbasheer@gmail.com

Abstract: According to the electronic Atlas [23], the group J_2 has an absolutely irreducible module of dimension 6 over $\mathbb{F}_4$. Therefore, a split extension group having the form $4^6{:}J_2 := \overline{G}$ exists. In this paper, we consider this group. Our purpose is to determine its conjugacy classes and character table using the methods of the coset analysis together with Clifford–Fischer theory. We determine the inertia factors of $\overline{G}$ by analyzing the maximal subgroups of J_2 and the maximal of the maximal subgroups of J_2 together with other various information. It turns out that the character table of $\overline{G}$ is a 53×53 real-valued matrix, while Fischer matrices of the extension are all integer-valued matrices with sizes ranging from 1 to 8.

Keywords: Group extensions, Janko sporadic simple group, Inertia groups, Fischer matrices, Character table.

1. Introduction

Visiting the history of the classification of finite simple groups, one can see that it was only a century after the establishment of the last Mathieu group that Z. Janko could construct a new sporadic simple group in 1964. This simple group has been named in his honor, is denoted by J_1, and has order 175560. Then Janko predicted the existence of other sporadic simple groups; namely, J_2, J_3, and J_4, which later are all proved to exist. According to Wilson [22], the original construction of the second Janko group J_2 was due to Marshall Hall (and thus, in some other papers, this group is referred to as Hall–Janko group HJ but here we use the more familiar notation J_2). Hall constructed this group as a permutation group acting on 100 points. Starting with the group $U_3(3)$, the group J_2 appears as a maximal normal subgroup of index 2 of the automorphism group of a graph Γ associated with $U_3(3)$ (for further details on the vertices and how are they connected, see the description given on page 224 of [22]).

The group J_2 has order $604800 = 2^7 \times 3^3 \times 5^2 \times 7$. It has Schur multiplier and outer automorphism groups both isomorphic to $\mathbb{Z}_2$. From the Atlas of Wilson [23], one can see that the group J_2 has a 6-dimensional absolutely irreducible module over $\mathbb{F}_4$. Therefore, a split extension group of the form $4^6{:}J_2 := \overline{G}$ exists. The present paper focuses on the group $\overline{G}$. Our purpose is to determine its conjugacy classes and the inertia factors of this extension with the fusions of their conjugacy classes into the classes of J_2. We will also find the character tables of these inertia factors and, finally, the full character table of the extension $\overline{G}$ under consideration. The methods used here to achieve the previous purpose are the coset analysis technique and the theory of Clifford–Fischer matrices. The most interesting part of this paper is the process of determining the inertia factor groups, where there are three inertia factor groups; namely, $H_1 = J_2$, H_2, and H_3. The main technique used for determining the structures of H_2 and H_3 is the analysis of the maximal subgroups of J_2 and the maximal subgroups of these maximal subgroups. There are many possibilities for H_2 and H_3, and combining all of them leads to contradictions except for only one possibility where we find that $H_2 = 2^{2+4}{:}S_3$ and $H_3 = 2^2 \times A_5$. We use a method of the coset analysis together with Clifford–Fischer theory to construct the character tables of H_2 and H_3, but we organize the columns of the character tables of these inertia factors according to the centralizers sizes. This paper determines all Fischer matrices of $\overline{G}$; their sizes vary between 1 and 8. The character table of $\overline{G}$ is a 53×53 real-valued matrix, which will be divided into 63 parts corresponding to 3 inertia factors and 21 conjugacy classes of $G = J_2$.

If $\overline{G} = N{\cdot}G$ is a group extension (here, N is the kernel of the extension and G is isomorphic to $\overline{G}/N$), then the character table of G produced using the coset analysis and Clifford–Fischer theory is in a special format that cannot be obtained by the direct computations using GAP [18] or Magma [15]. Another interesting point is the interplay between the coset analysis and Clifford–Fischer theory. This can be seen at the size of each Fischer matrix, where it is equal to the number of $\overline{G}$-classes corresponding to $[g_i]_G$ obtained via the coset analysis technique. In other words, computations of the conjugacy classes of $\overline{G}$ using the coset analysis technique will determine the sizes of all Fischer matrices.

From the Atlas [23], we can see that J_2 has an absolutely irreducible module of dimension 6 over the field $\mathbb{F}_4$. With α being a generator of the field $\mathbb{F}_4$, the following two elements g_1 and g_2 are 6×6 matrices over $\mathbb{F}_4$ that generates J_2:

$$g_1 = \begin{pmatrix} \alpha^2 & \alpha^2 & 0 & 0 & 0 & 0 \\ 1 & \alpha^2 & 0 & 0 & 0 & 0 \\ 1 & 1 & \alpha^2 & \alpha^2 & 0 & 0 \\ \alpha & 1 & 1 & \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & \alpha^2 & \alpha^2 & 0 & \alpha \\ \alpha^2 & 1 & \alpha^2 & 0 & \alpha^2 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} \alpha & 1 & \alpha^2 & 1 & \alpha^2 & \alpha^2 \\ \alpha & 1 & \alpha & 1 & 1 & \alpha \\ \alpha & \alpha & \alpha^2 & \alpha^2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \alpha^2 & 1 & \alpha^2 & \alpha^2 & \alpha & \alpha^2 \\ \alpha^2 & 1 & \alpha^2 & \alpha & \alpha^2 & \alpha \end{pmatrix},$$

where $o(g_1) = 2$, $o(g_2) = 3$, and $o(g_1 g_2) = 7$.

Using the above two generators of J_2 together with few GAP commands, we were able to construct our split extension group $\overline{G} = 4^6{:}J_2$ in terms of 7×7 matrices over $\mathbb{F}_4$. With α being a generator of the field $\mathbb{F}_4$, the following elements $\overline{g}_1$ and $\overline{g}_2$ generate the group $\overline{G}$:

$$\overline{g}_1 = \begin{pmatrix} 0 & 1 & \alpha & \alpha & \alpha^2 & 0 & 0 \\ \alpha & \alpha^2 & 0 & \alpha & \alpha & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 & \alpha^2 & \alpha^2 & 0 & 0 \\ 1 & \alpha & 0 & \alpha^2 & 1 & 1 & 0 \\ 1 & 0 & 1 & \alpha & 0 & 0 & 0 \\ \alpha^2 & \alpha & 1 & 0 & 0 & \alpha^2 & 0 \\ \alpha^2 & \alpha^2 & \alpha & 0 & \alpha^2 & 1 & 1 \end{pmatrix}, \quad \overline{g}_2 = \begin{pmatrix} \alpha & 0 & 1 & 1 & 0 & \alpha^2 & 0 \\ 1 & 1 & \alpha^2 & \alpha^2 & \alpha^2 & 1 & 0 \\ 1 & \alpha & 0 & \alpha & 0 & \alpha^2 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ \alpha^2 & 0 & 1 & 0 & 1 & 0 & 0 \\ \alpha & 1 & \alpha & \alpha & 0 & \alpha^2 & 0 \\ \alpha & 1 & \alpha^2 & 1 & \alpha & 1 & 1 \end{pmatrix},$$

where $o(\overline{g}_1) = 6$, $o(\overline{g}_2) = 12$, and $o(\overline{g}_1\overline{g}_2) = 10$.

To make the computations easier, we used a few GAP commands to convert the matrix representation of $\overline{G}$ into permutation representation. We represented $\overline{G}$ in terms of the set $\{1, 2, \ldots, 4096\}$.

Using GAP, we see that the group $\overline{G}$ possesses only one proper normal subgroup of order 4096. This normal subgroup is an elementary abelian group isomorphic to N. In GAP, one can check for the complements of N in $\overline{G}$, where in our case we obtained four complements, all isomorphic to J_2, and each of these four complements, together with N, gives the split extension in consideration.

For the notation used in this paper and how Clifford–Fischer theory and the coset analysis techniques are used, we follow [1–14, 17].

2. Conjugacy classes of $\overline{G} = 4^6{:}J_2$

Here we compute the conjugacy classes of $\overline{G}$ using the coset analysis technique (see [2] by Basheer, [5, 6, 8] by Basheer and Moori, or [20] and [21] by Moori for more details) since we are interested in organizing the classes of $\overline{G}$ corresponding to the classes of J_2. Note that J_2 has 21 conjugacy classes (see the Atlas [16] or Atlas of Wilson [23]). Corresponding to these 21 classes of J_2, we obtained 53 classes in $\overline{G}$.

In Table 1, we list the conjugacy classes of $\overline{G}$, where in this table:

- k_i represents the number of orbits $Q_{i1}, Q_{i2}, \ldots, Q_{ik_i}$ for the action of N on the coset $N\overline{g}_i = Ng_i$, where g_i is a representative of a class of the complement J_2 of N in $\overline{G}$. In particular, the action of N on the identity coset N produces 4 096 orbits each consists of a single element. Therefore, for $\overline{G}$, we have $k_1 = 4\,096$.

- f_{ij} is the number of orbits fused under the action of $C_G(g_i)$ on $Q_1, Q_2, \ldots, Q_k$. In particular, the action of $C_G(1_G) = G = J_2$ on the orbits $Q_1, Q_2, \ldots, Q_k$ affords three orbits of lengths 1, 1 575, and 2 520 (with corresponding point stabilizers J_2, $2^{2+4}{:}S_3$, and $2^2 \times A_5$. Thus, $f_{11} = 1$, $f_{12} = 1\,575$, and $f_{13} = 2\,520$.

- m_{ij} are weights (attached to each class of $\overline{G}$). These weights are computed by the formula

$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N|\frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|},$$

where N is the kernel of an extension $\overline{G}$ in consideration.

Table 1. The conjugacy classes of $\overline{G}$.

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_1 = 1A$	$k_1 = 4\,096$	$f_{11} = 1$	$m_{11} = 1$	g_{11}	1	1	2 477 260 800
		$f_{12} = 1\,575$	$m_{12} = 1\,575$	g_{12}	2	1 575	1 570 864
		$f_{13} = 2\,520$	$m_{13} = 2\,520$	g_{13}	2	2 520	983 040
$g_2 = 2A$	$k_2 = 256$	$f_{21} = 1$	$m_{21} = 16$	g_{21}	2	5 040	491 520
		$f_{22} = 15$	$m_{22} = 240$	g_{22}	2	75 600	32 768
		$f_{23} = 120$	$m_{23} = 1\,920$	g_{23}	4	604 800	4 096
		$f_{24} = 120$	$m_{24} = 1\,920$	g_{24}	4	604 800	4 096
$g_3 = 2B$	$k_3 = 64$	$f_{31} = 1$	$m_{31} = 64$	g_{31}	2	161 280	15 360
		$f_{32} = 1$	$m_{32} = 64$	g_{32}	4	161 280	15 360
		$f_{33} = 1$	$m_{33} = 64$	g_{33}	4	161 280	15 360
		$f_{34} = 1$	$m_{34} = 64$	g_{34}	4	161 280	15 360
		$f_{35} = 15$	$m_{35} = 960$	g_{35}	4	2 419 200	1 024
		$f_{36} = 15$	$m_{36} = 960$	g_{36}	4	2 419 200	1 024
		$f_{37} = 15$	$m_{37} = 960$	g_{37}	4	2 419 200	1 024
		$f_{38} = 15$	$m_{38} = 960$	g_{38}	4	2 419 200	1 024
$g_4 = 3A$	$k_4 = 1$	$f_{41} = 1$	$m_{41} = 4\,096$	g_{41}	3	2 293 760	1 080
$g_5 = 3B$	$k_5 = 16$	$f_{51} = 1$	$m_{51} = 256$	g_{51}	3	4 300 800	576
		$f_{52} = 4$	$m_{52} = 768$	g_{52}	6	12 902 400	192
		$f_{53} = 12$	$m_{53} = 3\,072$	g_{53}	6	51 609 600	48
		$f_{61} = 1$	$m_{61} = 256$	g_{61}	4	1 612 800	1 536

continued on the next page

Table 1 (continued from the previous page)

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_6 = 4A$	$k_6 = 16$	$f_{62} = 3$	$m_{62} = 768$	g_{62}	4	4 838 400	512
		$f_{63} = 3$	$m_{63} = 768$	g_{63}	4	4 838 400	512
		$f_{64} = 3$	$m_{64} = 768$	g_{64}	4	4 838 400	512
		$f_{65} = 6$	$m_{65} = 1\,536$	g_{65}	4	9 676 800	256
$g_7 = 5A$	$k_7 = 16$	$f_{71} = 1$	$m_{71} = 256$	g_{71}	5	516 096	4 800
		$f_{72} = 15$	$m_{72} = 3\,840$	g_{72}	10	7 741 440	320
$g_8 = 5B$	$k_8 = 16$	$f_{81} = 1$	$m_{81} = 256$	g_{81}	5	516 096	4 800
		$f_{82} = 15$	$m_{82} = 3\,840$	g_{82}	10	7 741 440	320
$g_9 = 5C$	$k_9 = 1$	$f_{91} = 1$	$m_{91} = 4\,096$	g_{91}	5	49 545 216	50
$g_{10} = 5D$	$k_{10} = 1$	$f_{10,1} = 1$	$m_{10,1} = 4\,096$	$g_{10,1}$	5	49 545 216	50
$g_{11} = 6A$	$k_{11} = 1$	$f_{11,1} = 1$	$m_{11,1} = 4\,096$	$g_{11,1}$	6	103 219 200	24
$g_{12} = 6B$	$k_{12} = 4$	$f_{12,1} = 1$	$m_{12,1} = 1\,024$	$g_{12,1}$	6	51 609 600	48
		$f_{12,2} = 1$	$m_{12,2} = 1\,024$	$g_{12,2}$	12	51 609 600	48
		$f_{12,3} = 1$	$m_{12,3} = 1\,024$	$g_{12,3}$	12	51 609 600	48
		$f_{12,4} = 1$	$m_{12,4} = 1\,024$	$g_{12,4}$	12	51 609 600	48
$g_{13} = 7A$	$k_{13} = 1$	$f_{13,1} = 1$	$m_{13,1} = 4\,096$	$g_{13,1}$	7	353 894 400	7
$g_{14} = 8A$	$k_{14} = 4$	$f_{14,1} = 1$	$m_{14,1} = 1\,024$	$g_{14,1}$	8	77 414 400	32
		$f_{14,2} = 1$	$m_{14,2} = 1\,024$	$g_{14,2}$	8	77 414 400	32
		$f_{14,3} = 1$	$m_{14,3} = 1\,024$	$g_{14,3}$	8	77 414 400	32
		$f_{14,4} = 1$	$m_{14,4} = 1\,024$	$g_{14,4}$	8	77 414 400	32
$g_{15} = 10A$	$k_{15} = 4$	$f_{15,1} = 1$	$m_{15,1} = 1\,024$	$g_{15,1}$	10	30 965 760	80
		$f_{15,2} = 1$	$m_{15,2} = 1\,024$	$g_{15,2}$	20	30 965 760	80
		$f_{15,3} = 1$	$m_{15,3} = 1\,024$	$g_{15,3}$	20	30 965 760	80
		$f_{15,4} = 1$	$m_{15,4} = 1\,024$	$g_{15,4}$	20	30 965 760	80
$g_{16} = 10B$	$k_{16} = 4$	$f_{16,1} = 1$	$m_{16,1} = 1024$	$g_{16,1}$	10	30 965 760	80
		$f_{16,2} = 1$	$m_{16,2} = 1\,024$	$g_{16,2}$	20	30 965 760	80
		$f_{16,3} = 1$	$m_{16,3} = 1\,024$	$g_{16,3}$	20	30 965 760	80
		$f_{16,4} = 1$	$m_{16,4} = 1\,024$	$g_{16,4}$	20	30 965 760	80
$g_{17} = 10C$	$k_{17} = 1$	$f_{17,1} = 1$	$m_{17,1} = 4\,096$	$g_{17,1}$	10	247 726 080	10
$g_{18} = 10D$	$k_{18} = 1$	$f_{18,1} = 1$	$m_{18,1} = 4\,096$	$g_{18,1}$	10	247 726 080	10
$g_{19} = 12A$	$k_{19} = 1$	$f_{19,1} = 1$	$m_{19,1} = 4\,096$	$g_{19,1}$	12	206 438 400	12
$g_{20} = 15A$	$k_{20} = 1$	$f_{20,1} = 1$	$m_{20,1} = 4\,096$	$g_{20,1}$	15	165 150 720	15
$g_{21} = 15B$	$k_{21} = 1$	$f_{21,1} = 1$	$m_{21,1} = 4\,096$	$g_{21,1}$	15	165 150 720	15

3. Inertia factor groups of $\overline{G} = 4^6{:}J_2$

We have seen in Section 2 that the action of $\overline{G}$ on N produced three orbits of lengths 1, 1 575, and 2 520. By a theorem of Brauer (see, for example, [2, Theorem 5.1.1]), it follows that the action of $\overline{G}$ on $\mathrm{Irr}(N)$ will also produce three orbits of lengths 1, r, and s, where

$$1 + r + s = |\mathrm{Irr}(N)| = 4\,096;$$

that is

$$r + s = 4\,095. \tag{3.1}$$

The values of r and s will be determined through deep investigation on the maximal subgroups of J_2 or on the maximal of the maximal subgroups of J_2 together with various information including sizes of the Fischer matrices, fusions of the conjugacy classes of some subgroups into the group J_2, and other information. In Table 2, we supply the maximal subgroups of J_2 (see the Atlas [16]). We will need these subgroups to determine H_2 and H_3.

Table 2. The maximal subgroups of $G = J_2$.

M_i	$\lvert M_i\rvert$	$[J_2 : M_i]$
$U_3(3)$	6 048	100
$(3^{\cdot}A_6){:}2$	2 160	280
$2_{-}^{1+4}{:}A_5$	1 920	315
$2^{2+4}{:}(3 \times S_3)$	1 152	525
$A_4 \times A_5$	720	840
$A_5 \times D_{10}$	600	1 008
$L_3(2){:}2$	336	1 800
$5^2{:}D_{12}$	300	2 016
A_5	60	10 080

First, since 1, r, and s are the lengths of the orbits on the action of $\overline{G}$ on N (which can be reduced to the action of G on N), it follows that $[G : H_1] = 1$, $[G : H_2] = r$, and $[G : H_3] = s$, where H_1, H_2, and H_3 are the inertia factors in $G = J_2$. It follows that $H_1 = G = J_2$ and $r, s \mid |G|$; that is $r, s|604\,800$. Now, 604 800 has 192 positive divisors, where 140 divisors are less than 4 095. Out of these 140 divisors, only four pairs (r, s) satisfy (3.1). These are the pairs

$$(r, s) \in \big\{(63, 4\,032), (315, 3\,780), (945, 3\,150), (1\,575, 2\,520)\big\}. \tag{3.2}$$

Here, we do not distinguish between the pairs (r, s) and (s, r) and therefore we exclude the other four pairs $(4\,032, 63), (3\,780, 315), (3\,150, 945)$, and $(2\,520, 1\,575)$ from our consideration and restrict ourselves only to those in (3.2). Another point we put in mind is that since $\overline{G}$ is a split extension of 4^6 by J_2 and 4^6 is an elementary abelian group, it follows that the three character tables of H_1, H_2, and H_3, which we will use to construct the character table of $\overline{G}$, are ordinary. From the Atlas and Table 1, we have $|\mathrm{Irr}(\overline{G})| = 53$ and $|\mathrm{Irr}(H_1)| = |\mathrm{Irr}(G)| = |\mathrm{Irr}(J_2)| = 21$. Since

$$\sum_{i=1}^{3} |\mathrm{Irr}(H_i)| = |\mathrm{Irr}(\overline{G})| = 53,$$

we have $|\mathrm{Irr}(H_1)| + |\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| = |\mathrm{Irr}(\overline{G})| = 53$, that is

$$|\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| = 32. \tag{3.3}$$

Our next task is to show that $(r, s) = (1\,575, 2\,520)$ and the action of $\overline{G}$ on $\mathrm{Irr}(N)$ is dual to the action of $\overline{G}$ on classes of N. This will be achieved by excluding the other possible pairs by getting a contradiction to some fact in each case.

Proposition 1. $(r, s) \neq (63, 4\,032)$.

P r o o f. To obtain a contradiction, suppose that $(r, s) = (63, 4\,032)$, i.e., $r = 63$ and $s = 4\,032$ (or $[J_2 : H_2] = 63$ and $[J_2 : H_3] = 4\,032$) and consequently $|H_2| = 9\,600$ and $|H_3| = 150$. Since $|H_2| = 9\,600$ and the maximal subgroups of J_2 are given in Table 2, it follows that $|H_2|$ is bigger than the size of any maximal subgroup of J_2, a contradiction. Thus, (r, s) cannot be $(63, 4\,032)$. □

Proposition 2. $(r, s) \neq (315, 3\,780)$.

P r o o f. To obtain a contradiction, suppose that $(r, s) = (315, 3\,780)$, i.e., $r = 315$ and $s = 3\,780$ (or $[J_2 : H_2] = 315$ and $[J_2 : H_3] = 3\,780$) and, consequently, $|H_2| = 3\,840$ and $|H_3| = 320$. Since $|H_2| = 3\,840$ and the maximal subgroups of J_2 are given in Table 2, it follows that H_2 does not sit in any of the maximal subgroups of J_2, a contradiction. Thus, (r, s) cannot be $(315, 3\,780)$. □

Proposition 3. $(r, s) \neq (945, 3\,150)$.

P r o o f. To obtain a contradiction, suppose that $(r, s) = (945, 3150)$, i.e., $r = 945$ and $s = 3150$ (or $[J_2 : H_2] = 945$ and $[J_2 : H_3] = 3\,150$) and, consequently, $|H_2| = 1\,280$ and $|H_3| = 384$. Since $|H_2| = 1\,280$ and the maximal subgroups of J_2 are given in Table 2, we see that H_2 is not among the maximal subgroups of J_2 and does not sit in any of them. This contradiction proves that (r, s) cannot be $(945, 3\,150)$. □

Proposition 4. *The action of J_2 on $\mathrm{Irr}(4^6)$ is dual to the action of J_2 on the conjugacy classes of $N = 4^6$.*

P r o o f. We have seen in Section 2 that the action of J_2 on the conjugacy classes of $N = 4^6$ produced 3 orbits of lengths 1, 1 575, and 2 520. From (3.1), we have $r+s = 4\,095$, where r and s are the lengths of the second the third orbits on the action of J_2 on $\mathrm{Irr}(4^6)$. Further, by (3.2), we have $(r, s) \in \{(63, 4\,032), (315, 3\,780), (945, 3\,150), (1\,575, 2\,520)\}$. We also proved in Propositions 1, 2, and 3 that $(r, s) \notin \{(63, 4\,032), (315, 3\,780), (945, 3\,150)\}$. Therefore, $(r, s) = (1\,575, 2\,520)$ and the action of J_2 on $\mathrm{Irr}(4^6)$ is dual to the action of J_2 on the conjugacy classes of $N = 4^6$, as claimed. □

Proposition 5. *The inertia factor groups have the forms $2^{2+4}{:}S_3$ and $2^2 \times A_5$.*

P r o o f. From Proposition 4, we can see that the orbit lengths on the action of J_2 on $\mathrm{Irr}(4^6)$ are 1, 1 575, and 2 520. It follows that $[G : H_1] = 1$, $[G : H_2] = 1\,575$ and $[G : H_3] = 2\,520$ and, consequently, $H_1 = G = J_2$, $|H_2| = 384$, and $|H_3| = 240$. By (3.3), we also have $|\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| = 32$. Now we investigate the maximal subgroups of J_2 to locate H_2 and H_3. Since $|H_2| = 384$ and the maximal subgroups of J_2 are given in Table 2, it follows that H_2 is either an index 5 subgroup of $2_{-}^{1+4}{:}A_5$ or an index 3 subgroup of $2^{2+4}{:}(3 \times S_3)$. If $H_2 \leq 2_{-}^{1+4}{:}A_5$ is such that $[2_{-}^{1+4}{:}A_5 : H_2] = 5$, then H_2 must be a maximal subgroup in it since the index is a prime number. Now, $2_{-}^{1+4}{:}A_5$ has 4 maximal subgroups of orders 384, 320, 192, and 120. The maximal subgroup of order 384 has the structure $2^{2+4}{:}6$ and 19 ordinary irreducible characters. Also, if $H_2 \leq 2^{2+4}{:}(3 \times S_3)$ is such that $[2^{2+4}{:}(3 \times S_3) : H_2] = 3$, then H_2 must be a maximal subgroup in it since the index is a prime number. Now, $2^{2+4}{:}(3 \times S_3)$ has 4 maximal subgroups of orders 576, 384 (twice), and 72. The two maximal subgroups of order 384 have structures $2^{2+4}{:}S_3$ and $2^{1+4}{:}A_4$, where $|\mathrm{Irr}(2^{2+4}{:}S_3)| = 12$ and $|\mathrm{Irr}(2^{1+4}{:}A_4)| = 15$. Thus, we have

$$\begin{gathered} H_2 \in \{2^{2+4}{:}6, 2^{2+4}{:}S_3, 2^{1+4}{:}A_4\}, \\ |\mathrm{Irr}(2^{2+4}{:}6)| = 19, \quad |\mathrm{Irr}(2^{2+4}{:}S_3)| = 12, \quad |\mathrm{Irr}(2^{1+4}{:}A_4)| = 15. \end{gathered} \tag{3.4}$$

Next, consider H_3. Since $|H_3| = 240$ and the maximal subgroups of J_2 are given in Table 2, we deduce that H_3 is either

- an index 9 subgroup of $(3^{\cdot}A_6){:}2$,
- an index 8 subgroup of $2_{-}^{1+4}{:}A_5$, or
- an index 3 subgroup of $A_4 \times A_5$.

Consider each of these cases. Using GAP, one can see that the group $(3^{\cdot}A_6){:}2$ has four maximal subgroups of orders 1 080, 216, 60, and 48. Therefore, H_3 cannot be a subgroup of $(3^{\cdot}A_6){:}2$ since $[(3^{\cdot}A_6){:}2 : H_3] = 9$, which is impossible. Next, consider the case where H_3 is an index 8 subgroup of $2_{-}^{1+4}{:}A_5$. Checking the order of all maximal subgroups of $2_{-}^{1+4}{:}A_5$, which can be done using GAP, shows that $2_{-}^{1+4}{:}A_5$ has four maximal subgroups of orders 384, 320, 192, and 120. Therefore, $H_3 \not\leq 2_{-}^{1+4}{:}A_5$. Finally, we turn to the last case where we consider H_3 to be a subgroup of $A_4 \times A_5$ of index 3. The group $A_4 \times A_5$ has five maximal subgroups of orders 240, 180, 144, 120, and 72. The maximal subgroup of order 240 has the structure $2^2 \times A_5$ and 20 ordinary irreducible characters. We deduce that H_3 has the structure $2^2 \times A_5$ and $|\mathrm{Irr}(H_3)| = 20$. Using this together with (3.4), we conclude that $(H_2, H_3) = (2^{2+4}{:}S_3, 2^2 \times A_5)$ is the required pair of inertia factor groups since it consists of (3.3), and all other possibilities are exhausted and each lead to a contradiction, except $(H_2, H_3) = (2^{2+4}{:}S_3, 2^2 \times A_5)$. Hence, we have the result. □

Next, we construct the character tables of H_1, H_2, and H_3 and determine the fusions of the conjugacy classes of these groups into the classes of $H_1 = G = J_2$. The character table of the simple Janko group J_2 can be found at the Atlas. As subgroups of $G = J_2$ that generated by g_1 and g_2 given in Section 1, and α being a generator of $\mathbb{F}_4$, the two inertia factor groups $H_2 = 2^{2+4}{:}S_3$ and $H_3 = 2^2 \times A_5$ are generated as follows: $H_2 = \langle \alpha_1, \alpha_2 \rangle$ and $H_3 = \langle \beta_1, \beta_2 \rangle$, where

$$\alpha_1 = \begin{pmatrix} 1 & 1 & \alpha & \alpha & \alpha & 0 \\ \alpha^2 & \alpha^2 & 1 & \alpha & \alpha & \alpha^2 \\ \alpha^2 & \alpha & \alpha^2 & \alpha^2 & \alpha^2 & \alpha^2 \\ 1 & \alpha & 1 & 1 & 1 & 0 \\ \alpha^2 & \alpha & \alpha & \alpha^2 & \alpha & \alpha^2 \\ \alpha^2 & \alpha & \alpha & \alpha^2 & \alpha^2 & \alpha \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} \alpha & 1 & \alpha & 1 & \alpha^2 & \alpha \\ 0 & 1 & 0 & 0 & 0 & \alpha \\ \alpha & 0 & 1 & 0 & \alpha & 1 \\ \alpha & \alpha & 1 & \alpha^2 & \alpha^2 & \alpha \\ \alpha & \alpha^2 & \alpha & \alpha & \alpha & \alpha^2 \\ \alpha & 1 & \alpha^2 & \alpha^2 & 0 & \alpha^2 \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} 1 & \alpha & \alpha^2 & 0 & 1 & 1 \\ 0 & 0 & \alpha^2 & 1 & 1 & 1 \\ 0 & \alpha^2 & 0 & 1 & \alpha^2 & \alpha^2 \\ \alpha & \alpha & 1 & 1 & 1 & 1 \\ 1 & 0 & \alpha^2 & \alpha^2 & 0 & 0 \\ 0 & 1 & \alpha^2 & 0 & \alpha & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} \alpha & 0 & \alpha^2 & 0 & 1 & \alpha^2 \\ 1 & \alpha^2 & \alpha & \alpha & \alpha & 0 \\ \alpha^2 & \alpha & 1 & 0 & 1 & 1 \\ 0 & \alpha & \alpha^2 & \alpha & 0 & \alpha^2 \\ 0 & \alpha & \alpha^2 & 1 & 1 & 0 \\ 0 & 1 & \alpha & \alpha & 0 & \alpha^2 \end{pmatrix}.$$

We recursively use Clifford–Fischer theory to construct the character table of H_2. The action of S_3 on the set $\mathrm{Irr}(2^{2+4})$ produced 6 orbits of lengths 1, 3, 3, 3, 3, and 6 with the corresponding inertia factor groups S_3, $\mathbb{Z}_2$ (four times), and the identity group. Also, H_3 is the direct product of the elementary abelian group 2^2 by A_5. Thus, the character table of H_3 is easy to construct since we know the character tables of both 2^2 and A_5. In this paper, we list the full character tables of H_2 and H_3 and organize the columns of the character tables according to the orders and the sizes of the centralizers.

Recall that H_2 and H_3 are not maximal subgroups of J_2, but they are maximal of some maximal subgroups of J_2 (H_2 is a maximal subgroup of $2^{2+4}{:}(3 \times S_3)$ while H_3 is a maximal subgroup of $A_4 \times A_5$). We determined the fusions of the conjugacy classes of H_2 and H_3 into the classes J_2 using the permutation characters of J_2 on $2^{2+4}{:}(3 \times S_3)$ and $A_4 \times A_5$; the permutation characters of $2^{2+4}{:}(3 \times S_3)$ and $A_4 \times A_5$ on H_2 and H_3, respectively, together with the sizes of centralizers. The following proposition plays a great role in determining the fusions; its proof can be found in [2].

Proposition 6. *Let $K_1 \leq K_2 \leq K_3$, and let ψ be a class function on K_1. Then, $(\psi\uparrow_{K_1}^{K_2})\uparrow_{K_2}^{K_3} = \psi\uparrow_{K_1}^{K_3}$. More generally, if $K_1 \leq K_2 \leq \cdots \leq K_n$ is a nested sequence of subgroups of K_n and ψ is a class function on K_1, then $(\psi\uparrow_{K_1}^{K_2})\uparrow_{K_2}^{K_3} \cdots \uparrow_{K_{n-1}}^{K_n} = \psi\uparrow_{K_1}^{K_n}$.*

P r o o f. See Proposition 3.5.6 of [2]. □

We supply the full character tables of the inertia factor groups H_2 and H_3 together with the fusions of their conjugacy classes into the classes of J_2 in Tables 3 and 4.

Table 3. The character table of $H_2 = 2^{2+4}{:}S_3$.

$[g]_{H_2}$	$1a$	$2a$	$2b$	$2c$	$3a$	$4a$	$4b$	$4c$	$4d$	$8a$	$8b$	$8c$
$\lvert C_{H_2}(g)\rvert$	384	128	16	16	3	32	32	32	16	8	8	8
$\hookrightarrow J_2$	$1A$	$2A$	$2B$	$2A$	$3B$	$4A$	$4A$	$4A$	$4A$	$8A$	$8A$	$8A$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	−1	1	1	1	1	−1	−1	−1	−1
χ_3	2	2	2	0	−1	2	2	2	0	0	0	0
χ_4	3	3	−1	−1	0	3	−1	−1	−1	−1	1	1
χ_5	3	3	−1	−1	0	−1	3	−1	−1	1	1	−1
χ_6	3	3	−1	1	0	−1	3	−1	1	−1	−1	1
χ_7	3	3	−1	1	0	3	−1	−1	1	1	−1	−1
χ_8	3	3	−1	−1	0	−1	−1	3	−1	1	−1	1
χ_9	3	3	−1	1	0	−1	−1	3	1	−1	1	−1
χ_{10}	6	6	2	0	0	−2	−2	−2	0	0	0	0
χ_{11}	12	−4	0	−2	0	0	0	0	2	0	0	0
χ_{12}	12	−4	0	2	0	0	0	0	−2	0	0	0

4. Fischer matrices of $\overline{G} = 4^6{:}J_2$

We now calculate the Fischer matrices of $\overline{G} = 4^6{:}J_2$. Following Section 3 of [5], we label the top and bottom of the columns of the Fischer matrix $\mathcal{F}_i$ corresponding to g_i by the sizes of the centralizers of g_{ij}, $1 \leq j \leq c(g_i)$, in $\overline{G}$ and m_{ij}, respectively.

The rows of $\mathcal{F}_i$ are partitioned into parts $\mathcal{F}_{ik}$, $1 \leq k \leq t$, corresponding to the inertia factors $H_1, H_2, \ldots, H_t$, where each $\mathcal{F}_{ik}$ consists of $c(g_{ik})$ rows corresponding to the α_k^{-1}-regular classes (those are the H_k-classes that fuse to the class $[g_i]_G$). Thus, each row of $\mathcal{F}_i$ is labeled by a pair (k, m), where $1 \leq k \leq t$ and $1 \leq m \leq c(g_{ik})$. We list the values of $|C_{\overline{G}}(g_{ij})|$ and m_{ij}, $1 \leq i \leq 27$, $1 \leq j \leq c(g_i)$, in Table 1. The fusions of classes of H_2 and H_3 into classes of G are given in Tables 3 and 4, respectively. Since the size of the Fischer matrix $\mathcal{F}_i$ is $c(g_i)$, it follows from Table 1 that the sizes of the Fischer matrices of $\overline{G} = 4^6{:}J_2$ range between 1 and 8 for every $i \in \{1, 2, \ldots, 21\}$.

The Fisher matrices have interesting arithmetic properties (see Proposition 3.6 in [5]). We used these properties to calculate some entries of these matrices and construct systems of algebraic equations. We solved these systems of equations using the symbolic mathematical package Maxima [19] and, hence, computed all of the Fisher matrices $\overline{G}$ that we list below.

$\mathcal{F}_1$

g_1		g_{11}	g_{12}	g_{13}
$o(g_{1j})$		1	2	2
$\lvert C_{\overline{G}}(g_{1j})\rvert$		2 477 260 800	1 570 864	983 040
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$			
$(1,1)$	604 800	1	1	1
$(2,1)$	384	1 575	39	-25
$(3,1)$	240	2 520	-40	24
m_{1j}		1	1 575	2 520

$\mathcal{F}_2$

g_2		g_{21}	g_{22}	g_{23}	g_{24}
$o(g_{2j})$		2	2	4	4
$\lvert C_{\overline{G}}(g_{2j})\rvert$		491 520	32 768	4 096	4 096
(k,m)	$\lvert C_{H_k}(g_{2km})\rvert$				
$(1,1)$	1 920	1	1	1	1
$(2,1)$	128	15	15	-1	-1
$(2,2)$	16	120	-8	-8	8
$(3,1)$	16	120	-8	8	-8
m_{2j}		16	240	1 920	1 920

$\mathcal{F}_3$

g_3		g_{31}	g_{32}	g_{33}	g_{34}	g_{35}	g_{36}	g_{37}	g_{38}
$o(g_{3j})$		2	4	4	4	4	4	4	4
$\lvert C_{\overline{G}}(g_{3j})\rvert$		15 360	15 360	15 360	15 360	1 024	1 024	1 024	1 024
(k,m)	$\lvert C_{H_k}(g_{3km})\rvert$								
$(1,1)$	240	1	1	1	1	1	1	1	1
$(2,1)$	16	15	15	15	15	-1	-1	-1	-1
$(3,1)$	240	1	-1	-1	1	-1	-1	1	1
$(3,2)$	240	1	-1	1	-1	-1	1	1	-1
$(3,3)$	240	1	1	-1	-1	-1	1	-1	1
$(3,4)$	16	15	-15	15	-15	1	-1	1	-1
$(3,5)$	16	15	-15	-15	15	1	-1	-1	1
$(3,6)$	16	15	15	-15	-15	1	1	-1	-1
m_{3j}		64	64	64	64	960	960	960	960

$\mathcal{F}_4$

g_4		g_{41}
$o(g_{4j})$		3
$\lvert C_{\overline{G}}(g_{4j})\rvert$		1 080
(k,m)	$\lvert C_{H_k}(g_{4km})\rvert$	
$(1,1)$	1 080	1
m_{4j}		4 096

$\mathcal{F}_5$

g_5		g_{51}	g_{52}	g_{53}
$o(g_{5j})$		3	6	6
$\lvert C_{\overline{G}}(g_{5j})\rvert$		576	192	48
(k,m)	$\lvert C_{H_k}(g_{5km})\rvert$			
$(1,1)$	36	1	1	1
$(2,1)$	3	12	-4	0
$(3,1)$	12	3	3	-1
m_{5j}		256	768	3 072

$\mathcal{F}_6$

g_6		g_{61}	g_{62}	g_{63}	g_{64}	g_{65}
$o(g_{6j})$		4	4	4	4	4
$\lvert C_{\overline{G}}(g_{6j})\rvert$		1 536	512	512	512	256
(k,m)	$\lvert C_{H_k}(g_{6km})\rvert$					
$(1,1)$	96	1	1	1	1	1
$(2,1)$	32	3	−1	−1	3	−1
$(2,2)$	32	3	3	−1	−1	−1
$(2,3)$	32	3	−1	3	−1	−1
$(2,4)$	16	6	−2	−2	−2	2
m_{6j}		256	768	768	768	1 536

$\mathcal{F}_7$

g_7		g_{71}	g_{72}
$o(g_{7j})$		5	10
$\lvert C_{\overline{G}}(g_{7j})\rvert$		4 800	320
(k,m)	$\lvert C_{H_k}(g_{7km})\rvert$		
$(1,1)$	300	1	1
$(3,1)$	20	15	−1
m_{7j}		256	3 840

$\mathcal{F}_8$

g_8		g_{81}	g_{82}
$o(g_{8j})$		5	10
$\lvert C_{\overline{G}}(g_{8j})\rvert$		4 800	320
(k,m)	$\lvert C_{H_k}(g_{8km})\rvert$		
$(1,1)$	300	1	1
$(3,1)$	20	15	−1
m_{8j}		256	3 840

$\mathcal{F}_9$

g_9		g_{91}
$o(g_{9j})$		5
$\lvert C_{\overline{G}}(g_{9j})\rvert$		50
(k,m)	$\lvert C_{H_k}(g_{9km})\rvert$	
$(1,1)$	50	1
m_{9j}		4 096

$\mathcal{F}_{10}$

g_{10}		$g_{10,1}$
$o(g_{10j})$		5
$\lvert C_{\overline{G}}(g_{10j})\rvert$		50
(k,m)	$\lvert C_{H_k}(g_{10km})\rvert$	
$(1,1)$	50	1
m_{10j}		4 096

$\mathcal{F}_{11}$

g_{11}		$g_{11,1}$
$o(g_{11j})$		6
$\lvert C_{\overline{G}}(g_{11j})\rvert$		24
(k,m)	$\lvert C_{H_k}(g_{11km})\rvert$	
$(1,1)$	24	1
m_{11j}		4 096

$\mathcal{F}_{12}$

g_{12}		$g_{12,1}$	$g_{12,2}$	$g_{12,3}$	$g_{12,4}$
$o(g_{12j})$		6	12	12	12
$\lvert C_{\overline{G}}(g_{12j})\rvert$		48	48	48	48
(k,m)	$\lvert C_{H_k}(g_{12km})\rvert$				
$(1,1)$	12	1	1	1	1
$(3,1)$	12	1	−1	1	−1
$(3,2)$	12	1	1	−1	−1
$(3,3)$	12	1	−1	−1	1
m_{12j}		1 024	1 024	1 024	1 024

$\mathcal{F}_{13}$

g_{13}		$g_{13,1}$
$o(g_{13j})$		7
$\lvert C_{\overline{G}}(g_{13j})\rvert$		7
(k,m)	$\lvert C_{H_k}(g_{13km})\rvert$	
$(1,1)$	7	1
m_{13j}		4096

$\mathcal{F}_{14}$

g_{14}		$g_{14,1}$	$g_{14,2}$	$g_{14,3}$	$g_{14,4}$
$o(g_{14j})$		8	8	8	8
$\lvert C_{\overline{G}}(g_{14j})\rvert$		32	32	32	32
(k,m)	$\lvert C_{H_k}(g_{14km})\rvert$				
$(1,1)$	8	1	1	1	1
$(2,1)$	12	1	−1	1	−1
$(2,2)$	12	1	1	−1	−1
$(2,3)$	12	1	−1	−1	1
m_{14j}		1024	1024	1024	1024

$\mathcal{F}_{15}$

g_{15}		$g_{15,1}$	$g_{15,2}$	$g_{15,3}$	$g_{15,4}$
$o(g_{15j})$		10	20	20	20
$\lvert C_{\overline{G}}(g_{15j})\rvert$		80	80	80	80
(k,m)	$\lvert C_{H_k}(g_{15km})\rvert$				
$(1,1)$	8	1	1	1	1
$(3,1)$	12	1	-1	1	-1
$(3,2)$	12	1	1	-1	-1
$(3,3)$	12	1	-1	-1	1
m_{15j}		1 024	1 024	1 024	1 024

$\mathcal{F}_{16}$

g_{16}		$g_{16,1}$	$g_{16,2}$	$g_{16,3}$	$g_{16,4}$
$o(g_{16j})$		10	20	20	20
$\lvert C_{\overline{G}}(g_{16j})\rvert$		80	80	80	80
(k,m)	$\lvert C_{H_k}(g_{16km})\rvert$				
$(1,1)$	8	1	1	1	1
$(3,1)$	12	1	-1	1	-1
$(3,2)$	12	1	1	-1	-1
$(3,3)$	12	1	-1	-1	1
m_{16j}		1 024	1 024	1 024	1 024

$\mathcal{F}_{17}$

g_{17}		$g_{17,1}$
$o(g_{17j})$		10
$\lvert C_{\overline{G}}(g_{17j})\rvert$		10
(k,m)	$\lvert C_{H_k}(g_{17km})\rvert$	
$(1,1)$	10	1
m_{17j}		4 096

$\mathcal{F}_{18}$

g_{18}		$g_{18,1}$
$o(g_{18j})$		10
$\lvert C_{\overline{G}}(g_{18j})\rvert$		10
(k,m)	$\lvert C_{H_k}(g_{18km})\rvert$	
$(1,1)$	10	1
m_{18j}		4 096

$\mathcal{F}_{19}$

g_{19}		$g_{19,1}$
$o(g_{19j})$		12
$\lvert C_{\overline{G}}(g_{19j})\rvert$		12
(k,m)	$\lvert C_{H_k}(g_{19km})\rvert$	
$(1,1)$	12	1
m_{19j}		4 096

$\mathcal{F}_{20}$

g_{20}		$g_{20,1}$
$o(g_{20j})$		15
$\lvert C_{\overline{G}}(g_{20j})\rvert$		15
(k,m)	$\lvert C_{H_k}(g_{20km})\rvert$	
$(1,1)$	15	1
m_{20j}		4 096

$\mathcal{F}_{21}$

g_{21}		$g_{21,1}$
$o(g_{21j})$		15
$\lvert C_{\overline{G}}(g_{21j})\rvert$		15
(k,m)	$\lvert C_{H_k}(g_{21km})\rvert$	
$(1,1)$	15	1
m_{21j}		4 096

5. Character table of $\overline{G} = 4^6{:}J_2$

In Sections 2, 3, and 4, we have determined:

- the conjugacy classes of $\overline{G} = 4^6{:}J_2$ (Table 1);
- the inertia factors H_1, H_2, and H_3;
- the character tables of all inertia factor groups of G (the Atlas together with Tables 3 and 4); in these two tables, we also supplied the fusions of the classes of the inertia factors H_2 and H_3 into classes of G;
- the Fischer matrices of $\overline{G}$ (see Section 4).

Following [2, 5], without any difficulties, one can construct the full character table of $\overline{G}$ in the format of Clifford–Fischer theory. The table will be composed of 63 parts corresponding to 21 cosets and three inertia factor groups. The full character table of $\overline{G}$ is a 53×53 $\mathbb{R}$-valued matrix, and we give it in the format of Clifford–Fischer theory in Table 5. We conclude by remarking that the accuracy of this character table has been tested using GAP.

Acknowledgements

The author would like to thank Professor J. Moori, from whom he studied group theory and representation theory. The author would also like to thank the University of Limpopo for providing financial support.

REFERENCES

1. Ali F., Moori J. The Fischer–Clifford matrices and character table of a maximal subgroup of Fi_{24}. *Algebra Colloq.*, 2010. Vol. 17, No. 3. P. 389–414. DOI: 10.1142/S1005386710000386
2. Basheer A. B. M. *Clifford-Fischer Theory Applied to Certain Groups Associated with Symplectic, Unitary and Thompson Groups*. PhD Thesis. Pietermaitzburg: University of KwaZulu-Natal, 2012.
3. Basheer A. B. M. On a group involving the automorphism of the Janko group J_2. *J. Indones. Math. Soc.*, 2023. Vol. 29, No. 2. P. 197–216. DOI: 10.22342/jims.29.2.1371.197-216
4. Basheer A. B. M. On a group extension involving the Suzuki group $Sz(8)$. *Afr. Mat.*, 2023. Vol. 34, No. 4. Art. no. 96. DOI: 10.1007/s13370-023-01130-z
5. Basheer A. B. M., Moori J. Fischer matrices of Dempwolff group $2^{5\cdot}GL(5,2)$. *Int. J. Group Theory*, 2012. Vol. 1, No. 4. P. 43–63. DOI: 10.22108/IJGT.2012.1590
6. Basheer A. B. M., Moori J. On the non-split extension group $2^{6\cdot}Sp(6,2)$. *Bull. Iranian Math. Soc.*, 2013. Vol. 39, No. 6. P. 1189–1212.
7. Basheer A. B. M.,Moori J. On the non-split extension $2^{2n\cdot}Sp(2n,2)$. *Bull. Iranian Math. Soc.*, 2015. Vol. 41, No. 2. P. 499–518.
8. Basheer A. B. M., Moori J. On a maximal subgroup of the Thompson simple group. *Math. Commun.*, 2015. Vol. 20, No. 2. P. 201–218. URL: https://hrcak.srce.hr/149786
9. Basheer A. B. M., Moori J. A survey on Clifford-Fischer theory. In: *Groups St Andrews 2013, C.M. Campbell, M.R. Quick, E.F. Robertson, C.M. Roney-Dougal (eds.)* London Math. Soc. Lecture Note Ser., vol. 422. Cambridge University Press, 2015. P. 160–172. DOI: 10.1017/CBO9781316227343.009
10. Basheer A. B. M., Moori J. On a group of the form $3^7{:}Sp(6,2)$. *Int. J. Group Theory*, 2016. Vol. 5, No. 2. P. 41–59. DOI: 10.22108/IJGT.2016.8047
11. Basheer A. B. M., Moori J. On two groups of the form $2^8{:}A_9$. *Afr. Mat.*, 2017. Vol. 28. P. 1011–1032. DOI: 10.1007/s13370-017-0500-1
12. Basheer A. B. M., Moori J. On a group of the form $2^{10}{:}(U_5(2){:}2)$. *Ital. J. Pure Appl. Math.*, 2017. Vol. 37. P. 645–658. URL: https://ijpam.uniud.it/online_issue/201737/57-BasheerMoori.pdf
13. Basheer A. B. M., Moori J. Clifford-Fischer theory applied to a group of the form $2_-^{1+6}{:}((3^{1+2}{:}8){:}2)$. *Bull. Iranian Math. Soc.*, 2017. Vol. 43, No. 1. P. 41–52.

14. Basheer A. B. M., Moori J. On a maximal subgroup of the affine general linear group $GL(6, 2)$. *Adv. Group Theory Appl.*, 2021. Vol. 11. P. 1–30. DOI: 10.32037/agta-2021-001
15. Bosma W., Cannon J. J. *Handbook of Magma Functions.* Sydney: University of Sydney, 1994.
16. Conway J. H., Curtis R. T., Norton S. P., Parker R. A., Wilson R. A. *ATLAS of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford: Clarendon Press, 1985. 250 p.
17. Fray R. L., Monaledi R. L., Prins A. L. Fischer–Clifford matrices of $2^8{:}(U_4(2){:}2)$ as a subgroup of $O_{10}^+(2)$. *Afr. Mat.*, 2016. Vol. 27. P. 1295–1310. DOI: 10.1007/s13370-016-0410-7
18. *GAP - Groups, Algorithms, Programming - a System for Computational Discrete Algebra.* Version 4.4.10, 2007. URL: http://www.gap-system.org
19. *Maxima, A Computer Algebra System.* Version 5.18.1, 2009. URL: http://maxima.sourceforge.net
20. Moori J. *On the Groups G^+ and $\overline{G}$ of the form $2^{10}{:}M_{22}$ and $2^{10}{:}\overline{M}_{22}$.* PhD Thesis. Birmingham: University of Birmingham, 1975.
21. Moori J. On certain groups associated with the smallest Fischer group. *J. London Math. Soc.*, 1981. Vol. 2. P. 61–67.
22. Wilson R. A. *The Finite Simple Groups.* London: Springer-Verlag, 2009. XV, 298 p. DOI: 10.1007/978-1-84800-988-2
23. Wilson R. A., et al. *Atlas of Finite Group Representations. Version 3.* URL: http://brauer.maths.qmul.ac.uk/Atlas/v3/

Table 4. The character table of $H_3 = 2^2 \times A_5$

$[g]_{H_3}$	1a	2a	2b	2c	2d	2e	2f	2g	3a	5a	5b	6a	6b	6c	10a	10b	10c	10d	10e	10f
$\lvert C_{H_3}(g)\rvert$	240	240	240	240	16	16	16	16	12	20	20	12	12	12	20	20	20	20	10	10
$\hookrightarrow J_2$	1A	2B	2B	2B	2B	2B	2B	2A	3B	5B	5A	6B	6B	6B	10B	10A	10A	10B	10B	10A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	−1	−1	−1	−1	1	1	1	1	1	−1	1	−1	−1	1	−1	−1	1	−1
χ_3	1	−1	1	−1	1	−1	−1	1	1	1	1	−1	−1	1	1	−1	−1	−1	−1	1
χ_4	1	−1	−1	1	−1	1	−1	1	1	1	1	1	−1	−1	−1	−1	1	1	−1	−1
χ_5	3	3	3	3	−1	−1	−1	−1	0	A	A^*	0	0	0	A	A^*	A^*	A	A	A^*
χ_6	3	3	3	3	−1	−1	−1	−1	0	A^*	A	0	0	0	A^*	A	A	A^*	A^*	A
χ_7	3	3	−3	−3	1	1	−1	−1	0	A	A^*	0	0	0	$-A$	A^*	$-A^*$	$-A$	A	$-A^*$
χ_8	3	3	−3	−3	1	1	−1	−1	0	A^*	A	0	0	0	$-A^*$	A	$-A$	$-A^*$	A^*	$-A$
χ_9	3	−3	3	−3	−1	1	1	−1	0	A	A^*	0	0	0	A	$-A^*$	$-A^*$	$-A$	$-A$	A^*
χ_{10}	3	−3	3	−3	−1	1	1	−1	0	A^*	A	0	0	0	A^*	$-A$	$-A$	$-A^*$	$-A^*$	A
χ_{11}	3	−3	−3	3	1	−1	1	−1	0	A	A^*	0	0	0	$-A$	$-A^*$	A^*	A	$-A$	$-A^*$
χ_{12}	3	−3	−3	3	1	−1	1	−1	0	A^*	A	0	0	0	$-A^*$	$-A$	A	A^*	$-A^*$	$-A$
χ_{13}	4	4	4	4	0	0	0	0	1	−1	−1	1	1	1	−1	−1	−1	−1	−1	−1
χ_{14}	4	4	−4	−4	0	0	0	0	1	−1	−1	−1	1	−1	1	−1	1	1	−1	1
χ_{15}	4	−4	4	−4	0	0	0	0	1	−1	−1	−1	−1	1	−1	1	1	1	1	−1
χ_{16}	4	−4	−4	4	0	0	0	0	1	−1	−1	1	−1	−1	1	1	−1	−1	1	1
χ_{17}	5	5	5	5	1	1	1	1	−1	0	0	−1	−1	−1	0	0	0	0	0	0
χ_{18}	5	5	−5	−5	−1	−1	1	1	−1	0	0	1	−1	1	0	0	0	0	0	0
χ_{19}	5	−5	5	−5	1	−1	−1	1	−1	0	0	1	1	−1	0	0	0	0	0	0
χ_{20}	5	−5	−5	5	−1	1	−1	1	−1	0	0	−1	1	1	0	0	0	0	0	0

where in Table 4, $A = (1-\sqrt{5})/2$ and $A^* = (1+\sqrt{5})/2$.

Table 5. The character table of $\overline{G} = 4^6{:}J_2$.

$[g_i]_{J_2}$	1A			2A				2B								3A	3B		
$[g_{ij}]_{\overline{G}}$	1a	2a	2b	2c	2d	4a	4b	2e	4c	4d	4e	4f	4g	4h	4i	3a	3b	6a	6b
$\lvert C_{\overline{G}}(g_{ij})\rvert$	2477260800	1572864	983040	491520	32768	4096	4096	15360	15360	15360	15360	1024	1024	1024	1024	1080	576	192	48
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	14	14	14	−2	−2	−2	−2	2	2	2	2	2	2	2	2	5	−1	−1	−1
χ_3	14	14	14	−2	−2	−2	−2	2	2	2	2	2	2	2	2	5	−1	−1	−1
χ_4	21	21	21	5	5	5	5	−3	−3	−3	−3	−3	−3	−3	−3	3	0	0	0
χ_5	21	21	21	5	5	5	5	−3	−3	−3	−3	−3	−3	−3	−3	3	0	0	0
χ_6	36	36	36	4	4	4	4	0	0	0	0	0	0	0	0	9	0	0	0
χ_7	63	63	63	15	15	15	15	−1	−1	−1	−1	−1	−1	−1	−1	0	3	3	3
χ_8	70	70	70	−10	−10	−10	−10	−2	−2	−2	−2	−2	−2	−2	−2	7	1	1	1
χ_9	70	70	70	−10	−10	−10	−10	−2	−2	−2	−2	−2	−2	−2	−2	7	1	1	1
χ_{10}	90	90	90	10	10	10	10	6	6	6	6	6	6	6	6	9	0	0	0
χ_{11}	126	126	126	14	14	14	14	6	6	6	6	6	6	6	6	−9	0	0	0
χ_{12}	160	160	160	0	0	0	0	4	4	4	4	4	4	4	4	16	1	1	1
χ_{13}	175	175	175	15	15	15	15	−5	−5	−5	−5	−5	−5	−5	−5	−5	1	1	1
χ_{14}	189	189	189	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	0	0	0	0
χ_{15}	189	189	189	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	0	0	0	0
χ_{16}	224	224	224	0	0	0	0	−4	−4	−4	−4	−4	−4	−4	−4	8	−1	−1	−1
χ_{17}	224	224	224	0	0	0	0	−4	−4	−4	−4	−4	−4	−4	−4	8	−1	−1	−1
χ_{18}	225	225	225	−15	−15	−15	−15	5	5	5	5	5	5	5	5	0	3	3	3
χ_{19}	288	288	288	0	0	0	0	4	4	4	4	4	4	4	4	0	−3	−3	−3
χ_{20}	300	300	300	−20	−20	−20	−20	0	0	0	0	0	0	0	0	−15	0	0	0
χ_{21}	336	336	336	16	16	16	16	0	0	0	0	0	0	0	0	−6	0	0	0
χ_{22}	1 575	39	−25	135	7	−9	7	15	15	15	15	−1	−1	−1	−1	0	12	−4	0
χ_{23}	1 575	39	−25	−105	23	7	−9	15	15	15	15	−1	−1	−1	−1	0	12	−4	0
χ_{24}	3 150	78	−50	30	30	−2	−2	30	30	30	30	−2	−2	−2	−2	0	−12	4	0
χ_{25}	4 725	117	−75	−75	53	5	−11	−15	−15	−15	−15	1	1	1	1	0	0	0	0
χ_{26}	4 725	117	−75	165	37	−11	5	−15	−15	−15	−15	1	1	1	1	0	0	0	0
χ_{27}	4 725	117	−75	165	37	−11	5	−15	−15	−15	−15	1	1	1	1	0	0	0	0
χ_{28}	4 725	117	−75	165	37	−11	5	−15	−15	−15	−15	1	1	1	1	0	0	0	0
χ_{29}	4 725	117	−75	−75	53	5	−11	−15	−15	−15	−15	1	1	1	1	0	0	0	0
χ_{30}	4 725	117	−75	−75	53	5	−11	−15	−15	−15	−15	1	1	1	1	0	0	0	0
χ_{31}	9 450	234	−150	90	90	−6	−6	30	30	30	30	−2	−2	−2	−2	0	0	0	0
χ_{32}	18 900	468	−300	180	−76	−12	20	0	0	0	0	0	0	0	0	0	0	0	0
χ_{33}	18 900	468	−300	−300	−44	20	−12	0	0	0	0	0	0	0	0	0	0	0	0
χ_{34}	2 520	−40	24	120	−8	8	−8	48	−16	−16	−16	0	0	0	0	0	3	3	−1
χ_{35}	2 520	−40	24	120	−8	8	−8	−16	−16	−16	48	0	0	0	0	0	3	3	−1
χ_{36}	2 520	−40	24	120	−8	8	−8	−16	−16	48	−16	0	0	0	0	0	3	3	−1
χ_{37}	2 520	−40	24	120	−8	8	−8	−16	48	−16	−16	0	0	0	0	0	3	3	−1
χ_{38}	7 560	−120	72	−120	8	−8	8	−36	12	12	12	−4	12	−4	−4	0	0	0	0
χ_{39}	7 560	−120	72	−120	8	−8	8	−36	12	12	12	−4	12	−4	−4	0	0	0	0
χ_{40}	7 560	−120	72	−120	8	−8	8	12	12	12	−36	−4	−4	12	−4	0	0	0	0
χ_{41}	7 560	−120	72	−120	8	−8	8	12	12	12	−36	−4	−4	12	−4	0	0	0	0
χ_{42}	7 560	−120	72	−120	8	−8	8	12	12	−36	12	−4	−4	−4	12	0	0	0	0
χ_{43}	7 560	−120	72	−120	8	−8	8	12	12	−36	12	−4	−4	−4	12	0	0	0	0
χ_{44}	7 560	−120	72	−120	8	−8	8	12	−36	12	12	12	−4	−4	−4	0	0	0	0
χ_{45}	7 560	−120	72	−120	8	−8	8	12	−36	12	12	12	−4	−4	−4	0	0	0	0
χ_{46}	10 080	−160	96	0	0	0	0	12	−4	−4	−4	−4	12	−4	−4	0	3	3	−1
χ_{47}	10 080	−160	96	0	0	0	0	−4	−4	−4	12	−4	−4	12	−4	0	3	3	−1
χ_{48}	10 080	−160	96	0	0	0	0	−4	−4	12	−4	−4	−4	−4	12	0	3	3	−1
χ_{49}	10 080	−160	96	0	0	0	0	−4	12	−4	−4	12	−4	−4	−4	0	3	3	−1
χ_{50}	12 600	−200	120	120	−8	8	−8	60	−20	−20	−20	−4	12	−4	−4	0	−3	−3	1
χ_{51}	12 600	−200	120	120	−8	8	−8	−20	−20	−20	60	−4	−4	12	−4	0	−3	−3	1
χ_{52}	12 600	−200	120	120	−8	8	−8	−20	−20	60	−20	−4	−4	−4	12	0	−3	−3	1
χ_{53}	12 600	−200	120	120	−8	8	−8	−20	60	−20	−20	12	−4	−4	−4	0	−3	−3	1

Continued on next page

Table 5 (continued)

$[g_i]_{J_2}$	$4A$					$5A$		$5B$		$5C$	$5D$	$6A$	$6B$				$7A$	$8A$				$10A$				$10B$				$10C$	$10D$	$12A$	$15A$	$15B$
$[g_{ij}]_{\overline{G}}$	$4j$	$4k$	$4l$	$4m$	$4n$	$5a$	$10a$	$5b$	$10b$	$5c$	$5d$	$6c$	$6d$	$12a$	$12b$	$12c$	$7a$	$8a$	$8b$	$8c$	$8d$	$10c$	$20a$	$20b$	$20c$	$10d$	$20d$	$20e$	$20f$	$10e$	$10f$	$12d$	$15a$	$15b$
$\lvert C_{\overline{G}}(g_{ij})\rvert$	1536	512	512	512	256	4800	320	4800	320	50	50	24	48	48	48	48	7	32	32	32	32	80	80	80	80	80	80	80	80	10	10	12	15	15
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	2	2	2	2	2	A	A	A^*	A^*	G	G^*	1	-1	-1	-1	-1	0	0	0	0	0	$-F$	$-F$	$-F$	$-F$	$-F^*$	$-F^*$	$-F^*$	$-F^*$	F	F^*	-1	0	0
χ_3	2	2	2	2	2	A^*	A^*	A	A	G^*	G	1	-1	-1	-1	-1	0	0	0	0	0	$-F^*$	$-F^*$	$-F^*$	$-F^*$	$-F$	$-F$	$-F$	$-F$	F^*	F	-1	0	0
χ_4	1	1	1	1	1	B	B	B^*	B^*	H	H^*	-1	0	0	0	0	0	-1	-1	-1	-1	$-F^*$	$-F^*$	$-F^*$	$-F^*$	$-F$	$-F$	$-F$	$-F$	0	0	1	F	F^*
χ_5	1	1	1	1	1	B^*	B^*	B	B	H^*	H	-1	0	0	0	0	0	-1	-1	-1	-1	$-F$	$-F$	$-F$	$-F$	$-F^*$	$-F^*$	$-F^*$	$-F^*$	0	0	1	F^*	F
χ_6	4	4	4	4	4	-4	-4	-4	-4	1	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	1	-1	-1
χ_7	3	3	3	3	3	3	3	3	3	-2	-2	0	-1	-1	-1	-1	0	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0
χ_8	2	2	2	2	2	C	C	C^*	C^*	0	0	-1	1	1	1	1	0	0	0	0	0	F	F	F	F	F^*	F^*	F^*	F^*	0	0	-1	$-F$	$-F^*$
χ_9	2	2	2	2	2	C^*	C^*	C	C	0	0	-1	1	1	1	1	0	0	0	0	0	F^*	F^*	F^*	F^*	F	F	F	F	0	0	-1	$-F^*$	$-F$
χ_{10}	-2	-2	-2	-2	-2	5	5	5	5	0	0	1	0	0	0	0	-1	0	0	0	0	1	1	1	1	1	1	1	1	0	0	1	-1	-1
χ_{11}	2	2	2	2	2	1	1	1	1	1	1	-1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	-1	-1	-1	1	1
χ_{12}	0	0	0	0	0	-5	-5	-5	-5	0	0	0	1	1	1	1	-1	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	1	1
χ_{13}	-1	-1	-1	-1	-1	0	0	0	0	0	0	3	1	1	1	1	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	-1	0	0
χ_{14}	-3	-3	-3	-3	-3	A	A	A^*	A^*	G	G^*	0	0	0	0	0	0	1	1	1	1	$-F$	$-F$	$-F$	$-F$	$-F^*$	$-F^*$	$-F^*$	$-F^*$	$-F$	$-F^*$	0	0	0
χ_{15}	-3	-3	-3	-3	-3	A^*	A^*	A	A	G^*	G	0	0	0	0	0	0	1	1	1	1	$-F^*$	$-F^*$	$-F^*$	$-F^*$	$-F$	$-F$	$-F$	$-F$	$-F^*$	$-F$	0	0	0
χ_{16}	0	0	0	0	0	D	D	D^*	D^*	$-H^*$	$-H$	0	-1	-1	-1	-1	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	F^*	F
χ_{17}	0	0	0	0	0	D^*	D^*	D	D	$-H$	$-H^*$	0	-1	-1	-1	-1	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	F	F^*
χ_{18}	-3	-3	-3	-3	-3	0	0	0	0	0	0	0	-1	-1	-1	-1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{19}	0	0	0	0	0	3	3	3	3	-2	-2	0	1	1	1	1	1	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0
χ_{20}	4	4	4	4	4	0	0	0	0	0	0	1	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
χ_{21}	0	0	0	0	0	-4	-4	-4	-4	1	1	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	-1	-1
χ_{22}	15	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	-1	3	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{23}	3	3	3	3	-5	0	0	0	0	0	0	0	0	0	0	0	0	1	-3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{24}	18	2	2	2	-6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{25}	-3	-3	-3	13	-3	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	-3	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{26}	9	-7	-7	9	1	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	3	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{27}	9	-7	9	-7	1	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	3	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{28}	9	9	-7	-7	1	0	0	0	0	0	0	0	0	0	0	0	0	3	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{29}	-3	-3	13	-3	-3	0	0	0	0	0	0	0	0	0	0	0	0	1	1	-3	1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{30}	-3	13	-3	-3	-3	0	0	0	0	0	0	0	0	0	0	0	0	-3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{31}	-18	-2	-2	-2	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{32}	-12	4	4	4	-4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{33}	12	-4	-4	-4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{34}	0	0	0	0	0	15	-1	15	-1	0	0	0	3	-1	-1	-1	0	0	0	0	0	3	-1	-1	-1	3	-1	-1	-1	0	0	0	0	0
χ_{35}	0	0	0	0	0	15	-1	15	-1	0	0	0	-1	3	-1	-1	0	0	0	0	0	-1	3	-1	-1	-1	3	-1	-1	0	0	0	0	0
χ_{36}	0	0	0	0	0	15	-1	15	-1	0	0	0	-1	-1	3	-1	0	0	0	0	0	-1	-1	-1	3	-1	-1	3	-1	0	0	0	0	0
χ_{37}	0	0	0	0	0	15	-1	15	-1	0	0	0	-1	-1	-1	3	0	0	0	0	0	-1	-1	3	-1	-1	-1	-1	3	0	0	0	0	0
χ_{38}	0	0	0	0	0	E	$-F$	E^*	$-F^*$	0	0	0	0	0	0	0	0	0	0	0	0	A	$-F$	$-F$	$-F$	A^*	$-F^*$	$-F^*$	$-F^*$	0	0	0	0	0
χ_{39}	0	0	0	0	0	E^*	$-F^*$	E	$-F$	0	0	0	0	0	0	0	0	0	0	0	0	A^*	$-F^*$	$-F^*$	$-F^*$	A	$-F$	$-F$	$-F$	0	0	0	0	0
χ_{40}	0	0	0	0	0	E	$-F$	E^*	$-F^*$	0	0	0	0	0	0	0	0	0	0	0	0	$-F$	A	$-F$	$-F$	$-F^*$	A^*	$-F^*$	$-F^*$	0	0	0	0	0
χ_{41}	0	0	0	0	0	E^*	$-F^*$	E	$-F$	0	0	0	0	0	0	0	0	0	0	0	0	$-F^*$	A^*	$-F^*$	$-F^*$	$-F$	A	$-F$	$-F$	0	0	0	0	0
χ_{42}	0	0	0	0	0	E	$-F$	E^*	$-F^*$	0	0	0	0	0	0	0	0	0	0	0	0	$-F$	$-F$	$-F$	A	$-F^*$	$-F^*$	A^*	$-F^*$	0	0	0	0	0
χ_{43}	0	0	0	0	0	E^*	$-F^*$	E	$-F$	0	0	0	0	0	0	0	0	0	0	0	0	$-F^*$	$-F^*$	$-F^*$	A^*	$-F$	$-F$	A	$-F$	0	0	0	0	0
χ_{44}	0	0	0	0	0	E	$-F$	E^*	$-F^*$	0	0	0	0	0	0	0	0	0	0	0	0	$-F$	$-F$	A	$-F$	$-F^*$	$-F^*$	$-F^*$	A^*	0	0	0	0	0
χ_{45}	0	0	0	0	0	E^*	$-F^*$	E	$-F$	0	0	0	0	0	0	0	0	0	0	0	0	$-F^*$	$-F^*$	A^*	$-F^*$	$-F$	$-F$	$-F$	A	0	0	0	0	0
χ_{46}	0	0	0	0	0	-15	1	-15	1	0	0	0	3	-1	-1	-1	0	0	0	0	0	-3	1	1	1	-3	1	1	1	0	0	0	0	0
χ_{47}	0	0	0	0	0	-15	1	-15	1	0	0	0	-1	3	-1	-1	0	0	0	0	0	1	-3	1	1	1	-3	1	1	0	0	0	0	0
χ_{48}	0	0	0	0	0	-15	1	-15	1	0	0	0	-1	-1	3	-1	0	0	0	0	0	1	1	1	-3	1	1	-3	1	0	0	0	0	0
χ_{49}	0	0	0	0	0	-15	1	-15	1	0	0	0	-1	-1	-1	3	0	0	0	0	0	1	1	-3	1	1	1	1	-3	0	0	0	0	0
χ_{50}	0	0	0	0	0	0	0	0	0	0	0	0	-3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{51}	0	0	0	0	0	0	0	0	0	0	0	0	1	-3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{52}	0	0	0	0	0	0	0	0	0	0	0	0	1	1	-3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{53}	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	-3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

where in Table 5, $A = 3(1-\sqrt{5})/2$, $B = (7-\sqrt{5})/2$, $C = (1-\sqrt{5})/2$, $D = -1+2\sqrt{5}$, $E = 15(1-\sqrt{5})/2$, $F = 5(1-\sqrt{5})/2$, $G = (3-\sqrt{5})/2$, $H = 1-\sqrt{5}$ and if $K = a+b\sqrt{m}$ for some $a, b \in \mathbb{R}$ and $m \in \mathbb{N}$, then K^* denotes the number $a - b\sqrt{m}$.

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 44–60

DOI: 10.15826/umj.2024.1.004

SPERNER THEOREMS FOR UNRELATED COPIES OF POSETS AND GENERATING DISTRIBUTIVE LATTICES[1][2]

Gábor Czédli

Bolyai Institute, University of Szeged,
Aradi vértanúk tere 1, H-6720 Szeged, Hungary

czedli@math.u-szeged.hu

Abstract: For a finite poset (partially ordered set) U and a natural number n, let $S(U,n)$ denote the largest number of pairwise unrelated copies of U in the powerset lattice (AKA subset lattice) of an n-element set. If U is the singleton poset, then $S(U,n)$ was determined by E. Sperner in 1928; this result is well known in extremal combinatorics. Later, exactly or asymptotically, Sperner's theorem was extended to other posets by A. P. Dove, J. R. Griggs, G. O. H. Katona, D. J. Stahl, and W. T. Jr. Trotter. We determine $S(U,n)$ for all finite posets with 0 and 1, and we give reasonable estimates for the "V-shaped" 3-element poset and, mainly, for the 4-element poset with 0 and three maximal elements. For a lattice L, let $G_{\min}(L)$ denote the minimum size of generating sets of L. We prove that if U is the poset of the join-irreducible elements of a finite distributive lattice D, then the function $k \mapsto G_{\min}(D^k)$ is the left adjoint of the function $n \mapsto S(U,n)$. This allows us to determine $G_{\min}(D^k)$ in many cases. E.g., for a 5-element distributive lattice D, $G_{\min}(D^{2023}) = 18$ if D is a chain and $G_{\min}(D^{2023}) = 15$ otherwise. The present paper, another recent paper, and a 2021 one indicate that large direct powers of small distributive lattices could be of interest in cryptography.

Keywords: Sperner theorem for partially ordered sets, Antichain of posets, Unrelated copies of a poset, Incomparable copies of a poset, Distributive lattice, Smallest generating set, Minimum-sized Generating set, Cryptography with lattices.

1. Introduction

This paper belongs both to extremal combinatorics and lattice theory, and it is intended to be self-contained for those who know the concept of a free semilattice, that of a distributive lattice, and the relation between lattice orders and lattice operations.

Our main goal is to establish a *bridge* between the combinatorial topic of Sperner (type) theorems and the lattice theoretical topic of minimum generating sets of finite lattices; this goal is accomplished by Theorem 1 in Section 2. If we start from the Sperner (type) theorems proved by Griggs, Stahl, and Trotter [9], Dove and Griggs [6], and Katona and Nagy [10], then the just-mentioned "bridge" can lead only to asymptotic results, in which we are less interested, or to rather special distributive lattices. Hence, we modestly generalize their Sperner theorems, see Observation 1, and we give reasonable estimates for a particular case; see Proposition 1.

A poset (that is, partially ordered set) U is said to be *bounded* if it has a smallest element, denoted by $0 = 0_U$, and a largest element, $1 = 1_U$; these elements are uniquely determined if they both exist. In Section 3, we give an *exact formula* for the maximum number of pairwise unrelated isomorphic copies of a finite *bounded* poset among the subsets of an n-element set; see Observation 1, which is an easy generalization of a result of Griggs, Stahl, and Trotter [9] from chains to bounded posets. The situation becomes more exciting in Section 4, where we present estimates for two particular posets, V and W given in Fig. 1.

[1]This research was supported by the National Research, Development and Innovation Fund of Hungary, under funding scheme K 138892.

[2]*This paper is dedicated to my colleague Eszter K. Horváth, PhD, on her sixtieth birthday.*

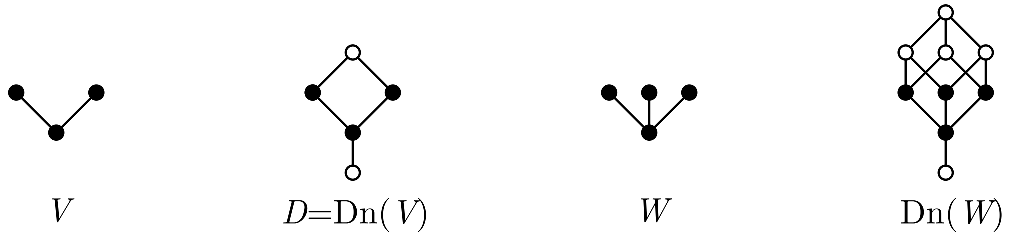


Figure 1. Two posets and the corresponding distributive lattices

The search for small generating sets has more than half a century-long history. Indeed, this topic goes back (at least) to Gelfand an Ponomarev [7]; see Zádori [14] for details of their result on subspace lattices. For small generating sets in some other lattices, see also the introductions and the bibliographic sections of Czédli [1–3]. Recently in [1] and [3], we have pointed out that large lattices and large powers of (small) lattices can have applications in cryptography provided that they have small generating sets. This led to the original motivation of the present paper: we wanted to determine how many elements are needed to generate a large direct power of a small distributive lattice.

Even though we prove only estimates rather than exact Sperner theorems in Section 4, they are sufficient to determine the minimum number of generators of direct powers of the corresponding distributive lattices with quite good accuracy and, in most of cases, exactly; this will be formulated in (4.7) and exemplified explicitly by (4.15) and implicitly by all collections of concrete data displayed in the paper. Note that even less accuracy would be satisfactory from a cryptographic point of view, in which the role of a small *minimum* number of generators is to indicate that there are *many small generating sets*. Hence, in addition to the exact lattice theoretical results that we can obtain by combining Theorem 1 with Observation 1 or (2.12), Section 4 also offers new possibilities for the cryptographic protocols given in [1] and [3].

2. A bridge between combinatorics and lattice theory

The purpose of this section is to generalize a result of Czédli [3] from finite Boolean lattices to finite direct powers of finite distributive lattices. To do so, we are going to borrow several concepts, notations, and ideas from [3] without further notice. Except for the sets $\mathbb{N}^+ := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}^+$, all sets and structures in this paper are assumed to be *finite* even when this is not explicitly mentioned.

Next, we recall some concepts and notations, and introduce a few new ones. For a real number x, the *lower integer part* and the *upper integer part* of x are denoted by $\lfloor x \rfloor$ and $\lceil x \rceil$, respectively. For $n \in \mathbb{N}_0$, note the rule: $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. A function $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ is *non-bounded* if for each $k \in \mathbb{N}_0$, there exists an $n \in \mathbb{N}_0$ such that $f(n) > k$. For a non-bounded function $f\colon \mathbb{N}_0 \to \mathbb{N}_0$, the *left adjoint* f^* of f is the function

$$f^*\colon \mathbb{N}_0 \to \mathbb{N}_0 \quad \text{defined by} \quad k \mapsto \min\{n \in \mathbb{N}_0 : k \le f(n)\}. \tag{2.1}$$

(The terminology "left adjoint", taken from Czédli [2], is explained by categorified posets, but we do not need this fact.) If $f(x) \le f(y)$ holds whenever $x \le y$, then f is an *increasing function*. For $\mathbb{N}_0 \to \mathbb{N}_0$ functions f_1 and f_2, $f_1 \le f_2$ means that $f_1(x) \le f_2(x)$ holds for every $x \in \mathbb{N}_0$. The following lemma follows straightforwardly from definitions and it belongs to folklore, so we do not prove it in the paper.

Lemma 1. *If f, f_1, and f_2 are increasing non-bounded $\mathbb{N}_0 \to \mathbb{N}_0$ functions then so are their left adjoints. Furthermore, for all $n, k \in \mathbb{N}_0$,*

$$k \leq f(n) \quad \textit{if and only if} \quad f^*(k) \leq n, \tag{2.2}$$

$$k > f(n) \quad \textit{if and only if} \quad f^*(k) > n, \tag{2.3}$$

$$f(n) = \max\{y \in \mathbb{N}_0 : f^*(y) \leq n\}, \quad \textit{and} \tag{2.4}$$

$$f_1 \leq f_2 \quad \textit{if and only if} \quad f_2^* \leq f_1^*. \tag{2.5}$$

For a poset U and a natural number $k \in \mathbb{N}^+$, let $kU = (kU, \leq)$ denote the *cardinal sum of k isomorphic copies* of U. That is, if $(U_1; \rho_1)$, ..., $(U_k; \rho_k)$ are pairwise disjoint isomorphic copies of $U = (U; \leq)$, then

$$(kU; \leq) := (U_1 \cup \cdots \cup U_k; \rho_1 \cup \cdots \cup \rho_k).$$

Then for $x \in U_i$ and $y \in U_j$, if $i \neq j$, then neither $x \leq y$ nor $y \leq x$, that is, x and y are *incomparable*, in notation, $x \parallel y$. In other words, U_i and U_j are *unrelated* for $i \neq j$. We obtain the (Hasse) diagram of kU by putting k copies of the diagram of U side by side. For $k \in \mathbb{N}_0$, the $(k+1)$-element chain will be denoted by C_k. Note that $k\mathsf{C}_0$ is the *k-element antichain*. For $n \in \mathbb{N}^+$, $[n]$ will stand for the set $\{1, \dots, n\}$ while $[0] := \emptyset$. For a set A, the *powerset lattice* (also called the *subset lattice*) of A is the lattice $(\{X : X \subseteq A\}; \subseteq)$. In this lattice, which we denote by $\mathsf{P}(A)$ or $(\mathsf{P}(A); \subseteq)$, the operations $\vee$ and $\wedge$ are $\cup$ and $\cap$, respectively. For an element y in a poset U, we denote $\{x \in U : x \leq y\}$ by $\downarrow y$ or, if confusion threatens, by $\downarrow_U y$. Similarly, $\uparrow y$ and $\uparrow_U y$ stand for $\{x \in U : y \leq x\}$.

For posets U_1 and U_2 and a function $\varphi \colon U_1 \to U_2$, φ is an *order embedding* if for all $x, y \in U_1$,

$$x \leq y \iff \varphi(x) \leq \varphi(y).$$

Let $\varphi \colon U_1 \hookrightarrow U_2$ denote that φ is an order embedding. Furthermore, let $U_1 \overset{\text{exists}}{\hookrightarrow} U_2$ denote that there exists an order embedding $\varphi \colon U_1 \hookrightarrow U_2$. For example, if U is a poset, then the function $U \to \mathsf{P}(U)$ defined by $y \mapsto \downarrow_U y$ is an order embedding. Thus,

$$\text{for any poset } U, \text{ we have that } U \overset{\text{exists}}{\hookrightarrow} \mathsf{P}([|U|]). \tag{2.6}$$

If $U_1 \subseteq U_2$ and the function $U_1 \to U_2$ defined by $x \mapsto x$ is an order-embedding, then U_1 is a *subposet* of U_2; this fact is denoted by $U_1 \leq U_2$. A poset cannot be empty by definition; the only exception is that for every poset U, $0U$ is a subposet of (and is embedded into) any other poset; the following definition needs this convention.

Definition 1. *Let U be a finite poset. For $k, n \in \mathbb{N}_0$, let*

$$S(U, n) := \max\{k \in \mathbb{N}_0 : kU \overset{\text{exists}}{\hookrightarrow} \mathsf{P}([n])\} \quad \textit{and} \tag{2.7}$$

$$S^*(U, k) := \min\{n \in \mathbb{N}_0 : kU \overset{\text{exists}}{\hookrightarrow} \mathsf{P}([n])\} = \min\{n \in \mathbb{N}_0 : k \leq S(U, n)\}; \tag{2.8}$$

(2.6) implies that the definition ":=" in (2.8) makes sense. For $n \in \mathbb{N}^+$, let

$$f_{\text{sb}}(n) := \binom{n}{\lfloor n/2 \rfloor} \quad \textit{and} \quad f_{\text{sb}}^*(k) := \min\{n \in \mathbb{N}^+ : k \leq f_{\text{sb}}(n)\}. \tag{2.9}$$

For the sake of better outlook and optical readability, let us agree that in in-line formulas, we often write $\mathrm{C}_{\text{bin}}(m, t)$ instead of $\binom{m}{t}$; especially when m or t is a complicated expression with subscripts. With this convention, $f_{\text{sb}}(n) = \mathrm{C}_{\text{bin}}(n, \lfloor n/2 \rfloor)$.

Remark 1. The notation in Definition 1 is coherent with (2.1) since the functions $S^*(U,-)\colon \mathbb{N}_0 \to \mathbb{N}_0$ defined by $k \mapsto S^*(U,k)$ and $f^*_{\rm sb}$ are the left adjoints of the functions $S(U,-)\colon \mathbb{N}_0 \to \mathbb{N}_0$ defined by $n \mapsto S(U,n)$ and $f_{\rm sb}$, respectively. (For $S^*(U,-)$, this follows immediately from $kU \overset{\text{exists}}{\hookrightarrow} \mathsf{P}([n]) \iff k \le S(U,n)$.)

The remark above enables us to benefit from Lemma 1. Note that the notation $f_{\rm sb}$ comes from Sperner's original Binomial coefficient as a Function. For subsets X and Y of $[n]$, using the terminology of Griggs, Stahl, and Trotter [9], we say that X and Y are *unrelated* if $x \parallel y$ for all $x \in X$ and $y \in Y$. So $S(U,n)$ is the maximum number of pairwise unrelated isomorphic copies of U in $\mathsf{P}([n])$.

With the notation introduced in Definition 1, Sperner's Theorem from [13] asserts that $S(\mathsf{C}_0,n) = f_{\rm sb}(n)$ while a Sperner theorem (i.e., a Sperner-type theorem) proved by Griggs, Stahl and Trotter [9, Theorem 2] asserts that

$$\text{for } t \in \mathbb{N}^+, \quad S(\mathsf{C}_t,n) = f_{\rm sb}(n-t), \quad \text{that is,} \quad S(\mathsf{C}_t,n) = \binom{n-t}{\lfloor (n-t)/2 \rfloor}. \tag{2.10}$$

Note that, by convention, $f_{\rm sb}(n-t) = 0$ for $n < t$. For later reference, some values of $S(\mathsf{C}_4,n)$ are as follows; here and later: the numbers of our tables in exponential forms are approximations in which the significands are correctly rounded to the given digits

n	17	18	2024	2025	2026
$S(\mathsf{C}_4,n)$	1 716	3 432	$2.137 \cdot 10^{606}$	$4.272 \cdot 10^{606}$	$8.544 \cdot 10^{606}$

. (2.11)

The *length* of a finite poset U is the largest t such that C_t is a subposet of U. The result cited in (2.10) has been generalized by Katona and Nagy [10, Theorem 4.3] to the following one.

If U is a finite poset of length t such that $S^*(U,1) = t$ then,

$$\text{for every} \quad n \in \mathbb{N}_0, \quad S(U,n) = f_{\rm sb}(n-t). \tag{2.12}$$

A *proper sublattice* of a lattice L is a nonempty subset X of L such that $X \neq L$ and X is closed with respect to $\vee$ and $\wedge$. A subset Y of L is a *generating set* of L if no proper sublattice of L includes Y. As L is assumed to be finite, the *least size of a generating set* of L makes sense; we denote it by

$$G_{\min}(L) := \min\{|Y| : Y \text{ is a generating set of } L\}. \tag{2.13}$$

In the k-th direct power $L^k := L \times \cdots \times L$ (k-fold direct product) of L, the lattice operations are performed component-wise; we are interested in $G_{\min}(L^k)$ for some distributive lattices L. The set of *join-irreducible* elements of L is denoted by $J(L)$; by definition, $x \in L$ belongs to $J(L)$ if and only if x covers exactly one element; in particular, the smallest element $0 = 0_L$ of L is not in $J(L)$. With the order inherited from L, $J(L) = (J(L); \le)$ is a poset.

Now that we have (2.13) and Definition 1, we can formulate the main result of the paper.

Theorem 1. *If D is a finite distributive lattice and $2 \le k \in \mathbb{N}^+$, then $G_{\min}(D^k) = S^*(J(D),k)$.*

P r o o f. We are going to use lots of ideas from Czédli [3], where the theorem was proved for the particular case when D is a finite Boolean lattice.

For $t \in \mathbb{N}^+$, denote by $F_{\rm meet}(t) = F_{\rm meet}(x_1,\dots,x_t)$ the free meet-semilattice with free generators $x_1, \dots, x_t$. We know from folklore and from §4 in Page 240 of McKenzie, McNulty and Taylor [12] (and it is not hard to see) that $F_{\rm meet}(t)$ is a subposet of $\mathsf{P}([t])$; in fact, $F_{\rm meet}(t)$ is (order isomorphic to) $\mathsf{P}([t]) \setminus \{[t]\}$.

Let $U := J(D)$. With $U_1 := U \times \{0\} \times \cdots \times \{0\}$, ..., $U_k := \{0\} \times \cdots \times \{0\} \times U$, it is clear that $U_1 \cup \cdots \cup U_k \subseteq J(D^k)$. As each element $\vec{x}$ of D^k is the join of some elements of $U_1 \cup \cdots \cup U_k$, we have that $J(D^k) = U_1 \cup \cdots \cup U_k \cong kU$.

To prove that $G_{\min}(D^k) \geq S^*(J(D), k)$, let $n := G_{\min}(D^k)$ and pick an n-element generating set $\{g_1, \dots, g_n\}$ of D^k. By (2.2), we need to show that $k \leq S(U, n)$. So, we need to embed kU into $\mathsf{P}([n])$. As $F_{\mathrm{meet}}(n) = F_{\mathrm{meet}}(x_1, \dots, x_n)$ is embedded into $\mathsf{P}([n])$ and $kU \cong J(D^k)$, it suffices to give an order embedding $J(D^k) \to F_{\mathrm{meet}}(n)$. In the *meet-semilattice reduct* $(D^k; \wedge)$ of the lattice $(D^k; \wedge, \vee)$, let $B := [g_1, \dots, g_n]_\wedge$ denote the meet-subsemilattice generated by $\{g_1, \dots, g_n\}$. By the distributivity of the *lattice* D^k, each $u \in J(D^k)$ is obtained so that we apply a disjunctive normal form to the generators $g_1, \dots, g_n$. That is, u is the join of some meets of the generators. By the join-irreducibility of u, the join is superfluous, and so u is the meet of some of the $g_1, \dots, g_n$. Thus, $u \in B$, and we have seen that $J(D^k) \subseteq B$. Since $F_{\mathrm{meet}}(n)$ is free, there exists a (unique) meet homomorphism $\varphi\colon F_{\mathrm{meet}}(n) \to B$ such that $\varphi(x_i) = g_i$ for all $i \in \{1, \dots, n\}$. Since each of the generators g_i of B is a φ-image, φ is surjective.

Define a function $\psi\colon B \to F_{\mathrm{meet}}(n)$ by the rule

$$\psi(b) := \bigwedge\{p \in F_{\mathrm{meet}}(n) : \varphi(p) = b\}.$$

Then, for every $b \in B$,

$$\varphi(\psi(b)) = \varphi\big(\bigwedge\{p \in F_{\mathrm{meet}}(n) : \varphi(p) = b\}\big) = \bigwedge\{\varphi(p) \in F_{\mathrm{meet}}(n) : \varphi(p) = b\} = b$$

shows that $\varphi(\psi(b)) = b$. Hence, $\psi(b)$ is the least preimage of b with respect to φ.

Now assume that $b_1, b_2 \in B$. If $b_1 \leq b_2$, then

$$\varphi(\psi(b_1) \wedge \psi(b_2)) = \varphi(\psi(b_1)) \wedge \varphi(\psi(b_2)) = b_1 \wedge b_2 = b_1$$

shows that $\psi(b_1) \wedge \psi(b_2)$ is a φ-preimage of b_1. As $\psi(b_1)$ is the smallest preimage, we obtain that $\psi(b_1) \leq \psi(b_1) \wedge \psi(b_2) \leq \psi(b_2)$, that is, ψ is order-preserving.

Conversely, if $\psi(b_1) \leq \psi(b_2)$, then

$$b_1 = \varphi(\psi(b_1)) = \varphi(\psi(b_1) \wedge \psi(b_2)) = \varphi(\psi(b_1)) \wedge \varphi(\psi(b_2)) = b_1 \wedge b_2 \leq b_2,$$

whereby $\psi\colon B \to F_{\mathrm{meet}}(n)$ is an order-embedding. Restricting ψ to $J(D^k)$, we obtain an embedding of $J(D^k)$ into $F_{\mathrm{meet}}(n)$, as required. Consequently, $G_{\min}(D^k) \geq S^*(J(D), k)$.

To prove the converse inequality, $G_{\min}(D^k) \leq S^*(J(D), k)$, now we change the meaning of n as follows: let $n := S^*(J(D), k)$. We have to show that D^k has an at most n-element generating set.

Let $U := J(D)$; then $kU \cong J(D^k)$ as in the first part of the proof. Furthermore, we know from (2.8) that kU is order embedded in $\mathsf{P}([n])$. Since $k \geq 2$, kU has no largest element. Thus, using that $F_{\mathrm{meet}}(n)$ is order isomorphic to $\mathsf{P}([n]) \setminus \{[n]\}$, kU is also embedded in $F_{\mathrm{meet}}(n) = F_{\mathrm{meet}}(x_1, \dots, x_n)$. So we assume that kU is a subposet of $F_{\mathrm{meet}}(n)$. A subset X of kU is called a *down-set* of kU if for every $y \in X$, $\downarrow_{kU} y \subseteq X$. The collection $\mathrm{Dn}(kU) = (\mathrm{Dn}(kU); \subseteq)$ of all down-sets of kU is a distributive lattice. Since $kU \cong J(D^k)$, we obtain by the well-known structure theorem of finite distributive lattices, see Grätzer [8, Theorem 107] for example, that $\mathrm{Dn}(kU) \cong D^k$. Hence, it suffices to find an (at most) n-element generating set of $\mathrm{Dn}(kU)$. For $i \in \{1, \dots, n\}$, define $Y_i := \{y \in kU : y \leq x_i$, understood in $F_{\mathrm{meet}}(n)\}$. Then $Y_i \in \mathrm{Dn}(kU)$, and we are going to show that $\{Y_1, \dots, Y_n\}$ generates $\mathrm{Dn}(kU)$. For every $X \in \mathrm{Dn}(kU)$, $X = \bigcup\{\downarrow_{kU} y : y \in X\} = \bigvee\{\downarrow_{kU} y : y \in X\}$. Therefore (since the meet in $\mathrm{Dn}(kU)$ is the intersection), it suffices to show that for each $u \in kU$,

$$\downarrow_{kU} y = \bigcap\{Y_i : u \in Y_i\}.$$

The "$\subseteq$" inclusion here is trivial since the Y_i's are down-sets. To verify the converse inclusion, assume that $v \in \bigcap\{Y_i : u \in Y_i\}$. This means that for all $i \in \{1, \dots, n\}$, if $u \in Y_i$, then $v \in Y_i$. In other words, for all $i \in \{1, \dots, n\}$, if $u \leq x_i$, then $v \leq x_i$. Thus, $v \leq \bigwedge\{x_i : u \leq x_i\}$. As each element of $F_{\mathrm{meet}}(n)$ is the meet of all elements above itself, $u = \bigwedge\{x_i : u \leq x_i\}$. By this equality and the just-obtained inequality, $v \leq u$, that is, $v \in \downarrow_{kU} u$. This shows the "$\supseteq$" inclusion and completes the proof. □

3. A Sperner type theorem

Let us repeat that a poset U is *bounded* if $0 = 0_U \in U$ and $1 = 1_U \in U$. Even though we have not seen the following statement in the literature, all the tools needed in its proof are present in Lubell [11], Griggs, Stahl, and Trotter [9], and Dove and Griggs [6]; this is why we call it an observation rather than a theorem.

Observation 1. *Let U be a finite poset, let $n, k \in \mathbb{N}_0$, and let $p := S^*(U, 1)$, that is,*

$$p = \min\{p' \in \mathbb{N}_0 : U \overset{\text{exists}}{\hookrightarrow} \mathsf{P}([p'])\}.$$

Then the following four assertions hold.

(a) *If $n \geq p$, then $S(U, n) \geq f_{\text{sb}}(n - p)$.*
(b) *If $k \geq 1$, then $S^*(U, k) \leq p + f^*_{\text{sb}}(k)$.*
(c) *If U is bounded and $n \geq p$, then $S(U, n) = f_{\text{sb}}(n-p)$, i.e., $S(U, n) = \mathrm{C}_{\text{bin}}(n-p, \lfloor (n-p)/2 \rfloor)$.*
(d) *If U is bounded and $k \geq 1$, then $S^*(U, k) = p + f^*_{\text{sb}}(k)$.*

If $|U| = 1$, then $p = 0$. Hence, Sperner's Theorem, see [13], is a particular case of Theorem 1. Clearly, so is (2.10), which we quoted from Griggs, Stahl and Trotter [9]. The forthcoming Table 1 shows that parts (c) and (d) would fail without assuming that U is bounded.

P r o o f. As we have already mentioned, all the ideas are taken from Lubell [11], Griggs, Stahl, and Trotter [9], and Dove and Griggs [6].

To prove part (a), let $B := \{n-p+1, n-p+2, \dots, n\}$. As $|B| = p$ and we can replace U with a poset isomorphic to it, we assume that $U \subseteq \mathsf{P}(B)$. The $\lfloor (n-p)/2 \rfloor$-element subsets of $\{1, \dots, n-p\}$ form a $k := f_{\text{sb}}(n-p)$-element antichain Φ in $\mathsf{P}([n-p])$. For $X_1, X_2 \in \Phi$ and $Y_1, Y_2 \in U$, if $X_1 \neq X_2$, then some $i \in \{1, \dots, n-p\}$ is in $X_1 \setminus X_2$ and so $i \in (X_1 \cup Y_1) \setminus (X_2 \cup Y_2)$. Hence, $(\{X \cup Y : X \in \Phi \text{ and } Y \in U\}; \subseteq) \cong (kU; \leq)$ is a subposet of $\mathsf{P}([n])$. Thus, $S(U, n) \geq k = f_{\text{sb}}(n-p)$, as required.

To prove part (b), observe that for $k \geq 1$, part (a) implies that

$$\{n : p \leq n \in \mathbb{N}_0 \text{ and } k \leq S(U, n)\} \supseteq \{n : p \leq n \in \mathbb{N}_0 \text{ and } k \leq f_{\text{sb}}(n-p)\}. \tag{3.1}$$

Observe also that, by (2.8), $k \leq S(U, n) \iff kU \overset{\text{exists}}{\hookrightarrow} \mathsf{P}([n])$. Hence, we can compute as follows; note that (3.1) will be used only once

$$S^*(U, k) \overset{(2.8)}{=} \min\{n : n \in \mathbb{N}_0 \text{ and } k \leq S(U, n)\} \tag{3.2}$$

$$\overset{k \geq 1}{=} \min\{n : p \leq n \in \mathbb{N}_0 \text{ and } k \leq S(U, n)\} \tag{3.3}$$

$$\overset{(3.1)}{\leq} \min\{n : p \leq n \in \mathbb{N}_0 \text{ and } k \leq f_{\text{sb}}(n-p)\} \tag{3.4}$$

$$= \min\{p + n' : n' \in \mathbb{N}_0 \text{ and } k \leq f_{\text{sb}}(n')\} \tag{3.5}$$

$$= p + \min\{n' : n' \in \mathbb{N}_0 \text{ and } k \leq f_{\text{sb}}(n')\} = p + f^*_{\text{sb}}(k). \tag{3.6}$$

To prove (c), assume that U is bounded. It suffices to verify that

$$S(U, n) \leq f_{\text{sb}}(n-p),$$

which is the converse of the inequality proved for part (a). With the notation $k := S(U, n)$, we know that there exists an order embedding $f \colon kU \to \mathsf{P}([n])$. Let $U_1, \dots, U_k$ be the pairwise disjoint isomorphic copies of U such that kU is the union of them. For $i \in [k]$, denote the restriction of f to U_i by f_i, and let $X_i := f_i(1_{U_i})$ and $Z_i := f_i(0_{U_i})$. Since the interval

$$[Z_i, X_i] = \{Y \in \mathsf{P}([n]) : Z_i \subseteq Y \subseteq X_i\}$$

is order isomorphic to $\mathsf{P}(X_i \setminus Z_i)$, it follows that $|X_i \setminus Z_i| \geq p$. Hence, we can pick a chain $Z_i = Y_0^{(i)} \subset Y_1^{(i)} \subset \cdots \subset Y_{p-1}^{(i)} \subset Y_p^{(i)} = X_i$. If we had that $Y_s^{(i)} \subseteq Y_t^{(j)}$ for some $i \neq j \in [k]$ and $s, t \in \{0, \dots, p\}$, then

$$f(0_{U_i}) = f_i(0_{U_i}) = Z_i = Y_0^{(i)} \subseteq Y_s^{(i)} \subseteq Y_t^{(j)} \subseteq Y_p^{(j)} = X_j = f_j(1_{U_j}) = f(1_{U_j})$$

and the fact that f is an order embedding would imply that $0_{U_i} \leq 1_{U_j}$, which is a contradiction. Hence $Y_s^{(i)}$ and $Y_t^{(j)}$ are incomparable for $i \neq j$. Therefore, letting

$$k\mathsf{C}_p = \bigcup_{i \in [k]} \{y_0^{(i)}, y_1^{(i)}, \dots, y_p^{(i)}\}$$

with $y_0^{(i)} \prec y_1^{(i)} \prec \cdots \prec y_p^{(i)}$, the "capitalizing map" $k\mathsf{C}_p \to \mathsf{P}([n])$ defined by $y_s^{(i)} \mapsto Y_s^{(i)}$ is an order embedding. Thus, it follows from Griggs, Stahl, and Trotter's result, quoted here in (2.10), that

$$S(U, n) = k \leq S(\mathsf{C}_p, n) = f_{\mathrm{sb}}(n - p),$$

as required. We have shown part (c).

To prove part (d), observe that in the argument for (b), part (a) yielded inequality (3.1), which was used only once in (3.2)–(3.6). Now that part (c) turns (3.1) into an equality, (3.2)–(3.6) turn into a computation proving the required equality $S^*(U, k) = p + f^*_{\mathrm{sb}}(k)$, completing the proof. □

4. Lower and upper estimates for non-bounded posets

For *any* finite poset U, Dove and Griggs [6] and Katona and Nagy [10], independently from each other, gave lower estimates and upper estimates of $S(U, n)$. Their estimates are asymptotically equal if n tends to infinity. Thus, $S(U, n)$ is asymptotically known[3] for each U. In general, however, this knowledge does not give us too much information on $S(U, n)$ for a *small* n. By parsing the arguments in Dove and Griggs [6] or Katona and Nagy [10], one can obtain some estimates for a small n but sometimes, putting generality aside, other constructions could be easier and could give better estimates. This will be exemplified by two small *concrete* posets; see Propositions 1 and 2 later. But first of all, let us agree that the set of all permutations of $[n]$ are denoted by Sym_n; its members are written in the form $\vec{\pi} = (\pi_1, \dots, \pi_n)$. For $\vec{\pi} \in \mathrm{Sym}_n$, $j \in [n]$ and $X \in \mathsf{P}([n]) \setminus \{\emptyset\}$, we denote by

$$\mathrm{Is}(j, \vec{\pi}) := \{\pi_m : 1 \leq m \leq j\}, \quad \mathrm{Lp}(X, \vec{\pi}) := \max\{m \in [n] : \pi_m \in X\}, \tag{4.1}$$

$$\text{and} \quad \Gamma(X) := \{\vec{\pi} \in \mathrm{Sym}_n : \mathrm{Is}(\mathrm{Lp}(Z_i, \vec{\pi}), \vec{\pi}) \subseteq X\} \tag{4.2}$$

the j*-th* ***initial set*** of $\vec{\pi}$, the ***last position*** of X in $\vec{\pi}$, and the *set of permutations associated with* X, respectively. We let $\mathrm{Lp}(\emptyset, \vec{\pi}) := 0$ and $\mathrm{Is}(0, \vec{\pi}) = \emptyset$. Of course, we can change "⊆" in (4.2) into "=".

[3] When writing arXiv:2308.15625v2, the earlier version of this paper, I did not know about Dove and Griggs [6] and Katona and Nagy [10]; thank goes to Dániel Nagy (the second author of [10]) to call my attention to these two papers.

The following statement is due to Lubell [11] and (apart from terminological changes) was used successfully by Dove and Griggs [6], Griggs, Stahl, and Trotter [9], and Katona and Nagy [10]:

$$\text{if } X, Y \in \mathsf{P}([n]) \text{ such that } X \parallel Y, \text{ then } \Gamma(X) \cap \Gamma(Y) = \emptyset \text{ and} \tag{4.3}$$

$$\text{for every } X \in \mathsf{P}([n]), \text{ we have that } |\Gamma(X)| = |X|! \cdot (n - |X|)!. \tag{4.4}$$

Next, let W denote the 4-element poset W with 0 and three maximal elements, see Fig. 1. For $n \in \mathbb{N}^+$ we define

$$^{\mathrm{up}}S(W, n) := \left\lfloor \frac{n}{3n - 2 - 2\lfloor n/2 \rfloor} \cdot f_{\mathrm{sb}}(n - 1) \right\rfloor. \tag{4.5}$$

With the convention that $\mathrm{C_{bin}}(n_1, n_2) = 0$ unless $0 \le n_2 \le n_1$, let

$$_{\mathrm{lo}}S(W, n) := \begin{cases} \displaystyle\sum_{i=0}^{\lfloor n/3 \rfloor - 1} \sum_{j=0}^{i} 3^j \binom{i}{j} \binom{n - 3i - 3}{\lfloor (n-1)/2 \rfloor + j - 3i} & \text{if } \ n \notin \{3, 5, 7\}, \\ \displaystyle\sum_{i=0}^{\lfloor n/3 \rfloor - 1} \sum_{j=0}^{i} 3^j \binom{i}{j} \binom{n - 3i - 3}{(n-3)/2 + j - 3i} & \text{if } \ n \in \{3, 5, 7\}. \end{cases} \tag{4.6}$$

Note that $_{\mathrm{lo}}S(W, 1) = S(W, 1)$ and $_{\mathrm{lo}}S(W, 2) = S(W, 2)$. Hence, we can often assume that $n \ge 3$. The *natural density* of a subset X of $\mathbb{N}^+$ is defined to be $\lim_{n \to \infty} |X \cap [n]|/n$, provided that this limit exists.

Proposition 1. *For* $3 \le n \in \mathbb{N}^+$, $^{\mathrm{up}}S(W, n)$ *and* $_{\mathrm{lo}}S(W, n)$ *defined in* (4.5) *and* (4.6) *are an upper estimate and a lower estimate of* $S(W, n)$*, that is,*

$$_{\mathrm{lo}}S(W, n) \le S(W, n) \le {}^{\mathrm{up}}S(W, n).$$

The functions $_{\mathrm{lo}}S(W, -)$, $S(W, -)$, $^{\mathrm{up}}S(W, -)$, *and* $1/4 \cdot f_{\mathrm{sb}}(-)$ *are asymptotically equal. Furthermore, denoting the left adjoints of the functions* $_{\mathrm{lo}}S(W, -)$ *and* $^{\mathrm{up}}S(W, -)$ *by* $_{\mathrm{lo}}S^*(W, -)$ *and* $^{\mathrm{up}}S^*(W, -)$*, respectively,*

$$^{\mathrm{up}}S^*(W, k) \le S^*(W, k) \le {}_{\mathrm{lo}}S^*(W, k) \quad \text{and} \quad 0 \le {}_{\mathrm{lo}}S^*(W, k) - {}^{\mathrm{up}}S^*(W, k) \le 1 \tag{4.7}$$

for all $k \in \mathbb{N}^+$*, and the natural density of the set*

$$\{k \in \mathbb{N}^+ : {}^{\mathrm{up}}S^*(W, k) = {}_{\mathrm{lo}}S^*(W, k)\}$$

is 1.

The proof below uses lots from the proofs in Dove and Griggs [6] and Katona and Nagy [10]; we are going to discuss the differences in Remark 2.

P r o o f. First, we deal with $^{\mathrm{up}}S(W, n)$. Let $k := S(W, n)$, and let $W_1, \ldots, W_k$ be pairwise unrelated copies of W in $\mathsf{P}([n])$. In particular, $(W_i, \subseteq)$ is order isomorphic to W. The assumption $n \ge 3$ gives that $^{\mathrm{up}}S(W, n) \ge 1$. Thus, we can assume that $k \ge 2$ as otherwise

$$S(W, n) = k \le {}^{\mathrm{up}}S(W, n)$$

is obvious. In accordance with Figure 1, we use the notation $W_i = \{Z_i, C_i, D_i, E_i\}$ where $Z_i \subset C_i$, $C_i \parallel D_i$, etc., and $Z_i \parallel E_j$ for $i \ne j$, etc. As it is trivial (and used also in Dove and Griggs [6] and Katona and Nagy [10]), if $i \ne j \in [k]$, $Y \in \mathsf{P}([n])$, $Y', Y'' \in W_i$, and $Y' \subseteq Y \subseteq Y''$, then $W_i \cup \{Y\}$

is still unrelated to W_j; we are going to use this "convexity principle" implicitly. As its first use, we can assume that Z_i equals the intersection $C_i \cap D_i \cap E_i$ as otherwise we could replace Z_i by this intersection.

We claim that with some pairwise distinct elements $c_i, d_i, e_i \in [n] \setminus Z_i$, we can change W_i to $W_i' = \{Z_i, Z_i \cup \{c_i\}, Z_i \cup \{d_i\}, Z_i \cup \{e_i\}\}$ such that $W_1, \dots, W_{i-1}, W_i', W_{i+1}, \dots, W_k$ still form a system of pairwise unrelated copies of W. Let $C_i' = C_i \setminus Z_i$, $D_i' = D_i \setminus Z_i$, and $E_i' = E_i \setminus Z_i$. If at least one of C_i', D_i' and E_i' is not a subset of the union of the other two, say, $C_i' \nsubseteq D_i' \cup E_i'$, then any choice of $c_i \in C_i' \setminus (D_i' \cup E_i')$, $d_i \in D_i' \setminus E_i'$, and $e_i \in E_i' \setminus D_i'$ does the job by the convexity principle. So we can assume that each of C_i', D_i' and E_i' is a subset of the union of the other two. Take an element from $C_i' \setminus D_i'$. As $C_i' \subseteq D_i' \cup E_i'$, this element is in E_i'; we denote it by $x_{C,\neg D,E}$. The meaning of its subscripts is that $x_{C,\neg D,E}$ belongs to C_i' and E_i' but not to D_i'. By symmetry, we obtain elements $x_{C,D,\neg E} \in (C_i' \cap D_i') \setminus E_i'$ and $x_{\neg C,D,E} \in (D_i' \cap E_i') \setminus C_i'$. The subscripts show that these three elements are pairwise distinct. This fact and the convexity principle imply that W_i can be changed to the required form with $c_i := x_{C,\neg D,E}$, $d_i := x_{C,D,\neg E}$, and $e_i := x_{\neg C,D,E}$. Therefore, in the rest of the proof, we assume that for all $i \in [k]$,

$$W_i = \{Z_i, Z_i \cup \{c_i\}, Z_i \cup \{d_i\}, Z_i \cup \{e_i\}\}.$$

Letting

$$\Gamma_i := \Gamma(Z_i) \cup \Gamma(Z_i \cup \{c_i\}) \cup \Gamma(Z_i \cup \{d_i\}) \cup \Gamma(Z_i \cup \{e_i\}),$$

our next task is to find a reasonable lower bound on $|\Gamma_i|$. With the notation $z_i := |Z_i|$, we can order the first z_i components of a

$$\vec{\pi} = (\pi_1, \dots, \pi_n) \in \Gamma(Z_i) \cap \Gamma(Z_i \cup \{c_i\}),$$

which form the set Z_i, in $z!$ ways. We have that $\pi_{z_i+1} = c_i$, and the last $n - z_i - 1$ components can be ordered in $(n - z_i - 1)!$ ways. Hence,

$$|\Gamma(Z_i \cup \{c_i\})| = z_i!(n - z_i - 1)!,$$

and the same is true for $|\Gamma(Z_i \cup \{d_i\})|$ and $|\Gamma(Z_i \cup \{e_i\})|$. This fact, (4.3), (4.4), and the inclusion-exclusion principle yield that

$$\begin{gathered} |\Gamma_i| = g_0(z_i), \quad \text{where} \\ g_0(x) := x!(n - x)! + 3(x + 1)!(n - x - 1)! - 3x!(n - x - 1)! \\ = (n + 2x_i)x_i!(n - 1 - x_i)!\,. \end{gathered} \tag{4.8}$$

Note that $z_i \geq 1$ as otherwise $Z_i = \emptyset$ would be comparable with Z_j for $j \in [k] \setminus \{i\}$. (Here we used that $k \geq 2$.) We also have that $z_i \leq n - 1$ since $Z_i \cup \{c_i\} \in \mathsf{P}([n])$. Thus, we can use later that $x \in [n - 1] = \{1, \dots, n - 1\}$.

For the auxiliary function

$$g_1(x) := g_0(x) - g_0(x - 1),$$

we have that

$$g_1(x) = g_2(x) \cdot x!(n - 1 - x)!, \quad \text{where} \quad g_2(x) = 4x^2 - 2x - (n^2 - 2n).$$

The smaller root of the quadratic equation $g_2(x) = 0$ is negative while the larger one is strictly between $n/2 - 1/2$ and $n/2 - 1/4$. Hence the largest integer x for which $g_2(x)$ and so $g_1(x)$ are negative is $x = \lfloor (n - 1)/2 \rfloor$. Therefore, on the set $[n - 1]$, g_0 takes its minimum at $\lfloor (n - 1)/2 \rfloor$.

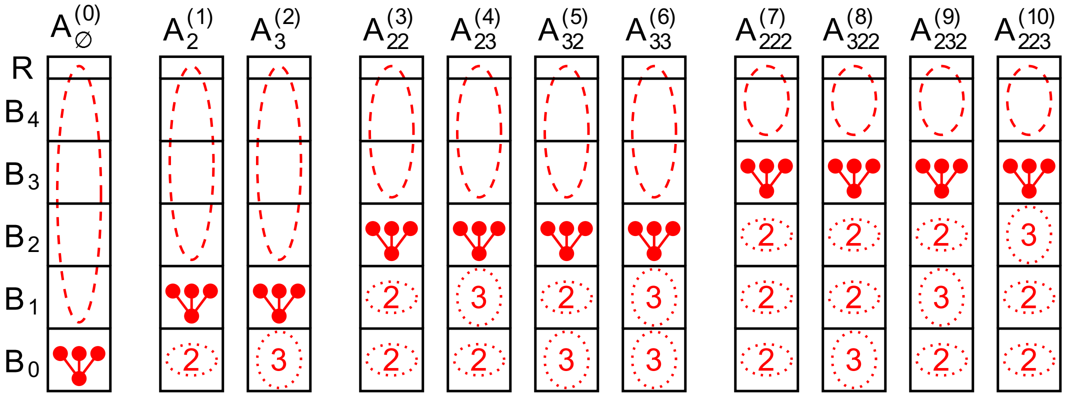


Figure 2. Copies of $A := [16]$. Here $|B_0| = \cdots = |B_4| = 3$. In the copy $A^{(0)}_\emptyset$, the oval stands for a 7-element subset. In each other copies, the total number of elements in the ovals is also 7.

Let $M := g_0(\lfloor (n-1)/2 \rfloor)$. Using that the Γ_i's are pairwise disjoint by (4.3) and $\Gamma_1 \cup \dots \Gamma_k \subseteq \mathrm{Sym}_n$, we obtain that

$$kM = \sum_{i=1}^{k} M \le \sum_{i=1}^{k} |\Gamma_i| \le |\mathrm{Sym}_n| = n!\,. \tag{4.9}$$

Dividing this inequality by M (and dealing with odd n's and even n'-s separately), we obtain the required inequality

$$S(W,n) = k \le {}^{\mathrm{up}}S(W,n).$$

Next, we turn our attention to ${}_{\mathrm{lo}}S(W,n)$. Let $m := \lfloor n/3 \rfloor$. For $n \notin \{3,5,7\}$, let $h := \lfloor (n-1)/2 \rfloor$. For $n \in \{3,5,7\}$, h stands for $(n-3)/2$. With $A := [n]$, let us fix pairwise disjoint 3-element subsets $B_0, B_1, \dots, B_{m-1}$ of A, and denote the "remainder set" $A \setminus (B_0 \cup \dots \cup B_{m-1})$ by R. These subsets are visualized in Figure 2, where $n = 16$, $m = 5$, $h = 7$, and A with subscripts and superscripts is drawn eleven times (in four groups separated by spaces). We can assume that $n \ge 3$. For $i \in \{0, \dots, m-1\}$, the elements of B_i are denoted as follow: $B_i = \{c_i, d_i, e_i\}$. For $i \in \mathbb{N}_0$, call a vector $\vec v = (v_0, \dots, v_{i-1}) \in \{2,3\}^i$ *eligible* if $i \le m-1$ and $v_0 + v_1 + \dots + v_{i-1} \le h$. Note that for $i = 0$, the empty vector is denoted by $\emptyset$ and it is eligible. As Figure 2 shows, there are exactly eleven eligible vectors for $n = 16$; they are the lower subscripts of the copies of A; because of space consideration, we write 232 instead of $(2,3,2)$, etc., in the figure. (The upper subscripts of A help to count the copies but play no other role.)

For each eligible $\vec v$, we define a family of copies of W in $\mathsf{P}([n]) = \mathsf{P}(A)$ as follows. Let i denote the dimension of $\vec v$, that is, $\vec v = (v_0, \dots, v_{i-1})$. For $j = 0, \dots, i-1$, pick a v_j-element subset X_j of B_j. In the figure, X_j is denoted by a dotted oval with v_j sitting in its middle. Furthermore, pick a subset X_i of $A \setminus (B_0 \cup B_1 \cup \dots \cup B_i)$ such that $X_i = h - v_0 - \dots - v_{i-1}$. In the figure, X_i is the dashed oval (without any number in its middle). Let us emphasize that $X_j \subseteq B_j$ holds only for $j < i$ but it never holds for $j = i$. Denote $(X_0, X_1, \dots, X_i)$ by $\vec X$, call it an *eligible set vector*, and let $Z_{\vec X} := X_0 \cup \dots \cup X_i$. Clearly,

$$\begin{array}{l}\text{regardless the choice of } \vec v \text{ and } \vec X, \text{ we have that}\\ |Z_{\vec X}| \text{ is always the same, namely, } \quad |Z_{\vec X}| = h.\end{array} \tag{4.10}$$

For convenience, let $\overline{c}_i := \{c_i\}$, $\overline{d}_i := \{d_i\}$, $\overline{e}_i := \{e_i\}$, and $\overline{z}_i := \emptyset$. Observe that $\{\overline{c}_i, \overline{d}_i, \overline{e}_i, \overline{z}_i\}$ is a copy of W in $\mathsf{P}(B_i)$; in each copy of A in the figure, this copy of W is indicated by its diagram for exactly one i. It follows that

$$W_{\vec X} := \{z_{\vec X},\ c_{\vec X},\quad d_{\vec X},\ e_{\vec X}\},$$
$$z_{\vec X} := \overline{z}_i \cup Z_{\vec X} = Z_{\vec X},\quad c_{\vec X} := \overline{c}_i \cup Z_{\vec X},\quad d_{\vec X} := \overline{d}_i \cup Z_{\vec X},\quad e_{\vec X} := \overline{e}_i \cup Z_{\vec X},$$

is also a copy of W but now in $\mathsf{P}(A) = \mathsf{P}([n])$.

To prove that ${}_{\mathrm{lo}}S(W,n) \le S(W,n)$, we need to show that ${}_{\mathrm{lo}}S(W,n)$ is the number of eligible set vectors $\vec X$ and for distinct eligible set vectors $\vec X \ne \vec X^\bullet$, the corresponding copies $W_{\vec X}$ and $W_{\vec X^\bullet}$ of W are unrelated.

First, we deal with the number of eligible set vectors $\vec X = (X_0,\dots,X_i)$. As each of the B_j's are 3-element and there are $\lfloor n/3\rfloor$ many of them, the largest value of i is at most $\lfloor n/3\rfloor - 1$, the upper limit of the outer summation index in (4.6). The eligible vector $\vec v$ that gives rise to $\vec X$ is uniquely determined by $\vec X$ since $\vec v = (|X_0|,\dots,|X_{i-1}|)$.

Let $j := |\{t \in \{0,\dots,i-1\} : v_t = 2\}|$. This j, which corresponds to the inner summation index in (4.6), is the number of 2's in dotted ovals in the figure. There are $\binom{i}{j}$ possibilities to choose the j-element set $\{t \in \{0,\dots,i-1\} : v_t = 2\}$; this is where the first binomial coefficient enters into (4.6). For each $t \in \{0,\dots,i-1\}$ such that $v_t = 2$, we can choose the 2-element subset X_t of B_t in 3 ways. As there are j such t's, this brings the power 3^j into (4.6). Since X_i is a subset of the $n-3i-3$-element set $A \setminus (B_0 \cup \dots \cup B_i)$ and

$$|X_i| = h - v_0 - \dots - v_{i-1} = h - 2j - 3(i-j) = h + j - 3i\,,$$

the second binomial coefficient in (4.6) gives how many ways we can choose X_i. Therefore, (4.6) precisely gives the number of eligible set vectors $\vec X$.

Next, assume that $\vec X = (X_0,\dots,X_i)$ and $\vec X^\bullet = (X_0^\bullet,\dots,X_{i^\bullet}^\bullet)$ are distinct eligible set vectors with corresponding (not necessarily different) eligible vectors $\vec v = (v_0,\dots,v_{i-1})$ and $\vec v^\bullet = (v_0^\bullet,\dots,v_{i^\bullet-1}^\bullet)$. Assume also that a and $a^\bullet$ are in W such that $(\vec X, a) \ne (\vec X^\bullet, a^\bullet)$. We need to show that $a_{\vec X} = \overline{a}_i \cup Z_X$ and $a^\bullet_{\vec X^\bullet} = \overline{a}_i^\bullet \cup Z_{\vec X^\bullet}$ are incomparable. There are two cases to consider; both can easily be followed by keeping an eye on Fig. 2 in addition to the formal argument.

First, assume that $i \ne i^\bullet$, say, $i < i^\bullet$. Observe that $|a^\bullet_{\vec X^\bullet} \cap B_i| = |X_i^\bullet| = v_i^\bullet \ge 2$ but $|a_{\vec X} \cap B_i| = |\overline{a}_i| \le 1$. So $|a^\bullet_{\vec X^\bullet} \cap B_i| > |a_{\vec X} \cap B_i|$. (Pictorially, a dotted oval, labeled by 2 or 3, has more elements than $|\overline{a}_i|$ symbolized by one of the vertices of the diagram of W drawn in B_i.) Hence, $a^\bullet_{\vec X^\bullet} \nsubseteq a_{\vec X}$. For the sake of contradiction, suppose that $a_{\vec X} \subseteq a^\bullet_{\vec X^\bullet}$. Then for every $j \in \{0,\dots,i-1\}$, $v_j = |B_j \cap a_{\vec X}| \le |B_j \cap a^\bullet_{\vec X^\bullet}| = v_j^\bullet$. Hence, we can compute as follows; the computation is motivated by comparing, say, $A_2^{(1)}$ and $A_{232}^{(9)}$ in Fig. 2:

$$\begin{aligned}|a_{\vec X} \cap (B_{i+1} \cup \dots \cup B_{m-1} \cup R)| &= |X_i| = h - v_0 - \dots - v_{i-1} \ge h - v_0^\bullet - \dots - v_{i-1}^\bullet\\ &= (h - v_0^\bullet - \dots - v_{i^\bullet-1}^\bullet) + (v_{i+1}^\bullet + \dots + v_{i^\bullet-1}^\bullet) + |\overline{a}_{i^\bullet}^\bullet| + (v_i^\bullet - |\overline{a}_{i^\bullet}^\bullet|)\\ &= |X_{i^\bullet}^\bullet| + (|X_{i+1}^\bullet| + \dots + |X_{i^\bullet-1}^\bullet| + |\overline{a}_{i^\bullet}^\bullet|) + (v_i^\bullet - |\overline{a}_{i^\bullet}^\bullet|)\\ &= |a^\bullet_{\vec X^\bullet} \cap (B_{i+1} \cup \dots \cup B_{m-1} \cup R)| + (v_i^\bullet - |\overline{a}_i^\bullet|) > |a^\bullet_{\vec X^\bullet} \cap (B_{i+1} \cup \dots \cup B_{m-1} \cup R)|.\end{aligned}$$

The strict inequality just obtained contradicts that $a_{\vec X} \subseteq a^\bullet_{\vec X^\bullet}$, and we conclude in the first case that $a_{\vec X} \parallel a^\bullet_{\vec X^\bullet}$, as required.

Second, assume that $i = i^\bullet$. If $X_j \parallel X_j^\bullet$ for some $j \in \{0,\dots,i\}$ or there are $s,t \in \{0,\dots,i\}$ such that[4] $X_s \subset X_s^\bullet$ but $X_t \supset X_t^\bullet$, then the validity of $a_{\vec X} \parallel a^\bullet_{\vec X^\bullet}$ is clear. Thus, we can assume that $X_j \subseteq X_j^\bullet$ for all $j \in \{0,\dots,i\}$. Then

$$h = |X_0| + \dots + |X_i| \le |X_0^\bullet| + \dots + |X_i^\bullet| = h$$

[4]According to the convention of lattice theory, "$\subset$" is the conjunction of "$\subseteq$" and "$\ne$".

Table 1. Some values of ${}_{\mathrm{lo}}S(W,n)$ and ${}^{\mathrm{up}}S(W,n)$; the *known* values of $S(W,n)$ are encircled.

n	3	4	5	6	7	8	9	10	11	12	13	14
${}_{\mathrm{lo}}S(W,n)$	①	①	②	⑥	9	17	36	66	120	234	456	876
${}^{\mathrm{up}}S(W,n)$	1	2	3	6	10	20	37	70	132	252	480	924

n	15	16	17	18	19	20	21
${}_{\mathrm{lo}}S(W,n)$	1 680	3 625	6 340	12 330	23 960	46 766	91 224
${}^{\mathrm{up}}S(W,n)$	1 775	3 432	6 630	12 870	24 967	48 620	94 631

n	22	23	24	25	26
${}_{\mathrm{lo}}S(W,n)$	178 388	348 656	683 130	1 337 896	2 625 364
${}^{\mathrm{up}}S(W,n)$	184 756	360 554	705 432	1 379 671	2 704 156

n	27	28	29	30
${}_{\mathrm{lo}}S(W,n)$	5 149 872	10 119 348	19 877 904	39 104 856
${}^{\mathrm{up}}S(W,n)$	5 298 418	10 400 600	20 410 200	40 116 600

together with $|X_j| \le |X_j^\bullet|$, for all $j \in \{0,\dots,i\}$, imply that $X_j = X_j^\bullet$ for all $j \in \{0,\dots,i\}$. Combining this equality with $X_j \subseteq X_j^\bullet$ for all $j \in \{0,\dots,i\}$, we obtain that $\vec{X} = \vec{X}^\bullet$, contradicting our assumption. We have shown that ${}_{\mathrm{lo}}S(W,n) \le S(W,n)$, as required.

It is well known that no matter how we fix two integers s and t,

$$\binom{n-s}{\lfloor n/2\rfloor - t} \quad \text{is asymptotically} \quad 2^{-s}\binom{n}{\lfloor n/2\rfloor} = 2^{-s} f_{\mathrm{sb}}(n) \quad \text{if} \quad n \to \infty; \tag{4.11}$$

this folkloric (and trivial) fact was used in Dove and Griggs [6] and Katona and Nagy [10], too. This fact and (4.5) yield that ${}^{\mathrm{up}}S(W,-)$ is asymptotically $1/4 \cdot f_{\mathrm{sb}}(-)$. Hence, to obtain the required asymptotic equations, it suffices to show that ${}_{\mathrm{lo}}S(W,-)$ is asymptotically $1/4 \cdot f_{\mathrm{sb}}(-)$, too. Let η and μ be small positive real numbers. As $\sum_{i=0}^{\infty} 2^{-i} = 2$, we can fix a $q \in \mathbb{N}^+$ such that $\sum_{i=0}^{q} 2^{-i} \ge 2 - \eta$. Using (4.11) and assuming that $i \le q$, we obtain that the second binomial coefficient in (4.6) is asymptotically $2^{-3i} f_{\mathrm{sb}}(n-3)$ or, rather, it is $1/8 \cdot 2^{-3i} f_{\mathrm{sb}}(n)$. So it is at least $1/8 \cdot 8^{-i}(1-\mu) f_{\mathrm{sb}}(n)$ for all but finitely many n. Hence, assuming that n is large enough and, in particular, $\lfloor n/3 \rfloor > q$,

$$\begin{aligned} {}_{\mathrm{lo}}S(W,n) &\ge \sum_{i=0}^{q}\sum_{j=0}^{i} 3^j \binom{i}{j} \cdot 8^{-i} \cdot \frac{1}{8}(1-\mu) f_{\mathrm{sb}}(n) = \frac{1}{8}(1-\mu) f_{\mathrm{sb}}(n) \sum_{i=0}^{q} 8^{-i} \sum_{j=0}^{i} \binom{i}{j} 3^j \cdot 1^{i-j} \\ &= \frac{1}{8}(1-\mu) f_{\mathrm{sb}}(n) \sum_{i=0}^{q} 8^{-i}(3+1)^i \ge \frac{1}{8}(1-\mu) f_{\mathrm{sb}}(n)(2-\eta) = \frac{(2-\eta)(1-\mu)}{8} f_{\mathrm{sb}}(n). \end{aligned} \tag{4.12}$$

As the last fraction in (4.12) can be arbitrarily close to $1/4$, it follows that ${}_{\mathrm{lo}}S(W,n)$ is asymptotically at least $1/4 \cdot f_{\mathrm{sb}}(n)$. It is asymptotically at most $1/4 \cdot f_{\mathrm{sb}}(n)$ since so is ${}^{\mathrm{up}}S(W,n)$ and we know that ${}_{\mathrm{lo}}S(W,n) \le S(W,n) \le {}^{\mathrm{up}}S(W,n)$. This completes the argument proving the "asymptotically equal" part of Proposition 1.

Next, we turn our attention to the left adjoints of our estimates. First of all, we claim that

$$\text{for every } n \in \mathbb{N}^+, \quad {}^{\mathrm{up}}S(W,n) \le {}_{\mathrm{lo}}S(W,n+1). \tag{4.13}$$

Let ${}^{\mathrm{up}+}S(W,-)$ be the same as ${}^{\mathrm{up}}S(W,-)$ except that we drop the outer "lower integer part" function from its definition. It suffices to prove (4.13) with ${}^{\mathrm{up}+}S(W,n+1)$ instead of ${}^{\mathrm{up}}S(W,n+1)$. We can assume that $n \geq 10$ as otherwise (4.13) is clear by Table 1[5]. Let $T(n)$ denote the sum of the two summands in the upper line of (4.6) that correspond to $(i,j) = (0,0)$ and $(i,j) = (1,1)$. After a straightforward but tedious calculation, if $n = 2m$, then

$$\frac{{}^{\mathrm{up}+}S(W,n)}{{}_{\mathrm{lo}}S(W,n+1)} \leq \frac{{}^{\mathrm{up}+}S(W,n)}{T(n+1)} = \frac{2m(2m-2)(2m-3)}{(m-1)^2(11m-12)}. \tag{4.14}$$

Subtracting the numerator from the denominator, we obtain $3m^3 - 14m^2 + 23m - 12$, which is clearly nonnegative for $5 \leq m \in \mathbb{N}^+$ (in fact, for all $m \in \mathbb{N}^+$), whence the fraction is at most 1 for $n = 2m \geq 10$. For an odd $n = 2n+1 \geq 10$, (4.14) turns into

$$\frac{{}^{\mathrm{up}+}S(W,n)}{{}_{\mathrm{lo}}S(W,n+1)} \leq \frac{{}^{\mathrm{up}+}S(W,n)}{T(n+1)} = \frac{4(2m+1)(2m-1)(2m-3)}{(4m+1)(11m^2-19m+6)},$$

and now the subtraction gives the polynomial $12m^3 - 17m^2 + 13m - 6$, which is clearly nonnegative for $2 \leq m \in \mathbb{N}^+$ (in fact, for all $m \in \mathbb{N}^+$). Thus, passing from m to n, the required inequality ${}^{\mathrm{up}+}S(W,n) \leq {}_{\mathrm{lo}}S(W,n+1)$ holds for all $10 \leq n \in \mathbb{N}^+$. We have shown the validity of (4.13).

Next, we deal with (4.7). By Table 1, the first few values of ${}^{\mathrm{up}}S^*(W,k)$ and those of ${}_{\mathrm{lo}}S^*(W,k)$ are as follows:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
${}^{\mathrm{up}}S^*(W,k)$	3	4	5	6	6	6	7	7	7	7	8	8	8	8	8
${}_{\mathrm{lo}}S^*(W,j)$	3	5	6	6	6	6	7	7	7	8	8	8	8	8	8

(4.15)

This implies (4.7) for $k \leq 15$ (in fact, for $k \leq 29$), so we can assume that $k > 15$. Using (4.15) and the obvious fact that ${}_{\mathrm{lo}}S(W,-)$ is a strictly increasing function on $\mathbb{N}^+ \setminus [7]$, there is a unique $7 \leq n \in \mathbb{N}^+$ such that

$${}_{\mathrm{lo}}S(W,n) < k \leq {}_{\mathrm{lo}}S(W,n+1).$$

Using (4.13) and the inequality ${}_{\mathrm{lo}}S(W,n) \leq {}^{\mathrm{up}}S(W,n)$, we obtain that

$$\text{either} \quad {}_{\mathrm{lo}}S(W,n) < k \leq {}^{\mathrm{up}}S(W,n) \tag{4.16}$$

$$\text{or} \quad {}^{\mathrm{up}}S(W,n) < k \leq {}_{\mathrm{lo}}S(W,n+1). \tag{4.17}$$

If (4.16), then ${}^{\mathrm{up}}S^*(V,k) = n$ and ${}_{\mathrm{lo}}S^*(W,k) = n+1$. If (4.17), then

$${}^{\mathrm{up}}S^*(W,k) = n+1 = {}_{\mathrm{lo}}S^*(V,k).$$

In both cases, $0 \leq {}_{\mathrm{lo}}S^*(W,k) - {}^{\mathrm{up}}S^*(W,k) \leq 1$, as required.

Next, for $t \in \mathbb{N}^+$, let

$$E_t := \{k \in [t] : {}^{\mathrm{up}}S^*(W,k) < {}_{\mathrm{lo}}S^*(W,k)\}.$$

To settle the last sentence of Proposition 1 about density, it suffices to show that $\lim_{t\to\infty}(|E_t|/t) = 0$. Let $\epsilon < 1/12$ be a positive real number; we are going to show that $|E_t|/t < \epsilon$ for all but finitely many t's. Asymptotic equalities will often be denoted by "$\sim$". As we have already proved that

$${}_{\mathrm{lo}}S(W,-) \sim \frac{1}{4} f_{\mathrm{sb}}(-),$$

[5]The table was obtained by the computer algebraic program Maple V Release 5, which ran on a desktop computer with AMD Ryzen 7 2700X Eight-Core Processor 3.70 GHz for 1/5 seconds.

(4.11) yields that ${}^{\mathrm{up}}S(W, n-1)/{}^{\mathrm{up}}S(W, n) \to 1/4$ as $n \to \infty$. This fact, $1/6 < 1/4 < 3^{-1}$, and ${}_{\mathrm{lo}}S(W, n) \sim {}^{\mathrm{up}}S(W, n)$ allow us to fix an $n_0 = n_0(\epsilon) \in \mathbb{N}^+$ such that for all $n \geq n_0$,

$$ {}^{\mathrm{up}}S(W, n)/6 < {}^{\mathrm{up}}S(W, n-1) < {}^{\mathrm{up}}S(W, n) \cdot 3^{-1}, \tag{4.18} $$

$$ {}^{\mathrm{up}}S(W, n) - {}_{\mathrm{lo}}S(W, n) < {}^{\mathrm{up}}S(W, n) \cdot \epsilon/12\,. \tag{4.19} $$

Later, it will be important that n_0 does not depend on t. Hence, from now on, we can assume that ${}^{\mathrm{up}}S(W, n_0) < t$. Since $\lim_{i\to\infty} {}^{\mathrm{up}}S(n_0 + i) = \infty$ in a strictly increasing way, there exists a unique $r = r(t) \in \mathbb{N}^+$ such that ${}^{\mathrm{up}}S(n_0 + r - 1) < t \leq {}^{\mathrm{up}}S(n_0 + r)$. Since ϵ is small, (4.18) and (4.19) yield that for all $i \in [r]$,

$$ \underbrace{{}^{\mathrm{up}}S(W, n_0 + i - 1) < {}_{\mathrm{lo}}S(W, n_0 + i)}_{\text{long good interval}} \leq {}^{\mathrm{up}}S(W, n_0 + i), \tag{4.20} $$

$$ {}^{\mathrm{up}}S(W, n_0 + r)/6 < t\,. \tag{4.21} $$

Observe that (4.20) and ${}_{\mathrm{lo}}S(W, n_0 + i - 1) \leq {}^{\mathrm{up}}S(W, n_0 + i - 1)$ imply that for every k in the left open and right closed interval $\big({}^{\mathrm{up}}S(W, n_0 + i - 1), {}_{\mathrm{lo}}S(W, n_0 + i)\big]$, which is under-braced in (4.20), ${}_{\mathrm{lo}}S^*(W, k) = {}^{\mathrm{up}}S^*(W, k) = n_0 + i$. So this interval is disjoint from $|E_t|$ for any $t \in \mathbb{N}^+$. Thus, letting

$$ c := {}^{\mathrm{up}}S(W, n_0 \quad \text{and} \quad d := {}^{\mathrm{up}}S(W, n_0 + r), $$

E_t is a subset of

$$ [1, c] \cup \bigcup_{i\in[r]} \big({}_{\mathrm{lo}}S(W, n_0 + i), {}^{\mathrm{up}}S(W, n_0 + i)\big]. $$

Hence,

$$ \begin{aligned} |E_t| &\leq c + \sum_{i\in[r]} \big({}^{\mathrm{up}}S(W, n_0 + i) - {}_{\mathrm{lo}}S(W, n_0 + i)\big) \\ &\overset{(4.19)}{\leq} c + \frac{\epsilon}{12} \cdot \sum_{i\in[r]} {}^{\mathrm{up}}S(W, n_0 + i) = c + \frac{\epsilon}{12} \cdot \sum_{i\in\{0,\dots,r-1\}} {}^{\mathrm{up}}S(W, n_0 + r - i) \\ &\overset{(4.18)}{\leq} c + \frac{\epsilon}{12} \sum_{i\in\{0,\dots,r-1\}} 3^{-i} d \leq c + \frac{\epsilon d}{12} \sum_{i\in\mathbb{N}_0} 3^{-i} = c + \frac{\epsilon d}{12} \cdot \frac{4}{3} = c + \frac{\epsilon d}{9}. \end{aligned} $$

This inequality and (4.21) yield that

$$ |E_t|/t \leq |E_t|/\big({}^{\mathrm{up}}S(W, n_0 + r)/6\big) \leq (c + \epsilon d/9)/(d/6) = 6c/d + 2\epsilon/3. $$

As $t \to \infty$, $r = r(t)$ and $d = {}^{\mathrm{up}}S(W, n_0 + r)$ also tend to ∞. So for all sufficiently large t, we have that $6c/d < \epsilon/3$, whereby $|E_t|/t \leq \epsilon/3 + 2\epsilon/3 = \epsilon$. Thus, $0 \leq |E_t|/t < \epsilon$ for all but finitely many t, and this is true for every positive $\epsilon \leq 1/12$. That is, $\lim_{t\in\mathbb{N}^+} |E_t|/t = 0$. Hence, the natural density of E is 0 and that of $\mathbb{N}^+ \setminus E$, which occurs in Proposition 1, is 1, as required. The proof or Proposition 1 is complete. □

Remark 2. (Differences from [6] and [10]) The differences we are going to summarize here are partly due to the fact that, naturally, more can be proved for a small particular poset than for all finite posets. When proving that $S(W, n) \leq {}^{\mathrm{up}}S(W, n)$, the only novelty is the argument between (4.8) and (4.9). More novelty occurs in our proof of ${}_{\mathrm{lo}}S(W, p) \leq S(W, n)$. As opposed to Dove and Griggs [6], where several "layers" are populated, we use no iteration and we have (4.10). Compared to Katona and Nagy [10], our construction performs better for small values of n; the following table shows what lower estimates could be extracted from [10]

n	10	50	100
by [10]:	21	14 833 897 694 226	12 229 253 884 310 811 313 310 605 728
${}_{\mathrm{lo}}S(W,n)$:	66	31 761 385 392 516	25 286 044 048 404 745 303 553 386 716

(We have no similar numerical comparison in case of [6].) Except for (2.12), which is quoted from [10] and does not apply for W, [6] and [10] give only asymptotic results but no concrete values of $S(U,n)$ for a poset U.

Remark 3. Even for a small n, the trivial algorithm for determining $S(W,n)$ is far from being feasible. For example, for $n = 10$, the "cover-preserving" copies of W in $\mathsf{P}([10])$ form a

$$\sum_{i=0}^{7} \mathrm{C}_{\mathrm{bin}}(10,i)\cdot \mathrm{C}_{\mathrm{bin}}(10-i,3) = 15\,360\text{-element set } \mathcal{H}.$$

All the $(S(W,10)+1)$-element subsets of $\mathcal{H}$ should be excluded, but no computer can exclude

$$\mathrm{C}_{\mathrm{bin}}(15\,360, S(10)+1) \geq \mathrm{C}_{\mathrm{bin}}(15\,360, 67) \geq 10^{185}$$

subsets; the first inequality here comes from Table 1.

Next, we investigate another small poset, V; see Fig. 1. Define

$${}_{\mathrm{lo}}S(V,n) := \sum_{i=0}^{\lfloor\lceil(n-2)/2\rceil/2\rfloor} \binom{n-2-2i}{\lceil(n-2)/2\rceil - 2i}, \tag{4.22}$$

$${}^{\mathrm{up}}S(V,n) := \Big(1+\frac{2n-3\lfloor n/2\rfloor -1}{2n-\lfloor n/2\rfloor -1}\Big)\cdot \binom{n-2}{\lfloor(n-2)/2\rfloor}. \tag{4.23}$$

Proposition 2 (Mostly from Katona and Nagy [10]). *For $2 \leq n \in \mathbb{N}^+$, Proposition 1 remains valid if we substitute V and $1/3\cdot f_{\mathrm{sb}}(-)$ for W and $1/4\cdot f_{\mathrm{sb}}(-)$, respectively.*

A few values of ${}_{\mathrm{lo}}S(V,n)$ and ${}^{\mathrm{up}}S(V,n)$ are listed below:

n	2	3	4	5	6	7	8	9	10	11	12	13
${}_{\mathrm{lo}}S(V,n)$	1	1	2	4	7	13	24	46	86	166	314	610
${}^{\mathrm{up}}S(V,n)$	1	1	2	4	7	14	25	48	90	173	326	632

, (4.24)

n	14	15	2022	2023
${}_{\mathrm{lo}}S(V,n)$	1 163	2 269	$\approx 2.848\,220\cdot 10^{606}$	$\approx 5.695\,500\cdot 10^{606}$
${}^{\mathrm{up}}S(V,n)$	1 201	2 340	$\approx 2.848\,846\cdot 10^{606}$	$\approx 5.696\,752\cdot 10^{606}$
${}^{\mathrm{up}}S/{}_{\mathrm{lo}}S \approx$	1.033	1.031	1.000 219 853	1.000 219 780

. (4.25)

We do not prove this proposition in the paper. It would be straightforward to simplify the proof of Proposition 1 to obtain a proof of Proposition 2. (The simplification means that $|B_i| = 2$ and all the eligible vectors $\vec{v}$ are of the form $(1,\dots,1)$ and so we do not need them.) Note that arXiv:2308.15625v2, the earlier version of this paper, contains a detailed proof of Proposition 2. However, our construction to prove that ${}_{\mathrm{lo}}S(V,n) \leq S(V,n)$ is included already in Katona and Nagy [10, last page], where ${}_{\mathrm{lo}}S(V,n) = S(V,n)$ is conjectured. Note the little typo in [10, equation (27)]; the upper limit of the summation should be $\lfloor(n+3)/2\rfloor$ rather than $\lfloor(n+2)/2\rfloor$. After that this typo is corrected, (27) in [10] is the same as (4.22).

The computation for the following mini-table took twelve minutes; see Footnote 5

n	2022	2023	2024
${}_{\mathrm{lo}}S(W,n) \approx$	$2.136\,194 \cdot 10^{606}$	$4.271\,332 \cdot 10^{606}$	$8.540\,554 \cdot 10^{606}$
${}^{\mathrm{up}}S(W,n) \approx$	$2.136\,987 \cdot 10^{606}$	$4.272\,916 \cdot 10^{606}$	$8.543\,720 \cdot 10^{606}$
${}^{\mathrm{up}}S/{}_{\mathrm{lo}}S \approx$	1.000 371 103	1.000 370 920	1.000 370 737

. (4.26)

It follows from Propositions 1 and 2, Table 1, (2.11), (4.15), (4.24), (4.25), and (4.26) that the minimum sizes of generating sets of the k-th direct powers of the lattices $\mathrm{Dn}(V)$ and $\mathrm{Dn}(W)$, drawn in Fig. 1, and the 5-element chain C_4 are given as follows

k	2022	2023	$3 \cdot 10^{606}$	$5 \cdot 10^{606}$
$G_{\min}(\mathsf{C}_4^k)$	18	18	2025	2026
$G_{\min}(D(V)^k)$	15	15	2023	2023
$G_{\min}(D(W)^k)$	16	16	2023	2024

.

5. Appendix: Maple worksheet

In this section, we present the Maple worksheet that computed Table 1; see Footnote 5. For the rest of the numerical data in the paper, either the two parameters in the "`for n from 3 to 30 do`" can be modified or a much simpler worksheet would do.

```
> restart;       time0:=time():
> #An upper bound for Sp(W,n):
> upSW:= proc(n) local s; s:=n/(3*n-2-2*floor(n/2));
>   floor(s*binomial(n-1, floor((n-1)/2)));
> end:
> # A lower bound for Sp(W,n):
> loSW:=proc(n) local s,i,j,ub,lb,h,summand,returnvalue;
>   s:=0;
>   if (n=3) or (n=5) or (n=7) then h:=floor((n-3)/2)
>                              else h:=floor((n-1)/2)
>   fi;
>   for i from 0 to ceil(n/3)-1 do ub:=n-3-3*i;
>    #ub: Upper number in the 2nd Binomial coefficient
>    if ub >= 0 then
>     for j from 0 to i do lb:=h-2*j-3*(i-j);# j: number of 2's,
>      #lb: Lower number in the 2nd Binomial coefficient
>      if (lb>=0) and (lb<=ub) then
>        summand:=binomial(i,j)*3^j*binomial(ub,lb);
>        s:=s+summand;
>      fi;#end of the "if (lb>=0) and (lb<=ub)" command
>     od; #end of the  j loop
>    fi; #end of the "if ub >= 0" command
>   od; #end of the i loop
>  returnvalue:=s; #the procedure returns with the last result
> end:
> for n from 3 to 30 do lower:=loSW(n):
>  upper:= upSW(n):
>  if lower>0 then ratio:=evalf(upper/lower) else ratio:=undefined fi :
>  print(`n=`, n, ` lower=` ,lower, ` upper=`,
>      upper, ` ratio=`, ratio);
>  if lower>10^6 then
>   print(`lg(lower)=`,evalf(log[10](lower)),
```

```
>         ‘lg(upper)=‘,evalf(log[10](upper))) fi;
> od:
> time2:=time():
> print(‘The total computation needed ‘, time2-time0,‘ seconds.‘);
```

Based on Theorem 1, some results analogous to Proposition 1 have recently been proved in [4] and [5].

REFERENCES

1. Czédli G. Four-generated direct powers of partition lattices and authentication. *Publ. Math. Debrecen,* 2021. Vol. 99. P. 447–472. DOI: 10.5486/PMD.2021.9024
2. Czédli G. Generating some large filters of quasiorder lattices. *Acta Sci. Math. (Szeged)*, 2024. DOI: 10.1007/s44146-024-00139-5
3. Czédli G. *Generating Boolean lattices by few elements and exchanging session keys.* 2023. 14 p. arXiv:2303.10790v3 [math.RA]
4. Czédli G. Generating the powers of a Boolean lattice with an extra 0. In: *Algebra and Model Theory 14: Proc. Internat. Conf., Novosibirsk—Erlagol.* Novosibirsk: Novosibirsk State Technical University, 2023. P. 25–40.
5. Czédli G. Minimum-sized generating sets of the direct powers of free distributive lattices. *Cubo*, 2024. Vol. 26, No. 2. P. 217–237. DOI: 10.56754/0719-0646.2602.217
6. Dove A. P., Griggs J. R. Packing posets in the Boolean lattice. *Order,* 2015. Vol 32. P. 429–438. DOI: 10.1007/s11083-014-9343-7
7. Gelfand I. M., Ponomarev V. A. Problems of linear algebra and classification of quadruples of subspaces in a finite dimensional vector space. In: *Hilbert Space Operators and Operator Algebras: Proc. Internat. Conf., Tihany, 1970. Colloq. Math. Soc. János Bolyai, vol. 5.* Amsterdam–London: North-Holland Publishing, 1972. P. 163–237.
8. Grätzer G. *Lattice Theory: Foundation.* Basel: Birkhäuser, 2011. XXX+614 p. DOI: 10.1007/978-3-0348-0018-1
9. Griggs J. R., Stahl J., Trotter W. T. Jr. A Sperner theorem on unrelated chains of subsets. *J. Combinatorial Theory Ser. A,* 1984. Vol. 36, No. 1. P. 124–127. DOI: 10.1016/0097-3165(84)90085-2
10. Katona G. O. H., Nagy D. T. Incomparable copies of a poset in the Boolean lattice. *Order,* 2015. Vol. 32. P. 419–427. DOI: 10.1007/s11083-014-9342-8
11. Lubell D. A short proof of Sperner's lemma. *J. Combinatorial Theory,* 1966. Vol. 1, No. 2. P. 299. DOI: 10.1016/S0021-9800(66)80035-2
12. McKenzie. R. N., McNulty G. F., Taylor W. F. *Algebras, Lattices, Varieties. I.* Monterey, California: Wadsworth & Brooks/Cole, 1987. XII+361 p. DOI: 10.1090/chel/383.H
13. Sperner E. Ein Satz über Untermengen einer endlichen Menge. *Math. Z.,* 1928. Vol. 27. P. 544–548. DOI: 10.1007/BF01171114 (in German)
14. Zádori L. Subspace lattices of finite vector spaces are 5-generated. *Acta Sci. Math. (Szeged),* 2008. Vol. 74, No. 3. P. 493–499.

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 61–67

DOI: 10.15826/umj.2024.1.005

PRICING POWERED α-POWER QUANTO OPTIONS WITH AND WITHOUT POISSON JUMPS

Javed Hussain[†], **Nisar Ali**[††]

Department of Mathematics, Sukkur IBA University,
Nisar Ahmed Siddiqui Road, Sukkur Sindh, Pakistan

[†]javed.brohi@iba-suk.edu.pk [††]nisar.msmaths19@iba-suk.edu.pk

Abstract: This paper deals with the problem of Black–Scholes pricing for the Quanto option pricing with power type powered and powered payoff underlying foreign currency is driven by Brownian motion and Poisson jumps, via risk-neutral probability measure. Our approach in this work is probabilistic, based on Feynman–Kac formula.

Keywords: Financial derivatives, Quanto option, Power payoff, Risk-neutral dynamics.

1. Introduction

This study focuses on the pricing of Quanto options with a powered-power payoff, where the underlying foreign currency is driven by a combination of Brownian motion and Poisson jumps, with the aim of avoiding arbitrage. Quanto options are derivatives that permit investors to acquire foreign assets without being exposed to the corresponding foreign exchange risk. These options are typically used when an investor wants to gain exposure to foreign assets without assuming foreign exchange risk [6]. For instance, if an investor wants to invest in a foreign market but does not want to take on the associated foreign exchange risk, they could utilize. Although, swap options remain a valuable strategic tool for the financial institutions by managing currency risks and exploring the opportunities from international markets. The main reason that traders buy and sell these assets is covering their risks in currency exposure, in addition to speculating that expected foreign currency appreciation will happen. So, this trading allows investors to take advantage of this real appreciation. Along with that, these financial instruments are multi-functional and cover more areas as portfolio diversification, tax optimization, the reduction of risk, etc. At times, a circumstance, where an investor tries to overcome currency problems and at the same time, manage their tax implications and portfolio diversification by using Quanto option which exists.

Conventionally, Quanto options have been solved using the Black–Scholes model as the underlying asset opinions under the guard condition of volatile constantly [1]. Although volatility imply method involves smiles and skews, however it is not a reason which leads to confusion. Addressing this, a series of local and volatility models are adopted, with a volatility, which is considered as a deterministic function of multiple factors such as the asset's price, underlying asset, current time, maturity, and option strike price. Local variability hypothesis of Quanto options addresses the accuracy of option price by overcoming the constraint of exogenously assumed volatility of options inherent in Black–Scholes model. In actual, Dupire [4] and Derman [3] were the leading researchers that developed and enhanced the permanent local volatility model as they identified a special diffusion process that is in line with the observed densities of the risk neutral probabilities which are derived from the implied volatility surfaces obtained from the European-style options in the market. The main advantage of local volatility models is their simplicity which is such that a randomness source is just one input, the price of underlying assets, thus giving the ability to easily calibrate. Here we have the power option that is a derivative in which the payoff is depended upon the underlying assets in the square root, cubic form, etc. Through this structure, the purchaser will be able to go for one side with respect to specific derivative or its volatility, or he will be left out depending on the trend observed in the Vanilla Options. Power options are commonly associated with the difference in the current price of the underlying instrument over fees that would exhibit intensity. Option in power call counterpart corresponds to cash flow of $\max(S_T^\alpha - K)$, while option in power put partners with $\max(K - S_T^\alpha)$, where $\alpha \geq 0, \alpha \in \mathbb{N}$. Taking Black–Scholes [1] leverage and diversification are the key discretionary using for only those investors who seek to acquire larger initial capital or premium, and, most possibly, this desire contributes to creation of their appeal.

Results are presented in this article are novel and have likely substantial value for the future comparisons of respective researches. Our work would encompass a variety of new findings at one point. The approach deals with gaps in the Black–Scholes risk-neutral valuation method, where the powered α-power Quanto call option prevails in the domestic currency which is fixed before and the use of the Feynman–Kac formula, both with and without Poisson jumps.

2. Price of Quanto option for a payoff at maturity

A foreign equity powered α-power Quanto call option, struck in a predetermined domestic currency, matures with a payoff given by

$$V_0 \left(\max\left(S_T^\alpha - K_f, 0\right)\right)^n = V_0 \left[\left(S_T^\alpha - K_f\right)^+\right]^n = V_0 \left[\left(S_T^\alpha - K_f\right)^n \mathbb{I}_{S{T^\alpha} > K_f}\right],$$

where V_0 represents a fixed exchange rate and K_f denotes the foreign currency strike price.

Assuming $n > 0$ is an integer, the payoff transforms into

$$V_0 \sum_{j=0}^{n} \binom{n}{j} \left(S_T^\alpha\right)^{n-j} \left(-K_f\right)^{1,j} \mathbb{I}_{\{S_T^\alpha > K_f\}}. \tag{2.1}$$

Theorem 1. *Let S_t represent the asset price in foreign currency X, and V_t denote the foreign exchange rate in foreign currency per unit of the domestic currency, both with constant volatilities σ_S and σ_V, respectively. We consider the risk-neutral dynamics (in domestic currency, cf. [5]) for a dividend-paying asset with rate q as follows*:

$$\begin{cases} dS_t = (r_f - q - \rho\sigma_S\sigma_V)S_t dt + \sigma_S S_t dB_t^{\mathbb{Q}^d}, \\ dV_t = (r_d - r_f)V_t dt + \sigma_V V_t dW_t^{\mathbb{Q}^d}, \end{cases} \tag{2.2}$$

where $B_t^{\mathbb{Q}^d}$ and $W_t^{\mathbb{Q}^d}$, $t \in [0, T]$, are $\mathbb{Q}^d$-standard Wiener processes. Then, for $\alpha > 0$, the price of a European power-α Quanto call option at time t in domestic currency with the payoff (2.1) *is given by*

$$C_q\left(t, S_t^\alpha\right) = V_0 e^{-r_d(T-t)} \sum_{j=0}^{n} \binom{n}{j} \left(-K_f\right)^{1,j} S_t^{\alpha(n-j)} e^{\alpha(n-j)\left\{r_f - q - \rho\sigma_S\sigma_V - (1-\alpha(n-j))\cdot\sigma_S^2/2\right\}\tau} N(d_{1,j}).$$

Here

$$d_{1,j} = \frac{\ln\left(S_t^\alpha / K_f\right) + \alpha\left(r_f - q - \rho\sigma_S\sigma_V - (1 + \alpha(n-j)/2)\,\sigma_S^2/2\right)\tau}{\alpha\sigma_S\sqrt{\tau}}.$$

P r o o f. Using the Feynman-Kac formula, as stated in Theorem 4.33 of the reference [2], the arbitrage price of a call option at time t, where t is less than or equal to the expiration date T, can be determined under the risk-neutral probability measure $\mathbb{Q}^d$,

$$C_q\left(t, S_t^\alpha\right) = V_0 e^{-r_d(T-t)} \sum_{j=0}^{n} \binom{n}{j} \left(-K_f\right)^{1,j} \mathbb{E}_{\mathbb{Q}^d}\left[\left(S_T^\alpha\right)^{n-j} \mathbb{I}_{\{S_T^\alpha > K_f\}} | \mathcal{F}_t\right]. \tag{2.3}$$

Hence, it remains to evaluate the conditional expectation in (2.3) for $0 \leq j < n$. In order to compute that, we must compute the solution for the SDE (2.2). Applying Ito's lemma on process $(\ln S_t)$ for $t \geq 0$, hence

$$d\left(\ln S_t\right) = \left(r_f - q - \rho\sigma_S\sigma_V - \frac{\sigma_S^2}{2}\right) dt + \sigma_S dB_t^{\mathbb{Q}^d}.$$

Integrating both sides, we get,

$$\int_t^T d\left(\ln S_u\right) = \int_t^T \left(r_f - q - \rho\sigma_S\sigma_V - \frac{\sigma_S^2}{2}\right) du + \int_t^T \sigma_S dB_u^{\mathbb{Q}^d},$$

$$\ln\left(\frac{S_T}{S_t}\right) = \left(r_f - q - \rho\sigma_S\sigma_V - \frac{\sigma_S^2}{2}\right)(T-t) + \sigma_S\left(B_T^{\mathbb{Q}^d} - B_t^{\mathbb{Q}^d}\right),$$

i.e.

$$S_T = S_t e^{\left\{r_f - q - \rho\sigma_S\sigma_V - \sigma_S^2/2\right\}(T-t) + \sigma_S\left(B_T^{\mathbb{Q}^d} - B_t^{\mathbb{Q}^d}\right)}.$$

We then have

$$\left(S_T^\alpha\right)^{n-j} = S_t^{\alpha(n-j)} e^{\alpha(n-j)\left(r_f - q - \rho\sigma_S\sigma_V - \sigma_S^2/2\right)\tau - \alpha(n-j)\sigma_S\sqrt{\tau}Z}, \tag{2.4}$$

where

$$T - t = \tau \quad \text{and} \quad Z = -\frac{B_T^{\mathbb{Q}^d} - B_t^{\mathbb{Q}^d}}{\sqrt{\tau}} \sim \mathcal{N}(0,\, 1),$$

which is independent of $\mathcal{F}_t$, we find that $S_T^\alpha > K_f$ if and only if

$$Z < \frac{\ln\left(S_t^\alpha / K_f\right) + \alpha\left(r_f - q - \rho\sigma_S\sigma_V - \sigma_S^2/2\right)\tau}{\alpha\sigma_S\sqrt{\tau}} =: -d_{2,j}. \tag{2.5}$$

It follows from (2.4), (2.5) and from the independence of Z with $\mathcal{F}_t$ that

$$\mathbb{E}_{\mathbb{Q}^d}\left[\left(S_T^\alpha\right)^{n-j} \mathbb{I}_{\{S_T^\alpha > K_f\}} | \mathcal{F}_t\right] = S_t^{\alpha(n-j)} e^{\alpha(n-j)\left(r_f - q - \rho\sigma_S\sigma_V - \sigma_S^2/2\right)\tau}$$
$$\times \mathbb{E}_{\mathbb{Q}^d}\left[e^{-\alpha(n-j)\sigma_S\sqrt{\tau}Z} \mathbb{I}_{\{Z < d_{2,j}\}} | \mathcal{F}_t\right] = g(\tau, S_t^\alpha),$$

where $g(\tau, x)$ is given by

$$g(\tau, x) = x^{\alpha(n-j)} e^{\alpha(n-j)\left(r_f - q - \rho\sigma_S\sigma_V - \frac{\sigma_S^2}{2}\right)\tau} \mathbb{E}_{\mathbb{Q}^d}\left[e^{-\alpha(n-j)\sigma_S\sqrt{\tau}Z} \mathbb{I}_{\{Z < d_{2,j}\}} | \mathcal{F}_t\right].$$

Since $Z \sim \mathcal{N}(0,\, 1)$, we obtain

$$g(\tau, x) = x^{\alpha(n-j)} e^{\alpha(n-j)\left(r_f - q - \rho\sigma_S\sigma_V - \sigma_S^2/2\right)\tau} \int_{-\infty}^{d_{2,j}} \frac{1}{\sqrt{2\pi}} e^{-\alpha(n-j)\sigma_S\sqrt{\tau}z - z^2/2} dz$$
$$= x^{\alpha(n-j)} e^{\alpha(n-j)\left\{r_f - q - \rho\sigma_S\sigma_V - (1 - \alpha(n-j))\sigma_S^2/2\right\}\tau} \int_{-\infty}^{d_{2,j}} \frac{1}{\sqrt{2\pi}} e^{-(z + \alpha(n-j)\sigma_S\sqrt{\tau})^2/2} dz.$$

Applying the substituting $v = z + \alpha(n-j)\sigma_S\sqrt{\tau}$ and setting

$$\begin{aligned} d_{1,j} := d_{2,j} + \alpha(n-j)\sigma_S\sqrt{\tau} &= \frac{\ln\left(S_t^\alpha/K_f\right) + \alpha\left(r_f - q - \rho\sigma_S\sigma_V - \sigma_S^2/2\right)\tau}{\alpha\sigma_S\sqrt{\tau}} + \alpha(n-j)\sigma_S\sqrt{\tau} \\ &= \frac{\ln\left(S_t^\alpha/K_f\right) + \alpha\left(r_f - q - \rho\sigma_S\sigma_V - \left(1+\alpha(n-j)/2\right)\sigma_S^2/2\right)\tau}{\alpha\sigma_S\sqrt{\tau}}, \end{aligned} \tag{2.6}$$

we get

$$\begin{aligned} g(\tau, x) &= x^{\alpha(n-j)} e^{\alpha(n-j)\left\{r_f - q - \rho\sigma_S\sigma_V - (1-\alpha(n-j))\sigma_S^2/2\right\}\tau} \int_{-\infty}^{d_{1,j}} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dz \\ &= x^{\alpha(n-j)} e^{\alpha(n-j)\left\{r_f - q - \rho\sigma_S\sigma_V - (1-\alpha(n-j))\sigma_S^2/2\right\}\tau} N(d_{1,j}). \end{aligned} \tag{2.7}$$

From (2.6) and (2.7), (2.3) becomes

$$C_q\left(t, S_t^\alpha\right) = V_0 e^{-r_d(T-t)} \sum_{j=0}^{n} \binom{n}{j} \left(-K_f\right)^{1,j} S_t^{\alpha(n-j)} e^{\alpha(n-j)\left\{r_f - q - \rho\sigma_S\sigma_V - (1-\alpha(n-j))\sigma_S^2/2\right\}\tau} N(d_{1,j}),$$

where

$$d_1 = \frac{\ln\left(S_t^\alpha/K_f\right) + \alpha\left(r_f - q - \rho\sigma_S\sigma_V - \left(1+\alpha(n-j)/2\right)\sigma_S^2/2\right)\tau}{\alpha\sigma_S\sqrt{\tau}}.$$

Fig. 1 depicts the progression of the Quanto expense concerning maturity time T and the strike price K_f.

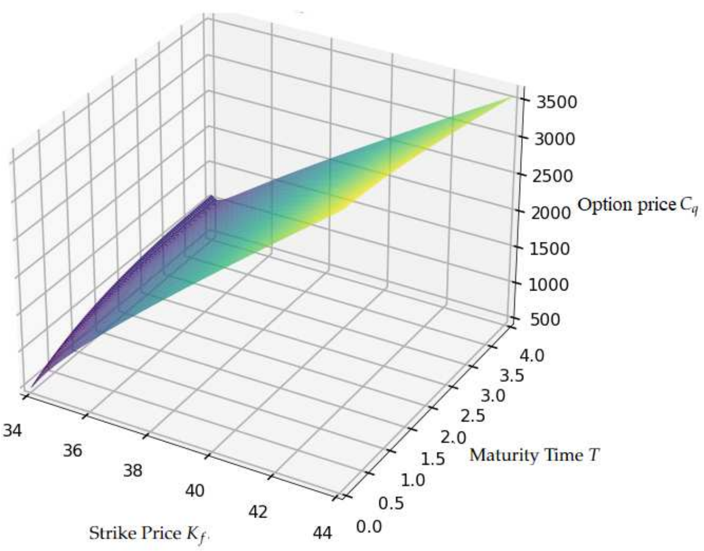


Figure 1. Powered power Quanto option value plotted against maturity time and strike price.

Now, our attention shifts to analyzing Quanto option premiums concerning the foreign currency strike price and maturity time. With $r_d = 0.5, V_0 = 30, \alpha = 5, r_f = 0.01, q = 0.1, \rho = 0.01, \sigma_s = 0.3$, and $\sigma_v = 0.2$, Fig. 1 illustrates the values of Quanto call option prices for $K_f \in [34, 44]$ and $T \in [0, 4]$. The plot reveals that while the evolution of Quanto option values isn't strictly monotonic, there's a discernible trend of increasing option prices with higher strike prices and longer maturities.

3. Pricing Quanto option with jumps

Theorem 2. *Suppose S_t represents the asset price in foreign currency X, where (N_t), $t \in \mathbb{R}+$, is a standard Poisson process with intensity $\lambda > 0$, independent of (B_t), $t \in \mathbb{R}+$, under a probability measure $\mathbb{Q}^d$. Let V_t denote the foreign exchange rate in foreign currency per unit of the domestic currency, both with constant volatilities σ_S and σ_V, respectively. We assume the following risk-neutral dynamics for a dividend-paying asset with rate q.*

$$\begin{cases} dS_t = (r_f - q - \rho\sigma_S\sigma_V)S_t dt + \sigma_S S_t dB_t^{\mathbb{Q}^d} + \eta S_{t-} dN_t, \\ dV_t = (r_d - r_f)V_t dt + \sigma_V V_t dW_t^{\mathbb{Q}^d}, \end{cases}$$

where $B_t^{\mathbb{Q}^d}$ and $W_t^{\mathbb{Q}^d}$, $t \in [0,T]$, are $\mathbb{Q}^d$ — standard Wiener processes. Then, for $\alpha > 0$, the price $C_q(t, S_t^\alpha)$ of a European power-α Quanto call option with jumps, at time t in domestic currency with the payoff (2.1), is given by,

$$C_q = V_0 e^{(\lambda - r_d)(T-t)} \sum_{j=0}^{n} \binom{n}{j} (-K_f)^{1,j} S_t^{\alpha(n-j)} e^{\alpha(n-j)\{r_f - q - \rho\sigma_S\sigma_V - (1-\alpha(n-j))\sigma_S^2/2\}\tau}$$
$$\times \sum_{n \geq 0} \frac{(\lambda(T-t))^n}{n!} N(d_{1,j}).$$

Here

$$d_{1,j} = \frac{\ln\left(S_t^\alpha (1+\eta)^n / K_f\right) + \alpha\left(r_f - q - \rho\sigma_S\sigma_V - (1+\alpha(n-j)/2)\,\sigma_S^2/2\right)\tau}{\alpha\sigma_S\sqrt{\tau}}.$$

P r o o f. As earlier, let us start by employing Feynman–Kac formula, as stated in [2, Theorem 4.33]. Under the risk-neutral probability measure $\mathbb{Q}^d$, the arbitrage price of the call option at time $t \leq T$ can be determined

$$C_q(t, S_t^\alpha) = V_0 e^{-r_d(T-t)} \sum_{j=0}^{n} \binom{n}{j} (-K_f)^{1,j} \mathbb{E}_{\mathbb{Q}^d}\left[(S_T^\alpha)^{n-j} \mathbb{I}_{\{S_T^\alpha > K_f\}} | \mathcal{F}_t\right], \tag{3.8}$$

where

$$S_T^\alpha = S_t^\alpha e^{\alpha\{r_f - q - \rho\sigma_S\sigma_V - \sigma_S^2/2\}(T-t) - \alpha\sigma_S(B_T^{\mathbb{Q}^d} - B_t^{\mathbb{Q}^d})} (1+\eta)^{N_T - N_t}.$$

We then have

$$(S_T^\alpha)^{n-j} = S_t^{\alpha(n-j)} e^{\alpha(n-j)\left(r_f - q - \rho\sigma_S\sigma_V - \sigma_S^2/2\right)\tau - \alpha(n-j)\sigma_S\sqrt{\tau}Z} (1+\eta)^{N_\tau}, \tag{3.9}$$

where

$$T - t = \tau, \quad Z = -\frac{B_T^{\mathbb{Q}^d} - B_t^{\mathbb{Q}^d}}{\sqrt{\tau}} \sim \mathcal{N}(0, 1),$$

which is independent of $\mathcal{F}_t$, we find that $S_T^\alpha > K_f$ if and only if

$$Z < \frac{\ln\left(S_t^\alpha (1+\eta)^n / K_f\right) + \alpha\left(r_f - q - \rho\sigma_S\sigma_V - \sigma_S^2/2\right)\tau}{\alpha\sigma_S\sqrt{\tau}} =: -d_{2,j}. \tag{3.10}$$

It follows form (3.9), (3.10) and the independence of Z with $\mathcal{F}_t$ that

$$\mathbb{E}_{\mathbb{Q}^d}\left[(S_T^\alpha)^{n-j} \mathbb{I}_{\{S_T^\alpha > K_f\}} | \mathcal{F}_t\right] = S_t^{\alpha(n-j)} e^{\alpha(n-j)\left(r_f - q - \rho\sigma_S\sigma_V - \sigma_S^2/2\right)\tau}$$
$$\times \mathbb{E}_{\mathbb{Q}^d}\left[e^{-\alpha(n-j)\sigma_S\sqrt{\tau}Z} (1+\eta)^{N_\tau} \mathbb{I}_{\{Z < d_{2,j}\}} | \mathcal{F}_t\right] = g(\tau, S_t^\alpha),$$

where $g(\tau,x)$ is given by

$$\begin{aligned} g(\tau,x) &= x^{\alpha(n-j)}e^{\alpha(n-j)\left(r_f-q-\rho\sigma_S\sigma_V-\sigma_S^2/2\right)\tau}\mathbb{E}_{\mathbb{Q}^d}\left[e^{-\alpha(n-j)\sigma_S\sqrt{\tau}Z}(1+\eta)^{N_\tau}\mathbb{I}_{\{Z<d_{2,j}\}}|\mathcal{F}_t\right] \\ &= x^{\alpha(n-j)}e^{\alpha(n-j)\left(r_f-q-\rho\sigma_S\sigma_V-\sigma_S^2/2\right)\tau}\sum_{n\geq 0}\mathbb{P}(N_\tau=n)\mathbb{E}_{\mathbb{Q}^d}\left[e^{-\alpha(n-j)\sigma_S\sqrt{\tau}Z}(1+\eta)^{n}\mathbb{I}_{\{Z<d_{2,j}\}}|\mathcal{F}_t\right]. \end{aligned}$$

Since $Z\sim\mathcal{N}(0,\,1)$, we obtain

$$\begin{aligned} g(\tau,x) &= x^{\alpha(n-j)}e^{\alpha(n-j)\left(r_f-q-\rho\sigma_S\sigma_V-\sigma_S^2/2\right)\tau} \\ &\times\sum_{n\geq 0}\mathbb{P}(N_\tau=n)(1+\eta)^n\int_{-\infty}^{d_{2,j}}\frac{1}{\sqrt{2\pi}}e^{-\alpha(n-j)\sigma_S\sqrt{\tau}z-z^2/2}dz \\ &= x^{\alpha(n-j)}e^{\alpha(n-j)\left\{r_f-q-\rho\sigma_S\sigma_V-(1-\alpha(n-j))\sigma_S^2/2\right\}\tau}e^{\lambda\tau} \\ &\times\sum_{n\geq 0}\frac{(\lambda(T-t))^n}{n!}\int_{-\infty}^{d_{2,j}}\frac{1}{\sqrt{2\pi}}e^{-(z+\alpha(n-j)\sigma_S\sqrt{\tau})^2/2}dz. \end{aligned}$$

Applying the substituting $v=z+\alpha(n-j)\sigma_S\sqrt{\tau}$ and setting

$$\begin{aligned} d_{1,j} &:= d_{2,j}+\alpha(n-j)\sigma_S\sqrt{\tau} \\ &= \frac{\ln\left(S_t^\alpha(1+\eta)^n/K_f\right)+\alpha\left(r_f-q-\rho\sigma_S\sigma_V-\sigma_S^2/2\right)\tau}{\alpha\sigma_S\sqrt{\tau}}+\alpha(n-j)\sigma_S\sqrt{\tau} \\ &= \frac{\ln\left(S_t^\alpha(1+\eta)^n/K_f\right)+\alpha\left(r_f-q-\rho\sigma_S\sigma_V-(1+\alpha(n-j)/2)\,\sigma_S^2/2\right)\tau}{\alpha\sigma_S\sqrt{\tau}}, \end{aligned} \tag{3.11}$$

we get

$$\begin{aligned} g(\tau,x) &= x^{\alpha(n-j)}e^{\alpha(n-j)\left\{r_f-q-\rho\sigma_S\sigma_V-(1-\alpha(n-j))\sigma_S^2/2\right\}\tau}e^{\lambda\tau}\sum_{n\geq 0}\frac{(\lambda(T-t))^n}{n!}\int_{-\infty}^{d_{1,j}}\frac{1}{\sqrt{2\pi}}e^{-v^2/2}dz \\ &= x^{\alpha(n-j)}e^{\lambda+\alpha(n-j)\left\{r_f-q-\rho\sigma_S\sigma_V-(1-\alpha(n-j))\sigma_S^2/2\right\}\tau}\sum_{n\geq 0}\frac{(\lambda(T-t))^n}{n!}N(d_{1,j}). \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), (3.8) becomes

$$\begin{aligned} C_q = V_0e^{(\lambda-r_d)(T-t)}&\sum_{j=0}^{n}\binom{n}{j}(-K_f)^{1,j}\,S_t^{\alpha(n-j)}e^{\alpha(n-j)\left\{r_f-q-\rho\sigma_S\sigma_V-(1-\alpha(n-j))\sigma_S^2/2\right\}\tau} \\ &\times\sum_{n\geq 0}\frac{(\lambda(T-t))^n}{n!}N(d_{1,j}), \end{aligned}$$

where

$$d_{1,j}=\frac{\ln\left(S_t^\alpha(1+\eta)^n/K_f\right)+\alpha\left(r_f-q-\rho\sigma_S\sigma_V-(1+\alpha(n-j)/2)\,\sigma_S^2/2\right)\tau}{\alpha\sigma_S\sqrt{\tau}}.$$

The diagram below illustrates the Quanto premium evolution with jumps concerning maturity time T and the strike price K_f.

Using the same dataset as before, with $r_d=0.5, V_0=30, \alpha=5, r_f=0.01, q=0.1,\ \rho=0.01$, $\sigma_s=0.3$, and $\sigma_v=0.2$. Additionally, setting $\eta=5, \lambda=5, n=6$ and $N=5$, Fig. 2 depicts the progression of Quanto option prices with jumps. It's noticeable that the Quanto option value exhibits an upward trend concerning both variables, maturity time and strike price.

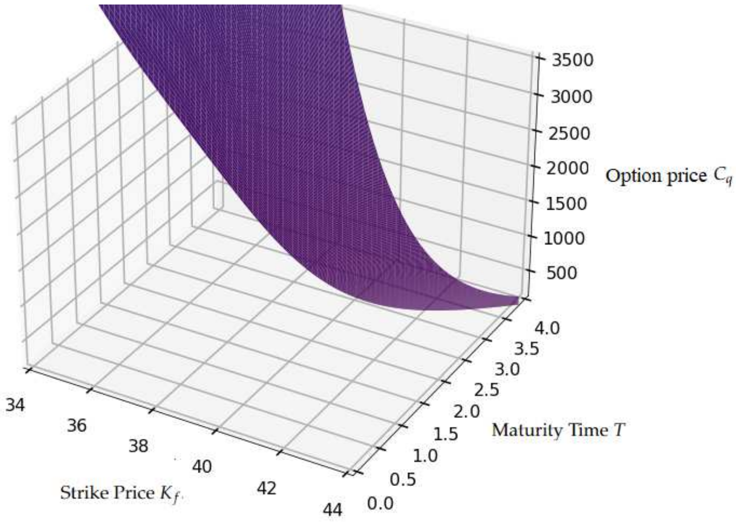


Figure 2. Powered power Quanto option call with jumps plotted against maturity time and strike price.

4. Conclusion

Quanto options are crucial tools for managing risk in the foreign exchange market. Determining their fair prices without arbitrage opportunities is essential. In this study, we have developed formulas to find the no-arbitrage prices for powered Quanto options. We considered scenarios where the underlying currencies follow Brownian motion and Brownian motion with jumps. We supported our theoretical framework with numerical simulations and results. We hope this research will inspire further exploration and interest in pricing exotic options.

REFERENCES

1. Black F., Scholes M. The pricing of options and corporate liabilities. *J. Political Economy*, 1973. Vol. 81, No. 3. P. 637–654. URL: http://www.jstor.org/stable/1831029
2. Capiński M., Kopp E., Traple J. *Stochastic Calculus for Finance.* Cambridge, UK: Cambridge University Press, 2012. 186 p.
3. Derman E., Kani I. Stochastic implied trees: arbitrage pricing with stochastic term and strike structure of volatility. *Int. J. Theor. Appl. Finance*, 1998. Vol. 1, No. 1. P. 61–110. DOI: 10.1142/s0219024998000059
4. Dupire B. Pricing with a smile. Risk, 1994. Vol. 7. P. 18–20.
5. Lee Y., Yoo H.-S., Lee J. Pricing formula for power Quanto options with each type of payoffs at maturity. *Global J. Pure Appl. Math.*, 2017. Vol. 13, No. 9. P. 6695–6702.
6. Teng L., Ehrhardt M., Günther M. The pricing of Quanto options under dynamic correlation. *J. Comput. Appl. Math.*, 2015. Vol. 275. P. 304–310. DOI: 10.1016/j.cam.2014.07.017

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 68–75

DOI: 10.15826/umj.2024.1.006

EXTREMAL VALUES ON THE MODIFIED SOMBOR INDEX OF TREES AND UNICYCLIC GRAPHS

Raghavendra H Kashyap[1], Yanamandram B. Venkatakrishnan[1], Rashad Ismail[2], Selvaraj Balachandran[1], Hari Naresh Kumar[1,†]

[1]School of Arts Sciences Humanities & Education, SASTRA Deemed University, Thanjavur, India – 613 401

[2]Department of Mathematics, Faculty of Science and Arts, King Khalid University, Mahayl Assir 61913, Saudi Arabia

[†]nareshhari1403@gmail.com

Abstract: Let $G=(V,E)$ be a simple connected graph. The modified Sombor index denoted by $mSo(G)$ is defined as

$$mSo(G)=\sum_{uv\in E}\frac{1}{\sqrt{d_u^2+d_v^2}},$$

where d_v denotes the degree of vertex v. In this paper we present extremal values of modified Sombor index over the set of trees and unicyclic graphs.

Keywords: Modified Sombor Index, Trees, Unicyclic graphs, Extremal values.

1. Introduction

A topological index is a real number derived from a structure of a graph that is not dependent on the way the vertices are labeled. A wide range of different topological indices have been employed in QSAR (Quantitative Structure – Activity Relationship) and QSPR (Quantitative Structure – Property Relationship) studies. Any topological indices belong to one of the two classes: they are either bond-additive, or distance based. Typical representation of bond-additive indices are two Zagreb indices, Harmonic index and Randić index.

Let $G=(V,E)$ be a simple connected graph. By the open neighborhood of a vertex v of G we mean the set

$$N_G(v)=\{u\in V: uv\in E\}$$

and by the closed neighborhood,

$$N_G[v]=N_G(v)\cup\{v\}.$$

The degree d_v of a vertex v is the cardinality of its open neighborhood. We denote by P_n and C_n a path and a cycle with n-vertices, respectively. A length of a cycle is the number of edges contained in the cycle. A star of order $n\geq 2$, denoted by S_n is a tree with at least $n-1$ leaves. A contraction of an edge $e=uv$ is the replacement of u and v with a single vertex such that edges incident to the new vertex are the edges other than e that were incident with u or v and the resulting graph is denoted by $G.uv$.

Recently, a degree based topological index called the Sombor index was introduced by Ivan Gutman in [4]. It is defined as

$$SO(G)=\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}$$

and further studied in [1–3, 6, 9, 10]. A variant of Sombor index namely, modified Sombor index, denoted by $mSo(G)$, is defined as

$$mSo(G) = \sum_{uv \in E} \frac{1}{\sqrt{d_u^2 + d_v^2}}.$$

In [8], a lower bound on a Modified Sombor index of unicyclic graphs with a given diameter is presented. In [7], bounds of modified Sombor index in terms of spectral radius and energy is given. A study on modified Sombor index matrix is done in [11]. An extreme value of the product of the Sombor index and the modified Sombor index is studied in [5].

In [7], to determine the extremal trees, unicyclic graphs, bicyclic graphs with respect to modified Sombor index were proposed. We determine the extremal graphs for the class of trees and unicyclic graphs, which answers the problem posed in [7]. In particular, we show that star and paths are the graphs with minimum and maximum modified Sombor index among all trees, and for unicyclic graphs we show that $U_n(n-1, 2, 2)$ and cycle are the graphs with minimum and maximum modified Sombor index.

2. Graph transformations

To begin with we present some graph transformations which will be useful to determine the extremal trees and unicyclic graphs.

Transformation A (*see Fig. 1*). Let G be a nontrivial connected graph and $u, v \in V(G)$, such that $d(v) \geq 3$ in G and $P_1 : uu_1u_2 \dots u_r$ and $P_2 : vv_1v_2 \dots v_s$ be two paths in G. Now we denote the graph H obtained from G by concatenating the paths P_1 and P_2.

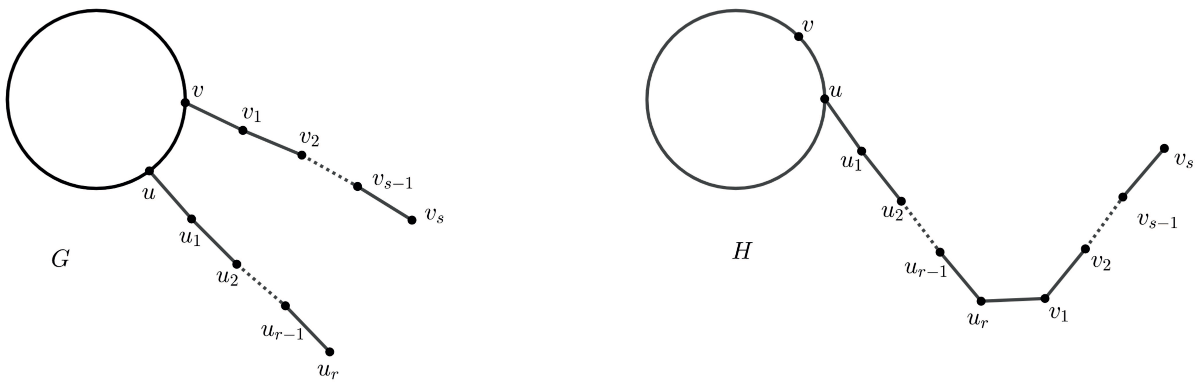


Figure 1. Transformation A.

Theorem 1. *Let H be the graph obtained from G using Transformation A, then $mSo(G) \leq mSo(H)$.*

P r o o f. The vertex v_1 in path P_2 is made adjacent to vertex u_r. Then

$$mSo(H) = mSo(G) - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{8}} - \frac{1}{\sqrt{d_v^2 + 4}}$$
$$- \sum_{\alpha \in N(v) \setminus v_1} \frac{1}{\sqrt{d_v^2 + d_\alpha^2}} + \sum_{\alpha \in N(v) \setminus v_1} \frac{1}{\sqrt{(d_v - 1)^2 + d_\alpha^2}}$$

$$= mSo(G) - \frac{1}{\sqrt{5}} + \frac{2}{\sqrt{8}} - \frac{1}{\sqrt{d_v^2+4}} + \sum_{\alpha \in N(v)\setminus v_1} \left(\frac{1}{\sqrt{(d_v-1)^2+d_\alpha^2}} - \frac{1}{\sqrt{d_v^2+d_\alpha^2}} \right)$$

$$mSo(H) \geq mSo(G) - \frac{1}{\sqrt{5}} + \frac{2}{\sqrt{8}} - \frac{1}{\sqrt{13}} + \sum_{\alpha \in N(v)\setminus v_1} \left(\frac{1}{\sqrt{(d_v-1)^2+d_\alpha^2}} - \frac{1}{\sqrt{d_v^2+d_\alpha^2}} \right)$$

$$mSo(H) \geq mSo(G) - \frac{1}{\sqrt{5}} + \frac{2}{\sqrt{8}} - \frac{1}{\sqrt{13}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{10}} > mSo(G).$$

□

Transformation B (see Fig. 2). Let G be a nontrivial connected graph and $u \in V(G)$, such that $d(u) \geq 3$ in G and $P : uu_1u_2 \ldots u_t$ be the path in G. The H is constructed from G by removing the leaf u_t in the path P and attaching it to the vertex u by an edge uu_t.

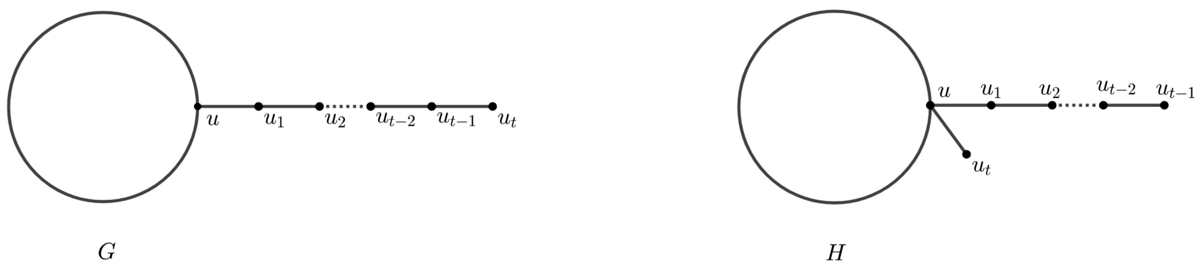


Figure 2. Transformation B.

Theorem 2. *Let H be the graph obtained from G using transformation B, then $mSo(H) \leq mSo(G)$.*

P r o o f. Applying Transformation B to graph G, we have

$$mSo(H) = mSo(G) - \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{(d_u+1)^2+1}}$$

$$- \sum_{\alpha \in N(u)} \frac{1}{\sqrt{d_u^2+d_\alpha^2}} + \sum_{\alpha \in N(u)} \frac{1}{\sqrt{(d_u+1)^2+d_\alpha^2}}$$

$$= mSo(G) - \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{(d_u+1)^2+1}} + \sum_{\alpha \in N(u)} \left(\frac{1}{\sqrt{(d_u-1)^2+d_\alpha^2}} - \frac{1}{\sqrt{d_u^2+d_\alpha^2}} \right)$$

$$mSo(H) \leq mSo(G) - \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{17}} < mSo(G).$$

□

Transformation C (see Fig. 3). Let G be a nontrivial connected graph, $uv \in E(G)$ with $N(v) \cap N(u) = \emptyset$. We denote the graph H obtained from $G.uv$ and making the vertex v adjacent to u by an edge uv.

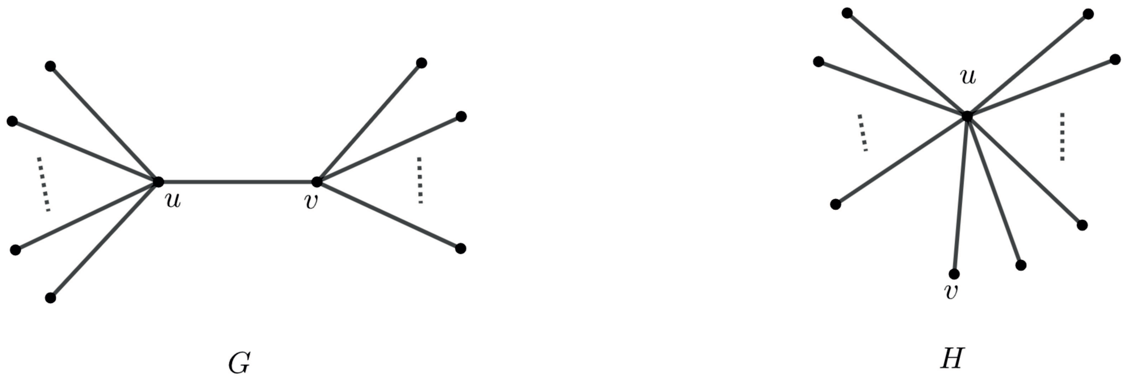


Figure 3. Transformation C.

Theorem 3. *Let H be the graph obtained from G using transformation C, then $mSo(H) \leq mSo(G)$.*

P r o o f. From the definition of Transformation C, we have $d_u, d_v \geq 2$. Then

$$mSo(H) = mSo(G) - \frac{1}{\sqrt{d_u^2 + d_v^2}} - \sum_{\alpha \in N(u) \setminus v} \frac{1}{\sqrt{d_u^2 + d_\alpha^2}} - \sum_{\alpha \in N(v) \setminus u} \frac{1}{\sqrt{d_v^2 + d_\alpha^2}}$$
$$+ \sum_{\alpha \in N(v) \cup N(u) \setminus \{u,v\}} \frac{1}{\sqrt{(d_v + d_u - 1)^2 + d_\alpha^2}} + \frac{1}{\sqrt{(d_v + d_u - 1)^2 + 1}}.$$

Since,

$$- \sum_{\alpha \in N(u) \setminus v} \frac{1}{\sqrt{d_u^2 + d_\alpha^2}} - \sum_{\alpha \in N(v) \setminus u} \frac{1}{\sqrt{d_v^2 + d_\alpha^2}} + \sum_{\alpha \in N(v) \cup N(u) \setminus \{u,v\}} \frac{1}{\sqrt{(d_v + d_u - 1)^2 + d_\alpha^2}} \leq 0,$$
$$-\frac{1}{\sqrt{d_u^2 + d_v^2}} + \frac{1}{\sqrt{(d_v + d_u - 1)^2 + 1}} \leq 0 \quad \text{for any} \quad d_u, d_v \geq 2.$$

Thus $mSo(H) \leq mSo(G)$. □

Transformation D (*see Fig. 4*). Let G be a unicyclic graph with cycle of length α, denoted by C_α and $u \in C_\alpha$, such that $d(u) = 3$ in G and $P : uu_1u_2 \dots u_t$ $(t \neq 2)$ be the path in G. Let w be the neighbour of u in C_α. The graph H is constructed from G by removing the leaf v_t and including it in the cycle C_α between the vertices u, w.

Theorem 4. *Let H be the graph obtained from G using transformation D, then $mSo(G) \leq mSo(H)$.*

P r o o f. From Transformation D, we have $d_u = 3$. Then
Case 1: $t \geq 3$

$$mSo(H) = mSo(G) - \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{13}} + \frac{1}{\sqrt{13}} + \frac{1}{\sqrt{8}} = mSo(G).$$

Case 2: $t = 1$

$$mSo(H) = mSo(G) - \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{13}} + \frac{3}{\sqrt{8}} \geq mSo(G).$$

□

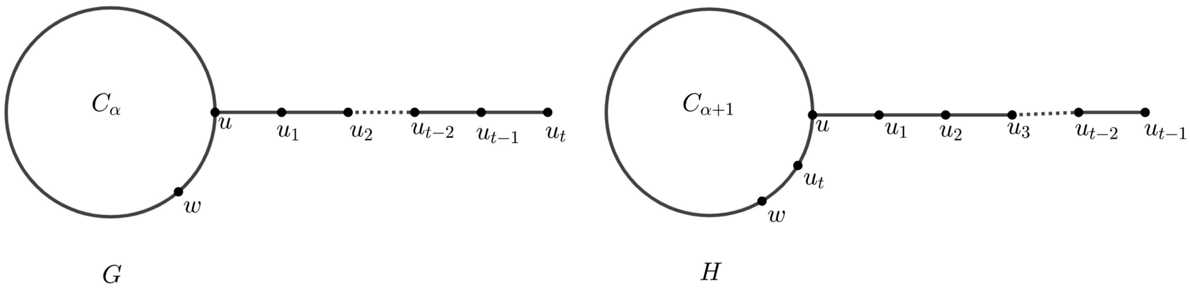


Figure 4. Transformation D.

Lemma 1. *Let G be a unicyclic path with cycle of length $n-2$, say C_{n-2} and $u \in C_{n-2}$ with a path uu_1u_2. Let H be the graph obtained from G by removing the vertices u_1 and u_2 and included in the cycle $C_{n-\alpha}$. Then $mSo(G) < mSo(H)$.*

P r o o f. Let the $d_u = 3$ in $V(G)$. Then,

$$mSo(H) = mSo(G) - \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{8}} - \frac{3}{\sqrt{13}} + \frac{5}{\sqrt{8}} > mSo(G).$$

□

Let $U_n(n_1, n_2, n_3)$ be the family of n-vertex unicyclic graph obtained from attaching n_1-2, n_2-2 and n_3-2 pendent vertices to the three vertices of a triangle respectively, where $n_1+n_2+n_3 = n+3$ and $n_1 \geq n_2 \geq n_3 \geq 2$.

Lemma 2. *For any $n \geq 5$, $n_1 + n_2 + n_3 = n + 3$ and $n_1 \geq n_2 \geq n_3 \geq 3$,*

$$mSo(U_n(n-1, 2, 2)) \leq mSo(U_n(n_1, n_2, n_3)).$$

P r o o f. Since $n_1 \geq n_2 \geq n_3 \geq 3$, we need to prove

$$mSo(U_n(n+1, n_2-1, n_3)) < mSo(U_n(n_1, n_2, n_3))$$

for $n_2 \geq 3$. Let

$$f(x) = \frac{x-2}{\sqrt{x^2+1}}, \quad x \geq 3.$$

Then

$$f''(x) = \frac{-4x^2-3x+2}{(x^2+1)^{5/2}} < 0$$

implies that $f(x+1) - f(x)$ is decreasing function for $x \geq 3$. Thus

$$\begin{aligned} &mSo(U_n(n_1+1, n_2-1, n_3)) - mSo(U_n(n_1, n_2, n_3)) \\ &= mSo(U_n(n_1+1, n_2-1, n_3)) - mSo(U_{n-1}(n_1, n_2-1, n_3)) \\ &-(mSo(U_n(n_1, n_2, n_3)) - mSo(U_{n-1}(n_1, n_2-1, n_3))) \end{aligned}$$

$$= \frac{n_2-2}{\sqrt{n_2^2+1}} - \frac{n_2-3}{\sqrt{(n_2-1)^2+1}} + \frac{1}{\sqrt{n_1^2+n_2^2}} - \frac{1}{\sqrt{n_1^2+(n_2-1)^2}} + \frac{1}{\sqrt{n_2^2+n_3^2}} - \frac{1}{\sqrt{(n_2-1)^2+n_3^2}}$$
$$-\left(\frac{n_1-1}{\sqrt{(n_1+1)^2+1}} - \frac{n_1-2}{\sqrt{n_1^2+1}} + \frac{1}{\sqrt{(n_1+1)^2+n_3^2}} - \frac{1}{\sqrt{n_1^2+n_3^2}} + \right.$$
$$\left. \frac{1}{\sqrt{(n_1+1)^2+(n_2-1)^2}} - \frac{1}{\sqrt{n_1^2+(n_2-1)^2}}\right).$$

Since

$$\frac{1}{\sqrt{n_1^2+n_2^2}} - \frac{1}{\sqrt{n_1^2+(n_2-1)^2}} < 0,$$
$$\frac{1}{\sqrt{n_2^2+n_3^2}} - \frac{1}{\sqrt{(n_2-1)^2+n_3^2}} < 0,$$
$$\frac{1}{\sqrt{(n_1+1)^2+n_3^2}} - \frac{1}{\sqrt{n_1^2+n_3^2}} < 0$$
$$\frac{1}{\sqrt{(n_1+1)^2+(n_2-1)^2}} - \frac{1}{\sqrt{n_1^2+(n_2-1)^2}} < 0,$$

then

$$mSo(U_n(n_1+1, n_2-1, n_3)) - mSo(U_n(n_1, n_2, n_3))$$
$$\leq f(n_2) - f(n_2-1) - (f(n_1+1) - f(n_1)) < 0.$$

□

3. Extremal trees and unicyclic graphs

In this section, we determine the extremal values of the modified Sombor index on the class of trees and unicyclic graphs.

Theorem 5. *Let T be a tree with n-vertices, where $n \geq 3$. Then*

$$mSo(S_n) \leq mSo(T) \leq mSo(P_n).$$

P r o o f. By repeated use of the Transformation A, any tree T can be transformed into a path. Thus by Theorem 1, $mSo(T) \leq mSo(P_n)$.

Now by repeated use of the Transformation C on T, we obtain a star. Thus by Theorem 3, $mSo(T) \geq mSo(S_n)$. □

Corollary 1. *Let T be a tree on n vertices, where $n \geq 3$, then*

$$\frac{n-1}{\sqrt{n^2-2n+2}} \leq mSo(T) \leq \frac{2}{\sqrt{5}} + \frac{n-2}{\sqrt{8}}.$$

Theorem 6. *Let G be an unicyclic graph with n-vertices, where $n \geq 4$. Then*

$$mSo(U_n(n-1, 2, 2)) \leq mSo(G) \leq mSo(C_n).$$

P r o o f. By repeated use of the transformation A, any unicyclic graph G can be transformed into a comet. Thus by Theorem 1, $mSo(G) \leq mSo(CO_{n-\alpha,\alpha})$. Furthermore by using Theorem 4 and Lemma 1, we get $mSo(CO_{n-\alpha,\alpha}) \leq mSo(C_n)$.

Now by repeated use of the Transformation B on G, we obtain a unicyclic graph G' with a cycle and remaining vertices as leaves. Thus by Theorem 2, $mSo(G) \geq mSo(G')$. Furthermore repeating the transformation C on G' we get $U_n(n_1, n_2, n_3)$. By Theorem 3, $mSo(G') \geq mSo(U_n(n_1, n_2, n_3))$. Furthermore using Lemma 2, we get $mSo(U_n(n_1, n_2, n_3)) \geq mSo(U_n(n-1, 2, 2))$. □

Corollary 2. *Let G be an unicyclic graph on n vertices, where $n \geq 4$, then*

$$\frac{n-3}{\sqrt{n^2-2n+2}} + \frac{2}{\sqrt{n^2-2n+5}} + \frac{1}{\sqrt{8}} \leq mSo(G) \leq \frac{n}{\sqrt{8}}.$$

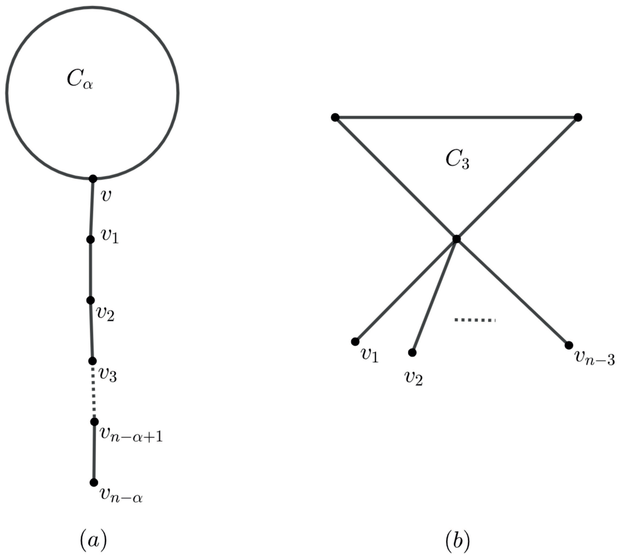


Figure 5. (a) Comet $CO_{n-\alpha,\alpha}$ (b) $U_n(n-1,2,2)$.

4. Conclusion

Bounds on modified Sombor index in terms of graph parameters are determined and various topological indices are compared with modified Sombor index in [7]. In [7] an open problem was proposed to determine the extremal trees, unicyclic graphs and bicyclic graphs with respect to modified Sombor index. Extremal trees and unicyclic graphs are determined here, which answers a part on the problem.

5. Acknowledgement

The authors are very greatful to all of the referees for their carefully reading and insightful comments.

REFERENCES

1. Cruz R., Gutman I., Rada J. Sombor index of chemical graphs. *Appl. Math. Comput.*, 2021. Vol. 399. Art. no. 126018. DOI: 10.1016/j.amc.2021.126018
2. Cruz R., Rada J. The path and the star as extremal values of vertex-degree-based topological indices among trees. *MATCH Commun. Math. Comput. Chem.*, 2019. Vol. 82. P. 715–732.
3. Deng H., Tang Z., Wu R. Molecular trees with extremal values of Sombor indices. *Int. J. Quantum Chem.*, 2021. Vol. 121, No. 11. Art. no. 26622. DOI: 10.1002/qua.26622
4. Gutman I. Geometric approach to degree-based topological indices: Sombor indices. *MATCH Commun. Math. Comput. Chem.*, 2021. Vol. 86. P. 11–16.
5. Gutman I., Redžepović I., Furtula B. On the product of Sombor and modified Sombor indices. *Open J. Discrete Appl. Math.*, 2023. Vol. 6, No. 2. P. 1–6.
6. Hernández J. C., Rodríguez J. M., Rosario O., Sigarreta J. M. Extremal problems on the general Sombor index of a graph. *AIMS Mathematics*, 2022. Vol. 7. No. 5. P. 8330–8343. DOI: 10.3934/math.2022464
7. Huang Y., Liu H. Bounds of modified Sombor index, spectral radius and energy. *AIMS Mathematics*, 2021. Vol. 6, No. 10. P. 11263–11274. DOI: 10.3934/math.2021653

8. Shooshtari H., Sheikholeslami S. M., Amjadi J. Modified Sombor index of unicyclic graphs with a given diameter. *Asian-Eur. J. Math.*, 2023. Vol. 16, No. 06. Art. no. 2350098. DOI: 10.1142/S1793557123500985

9. Wu F., An X., Wu B. Sombor indices of cacti. *AIMS Mathematics*, 2023. Vol. 8. No. 1. P. 1550–1565. DOI: 10.3934/math.2023078

10. Yang Ch., Ji M., Das K.C., Mao Y. Extreme graphs on the Sombor indices. *AIMS Mathematics*, 2022. Vol. 7, No. 10. P. 19126–19146. DOI: 10.3934/math.20221050

11. Zuo X., Rather B. A., Imran M., Ali A. On some topological indices defined via the modified Sombor matrix. *Molecules*, 2022. Vol. 27. No. 19. Art. no. 6772. DOI: 10.3390/molecules27196772

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 76–83

DOI: 10.15826/umj.2024.1.007

GRAPHS Γ OF DIAMETER 4 FOR WHICH $\Gamma_{3,4}$ IS A STRONGLY REGULAR GRAPH WITH $\mu = 4, 6$[1]

Alexander A. Makhnev[a,d†], **Mikhail P. Golubyatnikov**[a,c††], **Konstantin S. Efimov**[b,c,†††]

[a]Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, 620108, Russian Federation

[b]Ural State Mining University,
30 Kuibyshev Str., Ekaterinburg, 620144, Russian Federation

[c]Ural Federal University,
19 Mira Str., Ekaterinburg, 620002, Russian Federation

[d]University Hainan Province,
58 Renmin Av., Haikou 570228, Hainan, P.R. China

[†]makhnev@imm.uran.ru [††]mike_ru1@mail.ru [†††]konstantin.s.efimov@gmail.com

Abstract: We consider antipodal graphs Γ of diameter 4 for which $\Gamma_{1,2}$ is a strongly regular graph. A.A. Makhnev and D.V. Paduchikh noticed that, in this case, $\Delta = \Gamma_{3,4}$ is a strongly regular graph without triangles. It is known that in the cases $\mu = \mu(\Delta) \in \{2, 4, 6\}$ there are infinite series of admissible parameters of strongly regular graphs with $k(\Delta) = \mu(r+1) + r^2$, where r and $s = -(\mu + r)$ are nonprincipal eigenvalues of Δ. This paper studies graphs with $\mu(\Delta) = 4$ and 6. In these cases, Γ has intersection arrays $\{r^2+4r+3, r^2+4r, 4, 1; 1, 4, r^2+4r, r^2+4r+3\}$ and $\{r^2+6r+5, r^2+6r, 6, 1; 1, 6, r^2+6r, r^2+6r+5\}$, respectively. It is proved that graphs with such intersection arrays do not exist.

Keywords: Distance-regular graph, Strongly regular graph, Triple intersection numbers.

1. Introduction

We consider undirected graphs without loops or multiple edges.

Let Γ be a connected graph. The *distance* $d(a, b)$ between two vertices a and b of Γ is the length of a shortest path between a and b in Γ. Given a vertex a in a graph Γ, we denote by $\Gamma_i(a)$ the subgraph induced by Γ on the set of all vertices that are at distance i from a. The subgraph $[a] = \Gamma_1(a)$ is called the *neighbourhood of the vertex* a.

Let Γ be a graph and $a, b \in \Gamma$. Then the number of vertices in $[a] \cap [b]$ is denoted by $\mu(a, b)$ (by $\lambda(a, b)$) if a and b are at distance 2 (are adjacent) in Γ. Further, a subgraph induced by $[a] \cap [b]$ is called a *μ-subgraph* (a *λ-subgraph*). Let Γ be a graph of diameter d and $i, j \in \{1, 2, 3, \ldots, d\}$. A graph Γ_i has the same set of vertices as Γ and vertices u and w are adjacent in Γ_i if $d_\Gamma(u, w) = i$. A graph $\Gamma_{i,j}$ has the same set of vertices as Γ and vertices u and w are adjacent in Γ_i if $d_\Gamma(u, w) \in \{i, j\}$.

If vertices u and w are at distance i in Γ, then we denote by $b_i(u, w)$ (by $c_i(u, w)$) the number of vertices in the intersection $\Gamma_{i+1}(u)$ ($\Gamma_{i-1}(u)$) with $[w]$. A graph Γ of diameter d is called *distance-regular with intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if the values $b_i(u, w)$ and $c_i(u, w)$ are

[1]The study was supported by National Natural Science Foundation of China (12171126) and grant Laboratory of Endgeeniring Modelling and Statistics Calculations of Hainan Province.

independent of the choice of vertices u and w at distance i in Γ for any $i = 0, \dots, d$ [1]. Let $a_i = k_i - b_i - c_i$. Note that, for a distance-regular graph, b_0 is the degree of the graph and $c_1 = 1$.

Let Γ be a graph of diameter d, and let x and y be vertices of Γ. Denote by $p^l_{ij}(x, y)$ the number of vertices in the subgraph $\Gamma_i(x) \cap \Gamma_j(y)$ if $d(x, y) = l$ in Γ. In a distance-regular graph, the numbers $p^l_{ij}(x, y)$ are independent of the choice of vertices x and y, are denoted by p^l_{ij} and are called the *intersection numbers* of the graph Γ (see [1]).

Let Γ be a distance-regular graph of diameter $d \geq 3$. If Γ is an antipodal graph of diameter 4 with antipodality index r, then, by [1, Proposition 4.2.2], Γ has intersection array $\{k, k - a_1 - 1, (r - 1)c_2, 1; 1, c_2, k - a_1 - 1, k\}$.

Consider an antipodal distance-regular graph Γ of diameter 4 for which $\Gamma_{1,2}$ is a strongly regular graph. Makhnev and Paduchikh noticed in [3] that, in this case, $\Delta = \Gamma_{3,4}$ is a strongly regular graph without triangles and the antipodality index of Γ equals 2. It is known that in the cases $\mu = \mu(\Delta) \in \{2, 4, 6\}$ there arise infinite series of admissible parameters of strongly regular graphs with $k(\Delta) = \mu(r + 1) + r^2$, where r and $s = -(\mu + r)$ are nonprincipal eigenvalues of Δ.

In the present paper, we consider graphs with $\mu(\Delta) = 4$ and 6. In these cases, Γ has intersection arrays

$$\{r^2 + 4r + 3, r^2 + 4r, 4, 1; 1, 4, r^2 + 4r, r^2 + 4r + 3\}$$

and

$$\{r^2 + 6r + 5, r^2 + 6r, 6, 1; 1, 6, r^2 + 6r, r^2 + 6r + 5\},$$

respectively.

If $\mu(\Delta) = 4$, then Δ has parameters $(v, r^2 + 4r + 4, 0, 4)$, where

$$v = 1 + (r^2 + 4r + 4) + \frac{(r^2 + 4r + 4)(r^2 + 4r + 3)}{4}.$$

Further, Δ has nonprincipal eigenvalues r and $-(r + 4)$, and the multiplicity of r is equal to $(r + 3)(r + 2)(r^2 + 5r + 8)/8$.

Theorem 1. *A distance-regular graph with intersection array*

$$\{r^2 + 4r + 3, r^2 + 4r, 4, 1; 1, 4, r^2 + 4r, r^2 + 4r + 3\}$$

does not exist.

If $\mu(\Delta) = 6$, then Δ has parameters $(v, r^2 + 6r + 6, 0, 6)$, where

$$v = 1 + (r^2 + 6r + 6) + (r^2 + 6r + 6)(r^2 + 6r + 5)/6.$$

Further, Δ has nonprincipal eigenvalues r and $-(r + 6)$, and the multiplicity of r is equal to $(r + 5)(r^2 + 6r + 6)(r + 4)/12$. Therefore, r is even or congruent to 3 modulo 4.

Theorem 2. *A distance-regular graph with intersection array*

$$\{r^2 + 6r + 5, r^2 + 6r, 6, 1; 1, 6, r^2 + 6r, r^2 + 6r + 5\}$$

does not exist.

Corollary 1. *Distance-regular graphs with intersection arrays*

$$\{32, 27, 6, 1; 1, 6, 27, 32\}, \quad \{45, 40, 6, 1; 1, 6, 40, 45\}, \quad \{77, 72, 6, 1; 1, 6, 72, 77\},$$
$$\{96, 91, 6, 1; 1, 6, 91, 96\}, \quad \{117, 112, 6, 1; 1, 6, 112, 117\}$$

do not exist.

2. Triple intersection numbers

Let Γ be a distance-regular graph of diameter d. If u_1, u_2, and u_3 are vertices of the graph Γ and r_1, r_2, and r_3 are nonnegative integers not greater than d, then $\left\{ {u_1 u_2 u_3 \atop r_1 r_2 r_3} \right\}$ is the set of vertices $w \in \Gamma$ such that

$$d(w, u_i) = r_i, \quad \begin{bmatrix} u_1 u_2 u_3 \\ r_1 r_2 r_3 \end{bmatrix} = \left| \begin{Bmatrix} u_1 u_2 u_3 \\ r_1 r_2 r_3 \end{Bmatrix} \right|.$$

The numbers $\left[{u_1 u_2 u_3 \atop r_1 r_2 r_3} \right]$ are called triple intersection numbers. For a fixed triple u_1, u_2, u_3 of vertices, we will write $[r_1 r_2 r_3]$ instead of $\left[{u_1 u_2 u_3 \atop r_1 r_2 r_3} \right]$.

Unfortunately, there are no general formulas for numbers $[r_1 r_2 r_3]$. However, [2] suggests a method for calculating some numbers $[r_1 r_2 r_3]$.

Assume that u, v, and w are vertices of the graph Γ, $W = d(u, v)$, $U = d(v, w)$, and $V = d(u, w)$. Since there is exactly one vertex $x = u$ such that $d(x, u) = 0$, then the number $[0jh]$ is 0 or 1. Hence, $[0jh] = \delta_{jW}\delta_{hV}$. Similarly, $[i0h] = \delta_{iW}\delta_{hU}$ and $[ij0] = \delta_{iU}\delta_{jV}$.

Another set of equations can be obtained by fixing the distance between two vertices from $\{u, v, w\}$ and counting the number of vertices located at all possible distances from the third. Then, we get

$$\sum_{l=1}^{d} [ljh] = p^U_{jh} - [0jh], \quad \sum_{l=1}^{d} [ilh] = p^V_{ih} - [i0h], \quad \sum_{l=1}^{d} [ijl] = p^W_{ij} - [ij0]. \tag{2.1}$$

At the same time, some triples disappear. If $|i - j| > W$ or $i + j < W$, then $p^W_{ij} = 0$; therefore, $[ijh] = 0$ for all $h \in \{0, \dots, d\}$. Define

$$S_{ijh}(u, v, w) = \sum_{r,s,t=0}^{d} Q_{ri} Q_{sj} Q_{th} \begin{bmatrix} uvw \\ rst \end{bmatrix}.$$

If Krein's parameter q^h_{ij} is 0, then $S_{ijh}(u, v, w) = 0$.

3. A distance-regular graph with intersection array $\{r^2 + 4r + 3, r^2 + 4r, 4, 1; 1, 4, r^2 + 4r, r^2 + 4r + 3\}$

In this section, Γ is a distance-regular graph with intersection array

$$\{r^2 + 4r + 3, r^2 + 4r, 4, 1; 1, 4, r^2 + 4r, r^2 + 4r + 3\}.$$

Then, Γ has

$$1 + (r^2 + 4r + 3) + (r^2 + 4r + 3)(r^2 + 4r)/4 + (r^2 + 4r + 3) + 1$$

vertices and the spectrum

$$(r + 3)(r + 1) \quad \text{of multiplicity} \quad 1,$$
$$r + 3 \quad \text{of multiplicity} \quad \frac{(r^2 + 5r + 8)(r^2 + 3r + 4)(r + 1)}{16(r + 2)},$$
$$r - 1 \quad \text{of multiplicity} \quad \frac{(r^2 + 5r + 8)(r + 4)(r + 3)(r + 1)}{16(r + 2)},$$
$$-(r + 1) \quad \text{of multiplicity} \quad \frac{(r^2 + 5r + 8)(r^2 + 3r + 4)(r + 3)}{16(r + 2)},$$
$$-(r + 5) \quad \text{of multiplicity} \quad \frac{(r^2 + 3r + 4)(r + 3)(r + 1)r}{16(r + 2)}.$$

The multiplicity of $r+3$ is equal to

$$\frac{(r^2+5r+8)(r^2+3r+4)(r+1)}{16(r+2)}.$$

Further,

$$(r^2+5r+8, r+2) = (3r+8, r+2)$$

divides 2 and $(r+2, r^2+3r+4) = (r+2, r+4)$ divides 2; therefore $r+2$ divides 4. Consequently, $r=2$, a contradiction with the fact that the multiplicity of $r+3$ is equal to

$$(r^2+5r+8)(r^2+3r+4)(r+1)/(16(r+2)) = 22 \times 14 \times 3/64.$$

Theorem 1 is proved.

4. A distance-regular graph with intersection array $\{r^2+6r+5, r^2+6r, 6, 1; 1, 6, r^2+6r, r^2+6r+5\}$

In this section, Γ is a distance-regular graph with intersection array

$$\{r^2+6r+5, r^2+6r, 6, 1; 1, 6, r^2+6r, r^2+6r+5\}.$$

Then, Γ has

$$1+(r^2+6r+5)+(r^2+6r+5)(r^2+6r)/6+(r^2+6r+5)+1$$

vertices, the spectrum

$$\begin{gathered}(r+5)(r+1) \quad \text{of multiplicity} \quad 1,\\ r+5 \quad \text{of multiplicity} \quad f=(r+4)(r+3)(r+2)(r+1)/24,\\ r-1 \quad \text{of multiplicity} \quad (r+6)(r+5)(r+4)(r+1)/24,\\ -(r+1) \quad \text{of multiplicity} \quad (r+5)(r+4)(r+3)(r+2)/24,\\ -(r+7) \quad \text{of multiplicity} \quad (r+5)(r+2)(r+1)r/24,\end{gathered}$$

and the matrix Q (see [1]) of dual eigenvalues

$$\begin{pmatrix} 1 & f & \dfrac{f(r+6)(r+5)}{(r+2)(r+3)} & \dfrac{f(r+5)}{r+1} & \dfrac{f(r+5)r}{(r+4)(r+3)} \\ 1 & \dfrac{f}{r+1} & \dfrac{f(r+6)(r-1)}{(r+2)(r+3)(r+1)} & -\dfrac{f}{r+1} & -\dfrac{f(r+7)r}{(r+4)(r+3)(r+1)} \\ 1 & 0 & -r/2-2 & 0 & r/2+1 \\ 1 & -\dfrac{f}{r+1} & \dfrac{f(r+6)(r-1)}{(r+2)(r+3)(r+1)} & \dfrac{f}{r+1} & -\dfrac{f(r+7)r}{(r+4)(r+3)(r+1)} \\ 1 & -f & \dfrac{f(r+6)(r+5)}{(r+2)(r+3)} & -\dfrac{f(r+5)}{r+1} & \dfrac{f(r+5)r}{(r+4)(r+3)} \end{pmatrix}.$$

Lemma 1. *The intersection numbers are*

$$p^1_{11}=4,\quad p^1_{21}=r^2+6r,\quad p^1_{32}=r^2+6r,\quad p^1_{22}=r^4/6+2r^3+29r^2/6-7r,\quad p^1_{33}=0,\quad p^1_{34}=1;$$
$$p^2_{11}=6,\quad p^2_{12}=r^2+6r-7,\quad p^2_{13}=6,\quad p^2_{22}=r^4/6+2r^3+29r^2/6-7r+12,$$
$$p^2_{23}=r^2+6r-7,\quad p^2_{24}=1,\quad p^2_{33}=2;$$
$$p^3_{12}=r^2+6r,\quad p^3_{13}=4,\quad p^3_{14}=1,\quad p^3_{22}=r^4/6+2r^3+29r^2/6-7r,\quad p^3_{23}=r^2+6r,\quad p^3_{33}=0;$$
$$p^4_{13}=r^2+6r+5,\quad p^4_{22}=r^4/6+2r^3+41r^2/6+5r.$$

P r o o f. Direct calculations using formulas from [1, Lemma 4.1.7]. □

Fix vertices u, v, and w of the graph Γ and define

$$\{ijh\} = \left\{ \begin{matrix} uvw \\ ijh \end{matrix} \right\}, \quad [ijh] = \left[\begin{matrix} uvw \\ ijh \end{matrix} \right].$$

Let $\Delta = \Gamma_2(u)$, and let Λ be a graph with vertices from Δ in which two vertices are adjacent if they are at distance 2 in Γ. Then Λ is a regular graph of degree

$$p^2_{22} = r^4/6 + 2r^3 + 29r^2/6 - 7r + 12$$

on

$$k_2 = (r^2 + 6r + 5)(r^2 + 6r)/6 = r^4/6 + 2r^3 + 41r^2/6 + 5r$$

vertices.

Lemma 2. *Let $d(u, v) = d(u, w) = 2$ and $d(v, w) = 1$. Then, the triple intersection numbers are*

$$[111] = r_4, \quad [112] = [121] = -r_4 + 6, \quad [122] = r_3 + r_4 + r^2 + 6r - 19, \quad [123] = [132] = -r_3 + 6;$$
$$[211] = -r_3 - r_4 + 4, \quad [212] = [221] = r_3 + r_4 + r^2 + 6r - 12,$$
$$[222] = r^4/6 + 2r^3 + 17r^2/6 - 19r + 36,$$
$$[223] = [232] = r_3 + r_4 + r^2 + 6r - 12, \quad [233] = -r_3 - r_4 + 4, \quad [234] = [243] = 1;$$
$$[311] = r_3, \quad [312] = [321] = -r_3 + 6, \quad [322] = r_3 + r_4 + r^2 + 6r - 19, \quad [323] = [332] = -r_4 + 6;$$
$$[333] = r_4, \quad [422] = 1,$$

where $r_3 + r_4 \le 4$.

P r o o f. Simplification of formulas (2.1). □

By Lemma 2, we have

$$r^4/6 + 2r^3 + 17r^2/6 - 19r + 28$$
$$\le [222] = -2r_3 - 2r_4 + r^4/6 + 2r^3 + 17r^2/6 - 19r + 36 \le r^4/6 + 2r^3 + 17r^2/6 - 19r + 36.$$

Lemma 3. *Let $d(u, v) = d(u, w) = 2$ and $d(v, w) = 3$. Then, the triple intersection numbers are*

$$[112] = -r_{11} + 6, \quad [113] = r_{11},$$
$$[121] = -r_{12} + 6, \quad [122] = r_{11} + r_{12} + r^2 + 6r - 19, \quad [123] = -r_{11} + 6, \quad [132] = -r_{12} + 6;$$
$$[212] = [221] = r_{11} + r_{12} + r^2 + 6r - 12, \quad [213] = [231] = -r_{11} - r_{12} + 4, \quad [214] = [241] = 1,$$
$$[222] = -2r_3 - 2r_4 + r^4/6 + 2r^3 + 17r^2/6 - 19r + 36, \quad [223] = [232] = r_{11} + r_{12} + r^2 + 6r - 12;$$
$$[312] = -r_{12} + 6, \quad [313] = r_{12}, \quad [321] = -r_{11} + 6, \quad [322] = r_{11} + r_{12} + r^2 + 6r - 19,$$
$$[323] = -r_{12} + 6, \quad [331] = r_{11}, \quad [332] = -r_{11} + 6; \quad [422] = 1,$$

where $r_{11} + r_{12} \le 4$.

P r o o f. Simplification of formulas (2.1). □

By Lemma 3, we have

$$r^4/6 + 2r^3 + 17r^2/6 - 19r + 28$$
$$\leq [222] = -2r_3 - 2r_4 + r^4/6 + 2r^3 + 17r^2/6 - 19r + 36 \leq r^4/6 + 2r^3 + 17r^2/6 - 19r + 36.$$

Lemma 4. *Let $d(u,v) = d(u,w) = 2$ and $d(v,w) = 4$. Then, the triple intersection numbers are*

$$[113] = [131] = 6, \quad [122] = r^2 + 6r - 7;$$
$$[213] = [231] = r^2 + 6r - 7, \quad [222] = r^4/6 + 2r^3 + 29r^2/6 - 7r + 12;$$
$$[313] = [331] = 6, \quad [322] = r^2 + 6r - 7;$$
$$[422] = 1.$$

P r o o f. Simplification of formulas (2.1). □

By Lemma 4, we have

$$[222] = r^4/6 + 2r^3 + 29r^2/6 - 7r + 12.$$

Recall that

$$p^2_{12} = r^2 + 6r - 7, \quad p^2_{22} = r^4/6 + 2r^3 + 29r^2/6 - 7r + 12, \quad p^2_{23} = r^2 + 6r - 7, \quad p^2_{24} = 1.$$

Let v and w be vertices from Λ. Then the number d of edges between $\Lambda(v)$ and $\Lambda - (\{v\} \cup \Lambda(v))$ is

$$d = p^2_{12} \begin{bmatrix} uvx \\ 221 \end{bmatrix} + p^2_{32} \begin{bmatrix} uvy \\ 223 \end{bmatrix} + p^2_{42} \begin{bmatrix} uvz \\ 224 \end{bmatrix},$$

where x, y, and z are vertices from $\left\{ {uv \atop 2i} \right\}$ for $i = 1, 3$, and 4, respectively. Now, d satisfies the inequalities

$$(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 56) + r^4/6 + 2r^3 + 29r^2/6 - 7r + 12 \leq d$$
$$\leq (r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 72) + r^4/6 + 2r^3 + 29r^2/6 - 7r + 12.$$

On the other hand,

$$d = \sum_{w \in \Lambda(v)} (p^2_{22} - 1 - \lambda_\Lambda(v,w)) = k_\Lambda \Big(p^2_{22} - 1 - \frac{\sum_{w \in \Lambda(v)} \lambda_\Lambda(v,w)}{k_\Lambda}\Big).$$

So,

$$d = (r^4/6 + 2r^3 + 29r^2/6 - 7r + 12)(r^4/6 + 2r^3 + 29r^2/6 - 7r + 11 - \lambda),$$

where λ is the average value of degree of the vertex w in the graph Λ. Consequently,

$$\frac{(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 56)}{r^4/6 + 2r^3 + 29r^2/6 - 7r + 12} + 1 \leq \frac{r^4}{6} + 2r^3 + \frac{29r^2}{6} - 7r + 11 - \lambda$$
$$\leq \frac{(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 72)}{r^4/6 + 2r^3 + 29r^2/6 - 7r + 12} + 1$$

and

$$\frac{r^4}{6} + 2r^3 + \frac{29r^2}{6} - 7r + 10 - \frac{(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 72)}{r^4/6 + 2r^3 + 29r^2/6 - 7r + 12} \leq \lambda$$
$$\leq \frac{r^4}{6} + 2r^3 + \frac{29r^2}{6} - 7r + 10 - \frac{(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 56)}{r^4/6 + 2r^3 + 29r^2/6 - 7r + 12}.$$

Lemma 5. *Let $d(u,v)=d(u,w)=d(v,w)=2$. Then, the triple intersection numbers are*

$$[111]=r_9,\quad [112]=-r_7-r_9+6,\quad [113]=r_7,\quad [121]=-r_{10}-r_9+6,$$
$$[122]=r_7+r_8+r_9+r_{10}+r^2+6r-19,\quad [123]=-r_7-r_8+6,$$
$$[131]=r_{10},\quad [132]=-r_{10}-r_8+6,\quad [133]=r_8;$$
$$[211]=-r_8-r_9+6,\quad [212]=[221]=r_7+r_8+r_9+r_{10}+r^2+6r-19,$$
$$[213]=[231]=-r_{10}-r_7+6,\quad [222]=-2r_7-2r_8-2r_9-2r_{10}+r^4/6+2r^3+17r^2/6-19r+48,$$
$$[223]=[232]=r_7+r_8+r_9+r_{10}+r^2+6r-19,\quad [224]=[242]=1,\quad [233]=-r_8-r_8+6;$$
$$[311]=r_8,\quad [312]=-r_{10}-r_8+6,\quad [313]=r_{10},\quad [321]=-r_7-r_8+6,$$
$$[322]=r_7+r_8+r_9+r_{10}+r^2+6r-19,\quad [323]=-r_{10}-r_9+6,$$
$$[331]=r_7,\quad [332]=-r_7-r_9+6,\quad [333]=r_9;\quad [422]=1,$$

where

$$r_9+r_7,\ r_9+r_{10},\ r_7+r_8,\ r_{10}+r_8,\ r_8+r_9,\ r_7+r_{10}\le 6.$$

P r o o f. Simplification of formulas (2.1). □

By Lemma 5, we have

$$\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+24\le[222]=-2r_7-2r_8-2r_9-2r_{10}+\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+48$$
$$\le\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+48.$$

Let $d(u,v)=2$.

Let us count the number e_2 of pairs of vertices (s,t) at distance 2, where $s\in\left\{ {uv \atop 21} \right\}$ and $t\in\left\{ {uv \atop 22} \right\}$. On the one hand, by Lemma 2, we have

$$r^4/6+2r^3+17r^2/6-19r+28\le[222]\le r^4/6+2r^3+17r^2/6-19r+36,$$

so,

$$(r^2+6r-7)\Big(\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+28\Big)\le e_2\le(r^2+6r-7)\Big(\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+36\Big).$$

On the other hand, by Lemma 5, we have

$$[212]=r_7+r_8+r_9+r_{10}+r^2+6r-19$$

and

$$(r^2+6r-7)\Big(\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+28\Big)\le e_2$$
$$=-\sum_i(r_7^i+r_8^i+r_9^i+r_{10}^i)+(r^2+6r-19)\Big(\frac{r^4}{6}+2r^3+\frac{29r^2}{6}-7r+12\Big)$$
$$\le(r^2+6r-7)\Big(\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+36\Big).$$

In this way,

$$(r^2+6r-19)\Big(\frac{r^4}{6}+2r^3+\frac{29r^2}{6}-7r+12\Big)-(r^2+6r-7)\Big(\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+36\Big) \le (r_7^i+r_8^i+r_9^i+r_{10}^i)$$
$$\le (r^2+6r-19)\Big(\frac{r^4}{6}+2r^3+\frac{29r^2}{6}-7r+12\Big)-(r^2+6r-7)\Big(\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+28\Big).$$

Consequently,

$$(r_7^i+r_8^i+r_9^i+r_{10}^i)\le -145r^3/6-16r^2-96r-12,$$

a contradiction.

Theorem 2 is proved. □

The corollary follows from Theorems 1 and 2.

So, we have shown the nonexistence of graphs with intersection arrays

$$\{r^2+4r+3, r^2+4r, 4, 1;\ 1, 4, r^2+4r, r^2+4r+3\}$$

and

$$\{r^2+6r+5, r^2+6r, 6, 1;\ 1, 6, r^2+6r, r^2+6r+5\}.$$

In particular, distance-regular graphs with intersection arrays

$$\{32,27,6,1;\ 1,6,27,32\},\quad \{45,40,6,1;\ 1,6,40,45\},\quad \{77,72,6,1;1,6,72,77\},$$
$$\{96,91,6,1;\ 1,6,91,96\},\quad \{117,112,6,1;\ 1,6,112,117\}$$

do not exist.

REFERENCES

1. Brouwer A. E., Cohen A. M., Neumaier A. *Distance–Regular Graphs.* Berlin, Heidelberg: Springer–Verlag, 1989. 495 p. DOI: 10.1007/978-3-642-74341-2
2. Coolsaet K., Jurišić A. Using equality in the Krein conditions to prove nonexistence of certain distance-regular graphs. *J. Combin. Theory Ser. A*, 2018. Vol. 115, No. 6. P. 1086–1095. DOI: 10.1016/j.jcta.2007.12.001
3. Makhnev A. A., Paduchikh D. V. Inverse problems in the class of distance-regular graphs of diameter 4. *Proc. Steklov Inst. Math.*, 2022. Vol. 317, No. Suppl. 1. P. S121–S129. DOI: 10.1134/S0081543822030105

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 84–98

DOI: 10.15826/umj.2024.1.008

ARTINIAN M-COMPLETE, M-REDUCED, AND MINIMALLY M-COMPLETE ASSOCIATIVE RINGS

Leonid M. Martynov

Omsk State University of Ways of Communication,
35 Marksa Ppr., Omsk, 644046, Russian Federation

Tatiana V. Pavlova

University of Tyumen,
6 Volodarskogo Str., Tyumen, 625003, Russian Federation

t.v.pavlova@utmn.ru

Abstract: In 1996, the first author defined analogs of the concepts of complete (divisible), reduced, and periodic abelian groups, well-known in the theory of abelian groups, for arbitrary varieties of algebras. In 2021, the first author proposed a modification of the concepts of completeness and reducibility, which is more natural in the case of associative rings. The paper studies the modification of these concepts for associative rings. Artinian **M**-complete, **M**-reduced rings, and minimally **M**-complete associative nilpotent rings, simple rings with unity, and finite rings are characterized.

Keywords: Associative ring, Artinian ring, Finite ring, Complete ring, Reduced ring.

1. Introduction

In the theory of abelian groups, the notions of complete (divisible), reduced, and periodic (in particular, primary) groups are of great importance. In [16] (see also [17, 19]), some analogs of these notions were defined for arbitrary varieties of algebras. In the mentioned papers, the concepts of (atomic) complete, (atomic) reduced, and solvable algebra [30] (see also [31]) were defined by means of the atoms of these varieties and the Malcev products for these atoms [15]. Furthermore, the notions of periodic and primary algebra were defined using notions of (atomic) completeness, (atomic) reducibility, and solvability. In particular, an algebra is called periodic if each of its monogenic (i.e., one generated) subalgebras is finitely reduced. Note that a group or semigroup is periodic as a universal algebra (in our sense) if and only if it is periodic as a group or semigroup in the ordinary sense.

It is different for associative rings. In the theory of associative rings, a ring is called periodic if its multiplicative semigroup is periodic (in the ordinary sense). Any finite nonprime field is a periodic ring in the ordinary sense. On the other hand, such a field is monogenic, but it is not a finitely reduced ring; i.e., it is not a periodic algebra in the sense of papers [16, 17, 19]. To remove this difference, paper [21] suggests modifying the concepts of complete, reducible, periodic, and primary associative rings. This is done by using a special set **M** of subvarieties of the variety As of associative rings, where **M** is the union of the set of lattice atoms of subvarieties of As and the set of all varieties, each of which is generated by some finite nonprime field. In this case, **M**-periodic rings are rings with finite monogenic subrings (i.e., there is an analogy with groups and semigroups). Moreover, every finite field is both **M**-periodic and **M**-primary. Thus, the modification of the concepts discussed, given in [21], is more natural for associative rings.

In [21], properties of **M**-periodic and **M**-primary associative rings are studied. In addition, paper [21] characterizes the **M**-periodic, **M**-primary, and **M**-reduced varieties of associative rings. From the results of paper [21] (in particular, Remark 5.17), it follows that the class $\mathcal{MC}$ of all **M**-complete rings of As is closed with respect to homomorphic images, extensions, and direct sums in As. Furthermore, the class $\mathcal{MR}$ of all **M**-reduced rings of As is closed with respect to subrings, direct products, and extensions in As. Besides, the variety As is transverbal (in the sense of [15]) with respect to any variety belonging to **M**.

If we replace the set $At(L(As))$ by the set $\mathbf{M}$ in paper [18], the main result of [18] will change. From the modified result, we obtain that any ring R belonging to As contains the largest **M**-complete subring $C_{\mathbf{M}}(R)$; $C_{\mathbf{M}}(R)$ is a two-sided ideal of the ring R; the factor ring $R/C_{\mathbf{M}}(R)$ is an **M**-reduced ring. All of the above means that the basic properties of the modified concepts of completeness and reducibility for associative rings are saved. Further, let a mapping $r_{\mathbf{M}} : As \to As$ be such that $r_{\mathbf{M}}(R) = C_{\mathbf{M}}(R)$ for all $R \in As$. From the above, we obtain that $r_{\mathbf{M}}$ is a radical in the sense of Kurosh and Amitsur (see, for example, [1, p. 91] or [6, p. 27]). Here, $\mathcal{MC}$ is a radical class and $\mathcal{MR}$ is a semisimple class.

We say that the radical $r_{\mathbf{M}}$ is the **M**-*complete radical* and the ideal $C_{\mathbf{M}}(R)$ of R is the **M**-*complete radical of the ring* R. Note that $C_{\mathbf{M}}(R)$ contains any **M**-complete subring of the ring R. Therefore, $r_{\mathbf{M}}$ is a *strict radical* in the sense of Kurosh [13] (see also [6, p. 148]).

In papers [10, 11, 20, 24, 25], the complete radical of an associative ring was studied. It is easy to verify that analogous main results of these papers also hold for the **M**-complete radical. In papers [12, 22, 26–28], the structure of complete and reduced associative rings was studied. The main results of these papers are significantly modified if we replace the concepts of completeness and reducibility with the concepts of **M**-completeness and **M**-reducibility.

Recall that in the theory of abelian groups, the concept of a complete group coincides with that of a divisible group. Any minimal divisible abelian group is isomorphic to the (additive) quasi-cyclic group $\mathbb{C}_{p^\infty}$, where p is a prime, or to the additive group $\mathbb{Q}^+$ of the field $\mathbb{Q}$ of rational numbers. Minimal divisible abelian groups have significant importance since any divisible abelian group is a direct sum of minimal divisible abelian groups (see, for example, [4, Theorem 23.1, p. 124]). An associative ring R is called minimal **M**-complete if it is a nonzero **M**-complete ring, and all proper subrings of the ring R are **M**-reduced rings.

This paper aims to characterize **M**-complete, **M**-reduced associative Artinian rings, and minimal **M**-complete finite associative rings. Note that the class of Artinian rings contains all finite rings.

Nevertheless, in this paper, we do not limit ourselves to Artinian rings and provide some results, which are valid for all associative rings. We will use the results from papers [12, 22, 23, 26–29] on the study of complete and reduced associative rings and modify them for the concepts of **M**-completeness and **M**-reducibility. If the obtained results have fundamental changes, then proofs are provided. The modified formulations of the statements are given only with references to similar statements if the changes are insignificant. Before formulating and proving the basic results of the paper, we will provide and prove several lemmas. Some lemmas are of independent interest.

First, let us give some definitions, notations, and facts about associative rings.

2. Basic definitions, notations, and preliminary information

Further in the paper, by a *ring* we mean an associative ring (not necessarily with the unity), by an *ideal* we mean a two-sided ideal. Denote by $|M|$ the cardinal number of a set M. Positive integers are denoted by k, l, m, n (sometimes with subscripts), and primes are denoted by p and q. Denote by R^+ the additive group of the ring R. A ring with zero multiplication will be called an abelian ring. An abelian ring with an additive group R^+ is denoted by R^0. Denote by O the zero ideal of the ring R. A *simple ring* is a nonzero ring having no ideal besides O and itself. The smallest $n \in \mathbb{N}$ is said to be the *characteristic* of a ring R if $nR = O$ and is denoted by *char* R. If there is no such n, then *char* $R = 0$ is assumed.

The set of natural numbers is denoted by $\mathbb{N}$, and the set of primes is denoted by $\mathbb{P}$. Denote by $\mathbb{Z}$ the ring of integers, and by $\mathbb{Q}$ the field of rational numbers. Furthermore, let $\mathbb{Z}_n$ denote the ring of residue classes modulo $n > 1$. The finite field (Galois field) of p^m elements is denoted by $\mathbb{F}_{p^m}$. A *prime field* is a field, which has no proper subfields. Any prime field is isomorphic to the field $\mathbb{Q}$ of rational numbers or the finite field $\mathbb{F}_p$ of p elements.

If M is a nonempty subset of a ring R, then $\langle M\rangle$ and (M) denote the subring and the ideal of R generated by M, respectively. The subring generated by $a \in R$ is called *monogenic* and denoted by $\langle a\rangle$. An element $e \in R$ with the property $e^2 = e$ is called an *idempotent* of R. The idempotent $e \in R$ is called *basic* if $\sigma(e)$ is the unity of the factor ring $R/J(R)$, where $J(R)$ is the Jacobson radical of the ring R and σ is the natural homomorphism of R to $R/J(R)$. A ring R is called *idempotent* if $R^2 = R$, where $R^2 = \langle a \cdot b \mid a, b \in R\rangle$.

Denote by $M_n(R)$ the ring of square $n \times n$ matrices over a ring R. For a commutative ring R with unity, $R[x]$ denotes the ring of polynomials in x over R. Denote by $\mathbb{Z}\langle X\rangle$ a free (in As) ring over the infinite countable set $X = \{x_1, x_2, \dots\}$, i.e., the ring of polynomials with integer coefficients in noncommuting variables of X with zero free terms. An *identity* is a formal equality of the form $f(x_1, x_2, \dots, x_n) = 0$, where $f(x_1, x_2, \dots, x_n) \in \mathbb{Z}\langle X\rangle$.

Suppose that φ is the natural homomorphism of $\mathbb{Z}_{p^k}[x]$ onto $\mathbb{Z}_p[x]$ and $f(x) \in \mathbb{Z}_{p^k}[x]$ is a unitary polynomial of degree m such that $\varphi\big(f(x)\big)$ is an irreducible polynomial over $\mathbb{Z}_p$; then the factor ring $\mathbb{Z}_{p^k}[x]/\big(f(x)\big)$ is called a *Galois ring* of characteristic p^k and order p^{km}. The Galois ring, up to isomorphism, is defined by the numbers p, k, and n and is denoted by $GR(p^k, m)$. It is obvious that $GR(p^k, 1) \cong \mathbb{Z}_{p^k}$ and $GR(p, m) \cong \mathbb{F}_{p^m}$. Galois rings play a special role in the structural theory of finite associative rings.

Let $\operatorname{var}\Sigma$ denote the variety of rings defined by a system Σ of ring identities, and let $\operatorname{var}\mathcal{K}$ be the least variety of rings containing a class $\mathcal{K}$ of rings (in other words, $\operatorname{var}\mathcal{K}$ is the variety generated by $\mathcal{K}$). The free monogenic ring in As will be denoted by $\mathbb{Z}\langle x\rangle$. Below we use the following important notations:

$\mathcal{Z}_n^0 = \operatorname{var}\{nx = 0,\ xy = 0\} = \operatorname{var}\mathbb{Z}_n^0$;

$\mathcal{F}_{p^m} = \operatorname{var}\{px = 0,\ x^{p^m} = x\} = \operatorname{var}\mathbb{F}_{p^m}$.

Let $\mathcal{V}$ be a variety of rings. A ring R is called $\mathcal{V}$-*complete* if R has no homomorphisms onto nonzero rings from $\mathcal{V}$. Equivalently, $\mathcal{V}(R) = R$, where $\mathcal{V}(R)$ is the *verbal ideal* of R (i.e., $\mathcal{V}(R)$ is the least ideal in the set of all ideals I of the ring R such that the factor ring R/I belongs to $\mathcal{V}$). A ring is called $\mathcal{V}$-*solvable* if it has no nonzero $\mathcal{V}$-complete subrings.

Let $\mathbf{M}$ be the union of two sets $\mathbf{Z}$ and $\mathbf{F}$ of varieties of rings, where

$$\mathbf{Z} = \{\mathcal{Z}_p^0 \mid p \in \mathbb{P}\}, \quad \mathbf{F} = \{\mathcal{F}_{p^m} \mid p \in \mathbb{P},\ m \in \mathbb{Z}_+\},$$

i.e., $\mathbf{M} = \mathbf{Z} \cup \mathbf{F}$. Note that $\mathbf{M}$ contains the set $At(L(As))$, where $At(L(As))$ consists of the varieties $\mathcal{Z}_p^0$ and $\mathcal{F}_p$ for any prime p (see, for example, [9]). A ring R is called $\mathbf{M}$-*complete* if R is $\mathcal{M}$-complete for every $\mathcal{M} \in \mathbf{M}$. We call a ring R $\mathbf{M}$-*reduced* if R has no nontrivial $\mathbf{M}$-complete subrings. By analogy, the concepts of $\mathbf{Z}$-complete ($\mathbf{F}$-complete) ring and of $\mathbf{Z}$-reduced ($\mathbf{F}$-reduced) ring are defined. We point out the connection between the concepts of completeness and $\mathbf{M}$-completeness, as well as the concepts of reducibility and $\mathbf{M}$-reducibility. Obviously, any $\mathbf{M}$-complete ring is complete. But the converse statement, generally speaking, is incorrect. For example, any nonminimal finite field F is complete, while F is $\mathbf{M}$-reduced. On the other hand, any reduced ring is $\mathbf{M}$-reduced.

Recall that if a variety $\mathcal{V}$ is given by an identity system Σ, then the $\mathcal{V}$-verbal $\mathcal{V}(R)$ of a ring R coincides with the ideal of R generated by the values in R of all polynomials that are the left-hand sides of the identities of Σ. For varieties $\mathcal{Z}_p^0$ and $\mathcal{F}_{p^m}$ and a ring R, we indicate formulas to calculate the corresponding verbals:

$$\mathcal{Z}_p^0(R) = pR + R^2, \quad \mathcal{F}_{p^m}(R) = pR + R_{p^m},$$

where R_{p^m} is the ideal generated by the set

$$\{r^{p^m} - r \mid r \in R\}.$$

It is clear that a ring R is **M**-complete (**M**-reduced) if and only if $\mathcal{Z}_p^0(R) = R$ ($\mathcal{Z}_p^0(R) = O$) and $\mathcal{F}_{p^m}(R) = R$ ($\mathcal{F}_{p^m}(R) = O$) for all p and m, respectively.

It is clear that the **M**-complete radical $C_{\mathbf{M}}(R)$ of a ring R is equal to the sum of all **M**-complete subrings of R. A ring R is **M**-complete if and only if $C_{\mathbf{M}}(R) = R$. In particular, the **M**-complete radical $C_{\mathbf{M}}(R)$ of any ring R is an **M**-complete ideal of R. A ring R is **M**-reduced if and only if $C_{\mathbf{M}}(R) = O$. From a well-known fact for arbitrary radicals (see, for example, [1], Proposition 1, p. 91) it follows that the **M**-complete radical of a ring R is the intersection of all its ideals I such that the factor ring R/I is **M**-reduced.

Similarly, the **Z**-complete radical $C_{\mathbf{Z}}(R)$ and the **F**-complete radical $C_{\mathbf{F}}(R)$ of a ring R are defined by the sets **Z** and **F**, respectively. Recall that a radical r is called strict if the radical $r(R)$ of a ring R contains every r-radical subring A (i.e., a subring with the property $r(A) = A$) of R. As above, the **M**-complete radical is strict. It is clear that **Z**-complete and **F**-complete radicals are also strict. A ring is **M**-complete if and only if it is simultaneously **Z**-complete and **F**-complete. Denote by $\mathfrak{M}$ ($\mathfrak{Z}$, $\mathfrak{F}$) the class of all rings belonging to the varieties of rings from the set **M** (**Z**, **F**), respectively. It is clear that **M**-complete (**Z**-complete, **F**-complete) radicals are upper radicals defined by the class $\mathfrak{M}$ ($\mathfrak{Z}$, $\mathfrak{F}$), respectively. In addition, for any prime p, we need the notation $\mathfrak{F}_p$ for the class of rings of characteristic p from the class $\mathfrak{F}$.

Recall that the transverbality of the variety As over the subvariety $\mathcal{V}$ means that, for any ring R and an ideal I of R, $\mathcal{V}(I)$ is an ideal of R. As already noted, the transverbality of the variety As over any variety from the set **M** is proved in paper [21].

In additive notation, the atoms of the lattice $L(Ab)$ of subvarieties of the variety Ab of all abelian groups Ab are the varieties $\mathcal{A}_p = \operatorname{var}\{px = 0\}$ for all primes p. Note that the $\mathcal{A}_p$-completeness of an abelian group A means the validity of the equality $\mathcal{A}_p(A) = pA = A$. Further, the divisibility of an abelian group A is equivalent to its $\mathcal{A}_p$-completeness over all primes p, i.e., the completeness of A. It is well known that in every abelian group A, a divisible subgroup is always a direct summand, in A there is the largest divisible subgroup $C(A)$, and A is the direct sum of its complete and reduced subgroups. Moreover, as noted above, every divisible abelian group is the direct sum of some sets of isomorphic copies of the additive group $\mathbb{Q}^+$ of rationales and copies of quasi-cyclic groups C_{p^∞} for some primes p.

Recall that a ring is called a left Artinian ring if any decreasing chain of its left ideals stabilizes. Equivalently, the ring satisfies the minimum condition of left ideals. Further, left Artinian rings will be called Artinian rings. It is well known (see, for example, [8, Theorem 1, p. 63]) that the Jacobson radical of an Artinian ring is nilpotent. In addition, by the Wedderburn–Artin Theorem (see, for example, [8, p. 65]), any Artinian semisimple (in the sense of Jacobson radical) ring is isomorphic to the direct sum of finitely many full matrix rings over skew fields. It is well known that a factor ring of an Artinian ring is Artinian. Also, if the ideal I and the factor ring R/I of a ring R are both Artinian, then R itself is Artinian.

In conclusion of this section, we give some well-known statements that do not relate to the concepts of **M**-completeness and **M**-reducibility but are needed for the sequel.

Theorem 1. [5, Theorem 122.7, p. 350] *Every Artinian ring R is the ring direct sum $R = S \oplus T_{p_1} \oplus \cdots \oplus T_{p_k}$ of some torsion-free Artinian ring S and a finite number of Artinian p_i-rings T_{p_i} corresponding to various primes p_i.*

Theorem 2. [32, Proposition 6] *Let R be a finite ring with unity of characteristic p^k and radical $J(R)$. Then R contains a subring Q isomorphic to a direct sum of matrix rings over Galois rings such that $Q/pQ \cong R/J(R)$ and a (Q,Q)-submodule M of $J(R)$ such that $R = Q + M$ with $Q \cap M = O$.*

Theorem 3. [7, Theorem 1.4.3, p. 35] *If an additive group R^+ of a left Artinian ring R is a torsion-free group, then R possesses a left unity.*

Proposition 1. [28, Lemma 12] *An Artinian ring R is finite if and only if $mR = O$ for some $m \in \mathbb{N}$ and the factor ring $R/J(R)$ is finite.*

Lemma 1. [10, Lemma 3] *If I is an ideal of a ring R and K is a field, then any homomorphism $\varphi : I \to K$ can be extended to a homomorphism $\bar{\varphi} : R \to K$.*

Lemma 2. [24, Lemma] *For any ideal I of a ring R, the relation $M_n(R)/M_n(I) \cong M_n(R/I)$ holds.*

3. Artinian M-complete rings

This section aims to obtain a characterization of **M**-complete Artinian rings. Let us first give several lemmas. Some of them are valid for arbitrary rings.

Lemma 3. *If R is an* **M***-complete ring, then R^2 is an* **M***-complete ring.*

P r o o f. Let R be an **M**-complete ring. Since R is a $\mathcal{Z}_p^0$-complete ring for any prime p, the relation $\mathcal{Z}_p^0(R) = pR + R^2 = R$ holds for all primes p; i.e., $R = R^2 + pR$. We have

$$R^2 = (R^2 + pR)(R^2 + pR) = R^4 + pR^3 + p^2R^2 = R^4 + pR(R^2 + pR) = R^4 + pR^2;$$

i.e., $\mathcal{Z}_p(R^2) = R^2$. This equality means that the ring R^2 is $\mathcal{Z}_p^0$-complete.

We now show that R^2 is $\mathcal{F}_{p^m}$-complete for all p and m. Since R is a $\mathcal{Z}_p^0$-complete ring for any prime p, for each x of R, we find elements a, a_i, and b_i $(i = 1, \dots, n)$ of R such that

$$x = pa + \sum_{i=1}^{n} a_i b_i.$$

Then,

$$x^{p^m} - x = \Big(pa + \sum_{i=1}^{n} a_i b_i\Big)^{p^m} - \Big(pa + \sum_{i=1}^{n} a_i b_i\Big) = pz + \bigg(\Big(\sum_{i=1}^{n} a_i b_i\Big)^{p^m} - \Big(\sum_{i=1}^{n} a_i b_i\Big)\bigg)$$

for some $z \in R$. Therefore, $R_{p^m} \subseteq pR + (R^2)_{p^m}$. It follows that

$$(R_{p^m})^2 \subseteq (pR + (R^2)_{p^m})(pR + (R^2)_{p^m}) \subseteq pR^2 + (R^2)_{p^m}.$$

Considering this inclusion and the fact that the ring R is $\mathcal{F}_{p^m}$-complete, we get

$$R^2 = (pR + R_{p^m})(pR + R_{p^m}) \subseteq pR^2 + (R_{p^m})^2 \subseteq pR^2 + (R^2)_{p^m} = \mathcal{F}_{p^m}(R^2);$$

i.e., R^2 is $\mathcal{F}_{p^m}$-complete for all p and m. $\square$

Corollary 1. *If R is a minimally* **M**-*complete ring and $R^2 \neq O$, then $R^2 = R$.*

Lemma 4. *A simple ring R is either* **M**-*complete and does not belong to the set $\mathfrak{M}$ or* **M**-*reduced and isomorphic to a field $\mathbb{F}_{p^m}$ for some p and m.*

P r o o f. A simple ring R is a ring with nonzero multiplication, so $R \in \mathbf{F}$ for some prime p. As is well known, any nonzero ring of any variety from $\mathbf{F}$ is a subdirect product of finite fields (see, for example, [30]). □

Lemma 5. *Any nil ring R is an $\mathcal{F}$-complete ring.*

P r o o f. The homomorphic image of a nil ring is a nil ring and therefore cannot be a nonzero ring of a variety $\mathcal{F}_{p^m}$ for any p and m. □

Lemma 6. [26, Lemma 3] *If the ideal I of a ring R is contained in the kernel of any homomorphism of R onto rings from a variety $\mathcal{V}$ of rings, then R is a $\mathcal{V}$-complete ring if and only if R/I is a $\mathcal{V}$-complete ring.*

Repeating almost verbatim the proof of Lemma 5 from [11], one can verify the validity of the following statement.

Lemma 7. *A nilpotent ring R is* **M**-*complete if and only if its additive group R^+ is divisible.*

We omit the proof of the following statement analogous to Lemma 2 from [26], which corresponds almost verbatim to the proof of that lemma and uses the results mentioned in Section 1 on the Jacobson radical of Artinian rings, semisimple Artinian rings, the structure of divisible abelian groups, and Lemma 7.

Lemma 8. *The following conditions are equivalent for a ring R:*

(1) $R^+ \cong \bigoplus_{i=1}^{n} C_{p_i^\infty}$;
(2) *R is an* **M**-*complete abelian Artinian ring;*
(3) *R is an* **M**-*complete Artinian nilpotent ring.*

The main result of this section is a modification of two statements of Theorems 1 and 2 of [26].

Theorem 4. *An Artinian ring R is* **M**-*complete if and only if the following conditions hold for its ideal R^2:*

(1) *R^2 is an idempotent Artinian ring and if $R^2 \neq O$, then*

$$R^2/J(R^2) \cong \bigoplus_{i=1}^{k} M_{n_i}(K_i),$$

where K_i is a skew field and $M_{n_i}(K_i) \ncong \mathbb{F}_{p^m}$ for any prime p, $m \in \mathbb{N}$, and $i = 1, \ldots, k$;
(2) *if $R^2 \neq R$, then*

$$R/R^2 \cong \bigoplus_{j=1}^{n} C^0_{p_j^\infty}.$$

P r o o f. Let R be an $\mathbf{M}$-complete Artinian ring. It follows from Theorem 1 that R is the direct sum of its ideals S and T, where S is an $\mathbf{M}$-complete torsion-free Artinian ring and T is an $\mathbf{M}$-complete Artinian periodic ring. Therefore, it is enough to consider both rings separately.

By Theorem 3, the ring S possesses a left unity. Therefore, $S^2 = S$. Since S is an $\mathbf{M}$-complete ring, it follows that the factor ring $S/J(S)$ is an $\mathbf{M}$-complete ring. From the Wedderburn–Artin theorem and Lemma 4, it follows that the factor ring $S/J(S)$ is isomorphic to a direct sum of finitely many full matrix rings over skew fields and does not contain summands isomorphic to a finite field $\mathbb{F}_{p^m}$ for any prime p and $m \in \mathbb{N}$.

Further, consider a decreasing chain of ideals in T: $T \supseteq T^2 \supseteq T^3 \supseteq \dots$ Since T is an Artinian ring, we have $T^n = T^{n+1}$ for some n; i.e., the ideal T^n is an idempotent ring. Then, $\overline{T} = T/T^n$ is an $\mathbf{M}$-complete Artinian nilpotent ring and, by Lemma 8, $\overline{T}$ is an abelian ring. Hence, $xy \in T^n$ for all $x, y \in T$ and, therefore, $T^2 = T^n$; i.e., T^2 is an idempotent ring. In addition, by Lemma 8,

$$T/T^2 \cong \bigoplus_{j=1}^{n} C^0_{p_j^\infty}.$$

Let us show that the ideal T^2 is an Artinian ring. Let I be a left ideal of T^2. The group T^+ is periodic, therfore, for any $i \in I$, there exists $m \in \mathbb{N}$ such that $mi = 0$. By Lemma 8,

$$T/T^2 \cong \bigoplus_{i=1}^{k} C^0_{p_i^\infty},$$

it follows that there exists $t_1 \in T$ for any $t \in T$ such that $m\bar{t_1} = \bar{t}$ in the factor ring $\overline{T} = T/T^2$. Since $t - mt_1 \in T^2$, we have

$$ti = (t - mt_1 + mt_1)i = (t - mt_1)i + mt_1 i = (t - mt_1)i \in I.$$

This means that I is a left ideal of T.

Thus, T^2 is an Artinian and $\mathbf{M}$-complete ring by Lemma 3. Therefore, $T^2/J(T^2)$ also is $\mathbf{M}$-complete ring. It follows that

$$T^2/J(T^2) \cong \bigoplus_{i=1}^{k} M_{n_i}(K_i),$$

where K_i is a skew field and $M_{n_i}(K_i) \ncong \mathbb{F}_{p^m}$ for any prime p and $m \in \mathbb{N}$.

Conversely, let for the ideal R^2 of an Artinian ring R, conditions (1) and (2) of the theorem be satisfied. Four cases are possible:

(i) $R = R^2 = O$. Then, R is an $\mathbf{M}$-complete ring by definition.

(ii) $R \neq R^2 = O$ and $R/R^2 \cong \bigoplus_{j=1}^{n} C^0_{p_j^\infty}$. Then, $R = R/R^2$ is an $\mathbf{M}$-complete ring by Lemma 8.

(iii) $R = R^2 \neq O$ and $R/J(R) = R^2/J(R^2) \cong \bigoplus_{i=1}^{k} M_{n_i}(K_i)$, where K_i is a skew field and $M_{n_i}(K_i) \ncong \mathbb{F}_{p^m}$ for any prime p and $m \in \mathbb{N}$. In this case, R is a nonzero idempotent Artinian ring and $R/J(R)$ is an $\mathbf{M}$-complete ring. Then, $J(R)$ is an $\mathbf{M}$-complete ring by Lemma 5. This means that $J(R)$ is contained in the kernel of any homomorphism onto rings from a variety $\mathcal{F}_{p^m}$ for any p and m. Then, by Lemma 6, $R/J(R)$ is an $\mathcal{F}$-complete ring if and only if R is an $\mathcal{F}$-complete ring. The ring $R = R^2$ also is $\mathcal{Z}$-complete. Hence, R is an $\mathbf{M}$-complete ring.

(iv) $R \neq R^2$ and $R^2 \neq O$, where $R/R^2 \cong \bigoplus_{j=1}^{n} C^0_{p_j^\infty}$ and the ideal R^2 is an idempotent Artinian ring. Then, $R^2/J(R^2) \cong \bigoplus_{i=1}^{k} M_{n_i}(K_i)$, where K_i is a skew field and $M_{n_i}(K_i) \ncong \mathbb{F}_{p^m}$ for any prime p and $m \in \mathbb{N}$. The **M**-completeness of R^2 is proved similar to case (iii). The factor ring R/R^2 is **M**-complete by Lemma 8. In this case, it follows from Lemma 2.2 of [21] that the extension of the **M**-complete ring R^2 by the **M**-complete ring R/R^2 is an **M**-complete ring. Besides, it is known that an extension of an Artinian ring by an Artinian ring is also Artinian.

□

A special case of Theorem 4 is a modification of the result on complete finite rings from [12].

Corollary 2. *A finite nonzero ring R is **M**-complete if and only if the following conditions hold*:

(1) $R^2 = R$;
(2) $R/J(R)$ *is an **M**-complete ring and* $R/J(R) \cong \oplus_{i=1}^{n} M_{n_i}(\mathbb{F}_{p_i{}^{m_i}})$, *where* $n_i > 1$ *for all* $i = 1, 2, \dots, n$, $m_i \in \mathbb{N}$, *and* p_i *are primes.*

4. Artinian M-reduced rings

This section aims to characterize **M**-reduced Artinian rings.

The following statements are modifications for **M**-reduced rings of lemmas for reduced rings from [27]. Their proofs are easy to obtain if we replace the field $\mathbb{F}_p$ by the field $\mathbb{F}_{p^m}$ for any p and m.

Lemma 9. [27, Lemma 1] *For an Artinian nilpotent ring R, the following conditions are equivalent*:

(1) $mR = O$ *for some* $m \in \mathbb{N}$;
(2) R *is a finite ring*;
(3) R *is an **M**-reduced ring.*

From Lemma 9, it follows that all nilpotent **M**-reduced Artinian rings are finite.

Lemma 10. [27, Lemma 1] *Any Artinian **M**-reduced ring has characteristic* $m > 0$.

From Lemma 10 and Theorem 1, it follows that it is sufficient to characterize Artinian **M**-reduced rings of characteristic p^k.

Lemma 11. [27, Lemma 2] *Any ring R of characteristic p^k, where $k \in \mathbb{N}$ and p is a prime, is* $\mathcal{Z}_q^0$*-complete and* $\mathcal{F}_{q^m}$*-complete for any prime $q \neq p$ and $m \in \mathbb{N}$.*

Lemma 12. [27, Lemma 3] *A ring R of characteristic p^k for a prime p and $k \in \mathbb{N}$, is an **M**-complete ring if and only if the ring R/pR is **M**-complete.*

The following lemma describes the **M**-complete radical of an Artinian ring of characteristic p^k for a prime p and $k \in \mathbb{N}$.

Lemma 13. *For an Artinian ring R of characteristic p^k, where p is a prime and $k \in \mathbb{N}$, there exists $m, n \in \mathbb{N}$ such that the $\mathcal{F}$-complete radical $C_{\mathbf{F}}(R) = \mathcal{F}_{p^m}(R)$ and the **M**-complete radical $C_{\mathbf{M}}(R) = \mathcal{F}_{p^m}^n(R)$, where n is the idempotent degree of the verbal $\mathcal{F}_{p^m}(R)$; i.e., $\mathcal{F}_{p^m}^n(R) = \mathcal{F}_{p^m}^{n+1}(R)$.*

P r o o f. Consider the set of all verbals $\mathcal{F}_{p^d}(R)$ of the ring R, where $d \in \mathbb{N}$. Note that, for any $t \in \mathbb{N}$ and $x \in R$,

$$x^{p^{td}} - x = \left(x^{p^d} - x\right) \cdot \sum_{i=0}^{i=s} x^{p^{td} - p^d - i(p^d - 1)}, \quad \text{where} \quad s = \sum_{j=0}^{j=t} p^{(t-j)d}.$$

This means that if h is divisible by d, then $\mathcal{F}_{p^h}(R) \subseteq \mathcal{F}_{p^d}(R)$.

Since R is an Artinian ring, R contains the minimal verbal $\mathcal{F}_{p^m}(R)$. At the same time, $\mathcal{F}_{p^m}(R) \subseteq \mathcal{F}_{p^d}(R)$ for all $d \in \mathbb{N}$ (assuming that this is not the case, we get that the verbal $\mathcal{F}_{p^m}(R)$ is not minimal since it contains the verbal $\mathcal{F}_{p^l}(R) \subseteq \mathcal{F}_{p^m}(R) \cap \mathcal{F}_{p^d}(R) \neq \mathcal{F}_{p^m}(R)$, where l is the least common multiple of d and m).

By Lemma 11, the ring R of characteristic p^k is $\mathcal{F}_{q^t}$-complete for any prime $q \neq p$ and $t \in \mathbb{N}$. It follows from Lemma 1 that any ideal of an $\mathcal{F}_{q^t}$-complete ring, in particular, the ideal $\mathcal{F}_{p^m}(R)$, is also an $\mathcal{F}_{q^t}$-complete ring. From the same lemma and the fact that $\mathcal{F}_{p^m}(R) \subseteq \mathcal{F}_{p^k}(R)$ for all $k \in \mathbb{N}$, it follows that $\mathcal{F}_{p^m}(R)$ is an $\mathcal{F}_{p^k}$-complete ring for all $k \in \mathbb{N}$. Therefore, $\mathcal{F}_{p^m}(R)$ is an $\mathcal{F}$-complete ring; i.e., $\mathcal{F}_{p^m}(R) \subseteq C_{\mathbf{F}}(R)$; hence, $\mathcal{F}_{p^m}(R) = C_{\mathbf{F}}(R)$.

The decreasing chain of ideals $\mathcal{F}_{p^m}(R) \supseteq \mathcal{F}^2_{p^m}(R) \supseteq \mathcal{F}^3_{p^m}(R) \supseteq \dots$ of the ring R stabilizes at some step n. That is, $\mathcal{F}^n_{p^m}(R)$ is an idempotent ring; so it is $\mathcal{Z}$-complete. In addition, $\mathcal{F}^n_{p^m}(R)$ is an $\mathcal{F}$-complete ring by Lemma 1. So, $\mathcal{F}^n_{p^m}(R) \subseteq C_{\mathbf{M}}(R)$. Conversely, since $C_{\mathbf{M}}(R) \subseteq \mathcal{F}_{p^m}(R)$, we have $C^n_{\mathbf{M}}(R) \subseteq \mathcal{F}^n_{p^m}(R)$. Hence, $C_{\mathbf{M}}(R) = C^n_{\mathbf{M}}(R) \subseteq \mathcal{F}^n_{p^m}(R)$ and therefore $C_{\mathbf{M}}(R) = \mathcal{F}^n_{p^m}(R)$. □

Lemma 14. *A nonnilpotent Artinian ring R of characteristic p^k, where p is a prime and $k \in \mathbb{N}$, is an $\mathbf{M}$-reduced ring if and only if R is a finite ring and $C_{\mathbf{F}}(R) = J(R)$. In addition, the factor ring $R/J(R)$ is isomorphic to a finite direct sum of fields $\mathbb{F}_{p^{k_i}}$ for $k_i \in \mathbb{N}$.*

P r o o f. First, we show that $C_{\mathbf{F}}(R) = J(R)$. It follows from Lemma 5 that $J(R)$ is an $\mathcal{F}$-complete ring, so $J(R) \subseteq C_{\mathbf{F}}(R)$. Conversely, by Lemma 13, $C_{\mathbf{F}}(R) = \mathcal{F}_{p^m}(R)$ for some $m \in \mathbb{N}$. Since R is an $\mathbf{M}$-reduced ring, we have $C^n_{\mathbf{F}}(R) = \mathcal{F}^n_p(R) = O$ for some $n \in \mathbb{N}$. Thus, $\mathcal{F}_{p^m}(R)$ is a nilpotent ideal; hence, $C_{\mathbf{F}}(R) = \mathcal{F}_{p^m}(R) \subseteq J(R)$.

The factor ring $R/J(R)$ is isomorphic to a direct sum of finitely many full matrix rings over skew fields. Since $\mathcal{F}_{p^m}(R) = J(R)$, the factor ring $R/J(R)$ belongs to the variety $\mathcal{F}_{p^m}$. Therefore, each of these summands belongs to the variety $\mathcal{F}_{p^m}$ and is isomorphic to $\mathbb{F}_{p^{k_i}}$ for some $k_i \in \mathbb{N}$ by Lemma 4. Also, since $R/J(R)$ is a finite ring, R is also a finite ring by Proposition 1.

Conversely, if R is a finite ring, then $J(R)$ is an $\mathbf{M}$-reduced ring by Lemma 9. Hence, the ring R is $\mathbf{M}$-reduced as an extension of the $\mathbf{M}$-reduced ring $J(R)$ by the $\mathbf{M}$-reduced ring $R/J(R)$. □

The following statement, similar to Teorem 2 of [27], describes the structure of Artinian $\mathbf{M}$-reduced rings.

Theorem 5. *An Artinian ring R is an $\mathbf{M}$-reduced if and only if R is a finite ring with $\mathcal{F}$-complete radical $C_{\mathbf{F}}(R) = J(R)$ and either $R = J(R)$ or $R/J(R) \cong \bigoplus_{i=1}^{n} \mathbb{F}^{k_i}_{p_i}$, where p_i is a prime and $k_i \in \mathbb{N}$.*

P r o o f. By Lemma 10, for any Artinian $\mathbf{M}$-reduced ring R, there exists $m \in \mathbb{N}$ such that $mR = O$. Let $m = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_n^{k_n}$ be the canonical representation of the number m. Then, by Theorem 1, the ring R is a finite direct sum of its ideals R_i, where $p_i^{k_i} R_i = O$ for all $1 \leq i \leq n$.

It follows from the properties of a finite direct sum of rings that the rings R_i for all $1 \leq i \leq n$ are **M**-reduced Artinian rings. If the ring R_i is nonnilpotent, then it satisfies the conditions of Lemma 14, otherwise R_i satisfies the conditions of Lemma 9. If R_i is nilpotent, then $C_{\mathbf{F}}(R_i) = R_i$ by Lemma 5. In each case, R_i is a finite ring and $C_{\mathbf{F}}(R_i) = J(R_i)$. Thus, R is a finite ring and its $\mathcal{F}$-complete radical

$$C_{\mathbf{F}}(R) = \bigoplus_{i=1}^{n} C_{\mathbf{F}}(R_i) = \bigoplus_{i=1}^{n} \mathcal{F}_{p_i^{m_i}}(R_i) = \bigoplus_{i=1}^{n} J(R_i) = J(R).$$

Moreover, if $R \neq J(R)$, then the factor ring $R/J(R)$ is a finite direct sum of ideals isomorphic to finite fields.

Conversely, any finite ring R satisfying the conditions of the theorem is an extension of the **M**-reduced ring $J(R)$ by the **M**-reduced ring $R/J(R)$. This means that R is an **M**-reduced ring. □

5. Minimally M-complete Artinian rings

This section aims to characterize minimally **M**-complete Artinian rings. Before proving the main result, we formulate analogs of auxiliary statements from [22] and [28] and prove some of them.

The proofs of the following several statements almost verbatim correspond to the proofs of their analogs, so, we omit them.

Proposition 2. [25, Proposition 1] *For a basic idempotent e of a nonnilpotent Artinian ring R, $C_{\mathbf{M}}(eRe) = eC_{\mathbf{M}}(R)e$.*

For an **M**-complete radical, the requirement that the idempotent e is the basic idempotent of a nonnilpotent Artinian ring A is essential. For example, in the **M**-complete ring $R = M_2(\mathbb{F}_p)$, for the idempotent

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

the subring $eRe \cong \mathbb{F}_p$ is **M**-reduced; i.e., $C_{\mathbf{M}}(eRe) = O$. However, $eC_{\mathbf{M}}(R)e = eRe \neq O$.

Corollary 3. [25, Corollary 2] *A nonnilpotent minimally* **M***-complete Artinian ring contains a unit.*

Lemma 15. [22, Lemma 11] *If any decreasing chain of ideals of a ring R contained in the ideal I of this ring stabilizes at some finite step, then the* **M***-reducibility of the ring R implies the* **M***-reducibility of the ring R/I.*

Corollary 4. *Any homomorphic image of an Artinian* **M***-reduced ring is an* **M***-reduced ring.*

Corollary 5. *The homomorphic image of a minimally* **M***-complete finite ring is a minimally* **M***-complete ring.*

Lemma 16. [28, Lemma 11] *A finite idempotent ring R of characteristic p^k is minimally* **M***-complete if and only if R/pR is a minimally* **M***-complete ring.*

Lemma 17. [22, Lemma 15] *If I is a nilpotent ideal of a ring R of characteristic p^k and K is an* $\mathbf{M}$*-reduced subring of the ring R, then the homomorphic image $\overline{K}$ in the ring $\overline{R} = R/I$ is also an* $\mathbf{M}$*-reduced ring.*

Corollary 6. *If R is a minimally* $\mathbf{M}$*-complete Artinian ring of characteristic p^k, then the factor ring $R/J(R)$ is also a minimally* $\mathbf{M}$*-complete ring.*

Lemma 18. [22, Lemma 1] *A minimally* $\mathbf{M}$*-complete nilpotent ring is isomorphic to the ring $\mathbb{Q}^0$ or the ring $C^0_{p^\infty}$ for some prime p.*

Lemma 19. [22, Lemma 3] *A skew field K of characteristic zero is minimally* $\mathbf{M}$*-complete if and only if it is isomorphic to the field $\mathbb{Q}$ of rational numbers.*

Corollary 7. *The ring $\mathbb{Z}$ of integers and any of its subrings are* $\mathbf{M}$*-reduced.*

Proposition 3. [24, Proposition] *For any ring R and $n > 1$, the ring $M_n(R)$ is* $\mathbf{M}$*-complete if and only if the ring R is* $\mathbf{Z}$*-complete.*

Corollary 8. *For any idempotent ring R and $n > 1$, the ring $M_n(R)$ is* $\mathbf{M}$*-complete.*

The description of minimally $\mathbf{M}$-complete skew field of prime characteristic p differs significantly from the description of complete skew field of prime characteristic p obtained in Lemma 4 of [22].

Lemma 20. *A skew field K of prime characteristic p is minimally* $\mathbf{M}$*-complete if and only if K is isomorphic to the algebraic closure $\widehat{\mathbb{F}}_p$ of the field $\mathbb{F}_p$.*

P r o o f. Let K be the minimally $\mathbf{M}$-complete skew field of prime characteristic p. Then K contains the field $\mathbb{F}_p$ that obviously lies in the center of K.

Just as in the proof of Lemma 4 of [22], it can be shown that the existence of an element in K that is transcendent with respect to the field $\mathbb{F}_p$ is impossible.

Therefore, all elements of the skew field K are algebraic with respect to the field $\mathbb{F}_p$. It is clear that elements of K are algebraic with respect to a field $\mathbb{F}_{p^m}$ for any $m > 1$. Recall that the field $\mathbb{F}_{p^m}$ is $\mathbf{M}$-reduced by Lemma 4. Taking into account the well-known facts that, for any finite field $\mathbb{F}_q$ in the ring $\mathbb{F}_q[x]$, there exists an irreducible polynomial of any positive degree (see, for example, [14, Corollary 2.11, p. 70]) and an algebraic extension of $\mathbb{F}_q$ containing any of its roots is again a finite field, it is easy to understand that K must coincide with the union of a countable infinite strictly increasing sequences of corresponding finite fields, i.e., K is isomorphic to the algebraic closure $\widehat{\mathbb{F}}_p$ of the field $\mathbb{F}_p$.

Conversely, if a skew field K of prime characteristic p is the algebraic closure of the field $\mathbb{F}_p$, i.e., $A \cong \widehat{\mathbb{F}}_p$, then any proper nonzero subring F of K is a finite field $\mathbb{F}_{p^m}$ for some m and therefore it is $\mathbf{M}$-reduced. Thus, K is a minimally $\mathbf{M}$-complete ring. □

An analog of Lemma 5 from [22] for the $\mathbf{M}$-completeness also has significant changes.

Lemma 21. *The full ring $M_n(K)$ of matrices over a skew field K is minimally* $\mathbf{M}$*-complete if and only if $M_n(K)$ is isomorphic either to the field $\mathbb{Q}$ of rational numbers or the algebraic closure $\widehat{\mathbb{F}}_p$ of the field $\mathbb{F}_p$ or a ring $M_2(\mathbb{F}_p)$ for some prime p.*

P r o o f. Let $M_n(K)$ be a minimally **M**-complete ring of matrices of order n over a skew field K. Being **M**-complete, the ring $M_n(K)$ does not belong to any variety of **M** since the rings of the latter are **M**-reduced.

Let $n = 1$. In this case, $M_1(K) \cong K$. If $char\ K = 0$, then $K \cong \mathbb{Q}$ by Lemma 19. If $char\ K = p$, then $K \cong \widehat{\mathbb{F}}_p$ by Lemma 20.

Let now $n > 1$. The ring $M_n(K)$, in this case, contains the skew field K as its proper subring. Due to the minimal **M**-completeness of the ring $M_n(K)$, the skew field K must be **M**-reduced. By Lemma 4, being a simple ring, K must be isomorphic to a finite field $\mathbb{F}_{p^m}$ for some p and m. It is obvious that the ring $M_n(\mathbb{F}_{p^m})$ for $n \geq 3$ contains a proper subring isomorphic to an **M**-complete ring $M_2(\mathbb{F}_{p^m})$, and therefore is not **M**-minimally complete. Hence, $n = 2$. If $m > 1$ then the ring $M_2(\mathbb{F}_{p^m})$ contains the **M**-complete proper subring $M_2(\mathbb{F}_p)$. Thus, $m = 1$.

Finally, we show that the ring $M_2(\mathbb{F}_p)$ is minimally **M**-complete. Consider any proper nonzero **M**-complete subring R of a ring $M_2(\mathbb{F}_p)$. Being finite, and therefore Artinian, a semisimple factor ring $R/J(R)$, by the Wedderburn-Artin theorem, is a direct sum of a finite number of matrix rings over suitable skew fields. It is clear that, in our case, these skew fields must be finite fields. But then the orders of the matrices included in the decomposition of the ring must be equal to 1. Being a direct sum of **M**-reduced fields, by Lemma 2.5 of [21], the ring $R/J(R)$ must also be **M**-reduced. But then it is clear that the subring R is not **M**-complete. Thus, the ring $M_2(\mathbb{F}_p)$ has no proper nonzero **M**-complete subrings; therefore, it is a minimally **M**-complete ring. □

Note that, in [23], it is indicated that Lemma 5 of [22] describing minimally complete rings $M_n(K)$ of all $(n \times n)$-matrices over a skew field K, in the end, mistakenly states that such is the ring $M_2(\mathbb{F}_p)$ for any p. That this is not the case follows from the well-known representation of finite fields by matrices (see, e.g., [14], p. 90): elements of a finite field $\mathbb{F}_{p^n}$ of order p^n can be represented by square matrices of order n over the field $\mathbb{F}_p$. Consequently, the ring $M_2(\mathbb{F}_p)$ contains a subring isomorphic to the complete field $\mathbb{F}_{p^2}$ and, therefore, is not a minimally complete ring. As a consequence, rings of matrices of the form $M_2(\mathbb{F}_p)$ for all p should be excluded from the formulations of Lemma 5 and condition (2) of the theorem from [22]. Rings R for which the factor ring R/pR is isomorphic to $M_2(\mathbb{F}_p)$ also must be excluded from the formulation of condition (3) of the same theorem. The exact formulation of the theorem from [22] is Theorem 4 of paper [29].

Corollary 9. *A semisimple Artinian ring is minimally* **M**-*complete if and only if it is isomorphic either to the field* $\mathbb{Q}$ *of rational numbers or the algebraic closure* $\widehat{\mathbb{F}}_p$ *of the field* $\mathbb{F}_p$ *or a ring* $M_2(\mathbb{F}_p)$ *for some prime* p.

Lemma 22. *A minimally* **M**-*complete finite ring* R *of a prime characteristic is semisimple by Jacobson.*

P r o o f. A ring R satisfying the conditions of Lemma 22 is an algebra of finite dimension over a field $\mathbb{F}_p$. The ring $R/J(R)$ is minimally **M**-complete by Corollary 5. Therefore, by Corollary 9, $R/J(R)$ is the ring $M_2(\mathbb{F}_p)$ of square matrices of order 2 over a finite field $\mathbb{F}_p$ for some p; i.e., $R/J(R)$ is a central simple algebra. In any case, we get that the algebra is a separable algebra over a field $\mathbb{F}_p$. The field $\mathbb{F}_p$ is perfect; therefore, according to the Wedderburn–Maltsev theorem (see, for example, Theorem 13.18 in [3], p. 575), $R = J(R) \oplus S$, where S is a subalgebra of R and S is isomorphic to $R/J(R)$. Since the ring $R/J(R)$ is **M**-complete and R is minimally **M**-complete, we have $R = S$; i.e., R is a semisimple ring. □

Corollary 10. *In a minimally* **M**-*complete finite ring* R *of characteristic* p^k, *the equality* $J(R) = pR$ *is valid.*

P r o o f. In the ring R, the ideal pR is nilpotent; therefore, $pR \subseteq J(R)$. On the other hand, the ring R/pR is minimally **M**-complete by Lemma 5 and semisimple by Lemma 22. Hence, $J(R) \subseteq pR$; i.e., $J(R) = pR$. □

Lemma 23. *A minimally* **M***-complete ring R of all matrices of some order over the Galois ring is isomorphic to the ring $M_2(\mathbb{Z}_{p^k})$ for some prime p and $k \in \mathbb{N}$.*

P r o o f. For every Galois ring $GR(p^k, m)$, the factor ring

$$GR(p^k, m)/pGR(p^m, k) \cong GR(p, m) = \mathbb{F}_{p^m}.$$

Note, that any Galois ring has a unit and therefore is an idempotent ring. By Corollary 8, the matrix ring $M_n(GR(p^k, m))$ is **M**-complete for any $n \geq 2$. It follows that if $R = M_n(GR(p^k, m))$ is the minimally **M**-complete ring, then $n = 2$. By Lemma 2,

$$R/pR \cong M_2(GR(p^k, m))/M_2(pGR(p^k, m)) \cong M_2(GR(p, m)) = M_2(\mathbb{F}_{p^m}).$$

It follows from Lemma 21 that

$$R/pR \cong M_2(\mathbb{F}_p) \cong M_2(GR(p, 1)).$$

But then

$$R \cong M_2(GR(p^k, 1)) \cong M_2(\mathbb{Z}_{p^k}).$$

Conversely, let $R = M_2(\mathbb{Z}_{p^k})$ for some prime p and $k \in \mathbb{N}$. Then, R is a finite ring, for which $R^2 = R$ and $p^k R = O$. By Lemma 16, the ring R is minimally **M**-complete if and only if the ring R/pR is minimally **M**-complete. Since $R/pR \cong M_2(\mathbb{F}_p)$ is minimally **M**-complete by Lemma 21, we see that the ring $R = M_2(\mathbb{Z}_{p^k})$ is minimally **M**-complete. □

The main result of this section is the following modification of Theorem 1 from [22].

Theorem 6. (1) *Any minimally* **M***-complete nilpotent ring is isomorphic to the ring $\mathbb{Q}^0$ or the ring $C^0_{p^\infty}$ for some prime p.*
(2) *A simple ring with unit is minimally* **M***-complete if and only if it is isomorphic to the field $\mathbb{Q}$ of rational numbers or the algebraic closure $\widehat{\mathbb{F}}_p$ of the field $\mathbb{F}_p$ or a ring $M_2(\mathbb{F}_p)$ for some prime p.*
(3) *A finite ring is minimally* **M***-complete if and only if it is isomorphic to a matrix ring $M_2(\mathbb{Z}_{p^k})$ for some prime p and $k \in \mathbb{N}$.*

P r o o f. (1) This statement of Theorem 6 is the content of Lemma 18.

(2) Let a simple ring with unit is **M**-minimally complete. Then, it is an Artinian ring (see, for example, Corollary 4, [2], p. 196). But a simple Artinian ring is isomorphic to the ring of matrices $M_n(K)$ for some skew field K and a natural number n by the Wedderburn–Artin theorem. The rest follows from Lemma 9.

(3) Let R be a finite minimally **M**-complete ring. It follows that the additive group of the ring R is bounded. Then, by Theorem 1, R is a ring of characteristic p^k for some prime p. The ring R is nonnilpotent by Lemma 9. It follows that R is a ring with unity by Corollary 3.

By Corollary 6, the ring $R/J(R)$ is also minimally **M**-complete. Then, $R/J(R) \cong M_2(\mathbb{F}_p)$ by Corollary 9. By Theorem 2, the ring R contains a subring S isomorphic to the direct sum of full matrix rings over Galois rings such that $S/J(S) \cong R/J(R)$. In the ring S, the equality $J(S) = pS$

is valid. This means that $S/pS \cong M_2(\mathbb{F}_p)$. We obtain that S is a minimally **M**-complete subring of the ring R by Lemma 16. Therefore, $R = S$.

Minimally **M**-complete rings of all matrices of some order over Galois rings are described in Lemma 23. Hence, we get that $R \cong M_2(\mathbb{Z}_{p^k})$ for some prime p and $k \in \mathbb{N}$. □

6. Conclusion

The paper characterizes associative Artinian **M**-complete (Theorem 4), **M**-reduced (Theorem 5), and some classes of minimally **M**-complete associative Artinian rings (Theorem 6). For an exhaustive description of minimally **M**-complete Artinian rings, it is necessary to consider the remaining unexplored case of Artinian rings of characteristic p^k containing a subring isomorphic to the algebraic closure $\widehat{\mathbb{F}}_p$ of the field $\mathbb{F}_p$. As examples show, such rings exist for any prime p and $k \in \mathbb{N}$.

Acknowledgements

The preparation of this paper was overshadowed by the death of its first author. The second author dedicates this work to his memory, as a sign of gratitude for many years of fruitful collaboration. Leonid Matveevich Martynov always had an active life position, he was distinguished by high professionalism, decency, and love of life.

REFERENCES

1. Andrunakievich V. A., Ryabukhin Yu. M. *Radikaly algebr i strukturnaya teoriya* [Radicals of Algebra and Structure Theory]. Moscow: Nauka, 1979. 496 p. (in Russian)
2. Bourbaki N. *Algebra. Moduli, kol'ca, formy* [Algebra. Modules, Rings, Forms]. Moscow: Mir, 1966. 556 p.(in Russian)
3. Feis K. *Algebra: kol'ca, moduli i kategorii* [Algebra: Rings, Modules and Categories]. Vol. 1. Moscow: Mir, 1977. 688 p. (in Russian)
4. Fuchs L. *Beskonechnye abelevy gruppy* [Infinite Abelian Groups]. Vol. 1. Moscow: Mir, 1974. 335 p. (in Russian)
5. Fuchs L. *Beskonechnye abelevy gruppy* [Infinite Abelian Groups]. Vol. 2. Moscow: Mir, 1977. 416 p. (in Russian)
6. Gardner B. J., Wiegandt R. *Radical Theory of Rings.* New York: Marcel Dekker, 2004. 408 p. DOI: 10.1201/9780203913352
7. Herstein I. N. *Nekommutativnye kol'ca.* [Noncommutative Rings]. Moscow: Mir, 1972. 190 p. (in Russian)
8. Jacobson N. *Stroenie kolec.* [Structure of Rings]. Moscow: Izdat. Inostran. Liter., 1961. 392 p. (in Russian)
9. Kalicki J., Scott D. Equational completness of abstract algebras. *Indag. Math.*, 1955. Vol. 17. P. 650–659. DOI: 10.1016/S1385-7258(55)50089-8
10. Kornev A. I. Complete radicals of some group rings. *Sib. Math. J.*, 2007. Vol. 48. P. 857–862. DOI: 10.1007/s11202-007-0087-0
11. Kornev A. I., Pavlova T. V. Characterization of one radical of group rings over finite prime fields. *Sib. Math. J.*, 2004. Vol. 45. P. 504–512. DOI: 10.1023/B:SIMJ.0000028615.28396.f2
12. Kornev A. I., Pavlova T. V. Finite complete associative rings. In: *Matematika i Informatika: Nauka i Obrazovanie: collection of works.* Omsk: OmGPU, 2002. Vol. 1. P. 43–45. (in Russian)
13. Kurosh A. G. Radicals in the group theory. *Sib. Mat. Zh.*, 1962. Vol. 3, No. 6. P. 912–931. (in Russian)
14. Lidl R., Niederreiter G. *Konechnye polya* [Finite Fields]. Vol. 1. Moscow: Mir, 1988. 430 p. (in Russian)
15. Mal'tsev A. I. Multiplication of classes of algebraic systems. *Sib. Math. J.*, 1967. Vol. 8, No. 2. P. 254–267. DOI: 10.1007/BF02302476

16. Martynov L. M. On notions of completeness, solvability, primarity, reducibility and purity for arbitrary algebras. In: *Int. Conf. on Modern Algebra and Its Applications (Vanderbilt University, Tennessee, May 14–18, 1996. Schudule and Abstracts).* Nashville, 1996. P. 79–80.
17. Martynov L. M. On concepts of completeness, reducibility, primarity, and purity for arbitrary algebras. In: *Universal Algebra and its Applications: collection of works. Volgograd, September 6–11, 1999.* Volgograd: Peremena, 2000. P. 179–190. (in Russian)
18. Martynov L. M. One radical algebras with a property of transversality by minimal varieties. *Herald of Omsk University*, 2004. Vol. 2. P. 19–21. URL: https://elib.omsu.ru/issues/124/5973.php (in Russian)
19. Martynov L. M. Completeness, reducibility, primarity and purity for algebras: results and problems. *Sib. Èlektron. Mat. Izv.*, 2016. Vol. 13. P. 181–241. DOI: 10.17377/semi.2016.13.016 (in Russian)
20. Martynov L. M. On the complete radical of a monoid ring. *Herald of Omsk University*, 2017. Vol. 84, No. 2. P. 8–13.
21. Martynov L. M. About one modification of concepts of completely, reducibility, periodicity and primarity for associative rings. *Herald of Omsk University*, 2021. Vol. 26, No. 2. P. 12–22. DOI: 10.24147/1812-3996.2021.26(2).12-22
22. Martynov L. M., Pavlova T. V. On minimally complete associative rings. *Herald of Omsk University*, 2016. Vol. 79, No. 1. P. 6–13. (in Russian)
23. Martynov L. M., Pavlova T. V. Letter to the editor. *Herald of Omsk University*, 2019. Vol. 24, No. 2. P. 23–24. (in Russian)
24. Pavlova T. V. Complete radical of full ring of matrices over an arbitrary ring. In: *Matematika i Informatika: Nauka i Obrazovanie: collection of works.* Omsk: OmGPU, 2011. Vol. 10, P. 16–19. (in Russian)
25. Pavlova T. V. On Artinian rings with a split-off complete radical. *Herald of Omsk University*, 2018. Vol. 23, No. 4. P. 37–43. (in Russian)
26. Pavlova T. V. Complete associative Artinian rings. *Herald of Omsk University*, 2005. Vol. 1. P. 17–19. (in Russian)
27. Pavlova T. V. On reduced associative Artinian rings. In: *Problemy i perspektivy fiz-mat. i tekhn. obrazovaniya: collection of works.* Ishim, Filial TyumGU v Ishime, 2014. P. 40–46. (in Russian)
28. Pavlova T. V. Minimally complete associative Artinian rings. *Sib. Èlektron. Mat. Izv.*, 2017. Vol. 14. P. 1238–1247. DOI: 10.17377/semi.2017.14.105 (in Russian)
29. Pavlova T. V. Corrigendum: Minimally complete associative Artinian rings. *Sib. Èlektron. Mat. Izv.*, 2019. Vol. 16. P. 1913–1915. DOI: 10.33048/semi.2019.16.136 (in Russian)
30. Shevrin L. N., Martynov L. M. Attainable classes of algebras. *Siberian Math. J.*, 1971. Vol. 12, No. 6. P. 986–998.
31. Ševrin L. N., Martynov L. M. Attainability and solvability for classes of algebras. In: *Semigroups. Vol. 39: Coll. Math. Soc., J. Bolyai, G. Poliak, St. Schwarz, O. Steinfeld (eds.).* Amsterdam, Oxford, New York, 1985. P. 397–459.
32. Wilson R. S. On structure of finite rings. II *Pasific J. Math.*, 1974. Vol. 51, No. 1. P. 317–325. DOI: 10.2140/pjm.1974.51.317

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 99–111

DOI: 10.15826/umj.2024.1.009

STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES IN NEUTROSOPHIC 2-NORMED SPACES

Rahul Mondal

Department of Mathematics, Vivekananda Satavarshiki Mahavidyalaya,
Vidyasagar University,
Manikpara, Jhargram – 721513, West Bengal, India

imondalrahul@gmail.com

Nesar Hossain

Department of Mathematics, University of Burdwan,
Burdwan – 713104, West Bengal, India

nesarhossain24@gmail.com

Abstract: In this paper, we have studied the notion of statistical convergence for double sequences in neutrosophic 2-normed spaces. Also, we have defined statistically Cauchy double sequences and statistically completeness for double sequences and investigated some interesting results in connection with neutrosophic 2-normed space.

Keywords: Neutrosophic 2-normed space, Double natural density, Statistically double convergent sequence, Statistically double Cauchy sequence.

1. Introduction

In 1951, Fast [12] and Steinhaus [29] independently extended the concept of usual convergence of real sequences to statistical convergence of real sequences based on the natural density of a set. Later on, this idea has been studied in different directions and various spaces by many authors such as [8–10, 13, 14, 25, 26, 28, 31, 35], and many others.

After the introduction of the fuzzy set theory by Zadeh [37], there has been an extensive effort to find applications and fuzzy analogs of the classical theories and it is being applied in various branches of engineering and science [4, 15, 17, 19, 24]. Later on, the notion of the fuzzy set theory was developed effectively and generalized into new notions as its extensions like intuitionistic fuzzy set [1], interval-valued fuzzy set [36], interval-valued intuitionistic fuzzy set [2], and vague fuzzy set [3]. As a generalization of a crisp set, fuzzy set, intuitionistic fuzzy set, and Pythagorean fuzzy set, Smarandache [32] studied the concept of neutrosophic set. Later, Bera and Mahapatra introduced the notion of neutrosophic soft linear space [5] and neutrosophic soft normed linear space [6]. Recently, Kirişci and Şimşek [21] defined neutrosophic normed space and, in this space, many summability methods such as statistical convergence [21], statistical convergence of double sequences [18], ideal convergence [22], lacunary statistical convergence [23], deferred statistical convergence [11] etc.

Mursaleen and Edely [26] defined and studied statistical convergence and statistically Cauchy double sequences in $\mathbb{R}$. Sarabadan and Talebi [35] studied the notion of statistical convergence of double sequences in 2-normed spaces. Granados and Dhital [18] discussed statistical convergence and statistical Cauchy property for double sequences in neutrosophic normed spaces. Recently, Murtaza et al. [27] introduced neutrosophic 2-normed space and studied statistical convergence for single sequences. In the present paper, we study statistical convergence and statistically Cauchy double sequences in neutrosophic 2-normed spaces and prove some associated results in the line of investigations of them with respect to neutrosophic 2-norm.

2. Preliminaries

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ indicate the set of natural numbers and the set of reals, respectively; $|A|$ denotes the cardinality of the set A. First, we recall some basic definitions and notations.

Definition 1 [26]. *Let $\mathcal{K} \subseteq \mathbb{N}\times\mathbb{N}$ be a two-dimensional set of positive integers, and let $\mathcal{K}(m,n)$ be the number of (j,k) in $\mathcal{K}$ such that $j \le m$ and $k \le n$. Then, the two-dimensional analog of natural density can be defined as follows.*

The lower asymptotic density of the set $\mathcal{K} \subseteq \mathbb{N}\times\mathbb{N}$ is defined as

$$\underline{\delta_2}(\mathcal{K}) = \liminf_{m,n} \frac{\mathcal{K}(m,n)}{mn}.$$

In case the sequence $(\mathcal{K}(m,n)/(mn))$ has a limit in Pringsheim's sense, we say that $\mathcal{K}$ has double natural density defined as

$$\lim_{m,n} \frac{\mathcal{K}(m,n)}{mn} = \delta_2(\mathcal{K}).$$

Example 1. [26] Let

$$\mathcal{K} = \{(i^2, j^2) : i,j \in \mathbb{N}\}.$$

Then,

$$\delta_2(\mathcal{K}) = \lim_{m,n} \frac{\mathcal{K}(m,n)}{mn} \le \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0;$$

i.e., the set $\mathcal{K}$ has double natural density zero, while the set $\{(i,2j) : i,j \in \mathbb{N}\}$ has double natural density $1/2$.

Note that, setting $m = n$, we obtain the two-dimensional natural density due to Christopher [7].

Definition 2 [26]. *A real double sequence $\{l_{mn}\}$ is said to be statistically convergent to a number ξ if the set*

$$\{(m,n), m \le i, n \le j : |l_{mn} - \xi| \ge \varepsilon\}$$

has double natural density zero for all $\varepsilon > 0$.

Definition 3 [16]. *Let $\mathcal{Z}$ be a real vector space of dimension d, where $2 \le d < \infty$. A 2-norm on $\mathcal{Z}$ is a function $\|.,.\| : \mathcal{Z}\times\mathcal{Z} \to \mathbb{R}$ which satisfies the following conditions:*

(1) $\|x,y\| = 0$ *if and only if x and y are linearly dependent in $\mathcal{Z}$;*
(2) $\|x,y\| = \|y,x\|$ *for all x and y in $\mathcal{Z}$;*
(3) $\|\alpha x,y\| = |\alpha|\,\|x,y\|$ *for all α in $\mathbb{R}$ and for all x and y in $\mathcal{Z}$;*
(4) $\|x+y,z\| \le \|x,z\| + \|y,z\|$ *for all x, y, and z in $\mathcal{Z}$.*

Example 2. [34] Let $\mathcal{Z} = \mathbb{R}^2$. Define $\|\cdot,\cdot\|$ on $\mathbb{R}^2$ by $\|x,y\| = |x_1y_2 - x_2y_1|$, where $x = (x_1,x_2)$ and $y = (y_1,y_2) \in \mathbb{R}^2$. Then, $(\mathcal{Z}, \|\cdot,\cdot\|)$ is a 2-normed space.

Definition 4 [35]. *A double sequence* $\{l_{mn}\}$ *in a* 2*-normed space* $(\mathcal{Z}, \|.,.\|)$ *is called statistically convergent to* $\xi \in \mathcal{Z}$ *if, for all* $\varepsilon > 0$ *and all nonzero* $z \in \mathcal{Z}$*, the set*

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|l_{mn} - \xi, z\| \geq \varepsilon\}$$

has double natural density zero; i.e.,

$$\lim_{i,j} \frac{1}{ij} \left|\{(m,n), m \leq i, n \leq j : \|l_{mn} - \xi, z\| \geq \varepsilon\}\right| = 0.$$

Definition 5 [35]. *A double sequence* $\{l_{mn}\}$ *in a* 2*-normed space* $(\mathcal{Z}, \|.,.\|)$ *is called a statistically Cauchy double sequence if, for all* $\varepsilon > 0$ *and all* $z \in \mathcal{Z}$*, there exist* $n_0, m_0 \in \mathbb{N}$ *such that, for all* $m, p \geq n_0$ *and* $n, q \geq m_0$*, the set*

$$\{(m,n), m \leq i, n \leq j : \|l_{mn} - l_{pq}, z\| \geq \varepsilon\}$$

has double natural density zero.

Definition 6 [30]. *A binary operation* $\boxdot : [0,1] \times [0,1] \to [0,1]$ *is called a continuous* t*-norm if the following conditions hold:*

(1) $\boxdot$ *is associative and commutative;*
(2) $\boxdot$ *is continuous;*
(3) $x \boxdot 1 = x$ *for all* $x \in [0,1]$*;*
(4) $x \boxdot y \leq z \boxdot w$ *whenever* $x \leq z$ *and* $y \leq w$ *for all* $x, y, z, w \in [0,1]$.

Definition 7 [30]. *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous* t*-conorm if the following conditions are satisfied:*

(1) $*$ *is associative and commutative;*
(2) $*$ *is continuous;*
(3) $x * 0 = x$ *for all* $x \in [0,1]$*;*
(4) $x * y \leq z * w$ *whenever* $x \leq z$ *and* $y \leq w$ *for all* $x, y, z, w \in [0,1]$.

Example 3. [20] Here are examples of t-norms:

(1) $x \boxdot y = \min\{x, y\}$;
(2) $x \boxdot y = x.y$;
(3) $x \boxdot y = \max\{x + y - 1, 0\}$. This t-norm is known as Lukasiewicz t-norm.

Example 4. [20] Here are examples of t-conorms:

(1) $x * y = \max\{x, y\}$;
(2) $x * y = x + y - x.y$;
(3) $x * y = \min\{x + y, 1\}$. This is known as the Lukasiewicz t-conorm.

Lemma 1 [33]. *If* $\boxdot$ *is a continuous* t*-norm,* $*$ *is a continuous* t*-conorm, and* $r_i \in (0,1)$ *for* $1 \leq i \leq 7$*, then the following statements hold:*

(1) *if* $r_1 > r_2$*, then there are* $r_3, r_4 \in (0,1)$ *such that* $r_1 \boxdot r_3 \geq r_2$ *and* $r_1 \geq r_2 * r_4$*;*
(2) *if* $r_5 \in (0,1)$*, then there are* $r_6, r_7 \in (0,1)$ *such that* $r_6 \boxdot r_6 \geq r_5$ *and* $r_5 \geq r_7 * r_7$.

Now we recall the notion of neutrosophic 2-normed space.

Definition 8 [27]. *Let $\mathcal{Y}$ be a vector space, and let*

$$\mathcal{N}_2 = \{< (e,f), \Theta(e,f), \vartheta(e,f), \psi(e,f) >: (e,f) \in \mathcal{Y} \times \mathcal{Y}\}$$

be a 2-normed space such that

$$\mathcal{N}_2 : \mathcal{Y} \times \mathcal{Y} \times \mathbb{R}^+ \to [0,1].$$

Suppose that $\boxdot$ and $*$ are continuous t-norm and t-conorm, respectively. Then, the quadruple $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ is called a neutrosophic 2-normed space (N2-NS) if the following conditions hold for all $e, f, g \in \mathcal{Z}$, $\eta, \zeta > 0$, and $\beta \neq 0$:

(1) $0 \leq \Theta(e,f;\eta) \leq 1$, $0 \leq \vartheta(e,f;\eta) \leq 1$, *and* $0 \leq \psi(e,f;\eta) \leq 1$ *for every* $\eta > 0$;
(2) $\Theta(e,f;\eta) + \vartheta(e,f;\eta) + \psi(e,f;\eta) \leq 3$;
(3) $\Theta(e,f;\eta) = 1$ *iff e and f are linearly dependent*;
(4) $\Theta(\beta e,f;\eta) = \Theta(e,f;\eta/|\beta|)$ *for all* $\beta \neq 0$;
(5) $\Theta(e,f;\eta) \boxdot \Theta(e,g;\zeta) \leq \Theta(e,f+g;\eta+\zeta)$;
(6) $\Theta(e,f;\cdot) : (0,\infty) \to [0,1]$ *is a continuous nonincreasing function*;
(7) $\lim_{\eta\to\infty} \Theta(e,f;\eta) = 1$;
(8) $\Theta(e,f;\eta) = \Theta(f,e;\eta)$;
(9) $\vartheta(e,f;\eta) = 0$ *iff e and f are linearly dependent*;
(10) $\vartheta(\beta e,f;\eta) = \vartheta(e,f;\eta/|\beta|)$ *for all* $\beta \neq 0$;
(11) $\vartheta(e,f;\eta) * \vartheta(e,g;\zeta) \geq \vartheta(e,f+g;\eta+\zeta)$;
(12) $\vartheta(e,f;\cdot) : (0,\infty) \to [0,1]$ *is a continuous nonincreasing function*;
(13) $\lim_{\eta\to\infty} \vartheta(e,f;\eta) = 0$;
(14) $\vartheta(e,f;\eta) = \vartheta(f,e;\eta)$;
(15) $\psi(e,f;\eta) = 0$ *iff e and f are linearly dependent*;
(16) $\psi(\beta e,f;\eta) = \psi(e,f;\eta/|\beta|)$ *for each* $\beta \neq 0$;
(17) $\psi(e,f;\eta) * \psi(e,g;\zeta) \geq \psi(e,f+g;\eta+\zeta)$;
(18) $\psi(e,f;\cdot) : (0,\infty) \to [0,1]$ *is a continuous nonincreasing function*;
(19) $\lim_{\eta\to\infty} \psi(e,f;\eta) = 0$;
(20) $\psi(e,f;\eta) = \psi(f,e;\eta)$;
(21) *If* $\eta \leq 0$, $\Theta(e,f;\eta) = 0$, $\vartheta(e,f;\eta) = 1$, *and* $\psi(e,f;\eta) = 1$.

In this case, $\mathcal{N}_2 = (\Theta, \vartheta, \psi)$ is called neutrosophic 2-norm on $\mathcal{Y}$.

Definition 9 [27]. *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in an N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$. Choose $\varepsilon \in (0,1)$ and $\eta > 0$. Then, $\{l_n\}_{n\in\mathbb{N}}$ is called convergent if there exist $n_0 \in \mathbb{N}$ and $l_0 \in \mathcal{Y}$ such that*

$$\Theta(l_n - l_0, z;\eta) > 1 - \varepsilon, \quad \vartheta(l_n - l_0, z;\eta) < \varepsilon, \quad \psi(l_n - l_0, z;\eta) < \varepsilon$$

for all $n \geq n_0$ and $z \in \mathcal{Z}$; i.e.,

$$\lim_{n\to\infty} \Theta(l_n - l_0, z;\eta) = 1, \quad \lim_{n\to\infty} \vartheta(l_n - l_0, z;\eta) = 0, \quad \lim_{n\to\infty} \psi(l_n - l_0, z;\eta) = 0.$$

In this case, we write

$$\mathcal{N}_2 - \lim_{n\to\infty} l_n = l_0 \quad or \quad l_n \xrightarrow{\mathcal{N}_2} l_0$$

and l_0 is called an $\mathcal{N}_2$-limit of $\{l_n\}_{n\in\mathbb{N}}$.

Definition 10 [27]. *Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in an N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$. Choose $\varepsilon \in (0,1)$ and $\eta > 0$. Then, $\{l_k\}_{k\in\mathbb{N}}$ is called statistically convergent to ξ if the natural density of the set*

$$\mathcal{A}(\varepsilon,\eta) = \big\{k \leq n : \Theta(l_k - \xi, z;\eta) \leq 1 - \varepsilon \text{ or } \vartheta(l_k - \xi, z;\eta) \geq \varepsilon \text{ and } \psi(l_k - \xi, z;\eta) \geq \varepsilon\big\}$$

is zero for all $z \in \mathcal{Z}$, i.e., $\delta(\mathcal{A}(\varepsilon,\eta)) = 0$.

Definition 11 [27]. *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in an N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$. Choose $\varepsilon \in (0,1)$ and $\eta > 0$. Then, $\{l_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence if there exists $m_0 \in \mathbb{N}$ such that*

$$\Theta(l_n - l_m, z; \eta) > 1 - \varepsilon, \quad \vartheta(l_n - l_m, z; \eta) < \varepsilon, \quad \psi(l_n - l_m, z; \eta) < \varepsilon$$

for all $n, m \geq m_0$ and $z \in \mathcal{Z}$.

Definition 12 [27]. *Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in an N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$, $\varepsilon > 0$, and $\eta > 0$. Then, $\{l_k\}_{k\in\mathbb{N}}$ is called a statistical Cauchy sequence if there exists $n_0 \in \mathbb{N}$ such that*

$$\lim_n \frac{1}{n}\big|\{k \leq n : \Theta(l_k - l_{n_0}, z; \eta) \leq 1 - \varepsilon \text{ or } \vartheta(l_k - l_{n_0}, z; \eta) \geq \varepsilon \text{ and } \psi(l_k - l_{n_0}, z; \eta) \geq \varepsilon\}\big| = 0$$

for every $z \in \mathcal{Z}$ or, equivalently, the natural density of the set

$$\mathcal{A}(\varepsilon, \eta) = \{k \leq n : \Theta(l_k - l_{n_0}, z; \eta) \leq 1 - \varepsilon \text{ or } \vartheta(l_k - l_{n_0}, z; \eta) \geq \varepsilon \text{ and } \psi(l_k - l_{n_0}, z; \eta) \geq \varepsilon\}$$

is zero; i.e., $\delta(\mathcal{A}(\varepsilon, \eta)) = 0$.

3. Main results

Throughout this section, $\mathcal{Z}$ and $\delta_2(\mathcal{A})$ stand for neutrosophic 2-normed space and double natural density of the set $\mathcal{A}$ respectively unless otherwise stated. First, We define the following:

Definition 13. *A double sequence $\{l_{mn}\}$ in an N2-NS $\mathcal{Z}$ is said to be convergent to $\xi \in \mathcal{Z}$ with respect to $\mathcal{N}_2$ if, for all $\sigma \in (0,1)$ and $u > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\Theta(l_{mn} - \xi, z; u) > 1 - \sigma, \quad \vartheta(l_{mn} - \xi, z; u) < \sigma, \quad \psi(l_{mn} - \xi, z; u) < \sigma$$

for all $m, n \geq n_0$ and nonzero $z \in \mathcal{Z}$; i.e.,

$$\lim_{m,n\to\infty} \Theta(l_{mn} - \xi, z; u) = 1, \quad \lim_{m,n\to\infty} \vartheta(l_{mn} - \xi, z; u) = 0, \quad \lim_{m,n\to\infty} \psi(l_{mn} - \xi, z; u) = 0.$$

In this case, we write

$$\mathcal{N}_2 - \lim_{m,n\to\infty} l_{mn} = \xi \quad \text{or} \quad l_{mn} \xrightarrow{\mathcal{N}_2} \xi.$$

Definition 14. *A double sequence $\{l_{mn}\}$ in an N2-NS $\mathcal{Z}$ is said to be statistically convergent to $\xi \in \mathcal{Z}$ with respect to $\mathcal{N}_2$ if, for all $\sigma \in (0,1)$, $u > 0$, and nonzero $z \in \mathcal{Z}$,*

$$\delta_2\big(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \Theta(l_{mn} - \xi, z; u) \leq 1 - \sigma \text{ or } \vartheta(l_{mn} - \xi, z; u) \geq \sigma \text{ and } \psi(l_{mn} - \xi, z; u) \geq \sigma\}\big) = 0$$

or, equivalently,

$$\lim_{i,j} \frac{1}{ij}\big|\{m \leq i, n \leq j : \Theta(l_{mn} - \xi, z; u) \leq 1 - \sigma \text{ or } \vartheta(l_{mn} - \xi, z; u) \geq \sigma \text{ and } \psi(l_{mn} - \xi, z; u) \geq \sigma\}\big| = 0.$$

In this case, we write

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi \quad \text{or} \quad l_{mn} \xrightarrow{st_2(\mathcal{N}_2)} \xi$$

and ξ is called an $st_2(\mathcal{N}_2)$-limit of $\{l_{mn}\}$.

Lemma 2. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$. Then, for all $\sigma \in (0,1)$, $u > 0$, and nonzero $z \in \mathcal{Z}$, the following statements are equivalent:*

(1) $st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi$;
(2) $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u) \le 1-\sigma\}) = \delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(l_{mn}-\xi, z; u) \ge \sigma\}) = \delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(l_{mn}-\xi, z; u) \ge \sigma\}) = 0$;
(3) $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u) > 1-\sigma,\ \vartheta(l_{mn}-\xi, z; u) < \sigma,\ \psi(l_{mn}-\xi, z; u) < \sigma\}) = 1$;
(4) $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u) > 1-\sigma\}) = \delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(l_{mn}-\xi, z; u) < \sigma\}) = \delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(l_{mn}-\xi, z; u) < \sigma\}) = 1$;
(5) $st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} \Theta(l_{mn}-\xi, z; u) = 1$, $st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} \vartheta(l_{mn}-\xi, z; u) = 0$, *and* $st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} \psi(l_{mn}-\xi, z; u) = 0$.

Theorem 1. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$. If*

$$\mathcal{N}_2 - \lim_{m,n\to\infty} l_{mn} = \xi,$$

then

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi.$$

P r o o f. Let

$$\mathcal{N}_2 - \lim_{m,n\to\infty} l_{mn} = \xi.$$

Then, for all $\sigma \in (0,1)$ and $u > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\Theta(l_{mn}-\xi, z; u) > 1-\sigma, \quad \vartheta(l_{mn}-\xi, z; u) < \sigma, \quad \text{and} \quad \psi(l_{mn}-\xi, z; u) < \sigma$$

for all $m, n \ge n_0$ and nonzero $z \in \mathcal{Z}$. So, the set

$$\big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u) \le 1-\sigma \text{ or } \vartheta(l_{mn}-\xi, z; u) \ge \sigma \text{ and } \psi(l_{mn}-\xi, z; u) \ge \sigma\big\}$$

has at most finitely many terms. Since double natural density of a finite set is zero,

$$\delta_2\big(\big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u) \le 1-\sigma \text{ or } \vartheta(l_{mn}-\xi, z; u) \ge \sigma \text{ and } \psi(l_{mn}-\xi, z; u) \ge \sigma\big\}\big) = 0.$$

Therefore,

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi.$$

This completes the proof. □

But in the general case, the converse to Theorem 1 does not have to be true, as shown in the following example.

Example 5. Let $\mathcal{Y} = \mathbb{R}^2$ with $\|x, y\| = |x_1 y_2 - x_2 y_1|$, where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Define a continuous t-norm $\boxdot$ and a continuous t-conorm $*$ as $a \boxdot b = ab$ and $a * b = \min\{a+b, 1\}$ for $a, b \in [0,1]$, respectively. Take $\sigma \in (0,1)$, $x, y \in \mathcal{Y}$, and $u > 0$ such that $u > \|x, y\|$. Consider

$$\Theta(x, y; u) = \frac{u}{u + \|x, y\|}, \quad \vartheta(x, y; u) = \frac{\|x, y\|}{u + \|x, y\|}, \quad \psi(x, y; u) = \frac{\|x, y\|}{u}.$$

Then, $\mathcal{N}_2 = (\Theta, \vartheta, \psi)$ is a neutrosophic 2-norm on $\mathcal{Y}$ and the quadruple $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ becomes a neutrosophic 2-normed space. Define a double sequence $\{l_{mn}\} \in \mathcal{Z}$ by

$$l_{mn} = \begin{cases} (mn, 0), & m = s^2,\ n = t^2,\ s, t \in \mathbb{N}; \\ (0, 0), & \text{otherwise}. \end{cases}$$

Then, for nonzero $z \in \mathcal{Z}$, we have

$$\begin{aligned}
\mathcal{K}_{s,t}(\sigma,u) &= \big\{m \le s, n \le t : \Theta(l_{mn}, z; u) \le 1-\sigma \text{ or } \vartheta(l_{mn}, z; u) \ge \sigma \text{ and } \psi(l_{mn}, z; u) \ge \sigma\big\} \\
&= \Big\{m \le s, n \le t : \frac{u}{u + \|l_{mn}, z\|} \le 1-\sigma \text{ or } \frac{\|l_{mn}, z\|}{u + \|l_{mn}, z\|} \ge \sigma \text{ and } \frac{\|l_{mn}, z\|}{u} \ge \sigma\Big\} \\
&= \Big\{m \le s, n \le t : \|l_{mn}, z\| \ge \frac{u\sigma}{1-\sigma} \text{ or } \|l_{mn}, z\| \ge u\sigma\Big\} \\
&= \big\{m \le s, n \le t : l_{mn} = (mn, 0)\big\} \\
&= \big\{m \le s, n \le t : m = s^2,\ n = t^2,\ s,t \in \mathbb{N}\big\}
\end{aligned}$$

and

$$\frac{1}{st}|\mathcal{K}_{s,t}(\sigma,u)| \le \frac{1}{st}\big|\big\{m \le s, n \le t : m = s^2,\ n = t^2,\ s,t \in \mathbb{N}\big\}\big| \le \frac{\sqrt{s}\sqrt{t}}{st} \to 0 \quad \text{as} \quad s,t \to \infty;$$

i.e.,

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = 0.$$

But $\{l_{mn}\}$ is not convergent with respect to $\mathcal{N}_2$.

Theorem 2. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$. If $\{l_{mn}\}$ is statistically convergent with respect to $\mathcal{N}_2$, then an $st_2(\mathcal{N}_2)$-limit of $\{l_{mn}\}$ is unique.*

P r o o f. Suppose that

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi_1, \quad st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi_2,$$

where $\xi_1 \ne \xi_2$. Given $\sigma \in (0,1)$, choose $\lambda \in (0,1)$ such that

$$(1-\lambda) \boxdot (1-\lambda) > 1-\sigma, \quad \lambda * \lambda < \sigma.$$

Now, for all $u > 0$ and $z \in \mathcal{Z}$, we define the sets

$$\begin{aligned}
\mathcal{A}_{\Theta 1}(\lambda,u) &= \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn} - \xi_1, z; u/2) \le 1-\lambda\big\}, \\
\mathcal{A}_{\Theta 2}(\lambda,u) &= \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn} - \xi_2, z; u/2) \le 1-\lambda\big\}, \\
\mathcal{A}_{\vartheta 1}(\lambda,u) &= \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(l_{mn} - \xi_1, z; u/2) \ge \lambda\big\}, \\
\mathcal{A}_{\vartheta 2}(\lambda,u) &= \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(l_{mn} - \xi_2, z; u/2) \ge \lambda\big\}, \\
\mathcal{A}_{\psi 1}(\lambda,u) &= \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(l_{mn} - \xi_1, z; u/2) \ge \lambda\big\}, \\
\mathcal{A}_{\psi 2}(\lambda,u) &= \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(l_{mn} - \xi_2, z; u/2) \ge \lambda\big\}.
\end{aligned}$$

Since

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi_1, \quad st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi_2,$$

using Lemma 2, we get

$$\delta_2(\mathcal{A}_{\Theta 1}(\lambda,u)) = \delta_2(\mathcal{A}_{\vartheta 1}(\lambda,u)) = \delta_2(\mathcal{A}_{\psi 1}(\lambda,u)) = 0$$

and

$$\delta_2(\mathcal{A}_{\Theta 2}(\lambda,u)) = \delta_2(\mathcal{A}_{\vartheta 2}(\lambda,u)) = \delta_2(\mathcal{A}_{\psi 2}(\lambda,u)) = 0.$$

Now, let

$$\mathcal{A}_{\Theta,\vartheta,\psi}(\lambda,u) = [\mathcal{A}_{\Theta 1}(\lambda,u)\cup\mathcal{A}_{\Theta 2}(\lambda,u)]\cap[\mathcal{A}_{\vartheta 1}(\lambda,u)\cup\mathcal{A}_{\vartheta 2}(\lambda,u)]\cap[\mathcal{A}_{\psi 1}(\lambda,u)\cup\mathcal{A}_{\psi 2}(\lambda,u)].$$

Then, clearly, $\delta_2(\mathcal{A}_{\Theta,\vartheta,\psi}(\lambda,u)) = 0$; i.e., $\delta_2(\mathcal{A}^c_{\Theta,\vartheta,\psi}(\lambda,u)) = 1$.

Let $(p,q)\in\mathcal{A}^c_{\Theta,\vartheta,\psi}(\lambda,u)$. Then, the following three cases are possible.

Case i. If $(p,q)\in\mathcal{A}^c_{\Theta 1}(\lambda,u)\cap\mathcal{A}^c_{\Theta 2}(\lambda,u)$, then

$$\Theta(\xi_1-\xi_2,z;u)\geq\Theta(l_{pq}-\xi_1,z;u/2)\boxdot\Theta(l_{pq}-\xi_2,z;u/2)>(1-\lambda)\boxdot(1-\lambda)>1-\sigma.$$

Since $\sigma\in(0,1)$ is arbitrary, we have $\Theta(\xi_1-\xi_2,z;u)=1$, which yields $\xi_1=\xi_2$.

Case ii. If $(p,q)\in\mathcal{A}^c_{\vartheta 1}(\lambda,u)\cap\mathcal{A}^c_{\vartheta 2}(\lambda,u)$, then

$$\vartheta(\xi_1-\xi_2,z;u)\leq\vartheta(l_{pq}-\xi_1,z;u/2)*\vartheta(l_{pq}-\xi_2,z;u/2)<\lambda*\lambda<\sigma.$$

Since $\sigma\in(0,1)$ is arbitrary, we have $\vartheta(\xi_1-\xi_2,z;u)=0$, which yields $\xi_1=\xi_2$.

Case iii. If $(p,q)\in\mathcal{A}^c_{\psi 1}(\lambda,u)\cap\mathcal{A}^c_{\psi 2}(\lambda,u)$, then, similarly to Case *ii*, we get $\xi_1=\xi_2$.

Hence, an $st_2(\mathcal{N}_2)$-limit of $\{l_{mn}\}$ is unique. This completes the proof. □

Theorem 3. *Let $\mathcal{Y}$ be a real vector space, and let $\{l_{mn}\}$ and $\{w_{mn}\}$ be two double sequences in an N2-NS $\mathcal{Z}$. Then, the following statements hold*:

(1) *if $st_2(\mathcal{N}_2)-\lim_{m,n\to\infty} l_{mn}=\xi_1$ and $st_2(\mathcal{N}_2)-\lim_{m,n\to\infty} w_{mn}=\xi_2$, then*

$$st_2(\mathcal{N}_2)-\lim_{m,n\to\infty} l_{mn}+w_{mn}=\xi_1+\xi_2;$$

(2) *if $st_2(\mathcal{N}_2)-\lim_{m,n\to\infty} l_{mn}=\xi_1$ and $c\neq 0$, then $st_2(\mathcal{N}_2)-\lim_{m,n\to\infty} cl_{mn}=c\xi_1$.*

P r o o f. It is easy. So, we omit the details. □

Theorem 4. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$. Then,*

$$st_2(\mathcal{N}_2)-\lim_{m,n\to\infty} l_{mn}=\xi$$

if and only if there exists a subset

$$\mathcal{K}=\{m_1<m_2<\cdots<m_p<\cdots;n_1<n_2<\cdots<n_q<\cdots\}\subset\mathbb{N}\times\mathbb{N}$$

such that $\delta_2(\mathcal{K})=1$ and $\mathcal{N}_2-\lim_{p,q\to\infty} l_{m_p n_q}=\xi$.

P r o o f. First, suppose that $st_2(\mathcal{N}_2)-\lim_{m,n\to\infty} l_{mn}=\xi$. Now, for all $u>0$, $k\in\mathbb{N}$, and nonzero $z\in\mathcal{Z}$, define

$$\mathcal{A}_{\mathcal{N}_2}(k,u)=\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}:\Theta(l_{mn}-\xi,z;u)>1-\frac{1}{k},\ \vartheta(l_{mn}-\xi,z;u)<\frac{1}{k},\ \psi(l_{mn}-\xi,z;u)<\frac{1}{k}\Big\},\quad(3.1)$$

and

$$\mathcal{B}_{\mathcal{N}_2}(k,u)=\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}:\Theta(l_{mn}-\xi,z;u)\leq 1-\frac{1}{k}\text{ or }\vartheta(l_{mn}-\xi,z;u)\geq\frac{1}{k}\text{ and }\psi(l_{mn}-\xi,z;u)\geq\frac{1}{k}\Big\}.$$

Then, clearly, $\mathcal{A}_{\mathcal{N}_2}(k+1,u) \subset \mathcal{A}_{\mathcal{N}_2}(k,u)$ and, by our assumption, we have $\delta_2(\mathcal{B}_{\mathcal{N}_2}(k,u)) = 0$.

Also, from (3.1), we get $\delta_2(\mathcal{A}_{\mathcal{N}_2}(k,u)) = 1$. Now, let us show that, for $(m,n) \in \mathcal{A}_{\mathcal{N}_2}(k,u)$,

$$\mathcal{N}_2 - \lim_{m,n\to\infty} l_{mn} = \xi.$$

Suppose that $\{l_{mn}\}_{(m,n)\in\mathcal{A}_{\mathcal{N}_2}(k,u)}$ is not convergent with respect to $\mathcal{N}_2$. Then, for some $\sigma \in (0,1)$, we have

$$\Theta(l_{mn} - \xi, z; u) \le 1 - \sigma, \quad \vartheta(l_{mn} - \xi, z; u) \ge \sigma, \quad \psi(l_{mn} - \xi, z; u) \ge \sigma$$

except for at most finite number of terms $(m,n) \in \mathcal{A}_{\mathcal{N}_2}(k,u)$ and nonzero $z \in \mathcal{Z}$.

Define

$$\mathcal{C}_{\mathcal{N}_2}(\sigma,u) = \big\{(m,n)\in\mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u) > 1-\sigma \text{ and } \vartheta(l_{mn}-\xi, z; u) < \sigma,\ \psi(l_{mn}-\xi, z; u) < \sigma\big\},$$

where $\sigma > 1/k$. Clearly, $\delta_2(\mathcal{C}_{\mathcal{N}_2}(\sigma,u)) = 0$. Since $\sigma > 1/k$, we have $\mathcal{A}_{\mathcal{N}_2}(k,u) \subset \mathcal{C}_{\mathcal{N}_2}(\sigma,u)$ and, hence, $\delta_2(\mathcal{A}_{\mathcal{N}_2}(k,u)) = 0$, which contradicts $\delta_2(\mathcal{A}_{\mathcal{N}_2}(k,u)) = 1$. Therefore, for $(m,n) \in \mathcal{A}_{\mathcal{N}_2}(k,u)$, we have

$$\mathcal{N}_2 - \lim_{m,n\to\infty} l_{mn} = \xi.$$

Conversely, suppose that there exists a subset

$$\mathcal{K} = \{m_1 < m_2 < \cdots < m_p < \cdots ; n_1 < n_2 < \cdots < n_q < \cdots\} \subset \mathbb{N}\times\mathbb{N}$$

such that

$$\delta_2(\mathcal{K}) = 1, \quad \mathcal{N}_2 - \lim_{p,q\to\infty} l_{m_p n_q} = \xi.$$

Then, for all $\sigma \in (0,1)$ and $u > 0$, there exists $p_0 \in \mathbb{N}$ such that

$$\Theta(l_{m_p n_q} - \xi, z; u) > 1 - \sigma, \quad \vartheta(l_{m_p n_q} - \xi, z; u) < \sigma, \quad \psi(l_{m_p n_q} - \xi, z; u) < \sigma$$

for all $p, q \ge p_0$ and nonzero $z \in \mathcal{Z}$. Therefore,

$$\begin{aligned}\big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn} - \xi, z; u) \le 1 - \sigma \text{ or } \vartheta(l_{mn} - \xi, z; u) \ge \sigma \text{ and } \psi(l_{mn} - \xi, z; u) \ge \sigma\big\} \\ \subset \mathbb{N}\times\mathbb{N} \setminus \{m_{p_0+1} < m_{p_0+2}, \ldots ; n_{p_0+1} < n_{p_0+2}, \ldots\}.\end{aligned}$$

Hence,

$$\delta_2\big(\big\{(m,n)\in\mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u) \le 1-\sigma \text{ or } \vartheta(l_{mn}-\xi, z; u) \ge \sigma \text{ and } \psi(l_{mn}-\xi, z; u) \ge \sigma\big\}\big) = 0;$$

i.e., $st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi$. □

Definition 15. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$, $\sigma \in (0,1)$, and let $u > 0$. Then, $\{l_{mn}\}$ is called statistically Cauchy with respect to $\mathcal{N}_2$ if there exist $m_0 = m_0(\sigma)$ and $n_0 = n_0(\sigma) \in \mathbb{N}$ such that*

$$\begin{aligned}\delta_2\big(\big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn} - l_{m_0 n_0}, z; u) \le 1 - \sigma \text{ or } \vartheta(l_{mn} - l_{m_0 n_0}, z; u) \ge \sigma \\ \text{and } \psi(l_{mn} - l_{m_0 n_0}, z; u) \ge \sigma\big\}\big) = 0\end{aligned}$$

for nonzero $z \in \mathcal{Z}$.

Theorem 5. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$. If*

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi,$$

then $\{l_{mn}\}$ is statistically Cauchy with respect to $\mathcal{N}_2$.

P r o o f. Let

$$st_2(\mathcal{N}_2) - \lim_{m,n\to\infty} l_{mn} = \xi$$

and $\sigma \in (0,1)$ be given. Choose $\lambda \in (0,1)$ such that

$$(1-\lambda) \boxdot (1-\lambda) > 1-\sigma, \quad \lambda * \lambda < \sigma.$$

Then, for $\lambda \in (0,1)$, $u > 0$, and nonzero $z \in \mathcal{Z}$, we have $\delta_2(\mathcal{A}_{\mathcal{N}_2}(\lambda, u)) = 0$, where

$$\mathcal{A}_{\mathcal{N}_2}(\lambda, u) = \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-\xi, z; u/2) \le 1-\lambda \text{ or } \vartheta(l_{mn}-\xi, z; u/2) \ge \lambda \text{ and } \psi(l_{mn}-\xi, z; u/2) \ge \lambda\big\}.$$

Then, $\delta_2(\mathbb{N}\times\mathbb{N} \setminus \mathcal{A}_{\mathcal{N}_2}(\lambda, u)) = 1$. Let $(m_0, n_0) \in \mathcal{A}^c_{\mathcal{N}_2}(\sigma, u)$. So,

$$\Theta(l_{m_0n_0}-\xi, z; u/2) > 1-\lambda,\ \vartheta(l_{m_0n_0}-\xi, z; u/2) < \lambda \text{ and } \psi(l_{m_0n_0}-\xi, z; u/2) < \lambda.$$

Now, we define

$$\mathcal{B}_{\mathcal{N}_2}(\sigma, u) = \big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Theta(l_{mn}-l_{m_0n_0}, z; u) \le 1-\sigma \text{ or } \vartheta(l_{mn}-l_{m_0n_0}, z; u) \ge \sigma \text{ and } \psi(l_{mn}-l_{m_0n_0}, z; u) \ge \sigma\big\}$$

for every nonzero $z \in \mathcal{Z}$. Let us show that $\mathcal{B}_{\mathcal{N}_2}(\sigma, u) \subset \mathcal{A}_{\mathcal{N}_2}(\lambda, u)$. Let $(p,q) \in \mathcal{B}_{\mathcal{N}_2}(\sigma, u)$. Then, we get

$$\Theta(l_{pq}-l_{m_0n_0}, z; u) \le 1-\sigma,\ \vartheta(l_{pq}-l_{m_0n_0}, z; u) \ge \sigma \text{ and } \psi(l_{pq}-l_{m_0n_0}, z; u) \ge \sigma.$$

Case i. Consider $\Theta(l_{pq}-l_{m_0n_0}, z; u) \le 1-\sigma$. Let us show that

$$\Theta(l_{pq}-\xi, z; u/2) \le 1-\lambda.$$

Suppose that

$$\Theta(l_{pq}-\xi, z; u/2) > 1-\lambda.$$

Then, we have

$$1-\sigma \ge \Theta(l_{pq}-l_{m_0n_0}, z; u) \ge \Theta(l_{pq}-\xi, z; u/2) \boxdot \Theta(l_{m_0n_0}-\xi, z; u/2) > (1-\lambda) \boxdot (1-\lambda) > 1-\sigma,$$

which is impossible. Therefore,

$$\Theta(l_{pq}-\xi, z; u/2) \le 1-\lambda.$$

Case ii. Consider $\vartheta(l_{pq}-l_{m_0n_0}, z; u) \ge \sigma$. Let us show that

$$\vartheta(l_{pq}-\xi, z; u/2) \ge \lambda.$$

Suppose that

$$\vartheta(l_{pq}-\xi, z; u/2) < \lambda.$$

Then, we have

$$\sigma \le \vartheta(l_{pq}-l_{m_0n_0}, z; u) \le \vartheta(l_{pq}-\xi, z; u/2) \boxdot \vartheta(l_{m_0n_0}-\xi, z; u/2) < \lambda * \lambda < \sigma,$$

which is impossible. Therefore, we have

$$\vartheta(l_{pq} - \xi, z; u/2) \geq \lambda.$$

Case iii. If we consider $\psi(l_{pq} - l_{m_0 n_0}, z; u) \geq \sigma$, then, similarly to Case *ii*, we can show that

$$\psi(l_{pq} - \xi, z; u/2) \geq \lambda.$$

Therefore, $(p, q) \in \mathcal{A}_{\mathcal{N}_2}(\lambda, u)$. Hence, $\mathcal{B}_{\mathcal{N}_2}(\sigma, u) \subset \mathcal{A}_{\mathcal{N}_2}(\lambda, u)$. Since $\delta_2(\mathcal{A}_{\mathcal{N}_2}(\lambda, u)) = 0$, we have $\delta_2(\mathcal{B}_{\mathcal{N}_2}(\sigma, u)) = 0$. So, $\{l_{mn}\}$ is statistically Cauchy with respect to $\mathcal{N}_2$. □

Theorem 6. *Let $\{l_{mn}\}$ be a double sequence in an N2-NS $\mathcal{Z}$. If $\{l_{mn}\}$ is statistically Cauchy with respect to $\mathcal{N}_2$, then it is statistically convergent with respect to $\mathcal{N}_2$.*

P r o o f. Suppose that $\{l_{mn}\}$ is statistically Cauchy with respect to $\mathcal{N}_2$ but not statistically convergent to any $\xi \in \mathcal{Z}$ with respect to $\mathcal{N}_2$. Then, for $\sigma \in (0, 1)$, $u > 0$, and nonzero $z \in \mathcal{Z}$, there exist $m_0 = m_0(\sigma)$ and $n_0 = n_0(\sigma) \in \mathbb{N}$ such that $\delta_2(\mathcal{K}) = 0$, where

$$\mathcal{K} = \big\{(m, n) \in \mathbb{N} \times \mathbb{N} : \Theta(l_{mn} - l_{m_0 n_0}, z; u) \leq 1 - \sigma \text{ or } \vartheta(l_{mn} - l_{m_0 n_0}, z; u) \geq \sigma \text{ and } \psi(l_{mn} - l_{m_0 n_0}, z; u) \geq \sigma\big\},$$

and $\delta_2(\mathcal{M}) = 0$, where

$$\mathcal{M} = \big\{(m, n) \in \mathbb{N} \times \mathbb{N} : \Theta(l_{mn} - \xi, z; u/2) > 1 - \sigma \text{ or } \vartheta(l_{mn} - \xi, z; u/2) < \sigma \text{ and } \psi(l_{mn} - \xi, z; u/2) < \sigma\big\}.$$

Since

$$\Theta(l_{mn} - l_{m_0 n_0}, z; u) \geq 2\Theta(l_{mn} - \xi, z; u/2) > 1 - \sigma$$

and

$$\vartheta(l_{mn} - l_{m_0 n_0}, z; u) \leq 2\vartheta(l_{mn} - \xi, z; u/2) < \sigma,$$
$$\psi(l_{mn} - l_{m_0 n_0}, z; u) \leq 2\psi(l_{mn} - \xi, z; u/2) < \sigma,$$

if

$$\Theta(l_{mn} - \xi, z; \frac{u}{2}) > \frac{1 - \sigma}{2}$$

and

$$\vartheta(l_{mn} - \xi, z; \frac{u}{2}) < \frac{\sigma}{2}, \quad \psi(l_{mn} - \xi, z; u) < \frac{\sigma}{2},$$

we have

$$\delta_2\big(\big\{(m, n) \in \mathbb{N} \times \mathbb{N} : \Theta(l_{mn} - l_{m_0 n_0}, z; u) > 1 - \sigma \text{ and } \vartheta(l_{mn} - l_{m_0 n_0}, z; u) < \sigma,\ \psi(l_{mn} - l_{m_0 n_0}, z; u) < \sigma\big\}\big) = 0.$$

This gives $\delta_2(\mathcal{K}^c) = 0$ and so $\delta_2(\mathcal{K}) = 1$, a contradiction. Therefore, $\{l_{mn}\}$ is statistically convergent to some ξ. □

Definition 16. *An N2-NS $\mathcal{Z}$ is called statistically complete with respect to $\mathcal{N}_2$ if every statistically Cauchy sequence is statistically convergent with respect to $\mathcal{N}_2$.*

Remark 1. In the light of Theorems 5 and 6, we see that every N2-NS is statistically complete for double sequences.

Conclusion and future developments

In this paper, we have dealt with statistical convergent double sequences in an N2-NS and have shown that every N2-NS is statistically complete. Later on, these results may be the opening of new tools to generalize this notion in various directions such as $\mathcal{I}_2$-statistical and $\mathcal{I}_2$-lacunary statistical convergence with respect to $\mathcal{N}_2$. Also, this idea can be used in convergence-related problems in many branches of science and engineering.

Acknowledgements

We express great gratitude and deep respect to the reviewers and the managing editor for their valuable comments which improved the quality of the paper. The second author is grateful to The Council of Scientific and Industrial Research (CSIR), HRDG, India, for the award of Senior Research Fellow during the preparation of this paper.

REFERENCES

1. Atanassov K. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 1986. Vol. 20, No. 1 P. 87–96. DOI: 10.1016/S0165-0114(86)80034-3
2. Atanassov K. T., Gargov G. Interval valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 1989. Vol. 31, No. 3. P. 343–349. DOI: 10.1016/0165-0114(89)90205-4
3. Bustince H., Burillo P. Vague sets are intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 1996. Vol. 79, No. 3. P. 403–405. DOI: 10.1016/0165-0114(95)00154-9
4. Barros L. C., Bassanezi R. C., Tonelli P. A. Fuzzy modelling in population dynamics. *Ecological modelling*, 2000. Vol. 128, No. 1. P. 27–33. DOI: 10.1016/S0304-3800(99)00223-9
5. Bera T., Mahapatra N. K. On neutrosophic soft linear spaces. *Fuzzy Inf. Eng.,* 2017. Vol. 9, No. 3. P. 299–324. DOI: 10.1016/j.fiae.2017.09.004
6. Bera T., Mahapatra N. K. Neutrosophic soft normed linear space. *Neutrosophic Sets Syst.,* 2018. Vol. 23. P. 52–71.
7. Christopher J. The asymptotic density of some k-dimensional sets. *Amer. Math. Monthly*, 1956. Vol. 63, No. 6. P. 399–401. DOI: 10.2307/2309400
8. Connor J. S. The statistical and strong p-Cesaro convergence of sequences. *Analysis*, 1988. Vol. 8, No. 1–2. P. 47–63. DOI: 10.1524/anly.1988.8.12.47
9. Çakallı H., Savaş E. Statistical convergence of double sequences in topological groups. *J. Comput. Anal. Appl.*, 2010. Vol. 12, No. 1A. P. 421–426.
10. Nuray F., Dündar E., Ulusu U. Wijsman statistical convergence of double sequences of sets. *Iran. J. Math. Sci. Inform.*, 2021. Vol. 16, No. 1. P. 55–64.
11. Debnath S., Debnath S., Choudhury C. On deferred statistical convergence of sequences in neutrosophic normed spaces. *Sahand Commun. Math. Anal.,* 2022. Vol. 19, No. 4. P. 81–96. DOI: 10.22130/scma.2022.544537.1031
12. Fast H. Sur la convergence statistique. *Colloq. Math.,* 1951. Vol. 2, No. 3–4. P. 241–244. (in French)
13. Fridy J. A. On statistical convergence. *Analysis*, 1985. Vol. 5, No. 4. P. 301–313. DOI: 10.1524/anly.1985.5.4.301
14. Fridy J. A., Orhan C. Lacunary statistical convergence. *Pacific J. Math.,* 1993. Vol. 160, No. 1. P. 43–51. DOI: 10.2140/pjm.1993.160.43
15. Fradkov A. L., Evans R. J. Control of chaos: Methods and applications in engineering. *Annu. Rev. Control*, 2005. Vol. 29, No. 1. P. 33–56. DOI: 10.1016/j.arcontrol.2005.01.001
16. Gähler S. 2-normed spaces. *Math. Nachr.,* 1964. Vol. 28, No. 1–2. P. 1–43. DOI: 10.1002/mana.19640280102 (in German)
17. Giles R. A computer program for fuzzy reasoning. *Fuzzy Sets and Systems*, 1980. Vol. 4, No. 3. P. 221–234. DOI: 10.1016/0165-0114(80)90012-3
18. Granados C., Dhital A. Statistical convergence of double sequences in neutrosophic normed spaces. *Neutrosophic Sets Syst.*, 2021. Vol. 42. P. 333–344.

19. Hong L., Sun J.-Q. Bifurcations of fuzzy nonlinear dynamical systems. *Commun. Nonlinear Sci. Numer. Simul.*, 2006. Vol. 11, No. 1. P. 1–12. DOI: 10.1016/j.cnsns.2004.11.001
20. Klement E. P., Mesiar R., Pap E. Triangular norms. Position paper I: basic analytical and algebraic properties. *Fuzzy Sets and Systems*, 2004. Vol. 143, P. 5–26. DOI: 10.1016/j.fss.2003.06.007
21. Kirişci M., Şimşek N. Neutrosophic normed spaces and statistical convergence. *J. Anal.,* 2020. Vol. 28, P. 1059–1073. DOI: 10.1007/s41478-020-00234-0
22. Kişi Ö. Ideal convergence of sequences in neutrosophic normed spaces. *J. Intell. Fuzzy Syst.,* 2021. Vol. 41, No. 3. P. 2581–2590. DOI: 10.3233/JIFS-201568
23. Khan V. A., Khan M. D., Ahmad M. Some new type of lacunary statistically convergent sequences in neutrosophic normed space. *Neutrosophic Sets Syst.*, 2021. Vol. 42, No. 1. P. 241–252. URL: https://digitalrepository.unm.edu/nss_journal/vol42/iss1/15
24. Madore J. Fuzzy physics. *Ann. Physics*, 1992. Vol. 219, No. 1. P. 187–198. DOI: 10.1016/0003-4916(92)90316-E
25. Mursaleen M. λ-statistical convergence. *Math. Slovaca*, 2000. Vol. 50, No. 1. P. 111–115. URL: https://dml.cz/handle/10338.dmlcz/136769
26. Mursaleen M., Edely O. H. H. Statistical convergence of double sequences. *J. Math. Anal. Appl.,* 2003. Vol. 288, No. 1. P. 223–231. DOI: 10.1016/j.jmaa.2003.08.004
27. Murtaza S., Sharma A., Kumar V. Neutrosophic 2-normed spaces and generalized summability. *Neutrosophic Sets Syst.,* 2023. Vol. 55, No. 1. Art. no. 25. P. 415–426. URL: https://digitalrepository.unm.edu/nss_journal/vol55/iss1/25
28. Nuray F., Savaş E. Statistical convergence of sequences of fuzzy numbers. *Math. Slovaca*, 1995. Vol. 45, No. 3. P. 269–273. URL: https://dml.cz/handle/10338.dmlcz/129143
29. Steinhaus H. Sur la convergence ordinaire et la convergence asymptotique. *Colloq. Math.,* 1951. Vol. 2, No. 1. P. 73–74. (in French)
30. Schweizer B., Sklar A. Statistical metric spaces. *Pacific J. Math.,* 1960. Vol. 10, No. 1. P. 313–334. DOI: 10.2140/pjm.1960.10.313
31. Šalát T. On statistically convergent sequences of real numbers. *Math. Slovaca*, 1980. Vol. 30, No. 2. P. 139–150. URL: https://dml.cz/handle/10338.dmlcz/136236
32. Smarandache F. Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. *Int. J. Pure. Appl. Math.,* 2005. Vol. 24, No. 3. P. 287–297. URL: https://www.ijpam.eu/contents/2005-24-3/1/1.pdf
33. Saadati R., Park J. H. On the intuitionistic fuzzy topological spaces. *Chaos Solitons Fractals*, 2006. Vol. 27, No. 2. P. 331–344. DOI: 10.1016/j.chaos.2005.03.019
34. Şahiner A., Gürdal M., Saltan S., Gunawan H. Ideal convergence in 2-normed spaces. *Taiwanese J. Math.,* 2007. Vol. 11, No. 5. P. 1477–1484. URL: https://www.jstor.org/stable/43834569
35. Sarabadan S., Talebi S. Statistical convergence of double sequences in 2-normed spaces. *Int. J. Contemp. Math. Sciences*, 2011. Vol. 6, No. 8. P. 373–380.
36. Turksen I. B. Interval valued fuzzy sets based on normal forms. *Fuzzy Sets and Systems*, 1986. Vol. 20, No. 2. P. 191–210. DOI: 10.1016/0165-0114(86)90077-1
37. Zadeh L. A. Fuzzy sets. *Inform. Control*, 1965. Vol. 8, No. 3. P. 338–353. DOI: 10.1016/S0019-9958(65)90241-X

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 112–122

DOI: 10.15826/umj.2024.1.010

A TWO-FOLD CAPTURE OF COORDINATED EVADERS IN THE PROBLEM OF A SIMPLE PURSUIT ON TIME SCALES[1]

Elena S. Mozhegova[†], **Nikolai N. Petrov**[††]

Udmurt State University,
Universitetskaya Str., 1, Izhevsk, 426034, Russian Federation

[†]Mozhegovalena@yandex.ru [††]Kma3@list.ru

Abstract: In finite-dimensional Euclidean space, we study the problem of a simple pursuit of two evaders by a group of pursuers in a given time scale. It is assumed that the evaders use the same control and do not move out of a convex polyhedral set. The pursuers use counterstrategies based on information on the initial positions and on the prehistory of the control of evaders. The set of admissible controls of each of the participants is a sphere of unit radius with its center at the origin, and the goal sets are the origin. The goal of the group of pursuers is the capture of at least one evader by two pursuers. In terms of the initial positions and parameters of the game, a sufficient condition for capture is obtained. The study is based on the method of resolving functions, which makes it possible to obtain sufficient conditions for solvability of the pursuit problem in some guaranteed time.

Keywords: Differential game, Group pursuit, Evader, Pursuer, Time scale.

1. Introduction

The modern theory of differential pursuit-evasion games involves the development of methods for solving problems of conflict interaction of groups of pursuers and evaders [3, 6, 7, 10]. In particular, it is concerned with searching for new classes of problems which can be analyzed using the previously developed methods, for example, the method of resolving functions. It was pointed out in [1, 9] that some results obtained separately for the theories of differential and difference equations may be considered from a unified point of view if one admits the possibility of specifying dynamical systems on arbitrary closed subsets $\mathbb{R}^1$ called *time scales*. Time scales find applications in constructing various mathematical models [2, 4]. A nonantagonistic game of N persons in a time scale was considered in [11]. Sufficient conditions for the capture of one evader in the problem of a simple group pursuit in a given time scale were obtained in [15].

Ref. [14] addressed the problem of a simple pursuit of a group of rigidly coordinated evaders by a group of pursuers in a given time scale, where sufficient conditions for the capture of at least one evader were obtained. The problem of a multiple capture of a given number of evaders in time scales, under the condition that the evaders use programmed strategies, each pursuer catches no more than one evader and the motions of the players are simple was treated in [13].

Ref. [17] dealt with the problem of a simple pursuit of rigidly coordinated evaders in a given time scale, under the condition that the evaders do not move out of a convex polyhedral set. The goal of the pursuers was either the capture of one evader by two pursuers or the capture of two evaders. Sufficient conditions for capture were obtained.

In this paper we consider, in a given time scale, the problem of a simple pursuit of two evaders by a group of pursuers who use the same control, under the condition that the evaders do not move

[1]This work was supported by the Ministry of Science and Higher Education of the Russian Federation in the framework of state assignment, project FEWS-2024-0009.

out of a convex polyhedral set. The goal of the pursuers is the capture of at least one evader by two different pursuers. Sufficient conditions for capture are obtained.

2. Auxiliary definitions and facts

In this section we will outline the basic facts from time scale theory. All results presented below can be found, for example, in [5, 8].

Definition 1. *A nonempty closed subset $\mathbb{T} \subset \mathbb{R}^1$ such that $\sup\limits_{t\in\mathbb{T}} t = +\infty$ is called a time scale.*

Definition 2. *Let $\mathbb{T}$ be a time scale. A function $\sigma : \mathbb{T} \to \mathbb{R}^1$ of the form*

$$\sigma(t) = \inf\big\{s \in \mathbb{T} \mid s > t\big\}$$

is called a translation function.

Definition 3. *A function $f : \mathbb{T} \to \mathbb{R}^1$ is called Δ-differentiable at point $t \in \mathbb{T}$ if there exists a number $\gamma \in \mathbb{R}^1$ such that for any $\varepsilon > 0$ there exists a neighborhood W of point t such that the inequality*

$$|f(\sigma(t)) - f(s) - \gamma(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|$$

holds for all $s \in \mathbb{T} \cap W$.

In this case, the number γ is called the Δ-derivative of the function f at point t. The Δ-derivative of the function f at point t will be denoted by $f^{\Delta}(t) = \gamma$.

Definition 4. *A function $f : \mathbb{T} \to \mathbb{R}^n$, $f(t) = (f_1(t), \ldots, f_n(t))$ is called Δ-differentiable at point $t \in \mathbb{T}$ if all functions $f_1, \ldots, f_n$ are Δ-differentiable at point t.*

Let $\mathbb{T}$ be a time scale, $E \subset \mathbb{T}$. Denote

$$R(E) = \big\{t \in E \mid \sigma(t) > t\big\}.$$

Then the set $R(E)$ is no more than countable.

Definition 5. *The set $E \subset \mathbb{T}$ is called Δ-measurable if the set*

$$\tilde{E} = E \cup \bigcup_{t\in R(E)} (t, \sigma(t))$$

is measurable in the sense of Lebesgue.

Definition 6. *A function $f : \mathbb{T} \to \mathbb{R}^1$ is called Δ-measurable on a Δ-measurable set E if a function $\tilde{f}$ of the form*

$$\tilde{f}(t) = \begin{cases} f(t), & t \in E, \\ f(t_i), & t \in (t_i, \sigma(t_i)), \quad t_i \in R(E) \end{cases}$$

is measurable on the set $\tilde{E}$.

Definition 7. *A function $f : E \to \mathbb{R}^1$, $E \subset \mathbb{T}$ is called Δ-integrable on a Δ-measurable set E if the function $\tilde{f}$ is integrable in the sense of Lebesgue on the set $\tilde{E}$. If f is Δ-integrable on the set E, then we define $\int_E f(s)\Delta s$, assuming*

$$\int_E f(s)\Delta s = \int_{\tilde{E}} f d\mu,$$

where μ is the Lebesgue measure.

3. Formulation of the problem

Let a time scale $\mathbb{T}$, $t_0 \in \mathbb{T}$ be given.

In the space $\mathbb{R}^k (k \geqslant 2)$ we consider the differential game $\Gamma(n, 2)$ of $n+2$ persons: n pursuers $P_1, \dots, P_n$ and two evaders E_1, E_2 with laws of motion of the form

$$x_i^\Delta = u_i, \quad x_i(t_0) = x_i^0, \quad u_i \in V, \tag{3.1}$$

$$y_j^\Delta = v, \quad y_j(t_0) = y_j^0, \quad v \in V. \tag{3.2}$$

Here $x_i, y_j, x_i^0, y_j^0, u_i, v \in \mathbb{R}^k$, $i \in I = \{1, \dots, n\}$, $j \in J = \{1, 2\}$, $V = \{v \in \mathbb{R}^k \mid \|v\| \leqslant 1\}$. We assume that $x_i^0 \neq y_j^0$ for all $i \in I, j \in J$. Additionally, we assume that in the process of the game evaders E_1 and E_2 do not move out of a convex set D with a nonempty interior of the form

$$D = \big\{ y \in \mathbb{R}^k \mid (p_l, y) \leqslant \mu_l, \quad l = 1, \dots, r \big\},$$

where $p_1, \dots, p_r$ are unit vectors $\mathbb{R}^k$, $\mu_1, \dots, \mu_r$ are real numbers, and (a, b) is a scalar product. We also assume that $D = \mathbb{R}^k$ with $r = 0$.

Introduce new variables $z_{ij} = x_i - y_j$. Then instead of the systems (3.1) and (3.2) we obtain the system

$$z_{ij}^\Delta = u_i - v, \quad z_{ij}(t_0) = z_{ij}^0 = x_i^0 - y_j^0, \quad u_i, v \in V. \tag{3.3}$$

We will say the Δ-measurable function $v : \mathbb{T} \to \mathbb{R}^k$ is Δ-admissible if $v(t) \in V$, $y_j(t) \in D$ for all $t \in \mathbb{T}, j \in J$. Here $y_j(t)$ is a solution to the Cauchy problem (3.2) with a given function $v(\cdot)$.

We will say that the prehistory $v_t(\cdot)$ of the function v at time $t \in \mathbb{T}$ is a restriction of the function v to $[t_0, t) \cap \mathbb{T}$. Let

$$z^0 = \big\{ z_{ij}^0 \mid i \in I, \quad j \in J \big\}$$

denote the vector of initial positions.

The actions of the evaders can be interpreted as follows: there is a center which for all evaders E_1 and E_2 chooses the same control $v(t)$.

Definition 8. *We will say that a quasi-strategy $\mathcal{U}_i$ of pursuer P_i is given if a map U_i^0 is defined which associates the Δ-measurable function $u_i(t) = \mathcal{U}_i(z^0, t, v_t(\cdot))$ with values in V to the initial positions z^0, time $t \in \mathbb{T}$ and an arbitrary prehistory of the control $v_t(\cdot)$ of evaders E_1 and E_2.*

Definition 9. *A two-fold capture occurs in the game $\Gamma(n, 2)$ if there exist time $T_0 = T(z^0)$ and quasi-strategies $\mathcal{U}_1, \dots, \mathcal{U}_n$ of pursuers $P_1, \dots, P_n$ such that, for any measurable function $v(\cdot)$, $v(t) \in V$, $y(t) \in D$, $t \in [t_0, T_0] \cap \mathbb{T}$, there are numbers $l, m \in I$, $(m \neq l)$, $j \in \{1, 2\}$ and times τ_1, τ_2 $\in$ $[t_0, T_0] \cap \mathbb{T}$ such that $z_{lj}(\tau_1) = 0$, $z_{mj}(\tau_2) = 0$.*

4. Sufficient conditions for capture

Definition 10 [12]. *The vectors $a_1, a_2, \dots, a_m$ form a positive basis in $\mathbb{R}^k$ if for any $x \in \mathbb{R}^k$ there exist nonnegative real numbers $\alpha_1, \alpha_2, \dots, \alpha_m$ such that*

$$x = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m.$$

Let $\operatorname{Int} X$, $\operatorname{co} X$ denote, respectively, the interior and the convex hull of the set $X \subset \mathbb{R}^k$.

Theorem 1 [12]. *The vectors $a_1, a_2, \dots, a_m$ form a positive basis in $\mathbb{R}^k$ if and only if*

$$0 \in \operatorname{Intco}\{a_1, \dots, a_m\}.$$

Lemma 1. *Let $m \geqslant 3$, $a_1, \dots, a_m$, b_1, b_2, $p_1, \dots, p_r \in \mathbb{R}^k$ be such that*

1) *for each $q \in J_0 = \{1, \dots, m-2\}$*

$$0 \in \operatorname{Intco}\big\{a_i - b_1, a_i - b_2, i \in J_0 \setminus \{q\}, p_1, \dots, p_r\big\},$$

2) *$a_{m-1} - b_2 = t_1(b_1 - b_2)$, $a_m - b_2 = t_2(b_1 - b_2)$ for some $t_1 < 0$, $t_2 < 0$.*

Then for each $l \in J = \{1, \dots, m\}$ the following inclusion holds:

$$0 \in \operatorname{Intco}\big\{a_i - b_2, i \in J \setminus \{l\}, p_1, \dots, p_r\big\}. \tag{4.1}$$

P r o o f. If $m = 3$, then it follows from condition 1) of the lemma that

$$0 \in \operatorname{Intco}\{p_1, \dots, p_r\}.$$

Therefore, the condition (4.1) is satisfied automatically.

Let $m \geqslant 4$. Assume that there exists $q \in J$ for which

$$0 \notin \operatorname{Intco}\big\{a_i - b_2, i \in J \setminus \{q\}, p_1, \dots, p_r\big\}.$$

Then, by the separability theorem, there exists a unit vector $x \in \mathbb{R}^k$ such that

$$(a_i - b_2, x) \leqslant 0 \quad \text{for all} \quad i \in J \setminus \{q\}, \quad (p_j, x) \leqslant 0, \quad \text{for all} \quad j = 1, \dots, r. \tag{4.2}$$

It follows from condition 2) of the lemma that $(b_1 - b_2, x) \geqslant 0$. Then

$$(a_i - b_1, x) = (a_i - b_2, x) + (b_2 - b_1, x) \leqslant 0 \quad \text{for all} \quad i \in J \setminus \{q\}. \tag{4.3}$$

Inequalities (4.2) and (4.3) contradict condition 1) of the lemma. This proves the lemma. □

Let us introduce the following notation:

$$\lambda(h, v) = \sup\big\{\lambda \geqslant 0 \mid -\lambda h \in V - v\big\},$$

$$K(t) = \int_{t_0}^{t} \Delta s, \quad \Omega(J) = \big\{(i_1, i_2) \big| i_1, i_2 \in J, \ \ i_1 \neq i_2\big\},$$

where J is a finite set of natural numbers.

Lemma 2. *Let $m \geqslant 4$, $a_1, \dots, a_{m-2}$, c, $p_1 \in \mathbb{R}^k$ be such that for each $q \in J_0 = \{1, \dots, m-2\}$ the vectors $\{a_i, i \in J_0 \setminus \{q\}, c, p_1\}$ form a positive basis $\mathbb{R}^k$. Then for any $b_1, b_2 \in \mathbb{R}^k$ there exists $\rho_0 > 0$ such that for all $\rho > \rho_0$ the following inequality holds:*

$$\delta(\rho) = \min_{v \in V} \max\big\{\max_{\Lambda \in \Omega^0(J)} \min_{i \in \Lambda} \lambda(w_i, v), (p_1, v)\big\} > 0,$$

where $J = \{1, \dots, m\}$, $\Omega^0(J) = \Omega(J_0) \cup \{(m-1, m)\}$,

$$w_i = \begin{cases} a_i, & i \in J_0, \\ b_1 + \rho c, & i = m-1, \\ b_2 + \rho c, & i = m. \end{cases}$$

P r o o f. Assume that the statement of the lemma is false. Then there exist $b_1, b_2 \in \mathbb{R}^k$ such that for any $\rho_0 > 0$ there is $\rho > \rho_0$ for which $\delta(\rho) = 0$. It follows from the definition of $\delta(\rho)$ that there exists $v_\rho \in V$ such that $(p_1, v_\rho) \leqslant 0$ and for all $\Lambda \in \Omega^0(J)$

$$\min_{i\in\Lambda} \lambda(w_i, v_\rho) = 0, \quad \text{with} \quad \|v_\rho\| = 1.$$

From the last condition it follows that there exist $J(\rho) \subset J_0$, $|J(\rho)| = m-3$ and $j(\rho) \in \{m-1, m\}$ such that $\lambda(w_i, v_\rho) = 0$ for all $i \in J(\rho) \cup \{j(\rho)\}$.

Let $\rho_0 = 1$. Then there are $\rho_1 > \rho_0$, $v_1 \in V$, $J(\rho_1)$ for which

$$(p_1, v_1) \leqslant 0, \quad (w_i, v_1) \leqslant 0 \quad \text{for all} \quad i \in J(\rho_1) \cup \{j(\rho_1)\}, \quad \text{and} \quad \|v_1\| = 1.$$

For $\rho_0 = \rho_1 + 1$ there are $\rho_2 > \rho_0$, $v_2 \in V$, $J(\rho_2)$ for which

$$(p_1, v_2) \leqslant 0, \quad (w_i, v_2) \leqslant 0 \quad \text{for all} \quad i \in J(\rho_2) \cup \{j(\rho_2)\}, \quad \text{and} \quad \|v_2\| = 1.$$

Continuing this process further, we find that there exist sequences $\{\rho_s\}_{s=1}^{\infty}$,

$$\lim_{s\to+\infty} \rho_s = +\infty,$$

$\{v_s\}$, $\{J(\rho_s)\}$, $\{j(\rho_s)\}$ for which

$$(p_1, v_s) \leqslant 0, \quad (w_i, v_s) \leqslant 0 \quad \text{for all} \quad i \in J(\rho_s) \cup \{j(\rho_s)\}, \quad \text{and} \quad \|v_s\| = 1.$$

Consequently, there exists a subsequence $\{\rho_{s_i}\}$, $\lim\limits_{i\to+\infty} \rho_{s_i} = +\infty$ for which there are a subsequence $\{v_{s_i}\}$, a set J^0, $J^0 \subset J_0$, $|J^0| = m-3$, and an index $\hat{j} \in \{m-1, m\}$ such that for all j the following inequalities hold:

$$(p_1, v_{s_j}) \leqslant 0, \quad (w_i, v_{s_j}) \leqslant 0 \quad \text{for all} \quad i \in J^0 \cup \{\hat{j}\}, \quad \text{and} \quad \|v_{s_j}\| = 1.$$

From the sequence $\{v_{s_j}\}$ one can single out a subsequence $\{\overline{v}_l\}$ converging to v_0, with $\|v_0\| = 1$. Therefore, we have

$$(p_1, \overline{v}_l) \leqslant 0, \quad (w_i, \overline{v}_l) \leqslant 0 \quad \text{for all} \quad i \in J^0, \quad \Bigl(\frac{w_{\hat{j}}}{\rho_l} + c, \overline{v}_l\Bigr) \leqslant 0.$$

Passing in the last inequalities to the limit as $l \to +\infty$, we obtain

$$(p_1, v_0) \leqslant 0, \quad (w_i, v_0) \leqslant 0 \quad \text{for all} \quad i \in J^0, \quad (c, v_0) \leqslant 0.$$

Therefore, by virtue of Theorem 1 the set of vectors $\{w_i, i \in J^0, c, p_1\}$ does not form the positive basis $\mathbb{R}^k$, which contradicts the condition of the lemma. This proves the lemma. □

Lemma 3. *Let $a_1, \ldots, a_m, p_1 \in \mathbb{R}^k$ be such that*

$$\delta = \min_{v\in V} \max\Bigl\{ \max_{\Lambda\in\Omega^0(J)} \min_{j\in\Lambda} \lambda(a_j, v), (p_1, v)\Bigr\} > 0,$$

where $J_0 = \{1, \ldots, m-2\}$, $\Omega^0(J) = \Omega(J_0) \cup \{m-1, m\}$.

Then there exists $T_0 > t_0$, $T_0 \in \mathbb{T}$ such that for any admissible control $v(\cdot)$ of evaders there is $\Lambda = (\alpha, \beta) \in \Omega^0(J)$ such that

$$\int_{t_0}^{T_0} \lambda(a_\alpha, v(s))\Delta s \geqslant 1, \quad \int_{t_0}^{T_0} \lambda(a_\beta, v(s))\Delta s \geqslant 1.$$

P r o o f. Let $v(\cdot)$ be an admissible control of evaders. From [5] it follows that the functions $\lambda(a_j, v(t))$ are Δ-measurable and Δ-integrable. For each $t \in \mathbb{T}$ we define the sets

$$T_1(t) = \{t \in \mathbb{T} \mid (p_1, v(t)) \geqslant \delta\}, \quad T_2(t) = \{t \in \mathbb{T} \mid (p_1, v(t)) < \delta\}.$$

Since $(y_j(t), p_1) \leqslant \mu_1$ for all $t \in \mathbb{T}$, $j = 1, 2$, the following inequality holds:

$$\int_{t_0}^{t} (p_1, v(s))\Delta s \leqslant \mu_0 = \min\{\mu_1 - (p_1, y_1^0), \mu_1 - (p_1, y_2^0)\}.$$

Therefore,

$$\mu_0 \geqslant \int_{t_0}^{t} (p_1, v(s))\Delta s \geqslant \delta \int_{T_1(t)} \Delta s - \int_{T_2(t)} \Delta s, \quad K(t) = \int_{T_1(t)} \Delta s + \int_{T_2(t)} \Delta s.$$

The last two relations imply the validity of the inequality

$$\int_{T_2(t)} \Delta s \geqslant \frac{\delta K(t) - \mu_0}{1 + \delta}. \tag{4.4}$$

Next, we have

$$\max_{\Lambda \in \Omega^0(J)} \min_{j \in \Lambda} \int_{t_0}^{t} \lambda(a_j, v(s))\Delta s \geqslant \max_{\Lambda \in \Omega^0(J)} \int_{t_0}^{t} \min_{j \in \Lambda} \lambda(a_j, v(s))\Delta s. \tag{4.5}$$

For any nonnegative numbers $\gamma_\Lambda (\Lambda \in \Omega^0(J))$ we have

$$\max_{\Lambda \in \Omega^0(J)} \gamma_\Lambda \geqslant \frac{1}{N_0} \sum_{\Lambda \in \Omega^0(J)} \gamma_\lambda, \quad \text{where} \quad N_0 = 1 + \frac{(m-2)(m-3)}{2}.$$

Therefore,

$$\max_{\Lambda \in \Omega^0(J)} \int_{t_0}^{t} \min_{j \in \Lambda} \lambda(a_j, v(s))\Delta s \geqslant \frac{1}{N_0} \int_{t_0}^{t} \sum_{\Lambda \in \Omega^0(J)} \min_{j \in \Lambda} \lambda(a_j, v(s))\Delta s$$

$$\geqslant \frac{1}{N_0} \int_{t_0}^{t} \max_{\Lambda \in \Omega^0(J)} \min_{j \in \Lambda} \lambda(a_j, v(s))\Delta s \geqslant \frac{1}{N_0} \int_{T_2(t)} \max_{\Lambda \in \Omega^0(J)} \min_{j \in \Lambda} \lambda(a_j, v(s))\Delta s.$$

Hence, from (4.4) and (4.5) we obtain

$$\max_{\Lambda \in \Omega^0(J)} \min_{j \in \Lambda} \int_{t_0}^{t} \lambda(a_j, v(s))\Delta s \geqslant \frac{\delta}{N_0} \int_{T_2(t)} \Delta s \geqslant \frac{\delta}{N_0} \cdot \frac{\delta K(t) - \mu_0}{1 + \delta}.$$

Since

$$\lim_{t \in \mathbb{T}, t \to +\infty} K(t) = +\infty,$$

it follows from the last inequality that there exists $T_0 \in \mathbb{T}$ for which the following inequality holds:

$$\max_{\Lambda \in \Omega^0(J)} \min_{j \in \Lambda} \int\limits_{t_0}^{T_0} \lambda(a_j, v(s))\Delta s \geqslant 1,$$

which implies the validity of the statement of the lemma. This proves the lemma. □

Lemma 4 [17]. *Let $a_1, \ldots, a_m, p_1 \in \mathbb{R}^k$ be such that for each $q \in J = \{1, \ldots, m\}$ the vectors $\{a_i, i \in J \setminus \{q\}, p_1\}$ form a positive basis $\mathbb{R}^k$. Then*

$$\delta = \min_{v \in V} \max\big\{ \max_{\Lambda \in \Omega(J)} \min_{i \in \Lambda} \lambda(a_i, v), (p_1, v)\big\} > 0.$$

Lemma 5 [17]. *Let $a_1, \ldots a_m, p_1 \in \mathbb{R}^k$ be such that for each $q \in J = \{1, \ldots, m\}$ the vectors $\{a_i, i \in J \setminus \{q\}, p_1\}$ form a positive basis $\mathbb{R}^k$. Then there exists $T_0 > t_0$, $T_0 \in \mathbb{T}$ such that for any admissible control $v(\cdot)$ of evaders there is $\Lambda = (\alpha, \beta) \in \Omega(J)$,*

$$\int\limits_{t_0}^{T_0} \lambda(a_\alpha, v(s))\Delta s \geqslant 1, \quad \int\limits_{t_0}^{T_0} \lambda(a_\beta, v(s))\Delta s \geqslant 1.$$

Theorem 2. *Let $r = 1$ and suppose that there exists $j \in \{1, 2\}$ such that for any $q \in I$*

$$0 \in \operatorname{Intco}\big\{z^0_{ij}, i \in I \setminus \{q\}, p_1\big\}.$$

Then a two-fold capture occurs in the game $\Gamma(n, 2)$.

P r o o f. By virtue of Lemma 5

$$T^0 = \min\Big\{t \in \mathbb{T} \mid t > t_0, \inf_{v(\cdot)} \max_{\Lambda \in \Omega(J)} \min_{i \in \Lambda} \int\limits_{t_0}^{t} \lambda(z^0_{ij}, v(s))\Delta s \geqslant 1\Big\}$$

is finite. Let $v(\cdot)$ be an admissible control of evaders. Define the functions

$$h_i(t) = 1 - \int_{t_0}^{t} \lambda(z^0_{ij}, v(s))\Delta s.$$

Let pursuer P_i construct a control as follows. If the inequality $h_i(t) \geqslant 0$ is satisfied at time $t \in \mathbb{T}$, then we assume

$$u_i(t) = v(t) - \lambda(z^0_{ij}, v(t))z^0_{ij}.$$

If $\tau \in \mathbb{T}$ is the first time instant for which $h_i(\tau) = 0$, we assume that $\lambda(z^0_{ij}, v(t)) = 0$ for all $t \geqslant \tau$.

Let $\tau \in \mathbb{T}$ be the first time instant for which $h_i(\tau) < 0$, and let the inequality $h_i(t) > 0$ be satisfied for all $t \in \mathbb{T}$, $t < \tau$. Define the number

$$\tau^*_i = \sup\{t \in \mathbb{T} \mid h_i(t) > 0\}.$$

Then $(\tau_i^*, \tau) \cap \mathbb{T} = \emptyset$. Indeed, if there existed a time instant $t \in (\tau_i^*, \tau) \cap \mathbb{T}$, then the inequality $h_i(t) > 0$ would be satisfied, which is impossible by virtue of the definition of the number τ_i^*. In this case, we assume

$$u_i(\tau) = v(\tau) - \lambda^*(z_{ij}^0, v(\tau))z_{ij}^0, \quad \text{where} \quad \lambda^*(z_{ij}^0, v(\tau)) = \frac{h_i(\tau_i^*)}{\sigma(\tau_i^*) - \tau_i^*} = \frac{h_i(\tau_i^*)}{\tau - \tau_i^*}.$$

We note that in this case $\lambda^*(z_{ij}^0, v(\tau)) \leqslant \lambda(z_{ij}^0, v(\tau))$ and therefore $u_i(\tau) \in V$. Then

$$1 - \int_{t_0}^{\tau_i^*} \lambda(z_{ij}^0, v(s))\Delta s - \int_{\tau_i^*}^{\tau} \lambda^*(z_{ij}^0, v(s))\Delta s = h_i(\tau_i^*) - \int_{\tau_i^*}^{\tau} \frac{h_i(\tau_i^*)}{\tau - \tau_i^*}\Delta s = 0.$$

Then from the definition of the controls of the pursuers and the system (3.3) it follows that for all $t \in [t_0, T^0] \cap \mathbb{T}$ the equalities $z_{ij}(t) = z_{ij}^0 h_i(t), i \in I$, hold.

From Lemma 5 and the definition of the controls of the pursuers it follows that there exist numbers $l, m \in I$ such that $h_l(T^0) = 0$, $h_m(T^0) = 0$. This implies that pursuers P_l and P_m perform a capture of evader E_j. Consequently, a two-fold capture occurs in the game $\Gamma(n, 2)$. This proves the theorem. □

Theorem 3. *Let $r = 1$ and suppose that there exists a set $I_0 \subset I$, $|I_0| = n - 2$ such that for all $l \in I_0$*

$$0 \in \operatorname{Intco}\left\{z_{i1}^0, z_{i2}^0, i \in I_0 \setminus \{l\}, p_1\right\}. \tag{4.6}$$

Then a two-fold capture occurs in the game $\Gamma(n, 2)$.

P r o o f. By virtue of Theorem 1, it follows from condition (4.6) that for all $l \in I_0$ the set $\{z_{i1}^0, z_{i2}^0, i \in I_0 \setminus \{l\}, p_1\}$ forms a positive basis $\mathbb{R}^k$. Denote $c = y_1^0 - y_2^0$. Since

$$z_{i2}^0 = x_i^0 - y_2^0 = x_i^0 - y_1^0 + c = z_{i1}^0 + c,$$

for all $l \in I_0$ the positive basis $\mathbb{R}^k$ forms a set $\{z_{i1}^0, i \in I_0 \setminus \{l\}, c, p_1\}$.

We assume that $I_0 = \{1, \ldots, n-2\}$. It follows from Lemma 2 that there exists a number $\rho > 0$ such that for all $l \in I$ the vectors $\{w_i^0, i \in I \setminus \{l\}, p_1\}$ form a positive basis $\mathbb{R}^k$, where

$$w_i^0 = \begin{cases} z_{i1}^0, & \text{if} \quad i \in I_0, \\ z_{n-12}^0 + \rho c, & \text{if} \quad i = n-1, \\ z_{n2}^0 + \rho c, & \text{if} \quad i = n. \end{cases}$$

Hence, by virtue of Theorem 1, we find that for all $l \in I$

$$0 \in \operatorname{Intco}\left\{w_i^0, \ i \in I \setminus \{l\}, p_1\right\}.$$

It follows from Lemmas 2 and 3 that the number

$$T_0 = \min\left\{t \,\middle|\, t > t_0, t \in \mathbb{T}, \inf_{v(\cdot)} \max_{\Lambda \in \Omega^0(I)} \min_{j \in \Lambda} \int_{t_0}^{t} \lambda(w_j^0, v(s))\Delta s \geqslant 1\right\}$$

is finite. Let $v(\cdot)$ be an admissible control of the evaders. Define the functions

$$h_i(t) = 1 - \int_{t_0}^{t} \lambda(w_j^0, v(s))\Delta s.$$

Let pursuer P_i construct a control as follows. If the inequality $h_i(t) \geqslant 0$ is satisfied at time $t \in \mathbb{T}$, then we assume

$$u_i(t) = v(t) - \lambda(w_i^0, v(t))w_i^0.$$

If $\tau \in \mathbb{T}$ is the first time instant for which $h_i(\tau) = 0$, then we assume that $\lambda(w_i^0, v(t)) = 0$ for all $t \geqslant \tau$.

Let $\tau \in \mathbb{T}$ be the first time instant for which $h_i(\tau) < 0$, and let the inequality $h_i(t) > 0$ be satisfied for all $t \in \mathbb{T}$, $t < \tau$. Define the number

$$\tau_i^* = \sup\big\{t \in \mathbb{T} \mid h_i(t) > 0\big\}.$$

Then $(\tau_i^*, \tau) \cap \mathbb{T} = \emptyset$. Indeed, if there existed a time instant $t \in (\tau_i^*, \tau) \cap \mathbb{T}$, then the inequality $h_i(t) > 0$ would be satisfied, which is impossible by virtue of the definition of the number τ_i^*. In this case, we assume

$$u_i(\tau) = v(\tau) - \lambda^*(w_i^0, v(\tau))w_i^0, \quad \text{where} \quad \lambda^*(w_i^0, v(\tau)) = \frac{h_i(\tau_i^*)}{\sigma(\tau_i^*) - \tau_i^*} = \frac{h_i(\tau_i^*)}{\tau - \tau_i^*}.$$

We note that in this case $\lambda^*(w_i^0, v(\tau)) \leqslant \lambda(w_i^0, v(\tau))$ and therefore $u_i(\tau) \in V$. Then

$$1 - \int_{t_0}^{\tau_i^*} \lambda(w_i^0, v(s))\Delta s - \int_{\tau_i^*}^{\tau} \lambda^*(w_i^0, v(s))\Delta s = h_i(\tau_i^*) - \int_{\tau_i^*}^{\tau} \frac{h_i(\tau_i^*)}{\tau - \tau_i^*}\Delta s = 0.$$

Then from the definition of the controls of the pursuers and the system (3.3) it follows that for all $t \in [t_0, \hat{T}] \cap \mathbb{T}$ the following equalities hold:

$$z_{i1}(t) = z_{i1}^0 h_i(t), \quad i \in I_0,$$
$$z_{n-12}(t) = z_{n-12}^0 h_{n-1}(t) - \rho c(1 - h_{n-1}(t)),$$
$$z_{n2}(t) = z_{n2}^0 h_n(t) - \rho c(1 - h_n(t)).$$

From Lemma 3 and the definition of the controls of the pursuers it follows that there exist numbers $l, m \in I$ such that

$$h_l(T_0) = 0, \quad h_m(T_0) = 0. \tag{4.7}$$

Also, the following cases are possible.

1. $l, m \in I_0$. In this case, pursuers P_l and P_m perform a capture of evader E_1, which implies that a two-fold capture occurs in the game $\Gamma(n, 2)$.

2. Condition (4.7) is satisfied for $\Lambda = \{n-1, n\}$. Then

$$z_{n-12}(T_0) = -\rho c, \quad z_{n2}(T_0) = -\rho c. \tag{4.8}$$

We prove that in this case the following inclusion holds for any $l \in I_0$:

$$0 \in \operatorname{Intco}\big\{z_{i1}(T_0), z_{i2}(T_0), i \in I_0 \setminus \{l\}, p_1\big\}. \tag{4.9}$$

Let $l \in I_0$. We have

$$z_{i1}(T_0) = z_{i1}^0 h_i(T_0), \quad z_{i2}(T_0) = z_{i1}(T_0) + c = z_{i1}(T_0)h_i(T_0) + z_{i2}^0 - z_{i1}^0.$$

Therefore,

$$z_{i1}^0 = \frac{z_{i1}(T_0)}{h_i(T_0)}, \quad z_{i2}^0 = z_{i2}(T_0) + \frac{z_{i1}(T_0)(1 - h_i(T_0))}{h_i(T_0)}.$$

Since the set $\{z_{i1}^0, z_{i2}^0, i \in I_0 \setminus \{l\}, p_1\}$ forms a positive basis $\mathbb{R}^k$, the positive basis $\mathbb{R}^k$ is formed by the vectors

$$\left\{ \frac{z_{i1}(T_0)}{h_i(T_0)},\ z_{i2}(T_0) + \frac{z_{i1}(T_0)(1 - h_i(T_0))}{h_i(T_0)}, i \in I_0 \setminus \{l\}, p_1 \right\}.$$

From the condition $h_i(T_0) \in (0, 1]$, for all $i \in I_0$ we find that the positive basis $\mathbb{R}^k$ forms a set

$$\{z_{i1}(T_0), z_{i2}(T_0), i \in I_0 \setminus \{l\}, p_1\}.$$

By virtue of Theorem 1, the last relation implies the validity of (4.9).

From equations (4.8) we obtain

$$z_{n-12}(T_0) = -\rho(y_1(T_0) - y_2(T_0)), \quad z_{n2}(T_0) = -\rho(y_1(T_0) - y_2(T_0)).$$

By virtue of Lemma 1, we find that

$$0 \in \operatorname{Intco}\{z_{i2}(T_0), i \in I_0 \setminus \{l\}, p_1\}.$$

Taking T_0 to be the initial time and using Theorem 2, we find that there are pursuers P_r and P_q, $r \neq q$, that perform a capture of evader E_2. This proves the theorem. $\square$

Example 1. Let $k = 2$, $x_1^0 = (3; 1)$, $x_2^0 = (1; -2)$, $x_3^0 = (5; -2)$, $x_4^0 = (1; 3)$, $x_5^0 = (2; -3)$, $y_1^0 = (0; 0)$, $y_2^0 = (6; 0)$, $p_1 = (0; 1)$, $\mu_1 = 100$.

Then the condition for capture from Theorem 2 is not satisfied, and the condition for capture from Theorem 3 is satisfied for $I_0 = \{1, 2, 3\}$.

Theorem 4. *Let $D = \mathbb{R}^k$ and suppose that there exists $j \in \{1, 2\}$ such that for any $q \in I$*

$$0 \in \operatorname{Intco}\{z_{ij}^0, i \in I \setminus \{q\}\}.$$

Then a two-fold capture occurs in the game $\Gamma(n, 2)$.

This theorem is proved along the same lines as Theorem 2 using the results of [16].

Theorem 5. *Let $D = \mathbb{R}^k$ and suppose that there exists a set $I_0 \subset I$, $|I_0| = n - 2$ such that for all $l \in I_0$*

$$0 \in \operatorname{Intco}\{z_{i1}^0, z_{i2}^0, i \in I_0 \setminus \{l\}\}.$$

Then a two-fold capture occurs in the game $\Gamma(n, 2)$.

This theorem is proved along the same lines as Theorem 3 using the results of [16].

Theorem 6. *Let $r > 1$ and suppose that there exist $p \in \mathbb{R}^k$, $\mu \in \mathbb{R}^1$, $I_0 \subset I$, $|I_0| = n - 2$ such that $D \subset \{x \in \mathbb{R}^k \mid (p, x) \leqslant \mu\}$ and*

$$0 \in \operatorname{Intco}\{z_{i1}^0, z_{i2}^0, i \in I_0 \setminus \{l\}, p\}.$$

Then a two-fold capture occurs in the game $\Gamma(n, 2)$.

The validity of this theorem immediately follows from Theorem 3.

5. Conclusion

In the problem of a simple pursuit by a group of pursuers of two coordinated evaders on a given time scale, we obtained sufficient conditions for a two-fold capture, provided that the evaders didn't move out of a convex polyhedral set. To solve the problem, we used the method of resolving functions. The results obtained can be used in the study of new problems of conflict interaction between groups of pursuers and evaders on time scales.

REFERENCES

1. Aulbach B., Hilger S. Linear dynamic processes with inhomogeneous time scale. In: *Nonlinear Dynamics and Quantum Dynamical Systems: Contributions to the Int. Seminar ISAM-90, held in Gaussig (GDR), March 19–23, 1990.* Ser. Math. Res., vol. 59. Berlin, Boston: De Gruyter, 1990. P. 9–20. DOI: 10.1515/9783112581445-002
2. Benchohra M., Henderson J., Ntouyas S. *Impulsive Differential Equations and Inclusions.* New York: Hindawi Publishing, 2006. 380 p.
3. Blagodatskikh A. I., Petrov N. N. *Konfliktnoe vzaimodeistvie grupp upravlyaemykh ob"ektov* [Conflict Interaction of Groups of Controlled Objects]. Izhevsk: Udmurt State University, 2009. 266 p. (in Russian)
4. Bohner M., Peterson A. *Advances in Dynamic Equations on Time Scales.* Boston: Birkhäuser, 2003. 348 p. DOI: 10.1007/978-0-8176-8230-9
5. Cabada A., Vivero D. R. Expression of the Lebesgue Δ-integral on time scales as a usual Lebesgue integral; application to the calculus of Δ-antiderivatives. *Math. Comput. Model.*, 2006. Vol. 43, No. 1–2. P. 194–207. DOI: 10.1016/j.mcm.2005.09.028
6. Chikrii A. A. *Conflict-Controlled Processes.* Dordrecht, NL: Kluwer Academic Publishers, 1997. 424 p. DOI: 10.1007/978-94-017-1135-7
7. Grigorenko N. L. *Matematicheskie metody upravleniya neskol'kimi dinamicheskimi protsessami* [Mathematical methods of control over multiple dynamic processes]. Moscow: Moscow State University, 1990, 197 p. (in Russian)
8. Guseinov G. S. Integration on time scales. *J. Math. Anal. Appl.*, 2003. Vol. 285, No. 1. P. 107–127. DOI: 10.1016/S0022-247X(03)00361-5
9. Hilger S. Analysis on measure chains – a unified approach to continuous and discrete calculus. *Results in Mathematics,* 1990. Vol. 18. P. 18–56. DOI: 10.1007/BF03323153
10. Krasovskii N. N., Subbotin A. I. *Pozitsionnye differentsial'nye igry* [Positional Differential Games]. Moscow: Nauka, 1974. 456 p. (in Russian)
11. Martins N., Torres D. Necessary conditions for linear noncooperative N-player delta differential games on time scales. *Discuss. Math. Differ. Incl. Control Optim.*, 2011. Vol. 31, No. 1. P. 23–37. DOI: 10.7151/dmdico.1126
12. Petrov N. N. Controllability of autonomous systems. *Differ. Uravn.*, 1968. Vol. 4, No. 4. P. 606–617. URL: https://mi.mathnet.ru/eng/de328 (in Russian)
13. Petrov N. N. Multiple capture of a given number of evaders in the problem of simple pursuit with phase restrictions on timescales. *Dyn. Games Appl.*, 2022. Vol. 12, No. 2. P. 632–642. DOI: 10.1007/s13235-021-00387-y
14. Petrov N. N. On a problem of pursuing a group of evaders in time scales. *Trudy Inst. Mat. i Mekh. UrO RAN*, 2021. Vol. 27, No. 3. P. 163–171. DOI: 10.21538/0134-4889-2021-27-3-163-171 (in Russian)
15. Petrov N. N. The problem of simple group pursuit with phase constraints in time scales. *Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki*, 2020. Vol. 30, No. 2. P. 249–258. DOI: 10.35634/vm200208 (in Russian)
16. Petrov N. N., Mozhegova E. S. On a simple pursuit problem on time scales of two coordinated evaders. *Chelyabinsk Phys. Math. J.*, 2022. Vol. 7, No. 3. P. 277–286. DOI: 10.47475/2500-0101-2022-17302 (in Russian)
17. Petrov N. N., Mozhegova E. S. Simple pursuit problem with phase restrictions of two coordinated evaders on time scales. *Mat. Teor. Igr Pril.*, 2022. Vol. 14, No. 4. P. 81–95. URL: https://mi.mathnet.ru/eng/mgta314 (in Russian)

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 123–135

DOI: 10.15826/umj.2024.1.011

ALPHA LABELINGS OF DISJOINT UNION OF HAIRY CYCLES

G. Rajasekaran[†] **and L. Uma**

Department of Mathematics, Vellore Institute of Technology,
Vellore – 632014, Tamil Nadu, India

[†]rajasekaran.ganapathy@vit.ac.in

Abstract: In this paper, we prove the following results: (1) the disjoint union of $n \geq 2$ isomorphic copies of a graph obtained by adding a pendant edge to each vertex of a cycle of order 4 admits an α-valuation; (2) the disjoint union of two isomorphic copies of a graph obtained by adding $n \geq 1$ pendant edges to each vertex of a cycle of order 4 admits an α-valuation; (3) the disjoint union of two isomorphic copies of a graph obtained by adding a pendant edge to each vertex of a cycle of order $4m$ admits an α-valuation; (4) the disjoint union of two nonisomorphic copies of a graph obtained by adding a pendant edge to each vertex of cycles of order $4m$ and $4m-2$ admits an α-valuation; (5) the disjoint union of two isomorphic copies of a graph obtained by adding a pendant edge to each vertex of a cycle of order $4m-1$ $(4m+2)$ admits a graceful valuation (an α-valuation), respectively.

Keywords: Hairy cycles, Graceful valuation, α-valuation.

1. Introduction

Notation and terminology not defined here can be found in [2]. Throughout this paper, we denote by S_n and C_n a star on $n+1$ vertices and a cycle on n vertices, respectively.

If a labeling f on a graph G with p edges is a one-to-one function from the set of vertices of G to the set $\{0, 1, \ldots, p\}$ such that, for p pairs of adjacent vertices x and y, the values $|f(x)-f(y)|$ are distinct, then f is called a *graceful valuation* (a β-labeling or a β-valuation) of G. If, in addition, there exists an integer ℓ such that, for each edge $xy \in E(G)$, one of the values $f(x)$ and $f(y)$ does not exceed ℓ and the other is strictly greater than ℓ, then the labeling f is called an α-valuation of G with *critical value* ℓ. Note that a graph with an α-valuation is necessarily bipartite. As a result, such ℓ must be smaller than the smallest of the two vertex labels that yield the edge labeled 1. Let $\{A, B\}$ be stable sets (a partition) of vertices with $x \in A$ and $y \in B$. Without loss of generality, assume that

$$A = \{x \in V(G) : f(x) \leq \ell\}, \quad B = \{y \in V(G) : f(y) > \ell\}.$$

Clearly, every α-valuation is also a graceful labeling but not conversely. Rosa pioneered in 1966 [21] the concept of graph β-labeling. He also presented certain types of vertex labeling as an important tool for decomposing the complete graph K_{2p+1} into graphs with p edges.

Theorem 1 [21]. *Let a graph G with p edges has an α-valuation. Then, for $s \in \mathbb{N}$, there exists a G-decomposition of the complete graph K_{2ps+1}.*

Specifically, β-valuations were developed to challenge Ringel's conjecture [19] that K_{2n+1} can be decomposed into $2n+1$ subgraphs that are all isomorphic to a given tree with n edges. More results about graph labeling are collected and updated regularly in the survey by Gallian [9].

The *disjoint union of graphs* $H_1 = (V_1, E_1), H_2 = (V_2, E_2), \ldots, H_n = (V_n, E_n)$ is a graph $H = H_1 \cup H_2 \cup \cdots \cup H_n$ with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_n$ and edge set $E = E_1 \cup E_2 \cup \cdots \cup E_n$, where $V_1 \cap V_2 \cap \cdots \cap V_m = \varnothing$. Lakshmi and Vangipuram [13] proved that there is an α-valuation for the quadratic graph $Q(4, 4k)$ consisting of four cycles of length $4k$, $k \geq 1$. Abrham and Kotzig [1] proved that $C_m \cup C_n$ has an α-valuation if and only if both m and n are even and $m+n \equiv 0 \pmod 4$. Eshghi and Carter [6] showed several families of graphs of the form $C_{4_{n_1}} \cup C_{4_{n_2}} \cup \cdots \cup C_{4_{n_k}}$ that have α-valuations.

The *cartesian product* $G \square H$ of two graphs G and H is the graph with the vertex set

$$V(G \square H) = V(G) \square V(H)$$

and the edge set $E(G \square H)$ satisfying the following condition:

$$(x_1, x_2)(y_1, y_2) \in E(G \square H)$$

if and only if either $x_1 = y_1$ and $x_2 y_2 \in E(H)$ or $x_2 = y_2$ and $x_1 y_1 \in E(G)$.

The *corona* [7] of two graphs H_1 and H_2, denoted by $H_1 \odot H_2$, is the graph obtained by taking one copy of H_1, which has m vertices, and m copies of H_2, and then joining the kth vertex of H_1 with an edge to every vertex in the kth copy of H_2.

A unicyclic graph H (other than a cycle) is called a *hairy cycle* if the deletion of any edge e from the cycle of H results in a caterpillar. Thus, the coronas $C_n \odot mK_1$ are examples of hairy cycles. Kumar et. al. [11, 12, 15–18] proved that the hairy cycle $C_n \odot K_1$, $n \equiv 0 \pmod 4$, and graphs obtained by joining two graceful cycles by a path admit α-valuations. They also discussed that the subdivision of a cycle and pendant edges of $C_n \square K_4$, joining two isomorphic copies of $C_n \square K_4$, $C_n \odot K_1$, $n \equiv 0 \pmod 4$, and $C_n \odot K_1$, $n \equiv 3 \pmod 4$, are graceful. Moreover, they proved that $C_n \odot rK_1$, $n \equiv 3 \pmod 4$, and $C_n \odot K_1, n \equiv 0 \pmod 4$, are k-graceful. Graf [10] established that $C_n \odot K_1$ has a graceful valuation if $n \equiv 3$ or $4 \pmod 8$.

Barrientos [3, 4] showed that if G is a graceful graph with order greater than its size, then the graphs $G \odot nK_1$ and $G + nK_1$ are graceful. He also proved that helms (graphs obtained from a wheel by attaching one pendant edge to each vertex) are graceful. Minion and Barrientos, in [5] and [14], studied the gracefulness of $G \cup P_m$ and $C_r \cup G_n$, where G_n is a caterpillar of size n. Frucht and Salinas [8] analyzed the gracefulness of $C_m \cup P_n$, $n \geq 3$. Ropp [20] showed that the graph $(C_m \square P_2) \odot K_1$ is graceful. Truszczynski [22] conjectured that all unicyclic graphs except the cycle C_n, $n \equiv 1$ or $2 \pmod 4$, are graceful.

Labeled graphs are helpful mathematical models for coding theory, such as designing optimal radar, synch-set, missile guidance, and convolution codes with high auto-correlation. They make it easier to perform optimal nonstandard integer encoding.

This study focuses on graceful and α-valuation of some disconnected graphs. The concept of graceful and α-valuation in graph theory has attracted attention from many researchers during the past three decades. The earlier studies motivated us to research the problem that *the disjoint union of various hairy cycles* $C_{m_1}^{S_n} \cup C_{m_2}^{S_n} \cup \cdots \cup C_{m_k}^{S_n}$ *admits an α-valuation*, which we partially solve in the present paper.

2. Results

Theorem 2. *Let G be the graph obtained by the disjoint union of n isomorphic copies of the hairy cycle $C_4^{S_1}$. Then, G admits an α-valuation with exactly one missing number $\rho = 4n - 2$ and the critical value $\sigma = 4n$.*

Proof. Let $n \in \mathbb{N}$, and let G^k, $1 \leq k \leq n$, be the kth part of G. Let w_i^k and x_i^k, where $i = 1, 2, 3, 4$ and $k = 1, 2, \ldots, n$, denote vertices of the cycle and leaves of the kth part of G, respectively. Clearly, $|V(G)| = |E(G)| = 8n$.

To define $\Im : V(G) \to \{0,1,2,\dots,8n\}$, we label the vertices of G^1 as follows:

$$\Im(w_1^1)=0,\quad \Im(w_2^1)=8n-1,\quad \Im(w_3^1)=3,\quad \Im(w_4^1)=8n-3,$$
$$\Im(x_1^1)=8n,\quad \Im(x_2^1)=1,\quad \Im(x_3^1)=8n-2,\quad \Im(x_4^1)=4.$$

Next, we label the vertices of the remaining parts of G, $2\le k\le n$, as follows:

$$\Im(w_i^k)=\begin{cases} 4k+3(i-3)/2 & \text{if} \quad i=1,3,\\ 4(2n-k+1)-i/2 & \text{if} \quad i=2,4,\end{cases}$$
$$\Im(x_i^k)=\begin{cases} 4(k+1)-5i/2 & \text{if} \quad i=2,4,\\ 8n-4k+(11-3i)/2 & \text{if} \quad i=1,3.\end{cases}$$

Define the edge labeling $f^\star$ on $E(G^1)$ by

$$f^\star(wx)=|\Im(w)-\Im(x)|$$

for $wx\in E(G)$ as follows:

$$f^\star(w_1^1w_2^1)=8n-1,\quad f^\star(w_2^1w_3^1)=8n-4,\quad f^\star(w_3^1w_4^1)=8n-6,\quad f^\star(w_4^1w_1^1)=8n-3,$$
$$f^\star(w_1^1x_1^1)=8n,\quad f^\star(w_2^1x_2^1)=8n-2,\quad f^\star(w_3^1x_3^1)=8n-5,\quad f^\star(w_4^1x_4^1)=8n-7.$$

We label the remaining edges of G as follows:

$$f^\star(w_i^kw_{i+1}^k)=8(n-k+1)-2i \quad \text{for} \quad i=1,3,$$
$$f^\star(w_2^kw_3^k)=8(n-k)+3,$$
$$f^\star(w_4^kw_1^k)=8(n-k)+5,$$
$$f^\star(w_i^kx_i^k)=8(n-k)+10-3i \quad \text{for} \quad i=1,2,3,$$
$$f^\star(w_4^kx_4^k)=8(n-k+1).$$

It is clear that all the vertex and edge labels are distinct. Therefore, the graph G is graceful. Next, we prove that the graceful function $\Im$ is an α-valuation with the missing number $\rho=4n-2$ and the critical value $\sigma=4n$. Since the vertex set V of G is partitioned into two sets, $V=A\cup B$, we have

$$A=\{0,1,3,4,5,7,8,2,9,11,12,6,\dots,4n-3,4n-1,4n,4n-6\},$$
$$B=\{8n,8n-1,8n-2,8n-3,\dots,4n+1\}.$$

Clearly, A and B are independent sets. The number $\sigma=4n$ satisfies $f^\star(w)\le\sigma<f^\star(x)$ for every ordered pair $(w,x)\in A\times B$. Therefore, $\Im$ is an α-valuation of G (see $C_4^{S_1}\cup C_4^{S_1}\cup C_4^{S_1}$ in Fig. 1). □

Theorem 3. *The disjoint union of two isomorphic copies of $C_4^{S_n}$ admits an α-valuation with exactly one missing number $\rho=4(n+1)$ and the critical value $\sigma=4n+5$.*

P r o o f. Let $i=1,2,3,4$ and $j=1,2,\dots,n$. Denote by u_i (v_i) and u_{ij} (v_{ij}) the vertices of the cycle and leaves, respectively, in the first and second copies of $C_4^{S_n}$, respectively. Clearly,

$$|V(C_4^{S_n}\cup C_4^{S_n})|=|E(C_4^{S_n}\cup C_4^{S_n})|=8(n+1).$$

Define $\vartheta : V(C_4^{S_n}\cup C_4^{S_n})\to\{0,1,2,\dots,8(n+1)\}$ as follows: we label the vertices of the cycle of the first copy of $C_4^{S_n}$ by

$$\vartheta(u_1)=0,\quad \vartheta(u_2)=7n+8,\quad \vartheta(u_3)=n+2,\quad \vartheta(u_4)=6n+7$$

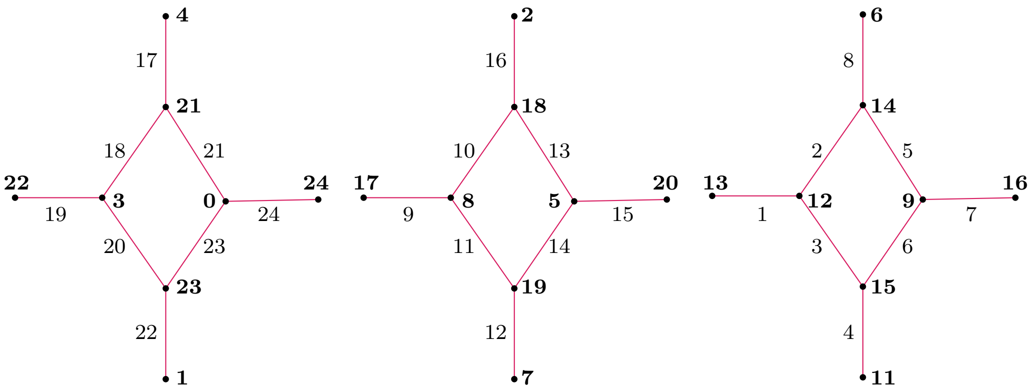


Figure 1. An α-valuation of $C_4^{S_1} \cup C_4^{S_1} \cup C_4^{S_1}$.

and the vertices of the cycle of the second copy of $C_4^{S_n}$ by

$$\vartheta(v_1) = 6(n+1), \quad \vartheta(v_2) = 3(n+1), \quad \vartheta(v_3) = 5(n+1), \quad \vartheta(v_4) = 4n+5,$$

respectively. Label the remaining vertices of the leaves in the graph $C_4^{S_n} \cup C_4^{S_n}$ as follows:

$$\vartheta(u_{ik}) = \begin{cases} 8(n+1) - (k-1) - \dfrac{(n+1)(i-1)}{2} & \text{if} \quad i = 1, 3, \\ k + \dfrac{(n+2)(i-2)}{2} & \text{if} \quad i = 2, 4, \end{cases}$$

$$\vartheta(v_{1k}) = 2(n+1), \quad \vartheta(v_{2k}) = 6(n+1) - k$$

for $1 \le k \le n$, and

$$\vartheta(v_{3k}) = 3n + k + 4, \quad \vartheta(v_{4k}) = 5(n+1) - k \quad \text{for} \quad 1 \le k < n,$$

$$\vartheta(v_{3n}) = n+1, \quad \vartheta(v_{4n}) = 3n+4.$$

It can be verified that all vertices of the graph are labeled and the labels are distinct. Now, we construct labels for the edge set E of the graph. Define a labeling f on $E(C_4^{S_n} \cup C_4^{S_n})$ by $f(uv) = |\vartheta(u) - \vartheta(v)|$ for $uv \in E$. We label the edges of the cycle of the first copy of $C_4^{S_n}$ by

$$f(u_1u_2) = 7n+8, \quad f(u_2u_3) = 6(n+1), \quad f(u_3u_4) = 5(n+1), \quad f(u_4u_1) = 6n+7$$

and the edges of the cycle of the second copy of $C_4^{S_n}$ by

$$f(v_1v_2) = 3(n+1), \quad f(v_2v_3) = 2(n+1), \quad f(v_3v_4) = n, \quad f(v_4v_1) = 2n+1.$$

Label the remaining edges of the leaves in the graph $C_4^{S_n} \cup C_4^{S_n}$ as follows:

$$f(u_iu_{ik}) = \begin{cases} 8(n+1) - (k-1) - (n+1)(i-1) & \text{if} \quad i = 1, 2, \\ 8(n+1) - k - (n+1)(i-1) & \text{if} \quad i = 3, 4, \end{cases}$$

for $1 \le k \le n$,

$$f(v_iv_{ik}) = \begin{cases} 4(n+1) - k - (n+1)(i-1) & \text{if} \quad i = 1, 2, \\ 4(n+1) - k - 1 - (n+1)(i-1) & \text{if} \quad i = 3, 4, \end{cases}$$

for $1 \leq k < n$, and

$$f(v_3 v_{3n}) = 4(n+1), \quad f(v_4 v_{4n}) = n+1.$$

It is quite clear that all the vertex and edge labels are distinct. Therefore, the graph $C_4^{S_n} \cup C_4^{S_n}$ is graceful. Next, we prove that this graceful function ϑ is an α-valuation with the missing number $\rho = 4(n+1)$ and the critical value $\sigma = 4n+5$. Since the vertex set V of $C_4^{S_n} \cup C_4^{S_n}$ is partitioned into two sets, $V = R \cup S$, we have

$$R = \{0, 1, 2, \ldots, n, n+2, n+3, \ldots, 2(n+1), 6(n+1), 6n+5, 6n+4, \\ \ldots, 5n+6, 2(2n+3), 4n+5, \ldots, 5n+4, 3n+4\}$$

and

$$S = \{8(n+1), 8n+7, 8n+6, \ldots, 7n+9, 7n+8, 7(n+1), \ldots, 2(3n+4), 2n+3, 2n+4, \\ \ldots, 3n+2, 3(n+1), 3n+5, 3n+6, \ldots, 4n+3, n+1\}.$$

Clearly, R and S are independent sets. The number $\sigma = 4n+5$ satisfies $f(u) \leq \sigma < f(v)$ for every ordered pair $(u, v) \in R \times S$. Therefore, ϑ is an α-valuation of $C_4^{S_n} \cup C_4^{S_n}$ (see $C_4^{S_3} \cup C_4^{S_3}$ in Fig. 2). □

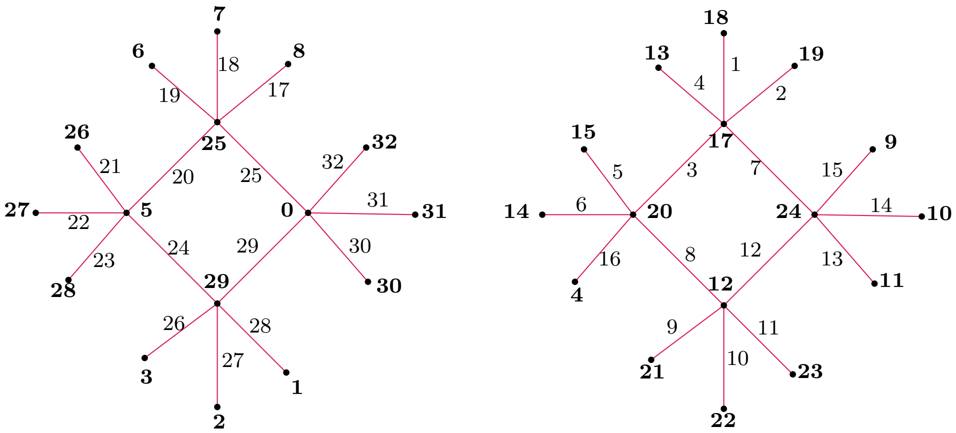


Figure 2. An α-valuation of $C_4^{S_3} \cup C_4^{S_3}$.

Theorem 4. *Let $C_{4m}^{S_1} \cup C_n^{S_1}$, $n \in \{4m, 4m-1, 4m-2\}$, be the disjoint union of hairy cycles. If there is a function*

$$\phi : V(C_{4m}^{S_1} \cup C_n^{S_1}) \to \{0, 1, 2, \ldots, 2(4m+n)\},$$

then

(i) *the graph $C_{4m}^{S_1} \cup C_n^{S_1}$, $n \in \{4m, 4m-2\}$, admits an α-valuation;*

(ii) *the graph $C_{4m}^{S_1} \cup C_{4m-1}^{S_1}$, $m \geq 1$, admits a graceful valuation.*

P r o o f. Let $C_{4m}^{S_1}$ be the graph (a hairy cycle) obtained by adding a pendant vertex to each vertex of the cycle of order $4m$. To prove this theorem, we need to prove the following claims.

Claim 1. The graph $C_{4m}^{S_1} \cup C_{4n}^{S_1}$, $m = n$ and $m \geq 1$, admits an α-valuation.

Claim 2. The graph $C_{4m}^{S_1} \cup C_{4m-2}^{S_1}$ admits an α-valuation.

Claim 3. The graph $C_{4m}^{S_1} \cup C_{4m-1}^{S_1}$, $m \geq 1$, admits a graceful valuation.

Before proving the claims, we fix a labeling of the hairy cycle $C_{4m}^{S_1}$ in $C_{4m}^{S_1} \cup C_n^{S_1}$. Because, throughout the proof, the labeling of the first part $C_{4m}^{S_1}$ is the same. For each $t \in \{4m, 4m-1, 4m-2\}$, we need to define a labeling ϕ on $C_{4m}^{S_1} \cup C_t^{S_1}$ and prove that this is an α-labeling. So, first, we define the labeling of the first part of the union as follows.

Let u_i and v_i, $i = 1, 2, \ldots, 4m$, be the vertices of the cycle and leaves of $C_{4m}^{S_1}$, respectively. Then,

$$\phi(u_i) = \begin{cases} i-1 & \text{if } i \leq 2m \text{ and } i \text{ is odd,} \\ i & \text{if } i > 2m \text{ and } i \text{ is odd,} \\ 2(4m+t)-(i-1) & \text{if } i \text{ is even,} \end{cases}$$

$$\phi(v_i) = \begin{cases} 2(4m+t)-(i-1) & \text{if } i \text{ is odd,} \\ i-1 & \text{if } i \leq 2m \text{ and } i \text{ is even,} \\ i & \text{if } i > 2m \text{ and } i \text{ is even.} \end{cases}$$

Next, we define the edge labeling g on the edges of $C_{4m}^{S_1}$ by

$$g(uv) = |\phi(u) - \phi(v)|$$

for $uv \in E$ as follows:

$$g(u_i u_{i+1}) = \begin{cases} 2(4m+t)-2i+1 & \text{if } i < 2m, \\ 2(4m+t)-2i & \text{if } 2m \leq i < 4m, \end{cases}$$

$$g(u_{4m}, u_1) = 4m + 2t + 1,$$

$$g(u_i v_i) = \begin{cases} 2(4m+t-i+1) & \text{if } i \leq 2m, \\ 2(4m+t)-2(i-1)-1 & \text{if } i > 2m. \end{cases}$$

Proof of Claim 1. This claim holds for only $m = n$.

Let x_i and y_i, $i = 1, 2, \ldots, 4n$, be the vertices of the cycle and leaves of $C_{4n}^{S_1}$ (the second part). Clearly,

$$|V(C_{4m}^{S_1} \cup C_{4m}^{S_1})| = |E(C_{4m}^{S_1} \cup C_{4m}^{S_1})| = 16m.$$

We now define the labeling of $C_{4m}^{S_1}$ as follows.

Case 1: *m is even.* We label the vertices by $\phi(x_{4m}) = 10m$,

$$\phi(x_i) = \begin{cases} 4m+i & \text{if } i \text{ is odd,} \\ 12m+1-i & \text{if } i \leq m \text{ and } i \text{ is even,} \\ 12m-i & \text{if } m < i < 2m \text{ and } i \text{ is even,} \\ 12m-1-i & \text{if } 2m \leq i \leq 4m-2 \text{ and } i \text{ is even,} \end{cases}$$

$$\phi(y_{3m-1}) = 11m, \quad \phi(y_{4m}) = 2m,$$

$$\phi(y_i) = \begin{cases} 12m-(i-1) & \text{if } i < m \text{ and } i \text{ is odd,} \\ 12m-i & \text{if } m \leq i < 2m \text{ and } i \text{ is odd,} \\ 12m-1-i & \text{if } 2m < i < 3m-1, \quad 3m-1 < i \leq 4m-1, \quad \text{and } i \text{ is odd,} \\ 4m+i & \text{if } i \leq 4m-2 \text{ and } i \text{ is even.} \end{cases}$$

It can be verified that all vertices of the graph are labeled and all labels are distinct. Label the set

E of edges in the graph as follows:

$$g(x_i x_{i+1}) = \begin{cases} 2(4m-i) & \text{if} \quad i \le m, \\ 2(4m-i)-1 & \text{if} \quad m < i < 2m-1, \\ 2(4m-i)-2 & \text{if} \quad 2m-1 \le i \le 4m-2, \end{cases}$$

$$g(x_{4m-1}x_{4m}) = 2m+1, \quad g(x_{4m}x_1) = 6m-1,$$

$$g(x_i y_i) = \begin{cases} 2(4m-i)+1 & \text{if} \quad i \le m, \\ 2(4m-i) & \text{if} \quad m < i < 2m, \\ 2(4m-i)-1 & \text{if} \quad 2m \le i < 3m-1, \quad 3m-1 < i \le 4m-1. \end{cases}$$

$$g(x_{3m-1}y_{3m-1}) = 4m+1, \quad g(x_{4m}y_{4m}) = 8m.$$

Suppose that $m = n$ and m is even. Then, the labeling of $C_{4m}^{S_1} \cup C_{4n}^{S_1}$ is a graceful valuation. Moreover, the labeling of $C_{4m}^{S_1} \cup C_{4m}^{S_1}$ is actually an α-valuation with the critical value $2m-1$, and the number $9m/4$ is not assigned to any vertex of $C_{4m}^{S_1} \cup C_{4m}^{S_1}$.

Case 2: *m is odd.* If $m = 1$, the labeling follows from Theorem 2. If $m \ge 2$, the labeling is defined as follows:

$$\phi(w_{3m}) = 7m+2, \quad \phi(w_{4m}) = 10m,$$

$$\phi(w_i) = \begin{cases} 4m+i & \text{if} \quad i \le m \quad \text{and} \quad i \quad \text{is odd,} \\ 4m+1+i & \text{if} \quad m < i < 3m, \quad 3m < i \le 4m-1, \quad \text{and} \quad i \quad \text{is odd,} \\ 12m+1-i & \text{if} \quad i \le 2m \quad \text{and} \quad i \quad \text{is even,} \\ 12m-i & \text{if} \quad 2m < i \le 4m-2 \quad \text{and} \quad i \quad \text{is even,} \end{cases}$$

$$\phi(z_{3m+1}) = 5m+1, \quad \phi(z_{4m}) = 2m,$$

$$\phi(z_i) = \begin{cases} 12m+1-i & \text{if} \quad i < 2m \quad \text{and} \quad i \quad \text{is odd,} \\ 12m-i & \text{if} \quad 2m < i \le 4m-1 \quad \text{and} \quad i \quad \text{is odd,} \\ 4m+i & \text{if} \quad i < m \quad \text{and} \quad i \quad \text{is even,} \\ 4m+1+i & \text{if} \quad m < i < 3m, \quad 3m+1 < i \le 4m-2, \quad \text{and} \quad i \quad \text{is even.} \end{cases}$$

It can be verified that all the vertices of the graph are labeled and the labels are distinct. We now construct labels for the set E of edges in the graph as follows:

$$g(w_{3m-1}w_{3m}) = 2m-1, \quad g(w_{3m}w_{3m+1}) = 2m-3, \quad g(w_{4m-1}w_{4m}) = 2m, \quad g(w_{4m}w_1) = 6m-1,$$

$$g(w_i w_{i+1}) = \begin{cases} 2(4m-i) & \text{if} \quad i \le m, \\ 2(4m-i)-1 & \text{if} \quad m < i \le 2m, \\ 2(4m-i)-2 & \text{if} \quad 2m < i < 3m-1, \quad 3m < i \le 4m-2, \end{cases}$$

$$g(w_{3m}z_{3m}) = 2(m-1), \quad g(w_{3m+1}z_{3m+1}) = 4m-2, \quad g(w_{4m}z_{4m}) = 8m,$$

$$g(w_i z_i) = \begin{cases} 2(4m-i)+1 & \text{if} \quad i \le m, \\ 2(4m-i) & \text{if} \quad m < i \le 2m, \\ 2(4m-i)-1 & \text{if} \quad 2m < i < 3m, \quad 3m+1 < i \le 4m-1. \end{cases}$$

Through the close examination of the above function ϕ, it can be seen that the induced edge labeling is bijective. It is clear that all the vertex labels are distinct. The edge labels are computed from these vertex labels and are also found to be distinct from 1 to $16m$. Therefore, $C_{4m}^{S_1} \cup C_{4n}^{S_1}$, where m is odd and $m = n$, is a graceful valuation. Moreover, the labeling of $C_{4m}^{S_1} \cup C_{4m}^{S_1}$ is actually an α-valuation with the critical value $2m$, and the number $7m + 4/4$ is not assigned to any vertex of $C_{4m}^{S_1} \cup C_{4m}^{S_1}$. This completes the proof of Claim 1.

Proof of Claim 2. Let a_i and b_i, $i = 1, 2, \dots, 4m-2$, be the vertices of the cycle and leaves of $C_{4m-2}^{S_1}$. Define the labeling of $C_{4m-2}^{S_1}$ as follows:

$$\phi(a_{2m}) = 6m+1,$$

$$\phi(a_i) = \begin{cases} 3(4m-1)-i & \text{if} \quad i \quad \text{is odd}, \\ 4m+i & \text{if} \quad 1 < i \leq 2m-1, \quad 2m < i \leq 4m-2, \quad \text{and} \quad i \quad \text{is even}, \end{cases}$$

$$\phi(b_{2m+1}) = 2m,$$

$$\phi(b_i) = \begin{cases} 4m+i & \text{if} \quad 1 \leq i \leq 2m-1, \quad 2m+1 < i < 4m-2, \quad \text{and} \quad i \quad \text{is odd}, \\ 3(4m-1)-i & \text{if} \quad i \quad \text{is even}. \end{cases}$$

Moreover, this produces the edge labels of $C_{4m-2}^{S_1}$:

$$g(a_{4m-2}a_1) = 4m-2, \quad g(a_{2m}b_{2m}) = 4(m-1), \quad g(a_{2m+1}b_{2m+1}) = 4m-2,$$

$$g(a_i a_{i+1}) = \begin{cases} 2(4m-2-i) & \text{if} \quad i < 2m-1, \quad 2m < i < 4m-2, \\ 8m-5-2i & \text{if} \quad 2m-1 \leq i \leq 2m, \end{cases}$$

$$g(a_i b_i) = 8m-3-2i \quad \text{if} \quad i \leq 2m-1, \quad i > 2m+1.$$

Through these combined labelings of the hairy cycles $C_{4m}^{S_1}$ (defined before Claim 1) and $C_{4m-2}^{S_1}$ bring out the labeling of $C_{4m}^{S_1} \cup C_{4m-2}^{S_1}$, and its induced edge labeling is bijective. It is clear that all the vertex labels are distinct. The edge labels are computed from these vertex labels and are also found to be distinct from 1 to $16m-4$. Therefore, $C_{4m}^{S_1} \cup C_{4m-2}^{S_1}$ is a graceful valuation. Moreover, the labeling of $C_{4m}^{S_1} \cup C_{4m-2}^{S_1}$ is actually an α-valuation with critical value $5m-2$, and the number $3m/2$ is not a label of $C_{4m}^{S_1} \cup C_{4m-2}^{S_1}$ (see $C_8^{S_1} \cup C_6^{S_1}$ in Fig. 3). This completes the proof of Claim 2.

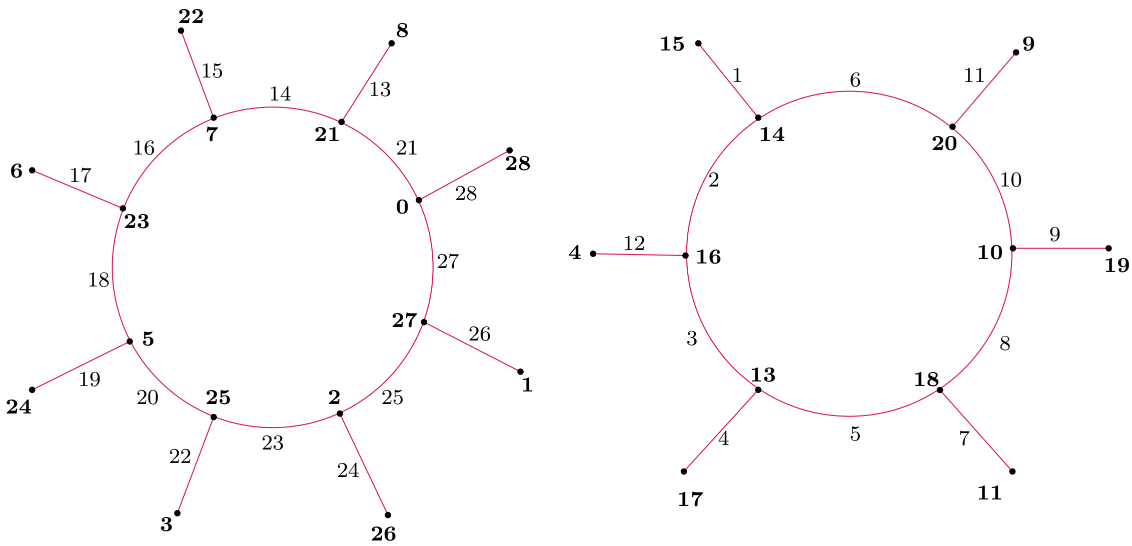


Figure 3. An α-valuation of $C_8^{S_1} \cup C_6^{S_1}$.

Proof of Claim 3. Let w_i and z_i, $i = 1, 2, \dots, 4m-1$, be the vertices of the cycles and leaves of $C_{4m-1}^{S_1}$. Clearly,

$$|V(C_{4m}^{S_1} \cup C_{4m-1}^{S_1})| = |E(C_{4m}^{S_1} \cup C_{4m-1}^{S_1})| = 16m-2.$$

Since the labeling of $C_{4m}^{S_1}$ is defined at the beginning of the proof, we only need to specify a labeling of $C_{4m-1}^{S_1}$ and $C_{4m-1}^{S_1}$, and we do this as follows.

Case 1: *m is even.* Define

$$\phi(x_{4m-1}) = 2(5m-1),$$

$$\phi(x_i) = \begin{cases} 4m+i & \text{if } \ i \le 4m-3 \ \text{ and } \ i \ \text{ is odd,} \\ 12m-1-i & \text{if } \ i \le m \ \text{ and } \ i \ \text{ is even,} \\ 2(6m-1)-i & \text{if } \ m < i < 2m \ \text{ and } \ i \ \text{ is even,} \\ 3(4m-1)-i & \text{if } \ i \ge 2m \ \text{ and } \ i \ \text{ is even,} \end{cases}$$

$$\phi(y_{3m-1}) = 11m-2, \quad \phi(x_{4m-1}) = 2m,$$

$$\phi(y_i) = \begin{cases} 12m-1-i & \text{if } \ i < m \ \text{ and } \ i \ \text{ is odd,} \\ 2(6m-1)-i & \text{if } \ m < i < 2m \ \text{ and } \ i \ \text{ is odd,} \\ 3(4m-1)-i & \text{if } \ 2m < i < 3m-1, \ \ 3m-1 < i < 4m-1, \ \text{ and } \ i \ \text{ is odd,} \\ 4m+i & \text{if } \ i \ \text{ is even.} \end{cases}$$

It can be verified that all the vertices of the graph are labeled and the labels are distinct. We now construct the set E of edge labels in the graph as follows:

$$g(x_{4m-2}x_{4m-1}) = 2m-1, \quad g(x_{4m-1}x_1) = 3(2m-1),$$

$$g(x_ix_{i+1}) = \begin{cases} 2(4m-1-i) & \text{if } \ i \le m, \\ 2(4m-i)-3 & \text{if } \ m < i < 2m-1, \\ 2(4m-2-i) & \text{if } \ 2m-1 \le i \le 4m-3, \end{cases}$$

$$g(x_{3m-1}y_{3m-1}) = 4m-1, \quad g(x_{4m-1}y_{4m-1}) = 2(4m-1),$$

$$g(x_iy_i) = \begin{cases} 2(4m-i)-1 & \text{if } \ i \le m, \\ 2(4m-1-i) & \text{if } \ m < i \le 2m-1, \\ 2(4m-i)-3 & \text{if } \ 2m-1 < i < 3m-1, \ \ 3m-1 < i \le 4m-2. \end{cases}$$

Case 2: *m is odd and* $m \ge 1$. Figure 4 shows a graceful valuation for $m = 1$.

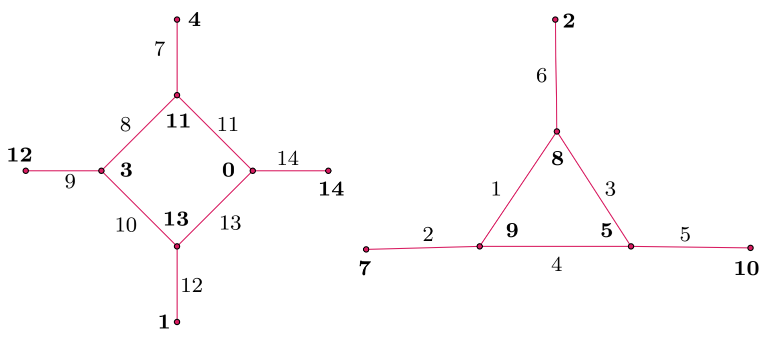


Figure 4. A graceful valuation of $C_4^{S_1} \cup C_3^{S_1}$.

If $m > 1$, then we define

$$\phi(w_{3m}) = 7m+2, \quad \phi(w_{4m-1}) = 10m-2,$$

$$\phi(w_i) = \begin{cases} 4m+i & \text{if } \ i \le m \ \text{ and } \ i \ \text{ is odd,} \\ 4m+1+i & \text{if } \ m < i < 3m, \ \ 3m < i < 4m-1, \ \text{ and } \ i \ \text{ is odd,} \\ 12m-1-i & \text{if } \ i \le 2m \ \text{ and } \ i \ \text{ is even,} \\ 2(6m-1)-i & \text{if } \ i > 2m \ \text{ and } \ i \ \text{ is even,} \end{cases}$$

$$\phi(z_{4m-1}) = 2m, \quad \phi(z_{3m+1}) = 5m+1,$$

$$\phi(z_i) = \begin{cases} 12m-1-i & \text{if } \ i < 2m \ \text{ and } \ i \ \text{ is odd,} \\ 2(6m-1)-i & \text{if } \ 2m < i \leq 4m-3 \ \text{ and } \ i \ \text{ is odd,} \\ 4m+i & \text{if } \ i < m \ \text{ and } \ i \ \text{ is even,} \\ 4m+i+1 & \text{if } \ m < i < 3m+1, \ \ 3m+1 < i \leq 4m-1, \ \text{ and } \ i \ \text{ is even.} \end{cases}$$

It can be verified that all the vertices of the graph are labeled and the labels are distinct. We now construct labels for the set E of edges in the graph as follows:

$$g(w_{3m-1}w_{3m}) = 2m-3, \quad g(w_{3m}w_{3m+1}) = 2m-5,$$

$$g(w_{4m-2}w_{4m-1}) = 2(m-1), \quad g(w_{4m-1}w_1) = 3(2m-1),$$

$$g(w_iw_{i+1}) = \begin{cases} 2(4m-1-i) & \text{if } \ i \leq m, \\ 2(4m-i)-3 & \text{if } \ m < i \leq 2m, \\ 2(4m-2-i) & \text{if } \ 2m < i < 3m-1, \ \ 3m < i < 4m-2, \end{cases}$$

$$g(w_{3m}z_{3m}) = 2(m-2), \quad g(w_{3m+1}z_{3m+1}) = 4(m-1), \quad g(w_{4m-1}z_{4m-1}) = 2(4m-1),$$

$$g(w_iz_i) = \begin{cases} 2(4m-i)-1 & \text{if } \ i \leq m, \\ 2(4m-1-i) & \text{if } \ m < i \leq 2m, \\ 2(4m-i)-3 & \text{if } \ 2m < i < 3m, \ \ 3m+1 < i < 4m-1. \end{cases}$$

We see that the labels of the edges of $C_{4m}^{S_1} \cup C_{4m-1}^{S_1}$ are distinct. Therefore, it can be easily shown that the graph $C_{4m}^{S_1} \cup C_{4m-1}^{S_1}$ has graceful valuations (see $C_8^{S_1} \cup C_7^{S_1}$ in Fig. 5). This completes the proof of Claim 3. □

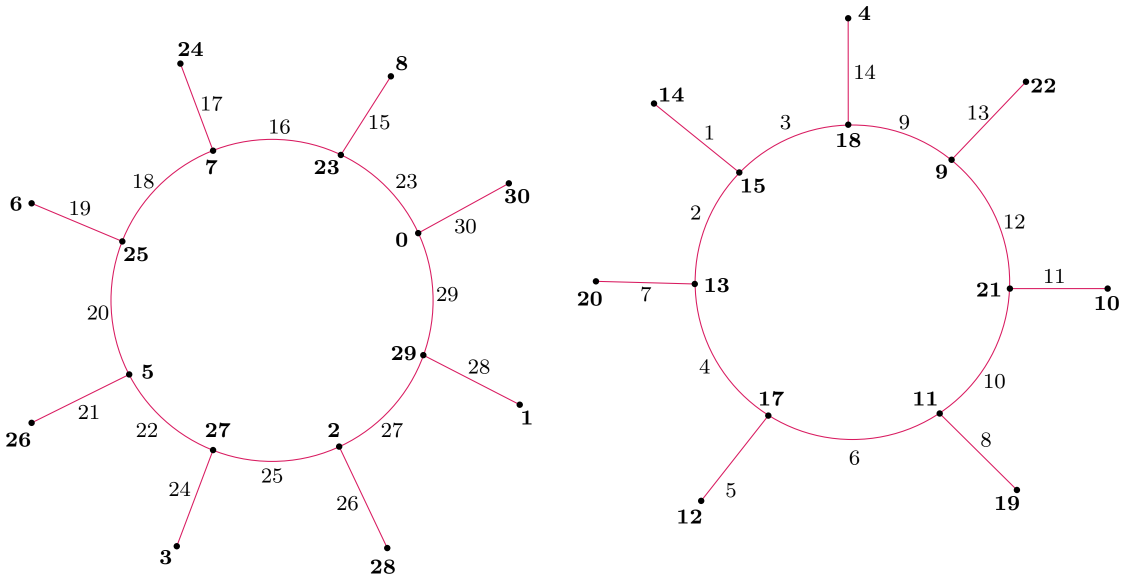


Figure 5. A graceful valuation of $C_8^{S_1} \cup C_7^{S_1}$.

Theorem 5. *The graph obtained by the disjoint union of two isomorphic copies of all hairy cycles $C_{4m+2}^{S_1}$ admits an α-valuation with exactly one missing number $\rho = 4(2m+1)$ and the critical value $\sigma = 2(3m+2)$.*

P r o o f. Let $i = 1, 2, 3, \ldots, 4m+2$, and let p_i (r_i) and q_i (s_i) denote the vertices of the cycle and leaves, respectively, in the first and second copies of $C_{4m+2}^{S_1}$, respectively. Clearly,

$$|V(C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1})| = |E(C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1})| \; = \; 8(2m+1).$$

Define a function $\xi : V(C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1}) \to \{0, 1, 2, \ldots, 8(2m+1)\}$ as follows. Figure 6 shows an α-valuation for $m = 1$.

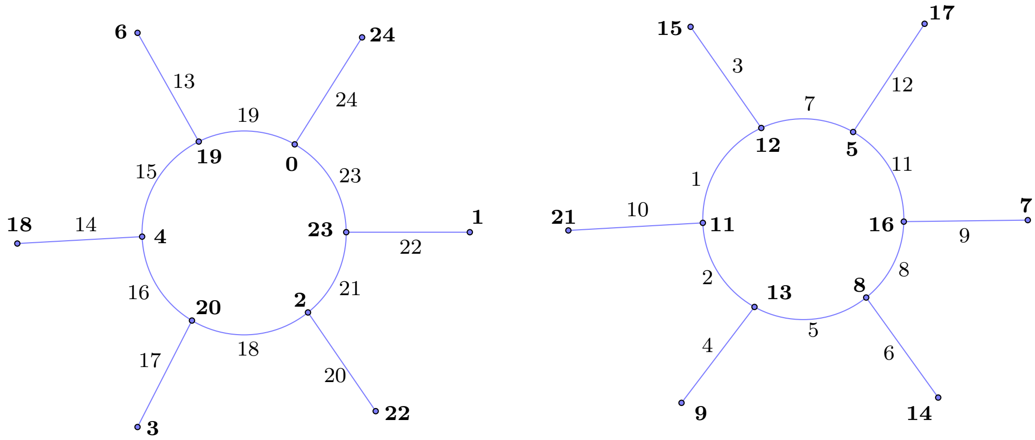


Figure 6. An α-valuation of $C_6^{S_1} \cup C_6^{S_1}$.

For $m > 1$, we label the vertices of $C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1}$ as follows:

$$\xi(p_{4m+2}) = 12m + 7,$$

$$\xi(p_i) = \begin{cases} 16m + 9 - i & \text{if} \quad i < 2m+1 \quad \text{and} \quad i \quad \text{is even}, \\ 16m + 8 - i & \text{if} \quad 2m+1 \le i \le 4m \quad \text{and} \quad i \quad \text{is even}, \\ i - 1 & \text{if} \quad i \quad \text{is odd}, \end{cases}$$

$$\xi(q_{4m+1}) = 12m + 6, \quad \xi(q_{4m+2}) = 4m + 2,$$

$$\xi(q_i) = \begin{cases} 16m + 9 - i & \text{if} \quad i \le 2m+1 \quad \text{and} \quad i \quad \text{is odd}, \\ 16m + 8 - i & \text{if} \quad 2m+1 < i \le 4m-1 \quad \text{and} \quad i \quad \text{is odd}, \\ i - 1 & \text{if} \quad i \le 4m \quad \text{and} \quad i \quad \text{is even}, \end{cases}$$

$$\xi(r_1) = 4m + 1, \quad \xi(r_{2m+3}) = 6m + 5, \quad \xi(r_{2m+2}) = 10m + 3,$$

$$\xi(r_i) = \begin{cases} 4m + 1 + i & \text{if} \quad 1 < i \le 2m+1, \quad 2m+3 < i \le 4m+1, \quad \text{and} \quad i \quad \text{is odd}, \\ 12m + 6 - i & \text{if} i < 2m+2, \quad 2m+2 < i \le 4m+2, \quad \text{and} \quad i \quad \text{is even}, \end{cases}$$

$$\xi(s_{2m+1}) = 10m + 4, \quad \xi(s_{2m+3}) = 14m + 7, \quad \xi(s_{2m+5}) = 10m + 5,$$

$$\xi(s_i) = \begin{cases} 12m + 6 - i & \text{if} \quad i < 2m+1 \quad \text{and} \quad i \quad \text{is odd}, \\ 12m + 8 - i & \text{if} \quad 2m+5 < i < 4m+2 \quad \text{and} \quad i \quad \text{is odd}, \\ 4m + 1 + i & \text{if} \quad 1 < i \le 2m+2 \quad \text{and} \quad i \quad \text{is even}, \\ 4m + 3 + i & \text{if} \quad 2m+2 < i \le 4m+2 \quad \text{and} \quad i \quad \text{is even}. \end{cases}$$

Clearly, ξ is injective. Now, we prove that the induced labeling

$$\ell : E(C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1}) \to \{1, 2, \ldots, 8(2m+1)\}$$

defined as $\ell(xy) = |\xi(x) - \xi(y)|$ for $xy \in E(C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1})$ is bijective.

The induced edge labeling ℓ has the following values:

$$\ell(p_{4m+2}p_1) = 12m+7, \quad \ell(p_{4m+1}p_{4m+2}) = 8m+7,$$

$$\ell(p_ip_{i+1}) = \begin{cases} 16m+9-2i & \text{if} \quad i < 2m+1, \\ 16m+8-2i & \text{if} \quad 2m+1 \le i \le 4m, \end{cases}$$

$$\ell(p_{4m+1}q_{4m+1}) = 8m+6, \quad \ell(p_{4m+2}q_{4m+2}) = 8m+5,$$

$$\ell(p_iq_i) = \begin{cases} 16m+10-2i & \text{if} \quad i \le 2m+1, \\ 16m+9-2i & \text{if} \quad 2m+1 < i \le 4m, \end{cases}$$

$$\ell(r_1r_2) = 8m+3, \quad \ell(r_{2m+2}r_{2m+3}) = 4m-2, \quad \ell(r_{4m+2}r_1) = 4m+3,$$

$$\ell(r_{2m+1}r_{2m+2}) = 4m+1, \quad \ell(r_{2m+3}r_{2m+4}) = 4m-3,$$

$$\ell(r_ir_{i+1}) = 8m+4-2i, \quad \text{for} \quad 1 < i < 2m+1, \quad 2m+4 \le i \le 4m+1,$$

$$\ell(r_1s_1) = 8m+4, \quad \ell(r_{2m+3}r_{2m+3}) = 8m+2, \quad \ell(r_{2m+5}s_{2m+5}) = 4m-1,$$

$$\ell(r_{4m+2}s_{4m+2}) = 1, \quad \ell(r_{2m+1}s_{2m+1}) = 4m+2, \quad \ell(r_{2m+2}s_{2m+2}) = 4m,$$

$$\ell(r_is_i) = \begin{cases} 8m+5-2i & \text{if} \quad 1 < i < 2m+1, \\ 8m-2i+3 & \text{if} \quad 2m+4 \le i < 4m+2 \quad \text{and} \quad i \quad \text{is even}, \\ 8m-2i+7 & \text{if} \quad i > 2m+5 \quad \text{and} \quad i \quad \text{is odd}. \end{cases}$$

It is clear that all the vertex and edge labels are distinct. Therefore, the graph $C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1}$ is graceful. Next, we prove that the above graceful function ξ is an α-valuation with the missing number $\rho = 4(2m+1)$ and the critical value $\sigma = 2(3m+2)$. Since the vertex set V of $C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1}$ is partitioned into two sets, $V = X \cup Y$, we have

$$X=\{0,1,2,\dots,4m,4m+2,4m+1,4m+3,4m+4,\dots,6m+3,6m+5,6m+7,6m+6,\dots,8m+5\}$$

and

$$Y=\{8(2m+1),16m+7,16m+6,\dots,14m+8,14m+6,\dots,12m+6,12m+7,12m+5,12m+4,$$
$$\dots,10m+6,10m+4,\dots,14m+7,10m+2,10m+5,\dots,8m+4\}.$$

Clearly, X and Y are independent sets. The number $\sigma = 2(3m+2)$ satisfies $f(x) \le \sigma < f(y)$ for every ordered pair $(x,y) \in X \times Y$. Therefore, ξ is an α-valuation of $C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1}$. □

3. Conclusion

This paper discussed graceful and α-valuations of certain disconnected graphs. Finding general characterizations for such graphs is an open problem. We also propose the following problem.

Problem 3.1. The disjoint union of various hairy cycles $C_{m_1}^{S_n} \cup C_{m_2}^{S_n} \cup \dots \cup C_{m_k}^{S_n}$ admits an α-labeling.

Acknowledgements

The authors wish to thank the referees for their valuable comments.

REFERENCES

1. Abrham J., Kotzig A. Graceful valuations of 2-regular graphs with two components. *Discrete Math.*, 1996. Vol. 150, No. 1–3. P. 3–15. DOI: 10.1016/0012-365X(95)00171-R
2. Balakrishnan R., Ranganathan K. *A Textbook of Graph Theory*, 2nd ed. NY: Springer, 2012. 292 p. DOI: 10.1007/978-1-4614-4529-6

3. Barrientos C. Equitable labeling of corona graphs. *J. Comb. Math. Comb. Comput.,* 2002. Vol. 41. P. 139–149.
4. Barrientos C. Graceful graphs with pendant edges. *Australas. J. Comb.*, 2005. Vol. 33. P. 99–107.
5. Barrientos C., Minion S. Constructing graceful graphs with caterpillars. *J. Algorithms Comput.* 2016. Vol. 48, No. 1. P. 117–125. DOI: 10.22059/jac.2016.7946
6. Eshghi K., Carter M. Construction of α-valuations of special classes of 2-regular graphs *Math. Comput. Sci. Eng.,* 2001. P. 139–154.
7. Frucht R., Harary F. On the corona of two graphs. *Aequationes Math.*, 1970. Vol. 4. P. 322–325. DOI: 10.1007/BF01844162
8. Frucht R. W., Salinas L. C. Graceful numbering of snakes with constraints on the first label. *Ars Comin.,* 1985. Vol. 20B. P. 143–157.
9. Gallian J. A. A dynamic survey of graph labeling. 23-th ed. *Electron. J. Combin.*, 2023. Art. no. #DS6. P. 1–644. DOI: 10.37236/27
10. Graf A. A new graceful labeling for pendant graphs. *Aequat. Math.,* 2014. Vol. 87. P. 135–145. DOI: 10.1007/s00010-012-0184-4
11. Kumar J., Mishra D., Kumar A., Kumar V. Alpha labeling of cyclic graphs. *Int. J. Appl. Comput.,* 2021. Vol. 7, Art. no. 151. P. 1–7. DOI: 10.1007/s40819-021-01084-5
12. Kumar A, Mishra D, Verma A., Srivastava V. K. Alpha labeling of cyclic graphs. I. *ARS Combinatorics,* 2021. Vol. 154. P. 257–263.
13. Lakshmi D. R., Vangipuram S. An α-valuation of quadratic graph $Q(4, 4k)$. *Proc. Nat. Acad. Sci., India, Sect. A*, 1987. Vol. 57, No. 4. P. 576–580.
14. Minion S., Barrientos C. Three Graceful Operations. *J. Alg. Comp.*, 2014. Vol. 45, No. 1. P. 13–24. DOI: 10.22059/jac.2014.7917.
15. Pradhan P., Kumar A. Graceful hairy cycles with pendent edges and some properties of cycles and cycle related graphs. *Bull. Calcutta Math. Soc.*, 2012. Vol. 104. P. 61–76.
16. Pradhan P., Kumar K. On graceful labeling of some graphs with pendant edges. *Gen. Math. Notes,* 2014. Vol. 23, No. 2. P. 51–62.
17. Pradhan P., Kumar K. On k-graceful labeling of some graphs. *J. Appl. Math. Inform.*, 2016. Vol. 34, No. 1–2. P. 9–17. DOI: 10.14317/jami.2016.009.
18. Pradhan P., Kumar A., Mishra D. On gracefulness of graphs obtained from hairy cycles. *J. Combin. Inform. Syst. Sci.*, 2010. Vol. 35, P. 471–480.
19. Ringel G. Problem 25. In: *Theory of Graphs and its Application. Proc. Symposium Smolenice 1963.* Prague, 1964.
20. Ropp D. Graceful labelings of cycles and prisms with pendant points. *Congr. Number,* 1990. Vol. 75. P. 218–234.
21. Rosa A. On certain valuations of the vertices of a graph. In: *Theory of Graphs: Int. Symposium, Rome, July 1966).* N.Y. and Dunod Paris: Gordon and Breach, 1967. P. 349–355.
22. Truszczyński M. Graceful unicyclic graphs. *Demonstratio Math.,* 1984. Vol. 17. P. 377–387.

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 136–146

DOI: 10.15826/umj.2024.1.012

IMPROVED FIRST PLAYER STRATEGY FOR THE ZERO-SUM SEQUENTIAL UNCROSSING GAME

Ksenia Rizhenko

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, 620108, Russian Federation

kseniarizhenko@gmail.com

Abstract: This paper deals with the known uncrossing zero-sum two-player sequential game, which is employed to obtain upper running time bound for the transformation of an arbitrary subset family of some finite set to an appropriate laminar one. In this game, the first player performs such a transformation, while the second one tries to slow down this process as much as possible. It is known that for any game instance specified by the ground set and initial subset family of size n and m respectively, the first player has a winning strategy of $O(n^4m)$ steps. In this paper, we show that the first player has a more efficient strategy, which helps him (her) to win in $O\big(\max\{n^2, mn\}\big)$ steps.

Keywords: Laminar family, Uncrossing game, Efficient winning strategy.

1. Introduction

A *laminar family* of subsets for a given finite set is a well-known concept in discrete mathematics widely exploited in algorithm design for combinatorial optimization problems. For instance, in [5] and [4], laminar families are used to construct polynomial time approximation algorithms for several modifications of the Tree Augmentation Problem (TAP). Recently [1], these results have been extended to efficient approximation algorithms for the Leaf-to-Leaf Connectivity Augmentation Problem (CAP).

On the other hand, *strongly laminar instances* of the Asymmetric Traveling Salesman Problem (ATSP) belong to the main building blocks of the first constant-ratio approximation algorithm for the ATSP proposed by the authors of the breakthrough papers [9] and [10]. By relying on this algorithm the authors of recent works [3, 6, 7] proved constant-ratio polynomial time approximation for several related asymmetric combinatorial problems including Prize-Collecting TSP and Capacitated Vehicle Routing Problem.

An advantage of employment the laminar families in approximation algorithms is based on the following two observations:

(i) for any finite set V of size $|V| = n$, the size $|\mathcal{L}|$ of an arbitrary laminar family $\mathcal{L}$ of its subsets is $O(n)$ [8];

(ii) any non-necessary laminar subset family $\mathcal{F}$ of size m can be transformed to some equivalent laminar family (see, e.g. [2]) in polynomial time with respect to n and m.

To obtain an upper bound for the running time of such a transformation, the authors of [2] introduced a sequential two-person zero-sum *uncrossing game*, where the first player is aimed to make the initial family $\mathcal{F}$ laminar, and the second player tries to slow down this process as much as possible. For each step of the game, its state is specified by the current family. The game ends with the first player winning as soon as the state becomes laminar family.

In [2], the authors showed that the first player has a strategy that allows he (she) to win in polynomial number of steps.

Theorem 1. *In the uncrossing game, the first player has a strategy to make the initial family* $\mathcal{F}$ *laminar in* $O(n^4 m)$ *steps.*

Theorem 2. *In the uncrossing game, the first player has a strategy to make the family* $\mathcal{F}$ *laminar in* $O(\max\{nm, n^2\})$ *steps.*

The rest of our paper is structured as follows. In Section 2, we give formulation of the uncrossing game and recall some its properties essential to our own constructions. Section 3 contains the proof of Theorem 2. Finally, in Section 4, we summarize our results and overview some directions of future research.

2. Problem statement and preliminaries

Let V be an arbitrary non-empty finite set. In the sequel, we call it *ground set.*

Definition 1. *Subsets* $X, Y \in 2^V$ *are called crossing and denoted* $X \nparallel Y$*, if all of the sets* $X \backslash Y$*,* $Y \backslash X$*,* $X \cap Y$*,* $V \backslash (X \cup Y)$ *are non-empty. Otherwise the sets* X *and* Y *are called laminar and denoted* $X \parallel Y$*.*

A subset family $\mathcal{F} \subseteq 2^V$ is called *laminar* if it contains no crossing subsets.

Definition 2. *A subset family* $\mathcal{I} \subseteq 2^V$ *is called cross-closed, if for any crossing sets* $X, Y \in \mathcal{I}$ *at least one of the pairs*

$$\{X \cap Y, X \cup Y\} \quad or \quad \{X \backslash Y, Y \backslash X\} \tag{2.1}$$

belongs to $\mathcal{I}$ *as well.*

Obviously, for an arbitrary $X \nparallel Y$, both pairs (2.1) are laminar. To simplify description of the *uncrossing game*, define two elementary *uncrossing operations* as follows. The first operation $\xrightarrow{\cap,\cup}$ substitutes some crossing subsets $X, Y \in \mathcal{F}$ by the subsets $X \cap Y, X \cup Y \in \mathcal{I}$, while the second one $\xrightarrow{\backslash,\backslash}$ makes the similar replacement by the subsets $X \backslash Y, Y \backslash X \in \mathcal{I}$.

Definition 3. *The uncrossing game is a sequential two-player zero-sum game, whose instance is specified by some non-empty finite set* V *and a family* $\mathcal{F}$*. It is assumed that* $\mathcal{F} \subseteq \mathcal{I}$ *for some cross-closed family* $\mathcal{I} \subseteq 2^V$ *given implicitly by a membership oracle.*

Each step of the game is defined as follows. If the family $\mathcal{F}$ *is laminar, then the first player wins. Otherwise, it performs one uncrossing operation for some subsets* $X, Y \in \mathcal{F}$*,* $X \nparallel Y$*. In turn, the second player returns back to* $\mathcal{F}$ *one of the subsets* X *or* Y*.*

In [2] several properties of the uncrossing game were proved. We remind some of them which are necessary for our own results.

Statement 1. *Let* $X, Y, Z \in \mathcal{F}$ *such that* $X \nparallel Y$*,* $X \parallel Z$*, and* $Y \parallel Z$*. The relation* $Q \parallel Z$ *holds for any* $Q \in \{X \backslash Y, Y \backslash X, X \cup Y, X \cap Y\}$*.*

Statement 1 allows to simplify the uncrossing game at each step, when the current family $\mathcal{F}$ contains a subset Z laminar to each other $X \in \mathcal{F}$.

Lemma 1. *The first player has a strategy to reduce the initial game instance to* $O(nm)$ *subinstances of the uncrossing game, each of them is of the smaller size.*

As it follows from the proof of Lemma 1, each of the obtained subinstances is equivalent to the uncrossing game specified by the ground set W and the family $\mathcal{R}$, where

$$W = \overline{1, r}, \quad \text{and} \quad \mathcal{R} = \{\overline{1,2},\ \overline{1,3}, \ldots, \overline{1, r-2},\ \overline{2, r-1}\} \tag{2.2}$$

for some $r \le n$, and the appropriate cross-closed family $\mathcal{I}^*$ induced by the family $\mathcal{I}$. Hereinafter, we use the standard notation $\overline{i,j}$ for the integer interval $\{i, i+1, \ldots, j-1, j\}$, whose entries are taken modulo r. In turn, it is easy to verify that this game is equivalent to the game

$$CG(r): \begin{cases} W = \overline{1, r}, \\ \mathcal{R} = \{\overline{1,2},\ \overline{2,3},\ \overline{2,4}, \ldots, \overline{2, r-1}\}, \end{cases} \tag{2.3}$$

which is the main object of the subsequent discussion.

Indeed, we can obtain game $CG(r)$ (see Fig. 1) by the cyclic shift on -1 to form a family

$$\mathcal{R} = \{\overline{r,1},\ \overline{1,2},\ \overline{1,3}, \ldots, \overline{1, r-2}\}$$

and taking the complement $W \setminus \overline{r,1} = \overline{2, r-1}$. The reverse transformation can be obtained by supplementing the complement $W \setminus \overline{1,2} = \overline{3,r}$ producing the family $\{\overline{3,r},\ \overline{2,3},\ \overline{2,4}, \ldots, \overline{2, r-1}\}$ by the cyclic shift on -1 resulting in the family

$$\mathcal{R} = \{\overline{1,2},\ \overline{1,3},\ , \ldots, \overline{1, r-2}, \overline{2, r-1}\}.$$

The game $CG(r)$ belongs to the known class of *cyclic* uncrossing games (see, e.g. [2]), each of them has the following properties:

(i) the family $\mathcal{R}$ is partitioned into two laminar subfamilies $\mathcal{R} = \mathcal{L}_1 \dot\sqcup \mathcal{L}_2$, such that $\overline{1,i} \in \mathcal{L}_1$ for $2 \leqslant i \leqslant r-2$ and $\overline{2,j} \in \mathcal{L}_2$;
(ii) the game is invariant to cyclic shifts and replacing each interval by its complement;
(iii) any two neighboring elements $i, i+1$ (modulo r) are *separated* by some set $X \in \mathcal{R}$, i.e. $|X \cap \{i, i+1\}| = 1$.

In the sequel, we propose a first player winning strategy for the $CG(r)$ based on transformation of the initial game to some simpler one. To this end, each time when the transformed family contains a pair $\{i, i+1\}$ violating property (iii), we can *contract* the interval $\overline{i, i+1}$ to a single point, which makes ground set W smaller and accordingly transforms $\mathcal{R}$ and $\mathcal{I}^*$.

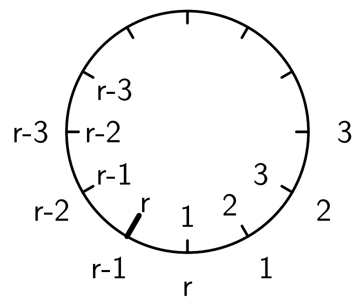


Figure 1. Towards the equivalence of game (2.2) and $CG(r)$.

Evidently, for $r < 4$ the game $CG(r)$ has a trivially solvable by the first player, since the initial family $\mathcal{R}$ is laminar. Consider the first non-trivial game $CG(4)$.

Lemma 2. *The first player can win in the $CG(4)$ after the single step.*

P r o o f. By construction of $CG(4)$, the family $\mathcal{R}$ consists of the only crossing pair $\overline{1,2}$ and $\overline{2,3}$. Since $\mathcal{I}^*$ is cross-closed, it contains either $\{1\}$ and $\{3\}$ or $\{2\}$ and $\{4\}$ (equivalent to $\overline{1,3}$). Then, the first player at the first step performs either the uncrossing operation $\overline{1,2} \nparallel \overline{2,3} \xrightarrow{\setminus,\setminus} \{1\},\{3\}$ or $\overline{1,2} \nparallel \overline{2,3} \xrightarrow{\cap,\cup} \{2\},\{4\}$, respectively. Thus, regardless to the behavior of the second player, the family $\mathcal{R}$ becomes laminar to the second step, since singletons are laminar to any other subset and can be excluded. Therefore, Lemma 2 follows.

3. Proof of Theorem 2

For subsequent discussions we will need to introduce one more family of cyclic uncrossing games. Each such a game has the form

$$CG(r,q): \begin{cases} W = \overline{1,r}, \\ \mathcal{R} = \{\overline{1,2},\ldots,\overline{1,q},\ \overline{2,q+1},\ldots,\overline{2,r-1}\}, \end{cases}$$

where $2 \le q \le r-2$ and the family $\mathcal{I}^*$ satisfies the additional constraints

$$\{1\},\{2\} \in \mathcal{I}^*, \quad \{3\},\ldots,\{q\},\ \{r\} \notin \mathcal{I}^*. \tag{3.1}$$

First, consider the special case, where $q = r-2$.

Lemma 3. *For an arbitrary $r \ge 4$, in the game $CG(r,r-2)$, the first player has a winning strategy within at most $r-3$ steps.*

P r o o f can be obtained by induction on r. In the base case $r = 4$, the game $CG(4,2)$ allows a single step winning strategy of the first player. To prove the induction step, assume that in $CG(r-1,r-3)$ there exists a winning strategy of the first player within $r-4$ steps and show that the claim holds for the game $CG(r,r-2)$.

At the first step of this game, the first player performs the uncrossing operation

$$\overline{1,r-2} \nparallel \overline{2,r-1} \xrightarrow{\setminus,\setminus} \{1\},\{r-1\},$$

which is admissible since $\{r\}$ does not belong to the cross-closed family $\mathcal{I}^*$. By construction, the second player has two options, to return to $\mathcal{R}$ either $\overline{1,r-2}$ or $\overline{2,r-1}$. In the first case, $\mathcal{R}$ becomes laminar and the first player wins immediately. Otherwise, the family $\mathcal{R}$ takes the form $\{\overline{1,2},\ldots\overline{1,r-3},\overline{2,r-1}\}$. Here elements $r-2$ and $r-1$ no longer separated. After the contraction, we obtain the game

$$CG(r-1,r-3): \begin{cases} W = \overline{1,r-1}, \\ \mathcal{R} = \{\overline{1,2},\ldots,\overline{1,r-3},\overline{2,r-2}\}. \end{cases}$$

Therefore, in the initial game $CG(r,r-2)$ the first player has a winning strategy of at most

$$1 + (r-4) = r-3$$

steps. Lemma 3 is proved. □

Lemma 4. *For the game $CG(r,q)$ first player has a strategy that allows him to win in at most $2r-q-5$ steps.*

P r o o f. The main idea of our proof consists of the following steps:

(i) we introduce an auxiliary edge-weighted digraph H with the node set

$$\mathcal{G} = \{CG(r, q) \colon 2 \leq q \leq r - 2\}.$$

An ordered pair $\big(CG(r_1, q_1), CG(r_2, q_2)\big)$ is an arc of the graph H, if the first player has a strategy to reduce $CG(r_1, q_1)$ to the game $CG(r_2, q_2)$ without visiting any other element of $\mathcal{G}$. Each arc is weighted with the appropriate number of steps. It is convenient to illustrate this graph on the integer lattice (Fig. 2).

(ii) we define linear order on the node set of the graph H as follows:

$$CG(r_2, q_2) \prec CG(r_1, q_1) \iff (r_2 < r_1) \vee ((r_1 = r_2) \wedge (q_2 > q_1)); \tag{3.2}$$

(iii) we show that this order is consistent to the digraph H in the following way: for an arbitrary arc $\big(CG(r_1, q_1), CG(r_2, q_2)\big)$ it holds; $CG(r_2, q_2) \prec CG(r_1, q_1)$.

(iv) finally, we complete the proof by induction regarding the order (3.2).

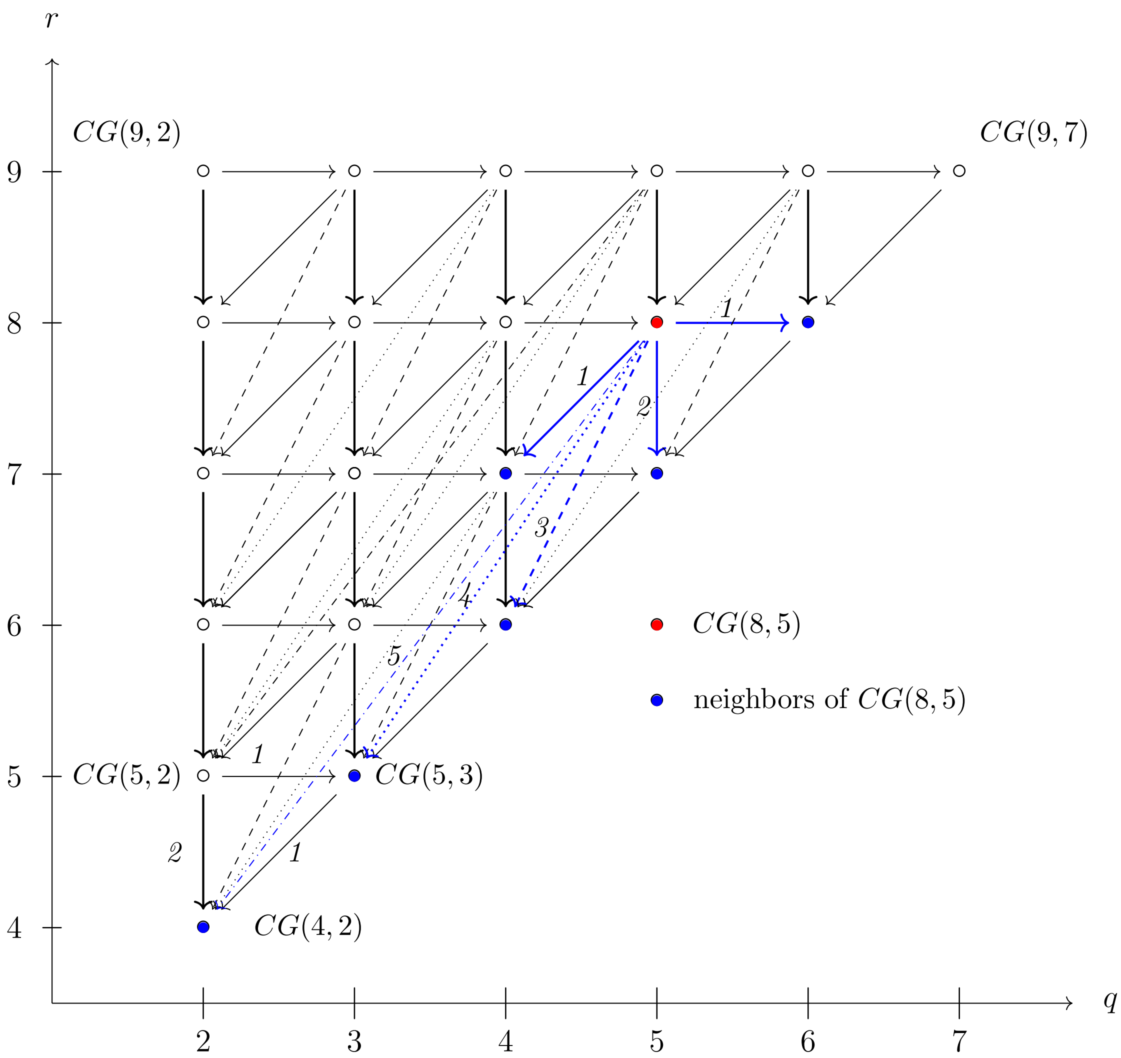


Figure 2. An auxiliary digraph H. Outgoing arcs and neighbors of $CG(8, 5)$ are highlighted.

Discuss the aforementioned points in detail.

Point (i). To define arc set of the graph H, fix an arbitrary node $CG(r,q) \in \mathcal{G}$ and show that its outgoing neighbors (i.e. the nodes $CG(r',q') \in \mathcal{G}$, for which $(CG(r,q), CG(r',q'))$ is an arc) are as follows:

$$\begin{gathered} \mathcal{N}(CG(r,q)) = \big\{CG(r-1,q-1), CG(r,q+1), CG(r-1,q), \\ CG(r-2,q-1), \dots, CG(r-q+1,2)\big\} \end{gathered} \tag{3.3}$$

In addition, we calculate a weight assigned to each arc connecting $CG(r,q)$ with its neighbors (Fig. 2).

By construction, for the family $\mathcal{I}^*$ specifying the game $CG(r,q)$ there exist two options, $\mathcal{I}^*$ can contain or not the singleton $\{q+1\}$.

Case $\{q+1\} \in \mathcal{I}^*$. At the first step of the game, the first player makes the uncrossing operation

$$\overline{1,q} \nparallel \overline{2,q+1} \xrightarrow{\setminus,\setminus} \{1\}, \{q+1\}.$$

If the second player returns $\overline{1,q}$, i. e.

$$\mathcal{R} = \{\overline{1,2},\ \overline{1,q},\ \overline{2,q+2}, \dots, \overline{2,r-1}\}$$

and then, after the contacting $\{q+1, q+2\}$ the initial game is transformed to

$$CG(r-1,q): \begin{cases} W = \overline{1,r-1}, \\ \mathcal{R} = \{\overline{1,2}, \dots, \overline{1,q},\ \overline{2,q+1}, \dots, \overline{2,r-2}\}, \end{cases}$$

since this actions obviously keep all the additional constraints

$$\{1\}, \{2\} \in \mathcal{I}^* \quad \text{and} \quad \{3\}, \dots, \{q\}, \{r-1\} \notin \mathcal{I}^* \tag{3.4}$$

for the transformed family $\mathcal{I}^*$. Thus, $CG(r-1,q)$ is an outgoing neighbor of the node $CG(r,q)$.

Else, if the second player returns $\overline{2,q+1}$, performing the similar contraction of $\{q, q+1\}$ transforms $CG(r,q)$ to the game

$$CG(r-1,q-1): \begin{cases} W = \overline{1,r-1}, \\ \mathcal{R} = \{\overline{1,2},\ \overline{1,q-1},\ \overline{2,q}, \dots, \overline{2,r-2}\}, \end{cases}$$

since constraints (3.4) are also remain valid.

In the *case* $\{q+1\} \notin \mathcal{I}^*$, we have $\overline{1,q+1}, \overline{2,q} \in \mathcal{I}^*$, since the crossing intervals $\overline{1,q}$ and $\overline{2,q+1}$ belong to the cross-closed family $\mathcal{I}^*$. By carrying out the similar argument for $\overline{1,q-1} \nparallel \overline{2,q}$, $\overline{1,q-2} \nparallel \overline{2,q-1}$ and so on and relying on equation (3.1), we obtain that in this case all the intervals $\overline{2,3}, \overline{2,4}, \dots, \overline{2,q}$ belong to $\mathcal{I}^*$.

At the first step of the game $CG(r,q)$, the first player performs the uncrossing operation

$$\overline{1,2} \nparallel \overline{2,q+1} \xrightarrow{\cap,\cup} \{2\}, \{1,q+1\}.$$

Next, if the second player returns $\overline{1,2}$ the game $CG(r,q)$ is transformed to

$$CG(r,q+1): \begin{cases} W = \overline{1,r}, \\ \mathcal{R} = \{\overline{1,2}, \dots, \overline{1,q},\ \overline{1,q+1},\ \overline{2,q+2}, \dots, \overline{2,r-1}\}, \end{cases}$$

since constraints (3.1) together with $\{q+1\} \notin \mathcal{I}^*$ hold.

Otherwise, if the second player returns $\overline{2, q+1}$ we have

$$\mathcal{R} = \{\overline{1,3}, \dots, \overline{1,q+1},\ \overline{2,q+1}, \dots, \overline{2,r-1}\}.$$

By contracting the interval $\{2,3\}$ to the singleton $\{2\}$, we got a new game G' specified by the novel ground set $W = \overline{1, r-1}$, subset family

$$\mathcal{R}' = \{\overline{1,2}, \dots, \overline{1,q},\ \overline{2,q}, \dots, \overline{2,r-2}\}$$

and the transformed cross-closed family $\mathcal{I}^*$ containing the singleton $\{2\}$. Observe that, although the constraints (3.4) are still satisfied, game G' does not belong to the game family $\mathcal{G}$.

Further, if $q = 2$, then the game G' coincides with $CG(r-1, 2)$. Otherwise, we need to proceed with the playing.

At the second step, the first player makes the uncrossing operation

$$\overline{1,2} \nparallel \overline{2,q} \xrightarrow{\cap,\cup} \{2\}, \overline{1,q}.$$

If the second player returns $\overline{1,2}$, the family $\mathcal{I}^*$ is kept unchanged and we obtain the game

$$CG(r-1,q): \begin{cases} W = \overline{1, r-1}, \\ \mathcal{R} = \{\overline{1,2}, \dots, \overline{1,q},\ \overline{2,q+1}, \dots, \overline{2,r-2}\} \end{cases}$$

since constraints (3.4) remain valid.

Else, if the second player returns $\overline{2,q}$ and transforms the family $\mathcal{R}$ to the form $\{\overline{1,3},\ \dots, \overline{1,q},\ \overline{2,q},\ \overline{2,q+1}, \dots, \overline{2,r-2}\}$, as in the previous step, by contracting interval $\{2,3\}$ to the singleton $\{2\}$, we obtain again the game

$$G': \begin{cases} W = \overline{1, r-2}, \\ \mathcal{R}' = \{\overline{1,2}, \dots, \overline{1,q-1},\ \overline{2,q-1}, \dots, \overline{2,r-3}\} \end{cases}$$

satisfying the conditions $\{1\}, \{2\} \in \mathcal{I}^*$ and $\{3\}, \dots, \{q-1\}, \{r-2\} \notin \mathcal{I}^*$ for the transformed family $\mathcal{I}^*$. Again, if $q = 3$, game G' is equivalent to $CG(r-2, q-1)$. Else, we proceed with playing the game G'.

Finally, to the beginning of the $(q-1)$-th step, the game G' will have the form

$$G': \begin{cases} W = \overline{1, r-q+2}, \\ \mathcal{R}' = \{\overline{1,2}, \overline{1,3},\ \overline{2,3}, \dots, \overline{2, r-q+1}\}, \end{cases}$$

where $\{1\}, \{2\} \in \mathcal{I}^*$ and $\{3\}, \{r-q+2\} \notin \mathcal{I}^*$. The first player makes the uncrossing operation

$$\overline{1,2} \nparallel \overline{2,3} \xrightarrow{\cap,\cup} \{2\}, \overline{1,3}.$$

If the second player returns $\overline{1,2}$, then we obtain the game $CG(r-q+2, 3)$. Otherwise, if he (she) returns $\overline{2,3}$ the game G' is transformed to the game $CG(r-q+1, 2)$. Indeed, after the second player move, we have $\mathcal{R} = \{\overline{1,3}, \overline{2,3}, \dots, \overline{2, r-q+1}\}$, which can be easily transformed to the form $\mathcal{R} = \{\overline{1,2}, \overline{2,4}, \dots, \overline{2, r-q+2}\}$ by contracting of the interval $\overline{2,3}$.

To this end, we proved that the set of outgoing neighbors of the node $CG(r,q)$ coincides with the set $\mathcal{N}(CG(r,q))$ defined by formula (3.3). As it follows from the argument above, weights of the appropriate arcs are as follows:

$$\begin{gathered} w\big(CG(r,q), CG(r,q+1)\big) = w\big(CG(r,q), CG(r-1,q-1)\big) = 1, \\ w\big(CG(r,q), CG(r-1,q)\big) = 2, \\ w\big(CG(r,q), CG(r-2,q-1)\big) = 3, \\ \dots, \\ w\big(CG(r,q), CG(r-q+1,2)\big) = q \end{gathered} \tag{3.5}$$

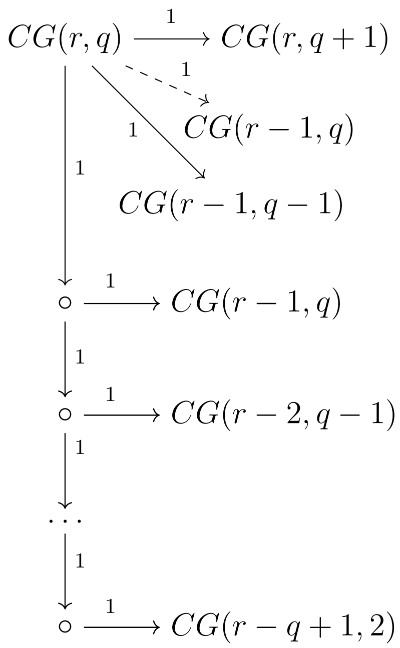


Figure 3. Illustration of the argument of point (i).

(see, also Fig. 3). Generally speaking, there are two parallel arcs connecting the nodes $CG(r,q)$ and $CG(r-1,q)$, but, to obtain the upper bound we take the longest one. Point (i) is proved.

In turn, point (iii) immediately follows from (3.2) and (3.3).

To complete the proof, we use $CG(4,2)$ as a base case of the induction. Since this game is a special case of the game $CG(4)$, the first player can win in a single step, by Lemma 2.

Induction step follows from the point (iii) and the following simple observation. Indeed, denote by $L(r,q)$ the number of steps in the first player winning strategy in the game $CG(r,q)$. By construction,

$$L(r,q) = \max\big\{w\big(CG(r,q), CG(r',q')\big) + L(r',q') \colon CG(r',q') \in \mathcal{N}(CG(r,q))\big\}.$$

By induction hypothesis, $L(r',q') = 2r' - q' - 5$. Taking into account (3.5), it is easy to verify that $L(r,q) \le 2r - q - 5$. Lemma 4 is proved. □

Let's get back to the game $CG(r)$.

Lemma 5. *For an arbitrary $r \ge 4$, in the game $CG(r)$ first player has a winning strategy of at most $2r-7$ steps.*

P r o o f. By construction, the game $CG(r)$ has the following form:

$$CG(r) : \begin{cases} W = \overline{1,r}, \\ \mathcal{R} = \{\overline{1,2},\ \overline{2,3}, \dots, \overline{2,r-1}\} \end{cases}$$

There exist two possible options.

Option 1. $\{1\},\{3\} \in \mathcal{I}^*$ or $\{2\},\{r\} \in \mathcal{I}^*$.

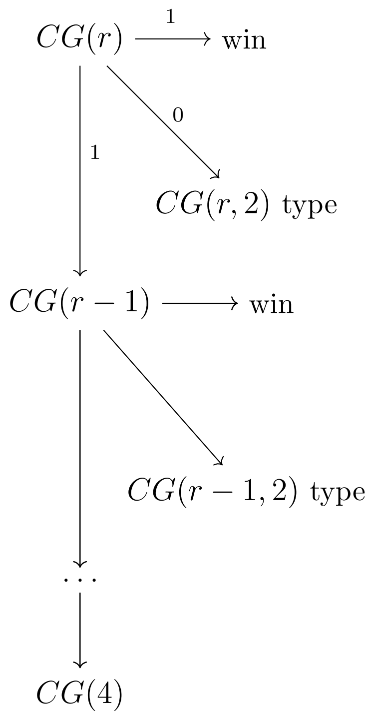


Figure 4. Scheme of possible transitions for game $CG(r)$.

It is easy to show that these conditions are symmetrical and can be considered in a similar way. Consider the case $\{1\},\{3\} \in \mathcal{I}^*$.

At the first step, the first player makes the following uncrossing operation:

$$\overline{1,2} \nparallel \overline{2,3} \xrightarrow{\setminus,\setminus} \{1\},\{3\}.$$

If the second player returns $\overline{2,3}$, then the first player wins immediately. Else, if he (she) returns $\overline{1,2}$, we obtain the game

$$CG(r-1):\begin{cases} W=\overline{1,r-1}, \\ \mathcal{R}=\{\overline{1,2},\ \overline{2,3},\dots,\overline{2,r-2}\} \end{cases}$$

after the contacting of $\{3,4\}$.

Option 2. In this case we necessarily have $\{1\},\{2\} \in \mathcal{I}^*$, $\overline{1,3} \in \mathcal{I}^*$, and $\{3\},\{r\} \notin \mathcal{I}^*$. These conditions leads that the considered game is $CG(r,2)$.

Complete the proof by induction by r (see Fig. 4). The base case is $CG(4)$, for which the claim follows from Lemma 2. Let Lemma 5 is valid for $r-1$. Prove it for r. Indeed, as it follows from the above argument, the game $CG(r)$ is either coincides with $CG(r,2)$ or can be reduced in one step to the game $CG(r-1)$. In the first case, the first player has a winning strategy of at most $2r-2-5=2r-7$, by Lemma 4, while in the second case such a strategy has at most

$$1+2(r-1)-7 \le 2r-7$$

steps. Lemma 5 follows. □

To the moment, we proved all the necessary technical lemmas and can return to proof of Theorem 2.

P r o o f of Theorem 2. The proposed strategy of the first player for the general uncrossing game specified by the triple $(V, \mathcal{F}, \mathcal{I})$, where $|V| = n$, $|\mathcal{F}| = m$ and $\mathcal{I}$ is given implicitly by the membership oracle, consists of following two stages.

First, we solve the cyclic games $CG(r, 2)$ and $CG(r)$ for $r \in \overline{4, n}$. As it follows from Lemma 4 and Lemma 5, all these games can be solved by at most $O(n^2)$ steps.

At the second stage, we employ Lemma 1, who guarantees that the first player has a strategy to reduce the initial game to $O(mn)$ games of form $CG(r)$ (for some $r \in \overline{4, n}$). For each such cyclic game, we take the solution (laminar family) obtained at the first stage (in constant time).

Thus, the overall strategy has $B = O(mn + n^2)$ steps. It is clear that

$$B = \begin{cases} O(mn), & \text{if } m \geq n, \\ O(n^2), & \text{otherwise.} \end{cases}$$

Theorem 2 is proved. □

It should be noticed that our strategy of the first player differs from the framework proposed in [2], where cyclic games obtained in Lemma 1 were solved ad hoc. By following to this framework and applying our Lemma 4 and 5, we can obtain another winning strategy of the first player, but with worse running time upper bound $O(n^2 m)$.

4. Conclusion

In this paper we proposed a more efficient first player winning strategy for the well-known uncrossing game. Proof of our result is entirely constructive and provides an algorithm to make laminar a given set family efficiently. Therefore, incorporating our result to approximation algorithms for combinatorial optimization problems relying on laminar set families can increase their performance.

In Lemma 5, for some special type of cyclic uncrossing games, we showed that the first player can win within at most linear number of steps. It seems interesting to generalize this result to more wide class of uncrossing games.

REFERENCES

1. Cecchetto F., Traub V., Zenklusen R. Better-than-4/3-approximations for leaf-to-leaf tree and connectivity augmentation. *Math. Program.*, 2023. 23 p. DOI: 10.1007/s10107-023-02018-3
2. Karzanov A. V. How to tidy up a symmetric set-system by use of uncrossing operations. *Theor. Comput. Sci.*, 1996. Vol. 157. No. 2. P. 215–225. DOI: 10.1016/0304-3975(95)00160-3
3. Khachay M.Yu., Neznakhina E.D., Ryzhenko K.V. Constant-factor approximation algorithms for a series of combinatorial routing problems based on the reduction to the asymmetric traveling salesman problem. *Proc. Steklov Inst. Math.*, 2022. Vol. 319. No. Suppl. 1. P. S140–S155. DOI: 10.1134/S0081543822060128
4. Kortsarz G., Nutov Z. LP-relaxations for tree augmentation. *Discr. Appl. Math.*, 2018. Vol. 239. P. 94–105. DOI: 10.1016/j.dam.2017.12.033
5. Maduel Y., Nutov Z. Covering a laminar family by leaf to leaf links. *Discr. Appl. Math.*, 2010. Vol. 158. No. 13. P. 1424–1432. DOI: 10.1016/j.dam.2010.04.002
6. Neznakhina E. D., Ogorodnikov Yu. Yu., Rizhenko K. V., Khachay M. Yu. Approximation algorithms with constant factors for a series of asymmetric routing problems. *Dokl. Math.*, 2023. Vol. 108. No. 3. P. 499–505. DOI: 10.1134/S1064562423701454

7. Rizhenko K., Neznakhina K., Khachay M. Fixed ratio polynomial time approximation algorithm for the prize-collecting asymmentric traveling salesman problem. *Ural Math. J.*, 2023. Vol. 9. No. 1. P. 135–146. DOI: 10.15826/umj.2023.1.012
8. Schrijver A. *Combinatorial Optimization. Polyhedra and Efficiency*. Berlin, Heidelberg: Springer, 2003. 1879 p. URL: https://link.springer.com/book/9783540443896
9. Svensson O., Tarnawski J., Végh L. A constant-factor approximation algorithm for the asymmetric traveling salesman problem. *J. ACM.*, 2020. Vol. 67. No. 6. Art. no. 37. P. 1–53. DOI: 10.1145/3424306
10. Traub V., Vygen J. An improved approximation algorithm for the asymmetric traveling salesman problem. *SIAM J. on Comput.*, 2022. Vol. 51. No. 1. P. 139–173. DOI: 10.1137/20M1339313

URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 147–158
DOI: 10.15826/umj.2024.1.013

ON G-VERTEX-TRANSITIVE COVERS OF COMPLETE GRAPHS HAVING AT MOST TWO G-ORBITS ON THE ARC SET[1]

Ludmila Yu. Tsiovkina

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, 620108, Russian Federation

tsiovkina@imm.uran.ru

Abstract: We investigate *abelian* (in the sense of Godsil and Hensel) distance-regular covers of complete graphs with the following property: *there is a vertex-transitive group of automorphisms of the cover which possesses at most two orbits in the induced action on its arc set.* We focus on covers whose parameters belong to some known infinite series of feasible parameters. We also complete the classification of arc-transitive covers with a non-solvable automorphism group and show that the automorphism group of any unknown edge-transitive cover induces a one-dimensional affine permutation group on the set of its antipodal classes.

Keywords: Antipodal cover, Distance-regular graph, Vertex-transitive graph, Arc-transitive graph.

1. Introduction

A distance-regular cover of a complete graph with parameters (n, r, μ) (or an (n, r, μ)-*cover*) is a connected graph (by "graph" hereafter we mean an undirected graph without loops or multiple edges) whose vertex set can be partitioned into n blocks (or *antipodal classes*) of equal size $r \geq 2$, such that each block induces an r-coclique, the union of any two different blocks induces a perfect matching, and any two non-adjacent vertices in different blocks have exactly $\mu \geq 1$ common neighbors. For an (n, r, μ)-cover Γ, we denote by $\mathcal{CG}(\Gamma)$ the group of all automorphisms of Γ that fix each of its antipodal classes setwise. If the group $\mathcal{CG}(\Gamma)$ is abelian and acts regularly on each antipodal class, then Γ is called an *abelian* (n, r, μ)-cover (see [9]).

Abelian covers form an intriguing, large subclass of covers that, by a Godsil-Hensel criterion, can be characterized in terms of certain matrices over group algebras [9]. Only a few general techniques for constructing such covers are known, and typically they require the existence of a related object (e.g., a crooked function, a generalized Hadamard matrix) that is again hard to construct or yield only covers with specific parameters. A promising task is to describe covers with "rich" automorphism groups. There are three known families of abelian covers with arc-transitive automorphism groups, namely, distance-transitive Taylor graphs, Thas-Somma covers, collinearity graphs of certain generalized quadrangles with a deleted spread, and related covers that come from the quotient construction due to Godsil and Hensel. A recent study [12, 14] shows that the list is almost complete, i.e., a new arc-transitive abelian (n, r, μ)-cover may be discovered only in a few open subcases. However, the general problem of classification of abelian (n, r, μ)-covers, whose automorphism group is vertex-transitive and has at most two orbits in its induced action on the arc set of the cover, is far from being resolved. In this paper, we will investigate this problem by focusing on covers with the triple of parameters (n, r, μ) belonging to some infinite series of feasible

[1]This work was supported by the Russian Science Foundation (project No. 20-71-00122).

parameters. We will also address some open subcases in classification of arc-transitive (n,r,μ)-covers and show that every arc-transitive (n,r,μ)-cover with a non-solvable automorphism group is known and is indeed a member of one of the above-mentioned families whenever it is abelian.

Note that if Γ is a G-vertex-transitive abelian (n,r,μ)-cover with $\mathcal{CG}(\Gamma) \leq G$ such that G induces a rank s permutation group G^Σ on the set Σ of antipodal classes of Γ, then the arc set of Γ is partitioned into a collection of $s-1$ orbitals of G. Therefore, the arc set of Γ is the union of at most two orbitals of G if and only if $s \leq 3$. This observation allows us to apply the classification of permutation groups of rank at most 3 in our arguments, as well as the classification results on abelian covers with primitive rank 3 groups G^Σ obtained in [15–17].

The first class of covers considered in this paper (see Section 2) is abelian (n,r,μ)-covers Γ with parameters of the form

$$(n,r,\mu) = \Big((t^2-1)^2, r, (t^2+t-1)\frac{(t-1)^2}{r}\Big),$$

where $-t$ is the smallest eigenvalue of the cover, and r is an odd divisor of $t-1$ with $\gcd(r,3)=1$. The study of covers with such parameters is motivated by the following result.

Theorem 1 (Coutinho, Godsil, Shirazi, Zhan [6]). *Let X be an abelian (n,r,μ)-cover with eigenvalues $n-1 > \theta > -1 > \tau$, where r is odd and $\delta = \lambda - \mu$. It yields a set of complex equiangular lines for which the absolute bound is attained if and only if either $r = \mu = 3 = \sqrt{n}$ or $t = -\tau \in \mathbb{N}$, $\gcd(3,r) = 1$, r divides $t-1$, and the eigenvalues θ, τ, and the parameters (n,r,μ,δ) are as follows*:

$$\theta = (t^2-2)t, \quad \tau = -t, \quad n = (t^2-1)^2, \quad r\mu = (t-1)^2(t^2+t-1) \quad \delta = (t^2-3)t,$$

where

$$m_\theta = (t^2-1)(r-1), \quad m_\tau = (t^2-2)(t^2-1)(r-1),$$

here m_σ denotes the multiplicity of the eigenvalue $\sigma \in \{\theta, \tau\}$.

In the case $(n,r,\mu) = (9,3,3)$, according to the result of Brouwer and Wilbrink (see [3, p. 386]), there exist exactly two non-isomorphic covers from the conclusion of Theorem 1; they can be constructed by removing one of two non-isomorphic spreads from the (unique) generalized quadrangle $\mathrm{GQ}(2,4)$. One of them is distance-transitive, so its automorphism group induces a rank 2 permutation group on the antipodal classes. The automorphism group of the other acts vertex-transitively but intransitively on the arc set, and induces a rank 3 permutation group on the set of antipodal classes. The existence of covers with parameters

$$(n,r,\mu) = \Big((t^2-1)^2, r, (t^2+t-1)\frac{(t-1)^2}{r}\Big)$$

from the conclusion of this theorem is an open question (see the discussion in [6]). We will describe some basic properties of automorphism groups of covers Γ with these parameters and apply them to investigate G-vertex-transitive covers Γ under restrictions on the rank s of the group G^Σ or on the parameter t: (i) $s \leq 3$ or (ii) $t \leq 11$.

The second class considered (see Section 3) is abelian covers Γ, for which the group G^Σ is an affine group of permutation rank $s = 2$. A classification of such pairs (Γ, G) was given in [14]; however, the question of a complete description of pairs (Γ, G) has remained open in the case when $|\Sigma|$ is even and G^Σ is an affine group of one-dimensional, symplectic or G_2 type. We will show that in this case each cover Γ with $r > 2$ and a non-solvable group G is known and isomorphic to a quotient of a certain distance-transitive Thas-Somma cover (see Theorem 2). As a result, we will complete the classification of arc-transitive (n, r, μ)-covers with a non-solvable automorphism group. We will also show that the automorphism group of any unknown edge-transitive (n, r, μ)-cover must induce a one-dimensional affine permutation group on the set of its antipodal classes (see Theorem 3).

Our terminology and notation are mostly standard and can be found in [1, 3] and [17].

2. Covers with parameters $\big((t^2-1)^2, r, (t^2+t-1)(t-1)^2/r\big)$

In this section, Γ is an abelian cover with parameters

$$\Big((t^2-1)^2, r, (t^2+t-1)(t-1)\frac{(t-1)}{r}\Big),$$

where t is a positive integer, r is an odd divisor of $t-1$ such that $\gcd(3, r) = 1$, and with eigenvalues

$$k = t^2(t^2-2), \quad \theta = (t^2-2)t, \quad -1, \quad \tau = -t$$

of multiplicities

$$1, \quad m_\theta = (t^2-1)(r-1), \quad k, \quad m_\tau = (t^2-2)(t^2-1)(r-1),$$

respectively. Let Σ be the set of antipodal classes of Γ, $n = |\Sigma|$, $v = nr$, $a \in F \in \Sigma$, and suppose that $K = \mathcal{CG}(\Gamma) \le G \le \mathrm{Aut}(\Gamma)$, G acts transitively on the vertices of Γ, and put $M = G_{\{F\}}$ and $C = G_F$.

Note that

$$\mu = (t^3-2t+1)(t-1)/r, \quad \lambda = (t^3-2t+1)(t-1)/r + t(t^2-3),$$

where λ denotes the number of common neighbors for two adjacent vertices.

Further, for an element $g \in G$, we will denote by $\alpha_i(g)$ the number of vertices x of Γ such that $\partial(x, x^g) = i$. By $\pi(l)$ we will denote the set of prime divisors of a positive integer l. For a finite group X, the set $\pi(|X|)$ is called its *prime spectrum* and is briefly denoted by $\pi(X)$. In what follows, if it is clear from the context, for a graph Φ, we denote by $[x]$ the adjacency of x in Φ, that is, $[x] = \Phi_1(x)$.

Lemma 1. *The following statements are true:*

1) $C = C_G(K) \cap G_a$ *and* $M = K : G_a$;
2) $|G : M| = (t^2-1)^2$ *and* $|G : G_a|$ *divides* $(t^2-1)^2(t-1)$;
3) $\pi(G) = \pi(nr) \cup \pi(G_a) \subseteq \pi\big(n! \cdot (t-1)\big)$;
4) $|\mathrm{Fix}(G_a)| = |N_G(G_a) : G_a|$ *divides* nr, *and* $|\mathrm{Fix}_\Sigma(M)| = |N_G(M) : M|$ *divides* n;
5) *if* $p \in \pi(G)$ *and* $p > t+1$, *then* $p \in \pi(G_F)$;
6) $G/C_G(K) \le \mathrm{Aut}(K)$ *and if* r *is prime, then* $G/C_G(K) \le Z_{r-1}$ *and every element* $g \in G$ *of prime order* $p \notin \pi(r-1)$ *either has no fixed points or* $\mathrm{Fix}(g)$ *is the union of some antipodal classes*;
7) *if* g *is an element of prime order* p *in* G, $p > r$ *and* $p \notin \pi((t^2-1)(t^2-2)t)$, *then* $r_3 = r-1$, $r_2 r_1 > 0$, *and* $|\Omega| \ge r(1+r_1) + r \cdot z$ *for some non-negative integer* z *which is a multiple of* p, *where* $r_i = |\Gamma_i(a)| \pmod p$, $i = 1, 2, 3$.

P r o o f straightforwardly follows from the assumptions that Γ is abelian and G is transitive on its vertices. □

Lemma 2. *Let $p \in \pi(G)$. If $p > t-1$ and p does not divide $t+1$, then*

$$\max\{p, |\mathrm{Fix}_\Sigma(g)|\} < \mu;$$

in particular, if $p > \mu/2$, then for any element $g \in G$ of order p, the subgraph $\mathrm{Fix}(g)$ *is an abelian* $(|\mathrm{Fix}_\Sigma(g)|, r, \mu')$*-cover with*

$$\mu' \equiv \mu \pmod p, \quad \mu > |\mathrm{Fix}_\Sigma(g)| > (r-1)\mu', \quad \pi(r) \subseteq \pi(|\mathrm{Fix}_\Sigma(g)|).$$

P r o o f. Let $g \in G$ and $|g| = p > t-1$. Suppose that p does not divide $t+1$. Then $\Omega := \mathrm{Fix}(g) \neq \varnothing$ and $|\Omega| = lr$, where $l = |\mathrm{Fix}_\Sigma(g)|$.

(1) Suppose $\alpha_2(g) > 0$. By definition, this means that $\partial(x, x^g) = 2$ for some vertex $x \in \Gamma - \Omega$. But then $[x] \cap [x^g]$ contains exactly l vertices from Ω, implying $l \leq \mu$.

Assume $p > \mu$. Then $[a] \cap [b] \subset \Omega$ for any non-adjacent vertices a and b from Ω belonging to different antipodal classes. But according to [9, Lemma 3.1], Ω is a (l, r, μ)-cover and thus $l > (r-1)\mu > \mu$, which is clearly impossible. Therefore, $p \leq \mu$ and since the number μ is composite, $p < \mu$. Notice that if $p > \mu/2$, then

$$0 < \mu' := |[a] \cap [b] \cap \Omega| \equiv \mu \pmod p < \mu/2 < p$$

for any non-adjacent vertices a and b from Ω, belonging to different antipodal classes. Then by [9, Lemma 3.1], Ω is an abelian (l, r, μ')-cover, implying $\mu > l > (r-1)\mu'$ and $\pi(r) \subseteq \pi(l)$ by [10, Theorem 2.5].

(2) Suppose $\alpha_2(g) = 0$. Then $\alpha_0(g) = lr = \alpha_0(g^i)$ for all $1 \leq i \leq p-1$ and $\alpha_1(g) = r(n-l)$. If $\alpha_2(g^s) > 0$ for some $1 < s \leq p-1$, then the desired result is obtained by reasoning as in part (1). Suppose that $\alpha_2(g) = \alpha_2(g^i) = 0$ for all $1 \leq i \leq p-1$. Then each $\langle g \rangle$-orbit in $\Gamma - \Omega$ is an $(n-l)$-clique. Since $n - l \leq \lambda + 2$, we have $l \geq (r-1)\mu \geq \lambda$. Since $|[x] \cap [x^g] \cap \Omega| = l$ for any vertices x and x^g from $\Gamma - \Omega$, we obtain $n - l = 2$, which means that each $\langle g \rangle$-orbit in $\Gamma - \Omega$ is an edge and $g^2 \in G_x$, so $|g| = 2$, a contradiction. The lemma is proven. □

Lemma 3. *Suppose g is an involution of G and $\Omega = \mathrm{Fix}(g) \neq \varnothing$. Let $f = |\Omega \cap F(z)|$ for $z \in \Omega$, $l = |\mathrm{Fix}_\Sigma(g)|$, and*

$$X = \{x \in \Gamma \setminus \Omega | \ [x] \cap \Omega \neq \varnothing\}.$$

Then Ω is a regular graph of degree $l-1$ on lf vertices and the following statements hold:

1) *if $l = 1$, then t is even, $\alpha_3(g) = 0$, $\alpha_1(g) + \alpha_2(g) = (n-1)r$, and Ω coincides with the antipodal class $F(z)$;*
2) *if $l > 1$, then $\alpha_3(g) = (r-f)l$, $\alpha_1(g) + \alpha_2(g) = (n-l)r$, each vertex in X is "on average" adjacent to $fl(n-l)/|X|$ vertices in Ω, and the number of such vertices in Ω does not exceed l (in particular, if $|X| = f(n-l)$, then $l \leq \lambda$) and*

$$f \leq \frac{|X|}{n-l} \leq |\Omega| \leq \frac{(\lambda-\mu)\alpha_1(g)}{n-l} + r\mu \leq r\lambda,$$

furthermore, if $F(z) \subset \Omega$, then $X = \Gamma \setminus \Omega$ and either (i) $\alpha_1(g) = (n-l)r$, $l \leq \lambda$, and each $\langle g \rangle$-orbit on $\Gamma \setminus \Omega$ is an edge, or (ii) $\alpha_2(g) > 0$ and $l \leq \mu$;
3) *if t is even, then $X = \{x \in \Gamma | \ \partial(x, x^g) \in \{1, 2\}\}$;*
4) *if $f = 1$ and $l > 1$, then Ω is an l-clique and $l \leq r\mu/(t-1) \leq \mu$;*
5) *if $f > 1$, t is even, and $l > 1$, then the diameter of the graph Ω is 3 and $|\Omega| \leq l + (l-1)^2(l-2)$.*

P r o o f. Let $f = |\Omega \cap F(z)|$ and assume $\Omega \neq \varnothing$ (which is automatically satisfied for even t). Then $|\Omega| = f \cdot l > 0$ and for any two vertices a and b in Ω, we have

$$l - 1 = |[a] \cap \Omega| = |[b] \cap \Omega|,$$

meaning that the graph Ω is regular of degree $l - 1$. Recall that

$$\mu = (t^2 + t - 1)\frac{(t-1)^2}{r}, \quad \lambda = (t^2 - 1)^2 - 2 - (r - 1)\mu.$$

Since r is odd and divides $t - 1$, we have

$$t - 1 \equiv l \equiv \lambda \equiv \mu \pmod 2.$$

Let Λ be the set of all vertices from antipodal classes that intersect Ω.

Note that if $\partial(x, x^g) = 3$, then $x \in \Lambda$, $|F(x) \cap \Omega| = f$, and $[x] \subset \Gamma \setminus \Omega$. And if $x \neq x^g$ and $\partial(x, x^g) < 3$, then $x \in \Gamma \setminus \Lambda$ and $|[x] \cap [x^g] \cap \Omega| \equiv \lambda \equiv \mu \pmod 2$. Therefore,

$$\Lambda = \Omega \cup \{x \in \Gamma |\ \partial(x, x^g) = 3\}, \quad \Gamma \setminus \Lambda = \{x \in \Gamma |\ \partial(x, x^g) \in \{1, 2\}\}.$$

From the above, we have

$$\alpha_3(g) = (r - f)l, \quad \alpha_1(g) + \alpha_2(g) = (n - l)r,$$

and for each vertex $y \in \Gamma \setminus \Lambda$, $|[y] \cap \Omega| \leq l$, so that either $|[y] \cap \Omega| \leq \mu$ and $\partial(y, y^g) = 2$, or $|[y] \cap \Omega| \leq \lambda$ and $\partial(y, y^g) = 1$. On the other hand, each vertex in Λ is adjacent exactly to $n - l$ vertices in $\Gamma \setminus \Lambda$. Let

$$X = \big\{x \in \Gamma \setminus \Omega |\ [x] \cap \Omega \neq \varnothing\big\}.$$

Then

$$\bigcup_{b \in F(z)} [b] \cap (\Gamma \setminus \Lambda) \subseteq X,$$

implying $|X| \geq (n - l)f$, and $X = \Gamma \setminus \Lambda$ for even t. Estimating the number of edges between X and Ω gives

$$|\Omega|(n - l) \leq \min\big\{l(\alpha_1(g) + \alpha_2(g)), \lambda\alpha_1(g) + \mu\alpha_2(g)\big\}.$$

And since

$$\lambda\alpha_1(g) + \mu\alpha_2(g) = (\lambda - \mu)\alpha_1(g) + \mu(n - l)r$$

and the number of vertices in X does not exceed the number of edges from Ω to X, we have $|X| \leq |\Omega|(n - l)$ and

$$\frac{|X|}{n - l} \leq |\Omega| = f \cdot l \leq \frac{(\lambda - \mu)\alpha_1(g)}{n - l} + r\mu \leq r \max\{\mu, \lambda\} = r\lambda.$$

Hence for even t we have $|\Omega| \geq r$, and in particular, for $l = 1$, we have $\Omega = \Lambda$.

Let $l > 1$ and Φ be a connected component of the graph Ω. Then Φ is an f-covering of the complete graph on $m = |\Phi|/f$ vertices, in particular, $\Phi = \Omega$ for even t. Since the number of edges from Ω to X is $|\Omega|(n - l)$, each vertex in X is adjacent "on average" to $fl(n - l)/|X|$ vertices in Ω. Recall that $\mu < \lambda$. Therefore, if $|X| = f(n - l)$, then $l \leq \lambda$.

If $f = 1$, then Ω is an l-clique and $l \leq r\mu/(t - 1)$. And if $f > 1$ and t is even, then the diameter of Ω is 3 and, due to the Moore bound, we have

$$|\Omega| \leq l + (l - 1)^2(l - 2).$$

Suppose $f = r$. If $\alpha_1(g) = (n - l)r$, then $l \leq \lambda$ and each $\langle g \rangle$-orbit on $\Gamma \setminus \Lambda$ is an edge. If $\alpha_1(g) < (n - l)r$ (which is equivalent to $\alpha_2(g) > 0$), then $l \leq \mu$. The lemma is proven. □

Lemma 4. *Suppose that G^{Σ} has permutation rank 3 with subdegrees $1 \le k_1 \le k_2$. Then*

1) *G_a has exactly two orbits on $[a]$, denoted X_1 and X_2, with lengths k_1 and k_2 respectively, satisfying*
$$k_1(\lambda - \lambda_1) = k_2(\lambda - \lambda_2), \tag{2.1}$$
where $\lambda_i = |[x_i] \cap X_i|$ for vertex $x_i \in X_i$, $i = 1, 2$;
2) *if G_a fixes a vertex $a^* \in F(a) - \{a\}$, then*
$$k_1(\mu - \mu_1) = k_2(\mu - \mu_2), \tag{2.2}$$
where $\mu_1 = |[y_1] \cap X_1|$ for vertex $y_1 \in X_1^*$, $\mu_2 = |[y_2] \cap X_2^*|$ for vertex $y_2 \in X_2$, and X_i^* is a G_a-orbit on $[a^*]$ of length k_i, $i = 1, 2$.

P r o o f. As the groups ${G_a}^{[a]}$ and ${G_{\{F\}}}^{\Sigma - \{F\}}$ are permutation isomorphic, (2.1) follows by counting the number of edges between X_1 and X_2 in two different ways, and (2.2) is obtained in a similar way by counting the number of edges between X_1^* and X_2.

□

Proposition 1. *The permutation rank s of the group G^{Σ} is not equal to 2 (so Γ cannot be arc-transitive).*

P r o o f. On the contrary, suppose $s = 2$. Then, since Γ is abelian and $K \le G$, we conclude that G acts transitively on the arcs of Γ. As n is not a prime power, by Burnside's theorem it follows that G^{Σ} is an almost simple group. But then the number $k = t^2(t^2 - 2)$ is a power of a prime (see [12]), a contradiction. □

Proposition 2. *If G^{Σ} is a primitive group of rank 3, then the group G^{Σ} is either almost simple and its socle cannot be an alternating group or a sporadic simple group, or $T \times T \trianglelefteq G^{\Sigma} \le T_0 \wr 2$, where T_0 is a 2-transitive group of degree $n_0 = t^2 - 1$ with a simple non-abelian socle T and T cannot be an alternating group of degree n_0.*

P r o o f. Suppose that G^{Σ} is a primitive group of rank 3 with nontrivial subdegrees k_1 and k_2, where $k_1 \le k_2$. Under this condition, according to [5, Theorem 2.6.14] and [2], the group G^{Σ} of degree $(t^2 - 1)^2$ has two self-paired orbitals on Σ.

Since $n = (t^2 - 1)^2$ is not a prime power, by the classification of primitive groups of rank 3 (e.g., see [4, Chapter 11, Theorem 11.1.1]), the group G^{Σ} is either almost simple or is of wreath product type. Furthermore, according to [15, 16], the socle $\mathrm{Soc}(G^{\Sigma})$ of the group G^{Σ} cannot be an alternating or a sporadic simple group for all t.

Suppose that the group G^{Σ} is of wreath product type, that is, $P = T \times T \trianglelefteq G^{\Sigma} \le T_0 \wr 2$, where T_0 is a 2-transitive group of degree n_0 with a simple non-abelian socle T and $n = n_0^2$. Then nontrivial subdegrees of the group G^{Σ} are $k_1 = 2(n_0 - 1)$ and $k_2 = (n_0 - 1)^2$. Simplifying the equation (2.1), we get
$$2(\lambda - \lambda_1) = (n_0 - 1)(\lambda - \lambda_2).$$
Since the group G_a contains a subgroup $A = S \times S$, where S is the stabilizer of a point in T, and A has exactly two orbits on X_1, say Y_1 and Y_2, of lengths $|Y_1| = |Y_2| = n_0 - 1$, and $A^{Y_i} \simeq S$, λ_1 must be a sum of subdegrees of A^{X_1}. Hence in the case $T \simeq \mathrm{Alt}_{n_0}$ we have $S \simeq \mathrm{Alt}_{n_0 - 1}$ and $\lambda_1 \in \{0, n_0 - 2, n_0 - 1, 2n_0 - 3\}$. But then the above equation implies that $n_0 - 1 = t^2 - 2$ divides

Table 1. Parameters of Γ with $t \le 11$ (see [6]).

n	r	μ	$\delta = \lambda - \mu$	θ	$\tau = -t$	m_θ	m_τ
1225	5	205	198	204	-6	140	4760
3969	7	497	488	496	-8	378	23436
14400	5	2620	1298	1309	-11	480	57120

one of 2λ or $2(\lambda - 1)$, which is impossible. So $T \not\simeq \mathrm{Alt}_{n_0}$ for all t. □

Next we will consider the case $t \le 11$, in which the parameters of Γ are given in Table 1.

Proposition 3. *If $t \le 11$ and G^Σ is a primitive group of rank s, then $s > 3$.*

P r o o f. Suppose that G^Σ is a primitive group of rank 3 with subdegrees $1, k_1$ and k_2, where $k_1 \le k_2$. Let us consider the case $t \le 11$.

Case $t = 6$. Suppose $t = 6$ (so $n = 1225 = 5^2 7^2$). According to [16, 17], it suffices to consider the case where the group G^Σ is of wreath product type with degree $n = n_0^2$. But then $n_0 = 35$, T_0 is a 2-transitive group of degree 35, and by the classification of finite 2-transitive permutation groups (e.g., see [4, Theorem 11.2.1]), $T \simeq \mathrm{Alt}_{n_0}$, which contradicts to Proposition 2.

Case $t = 8$. Suppose $t = 8$ (so $n = 3969 = 3^4 7^2$). The case of an almost simple group G^Σ is impossible, since according to the classification of primitive almost simple groups of rank 3 (e.g., see [4, Chapter 11] or [7, Tables 4–6]), $\mathrm{Soc}(G^\Sigma)$ cannot be an exceptional simple group or a classical simple group of degree n.

Suppose that the group G^Σ is of wreath product type with degree $n = n_0^2$. Then $n_0 = 63$ and T_0 is a 2-transitive group of degree 63. By Proposition 2, $T \not\simeq \mathrm{Alt}_{n_0}$, so the classification of 2-transitive groups (e.g., see [4, Theorem 11.2.1]) gives $T \simeq PSL_6(2)$. But if $T \simeq PSL_6(2)$, then the subdegrees of A^{Y_i} on Y_i are 1, 1 and 60. Therefore,

$$\lambda_1 \pmod{n_0 - 1} \in \{0, 1, n_0 - 3, n_0 - 2\},$$

and the equation (2.1) has no solution, a contradiction.

Case $t = 11$. Suppose $t = 11$ (so $n = 14400 = 2^6 3^2 5^2$). The case of an almost simple group G^Σ is impossible, since according to the classification of primitive almost simple groups of rank 3 (e.g., see [4, Chapter 11]), $\mathrm{Soc}(G^\Sigma)$ cannot be an exceptional simple group or a classical simple group of degree n.

Suppose the group G^Σ is of wreath product type of degree $n = n_0^2$. By Proposition 2, $T \not\simeq \mathrm{Alt}_{n_0}$, so by [4, Theorem 11.2.1] we have $T \simeq PSp_8(2)$. But if $T \simeq PSp_8(2)$, then the subdegrees of A^{Y_i} on Y_i are $1, 54$ and 64. Therefore,

$$\lambda_1 \pmod{n_0 - 1} \in \{0, 54, 64, n_0 - 1, n_0 - 2, 2n_0 - 3\}.$$

But $\lambda = \mu + \delta = 3918$, and the equation (2.1) has no solution, a contradiction.

The proposition is proven. □

Proposition 4. *If $t = 6$, $\overline{G} := G^\Sigma$ is an imprimitive group of rank s and $\mathcal{B}$ is its imprimitivity system, then either $s > 3$ or $s = 3$ and $\{|B|, |\mathcal{B}|\} = \{25, 49\}$, where $B \in \mathcal{B}$, $\overline{G}_B^B$ and $\overline{G}^{\mathcal{B}}$ are affine 2-transitive groups.*

P r o o f. Suppose that $\overline{G}$ is a rank 3 imprimitive group with subdegrees 1, k_1, and k_2, where $k_1 \leq k_2$. Let $\mathcal{B}$ be the unique nontrivial system of imprimitivity of the group $\overline{G}$ (see [8, Lemma 3.3]), and fix arbitrarily its block B (of size k_1+1). Then k_1+1 divides

$$\gcd(k_2,(t^2-1)^2), \quad \overline{G} \leq \overline{G}_B{}^B \wr \mathrm{Sym}(\mathcal{B}),$$

$\overline{G}_B$ acts 2-transitively on B, and G acts 2-transitively on $\mathcal{B}$ (e.g., see [8, Lemma 3.1]). Let $T_0 = \overline{G}_B{}^B$ and $T = \mathrm{Soc}(T_0)$. Let S be the kernel of the action of G on $\mathcal{B}$, W be the full pre-image of $\overline{G}_B$ in G and N be the kernel of the action of W on B. Without loss of generality, we assume that $a \in F \in B$. Then $K \leq N \cap S \trianglelefteq W$ and $N = K : N_a$.

Since $n = 35^2$, then $|B| \in \{5,7,25,35,49,175,245\}$. If $|B| = 35, 175$ or 245, then applying the classification of finite 2-transitive groups (e.g., see [4, Theorem 11.2.1]) we obtain $T \simeq \mathrm{Alt}(B)$. But Lemma 2 implies $\max(\pi(G)) \leq 101$, so $|B| = 35 = |\mathcal{B}|$ and $T \simeq \mathrm{Soc}(G^{\mathcal{B}}) \simeq \mathrm{Alt}(\mathcal{B})$. In this case $\mathrm{Alt}_{34} \leq W/S \trianglelefteq \mathrm{Sym}_{34}$. On the other hand, $W/N \trianglelefteq \mathrm{Sym}_{35}$, in particular, W/N contains an element of order 35. Since the group NS is normal in W, then either $S \leq N$ or $NS/N \geq \mathrm{Soc}(W/N) \simeq T$. If $S \leq N$, then $\mathrm{Alt}_{35} \leq (W/S)/(N/S) \simeq W/N$, a contradiction. Hence $|W : NS| \leq 2$ and the group NS acts 3-transitively on B, so $|G_{\{F\}} : NS_{\{F\}}| \leq 2$. But then, simplifying the equation (2.1), we get

$$(\lambda - \lambda_1) = (k_1+1)(\lambda - \lambda_2),$$

which contradicts the fact that $\lambda_1 \in \{0, k_1 - 1\}$ in this case.

If $|B| = 5$ or 7, then $|\mathcal{B}| = 175$ or 245, and in view of [4, Theorem 11.2.1] we obtain $\mathrm{Soc}(G^{\mathcal{B}}) \simeq \mathrm{Alt}(\mathcal{B})$. But $\max(\pi(G)) \leq 101$ by Lemma 2, a contradiction. So $\{|B|, |\mathcal{B}|\} = \{25, 49\}$.

1. Let $|B| = 25$. Then simplifying the equation (2.1), we get

$$(\lambda - \lambda_1) = 50(\lambda - \lambda_2) > 0, \quad \lambda_1\lambda_2 > 0.$$

   In view of [4, Theorem 11.2.1] we have either $T \simeq \mathrm{Alt}(B)$ or $T \simeq E_{25}$ and W^B is an affine group. But in the first case G_a is 2-transitive on X_1, hence $\lambda_1 \in \{0, k_1 - 1\}$ and the above equation has no solution, a contradiction. Hence $T \simeq E_{25}$. Applying [4, Theorem 11.2.1] again, we obtain that either $\mathrm{Soc}(G^{\mathcal{B}}) \simeq \mathrm{Alt}(\mathcal{B})$ or $\mathrm{Soc}(G^{\mathcal{B}}) \simeq E_{49}$ and $G^{\mathcal{B}}$ is an affine group.

   Suppose that $\mathrm{Soc}(G^{\mathcal{B}}) \simeq \mathrm{Alt}(\mathcal{B})$. Then $\mathrm{Alt}_{48} \leq W/S \leq \mathrm{Sym}_{48}$ and $W - S$ contains an element g of order 37, which fixes pointwise 11 blocks in $\mathcal{B} \setminus \{B\}$. In view of Lemma 1 $g \in G_F \cap N$ and $|\mathrm{Fix}_\Sigma(g)| \geq 25 \cdot 12 = 300$, which is a contradiction to Lemma 2.

2. Let $|B| = 49$. Then $k_1 = 48$ and $k_2 = 49 \cdot 24$, so the equation (2.1) implies

$$2(\lambda - \lambda_1) = 49(\lambda - \lambda_2) > 0, \quad \lambda_1\lambda_2 > 0.$$

   In view of the classification of 2-transitive groups (e.g., see [4, Theorem 11.2.1]) we obtain either $T \simeq \mathrm{Alt}(B)$ or $T \simeq E_{49}$ and W^B is an affine group. But in the first case G_a is 2-transitive on X_1, so $\lambda_1 \in \{0, k_1 - 1\}$ and the equation above has no solution, a contradiction. Hence $T \simeq E_{49}$. Applying again [4, Theorem 11.2.1], we obtain either $\mathrm{Soc}(G^{\mathcal{B}}) \simeq \mathrm{Alt}(\mathcal{B})$ or $\mathrm{Soc}(G^{\mathcal{B}}) \simeq E_{25}$ and $G^{\mathcal{B}}$ is an affine group.

   Suppose $\mathrm{Soc}(G^{\mathcal{B}}) \simeq \mathrm{Alt}(\mathcal{B})$. Then $\mathrm{Alt}_{24} \leq W/S \leq \mathrm{Sym}_{24}$. On the other hand, $W/N \leq AGL_2(7)$ and $G_{\{F\}}/N \leq GL_2(7)$. Since $|GL_2(7)| = 2^5 3^2 7$ and $|B| = 49$, N contains an element g of order 13, which fixes pointwise 11 blocks in $\mathcal{B} \setminus \{B\}$. In view of Lemma 1 $g \in G_F \cap N$ and $|\mathrm{Fix}_\Sigma(g)| \geq 49 \cdot 12 = 588$, which is a contradiction to Lemma 2.

The proposition is proven. □

3. Arc-transitive affine covers

Throughout this section, Γ is an arc-transitive $\big(2^e, r, 2^e/r\big)$-cover, where $r > 2$, with eigenvalues $k = 2^e - 1, \theta, -1, \tau$ of multiplicities $1, m_\theta, k, m_\tau$ respectively. Let Σ be the set of antipodal classes of the graph Γ, $2^e = n = |\Sigma| = r\mu$, $v = nr$, $a \in F \in \Sigma$, and $K = \mathcal{CG}(\Gamma)$. Suppose $|K| = r$, the group $G \leq \mathrm{Aut}(\Gamma)$ acts transitively on the arc set of Γ and induces an affine 2-transitive permutation group G^Σ on Σ. Let $C = G_F$, T be the full preimage of the socle of the group G^Σ in G and $L = {G_a}^\infty$. In [14], it was shown that under these assumptions, the pair (Γ, G) satisfies one of the following conditions:

1) $e = 2dc$, $d = 3$, r divides 2^c, T is an elementary abelian group of order $2^e r$, not containing any subgroup of order 2^e that is normal in G, and $G_2(2^c) \simeq L \trianglelefteq G_a$;
2) $e = 2dc$, $d \geq 1$, r divides 2^c, T is an elementary abelian group of order $2^e r$, not containing any subgroup of order 2^e that is normal in G, and $Sp_{2d}(2^c) \simeq L \trianglelefteq G_a$;
3) $G_a \leq \Gamma L_1(2^e)$.

In the corresponding cases, we will say that the group G_a is of type G_2, of symplectic type, or of one-dimensional type. Moreover, by [14], $L \leq C$ whenever the group G_a is of symplectic or of G_2 type.

Recall that for each subgroup $1 < N < K$ the *quotient graph* Γ^N, that is the graph on the set of N-orbits where two orbits are adjacent if and only if there is an edge of Γ joining them, is a $(n, r/|N|, \mu|N|)$-cover [9].

Theorem 2. *Suppose $r > 2$ and the group G_a is of symplectic or of G_2 type. Then the $\big(2^e, r, 2^e/r\big)$-cover Γ is isomorphic to a quotient Φ^U of a distance-transitive Thas–Somma $\big(2^e, 2^c, 2^{c(2d-1)}\big)$-cover Φ for some subgroup $U \leq \mathcal{CG}(\Phi)$ of index r. Furthermore, if $r = 2^c$, then $\Gamma \simeq \Phi$ and Γ is characterized by its parameters in the class of arc-transitive distance-regular covers of complete graphs with a non-solvable automorphism group.*

P r o o f. Let $L \simeq G_2(2^c)$ or $Sp_{2d}(2^c)$, and $r = 2^s \leq 2^c$. First of all, we will determine possible complements to T in TL, then we will find how many classes of conjugate complements to T the group TL can contain, and to which one the group L can belong. From the conditions, it follows that TL centralizes K and $(TL/K)' = TL/K$. This means that the group TL is a central extension of the group TL/K (e.g., see [1, Ch. 33]).

Let us consider the group T as a $\mathbb{GF}(2)L$-module and K as its L-invariant subspace. Then, with respect to a suitable $\mathbb{GF}(2)$-basis in T (e.g., see [1, 13.2]), the matrices of elements g of the group L, considered as a subgroup of $\mathrm{Aut}(T) \simeq GL_{e+s}(2)$, have the following form:

$$\begin{pmatrix} \varphi(g) & 0 \\ \psi(g) & I_s \end{pmatrix},$$

where $\varphi : L \to GL_e(2)$ is a homomorphism and I_s is the identity matrix of order $s \times s$. Since T is irreducible as a $\mathbb{GF}(2)L$-module, we have $\psi \not\equiv 0$. Let $\widetilde{V} = T/K$ and let us fix a nonzero vector $w \in K$. Define the map $\gamma_w : L \to \widetilde{V}$ by the rule $g\gamma_w = w\psi(g) + K$ for all $g \in L$. Then for all $g, h \in L$, we have

$$(gh)\gamma_w = (g\gamma_w)^h + h\gamma_w,$$

which means γ_w is a cocycle (e.g., see [1, Ch. 6]). Recall that according to [11], the first cohomology group $\mathrm{H}^1(L, \widetilde{V})$ of the group L in $\widetilde{V}$ is one-dimensional as a $\mathbb{GF}(2^c)$-space. Therefore (e.g., see [1, 17.7]), multiplication by a scalar $f \in \mathbb{GF}(2^c)$ gives a set of functions $(fI_s)\psi(g)$, which determines representatives R of all classes of conjugate complements to T in TL.

Case $r = 2^c$. Since $[T, L] = T$ and $K \le Z(TL)$, by [1, 17.12] when $r = 2^c$, the $\mathbb{GF}(2)L$-module T is the largest among $\mathbb{GF}(2)L$ extensions V of the module K by $\widetilde{V}$ with the property $[V, L] = V$ and $K \le C_V(G)$. Representatives R_1 and R_2 of any two classes of complements to T in TL are conjugate in $\mathrm{Aut}(T)$ by matrices of the form $diag(I_e, fI_s)$, where $f \in \mathbb{GF}(2^c)$. This implies that the set of orbital $(2^e, r, 2^e/r)$-covers of the group TL with vertex stabilizer R_1 gives the set of orbital $(2^e, r, 2^e/r)$-covers of the group TL with vertex stabilizer R_2, where each graph from the first collection is isomorphic to some graph from the second collection and vice versa. Therefore, it is enough to find the set of all orbital $(2^e, r, 2^e/r)$-covers of the group TL with vertex stabilizer L. One of such covers is the distance-transitive $(2^e, r, 2^e/r)$-cover Φ from the Thas–Somma construction [10, Example 3.6, Proposition 6.2]. According to [10], it is characterized by its parameters in the class of distance-transitive covers. Next we will show that it is also characterized by its parameters in the class of arc-transitive covers with a non-solvable group of automorphisms. Note that $G_2(2^c) \le Sp_6(2^c)$ (e.g., see [18, Theorem 3.7]), and only in the case when $d = 3$ two different types are permissible for the group G_a.

Let $L \simeq Sp_{2d}(2^c)$. Since the groups $L^{[a]}$ and $L^{\Sigma-\{F\}}$ are permutation isomorphic, in order to determine the subdegrees of the group G_a on $[a]$, it suffices to consider the action of the group $Sp_{2d}(2^c)$ on its natural module W. Recall that it acts transitively on the set of all hyperbolic pairs in the space W, and for each non-zero vector w in W, there are exactly q^{2d-1} hyperbolic pairs of the form (w, u). Therefore, the stabilizer of the vector w fixes pointwise $q-1$ non-zero vectors from $\langle w \rangle$ and has $q-1$ orbits on W of length q^{2d-1}. If a vertex $b \in [a]$ is associated with the vector w, then, since each vertex b^* from $F(b) - \{b\}$ is adjacent to exactly $\mu = q^{2d-1}$ vertices from $[a] - [b]$ and $\lambda - \mu = -2$, the last statement implies that the group $L_{a,b} = L_{a,b^*}$ acts transitively on each μ-subgraph of the form $[a] \cap [b^*]$.

Suppose $\Gamma \ne \Phi$. Let Q and S be the self-paired orbitals of the group TL, corresponding to the arc sets of the graphs Φ and Γ, respectively. Then $Q(a) \ne S(a)$ and $\Gamma_1(a^*) = S(a^*) = Q(a) = \Phi_1(a)$ for some vertex $a^* \in F(a)$. Since T is an elementary abelian group, regular on vertices, for some involutions $g \in T - K$ and $h \in K$, we have $a^g \in Q(a)$ and $a^{gh} \in S(a)$. Without loss of generality, we may assume that $a^* = a^h$, so that $(a^*)^g = a^{gh}$. Then $\Gamma_1(a) = S(a) = Q(a^*) = \Phi_1(a^*)$. Since $|Q(a^{gh}) \cap Q(a^*)| = |S(a^{gh}) \cap S(a)| = \lambda$ and $\lambda < \mu$, from the action of the group $L_{a,a^g} = L_{a^*,(a^*)^g}$ on Σ, we obtain $Q(a^{gh}) \cap Q(a^*) = S(a^{gh}) \cap S(a)$, which implies $Q \cap S \ne \varnothing$, a contradiction.

Let $L \simeq G_2(2^c)$. Then, to determine the subdegrees of the group G_a on $[a]$, it suffices to consider the action of the group $G_2(2^c)$ as a subgroup of $Sp_6(2^c)$ on its natural module W (e.g., see [18, 4.3]). Recall that the stabilizer of a 1-dimensional subspace in L has exactly four orbits on the 1-dimensional subspaces of W, and their lengths are 1, $q(q+1)$, $q^3(q+1)$, and q^5. Therefore, the stabilizer of a non-zero vector w in L fixes pointwise $q-1$ non-zero vectors from $\langle w \rangle$ and has $q-1$ orbits on W of length q^5. By reasoning as above, we obtain the required statement.

Further, according to [9, Theorem 6.2], for any subgroup U in K of index $2^l := |K : U| > 1$, the quotient graph Φ^U is a $(2^e, 2^l, 2^{c-l}\mu)$-cover, and the group TL/U acts as an arc-transitive group of automorphisms of this graph. Next we will show that these covers exhaust all possibilities for Γ when $s < c$.

Case $r < 2^c$. In this case, $s < c$. Let us show that the only candidates for Γ are the quotient graphs Φ^U of the distance-transitive Thas-Somma $(2^e, 2^c, 2^{c(2d-1)})$-cover Φ defined for subgroups $U \le \mathcal{CG}(\Phi)$ of index 2^s. Let $\Psi = \Phi^U$ be such a quotient graph, and let Q and S be the self-paired orbitals of the group TL, corresponding to the arc sets of the graphs Ψ and Γ, respectively.

Suppose $\Gamma \ne \Psi$. A contradiction is obtained by a similar argument as before. Indeed, then $Q(a) \ne S(a)$ and $\Gamma_1(a^*) = S(a^*) = Q(a) = \Psi_1(a)$ for some vertex $a^* \in F(a)$. Since T is an elementary abelian group, regular on vertices, for some involutions $g \in T - K$ and $h \in K$, we have $a^g \in Q(a)$ and $a^{gh} \in S(a)$. Without loss of generality, we may assume that $a^* = a^h$, so that

$(a^*)^g = a^{gh}$. Then $\Gamma_1(a) = S(a) = Q(a^*) = \Psi_1(a^*)$. Since

$$|Q(a^{gh}) \cap Q(a^*)| = |S(a^{gh}) \cap S(a)| = \lambda = \mu - 2 = q^{2d-1}2^{c-s} - 2,$$

and λ is not divisible by q^{2d-1}, from the action of the group $L_{a,a^g} = L_{a^*,(a^*)^g}$ on Σ, we obtain

$$\big(Q(a^{gh}) \cap Q(a^*)\big) \cap \big(S(a^{gh}) \cap S(a)\big) \neq \varnothing,$$

which implies $Q \cap S \neq \varnothing$, a contradiction.

The rest statements of theorem follow from the classification of edge-transitive covers in the almost simple case (see [12]) and in the affine case (see [14]). □

Theorem 2 together with the results of [13, 14] and [12] yields the main theorem of this section.

Theorem 3. *If Φ is an unknown edge-transitive (n, r, μ)-cover, then the group* $\mathrm{Aut}(\Phi)$ *induces a one-dimensional affine permutation group on the set of its antipodal classes. Moreover, if the automorphism group of the cover Φ acts transitively on its arc set, then it is solvable, $\mu > 1$, $n = r\mu$ is a prime power, and $|\mathcal{CG}(\Phi)| = r$.*

4. Concluding remarks

This paper finalizes a major part of the project of classifying edge-transitive distance-regular covers of complete graphs, providing a complete classification in the case when the automorphism group of such cover is non-solvable and arc-transitive.

In general, a larger class of G-vertex-transitive covers having at most two G-orbits in the induced action of G on the arc set is still not well understood, and its study remains relevant in the context of finding new constructions of covers with various parameters of interest, e.g., of abelian covers with parameters $((t^2-1)^2, r, (t^2+t-1)(t-1)^2/r)$ whose study is motivated by important problems in discrete geometry [6]. In this paper, we established some basic properties of automorphism groups of abelian covers Γ with these parameters and applied them to investigate G-vertex-transitive covers Γ under restrictions on the rank s of the group G^{Σ} or the parameter t: $s \leq 3$ or $t \leq 11$. This work will be extended in a forthcoming publication of the author for covers with arbitrary values of t.

Next we list a few open questions:

1) Classify arc-transitive covers with a solvable automorphism group.
2) Classify half-transitive covers.
3) Is there any G-vertex-transitive cover with parameters $((t^2-1)^2, r, (t^2+t-1)(t-1)^2/r)$ having precisely two G-orbits in the induced action of G on the arc set?

REFERENCES

1. Aschbacher M. *Finite Group Theory*, 2nd ed. Cambridge: Cambridge University Press, 2000. 305 p. DOI: 10.1017/CBO9781139175319
2. Berggren J. L. An algebraic characterization of finite symmetric tournaments. *Bull. Austral. Math. Soc.*, 1972. Vol. 6, No. 1. P. 53–59. DOI: 10.1017/S0004972700044257
3. Brouwer A. E., Cohen A. M., Neumaier A. *Distance–Regular Graphs*. Berlin etc: Springer–Verlag, 1989. 494 p. DOI: 10.1007/978-3-642-74341-2
4. Brouwer A. E., Van Maldeghem H. *Strongly Regular Graphs*. Cambridge: Cambridge University Press, 2022. 462 p. DOI: 10.1017/9781009057226
5. Chen G., Ponomarenko I. *Lecture Notes on Coherent Configurations*. 2023. 356 p. URL: https://www.pdmi.ras.ru/ inp/ccNOTES.pdf

6. Coutinho G., Godsil C., Shirazi M., Zhan H. Equiangular lines and covers of the complete graph. *Lin. Alg. Appl.*, 2016. Vol. 488. P. 264–283. DOI: 10.1016/j.laa.2015.09.029
7. Coutts H. J., Quick M. R., Roney-Dougal C. M. The primitive groups of degree less than 4096. *Comm. Algebra*, 2011. Vol. 39. P. 3526–3546. DOI: 10.1080/00927872.2010.515521
8. Devillers A., Giudici M., Li C.H., Pearce G., Praeger Ch. E. On imprimitive rank 3 permutation groups. *J. London Math. Soc.*, 2011. Vol. 84. P. 649–669. DOI: 10.1112/jlms/jdr009
9. Godsil C. D., Hensel A. D. Distance regular covers of the complete graph. *J. Comb. Theory Ser. B.*, 1992. Vol. 56, No. 1. P. 205–238. DOI: 10.1016/0095-8956(92)90019-T
10. Godsil C. D., Liebler R. A., Praeger C. E. Antipodal distance transitive covers of complete graphs. *Europ. J. Comb.*, 1998. Vol. 19, No. 4. P. 455–478. DOI: 10.1006/eujc.1997.0190
11. Jones W., Parshall B. On the 1-cohomology of finite groups of Lie type. In: *Proc. Conf. on Finite Groups, eds: W.R. Scott and F. Gross*. New York: Academic Press, 1976. P. 313–327. DOI: 10.1016/B978-0-12-633650-4.50022-9
12. Tsiovkina L. Yu. Covers of complete graphs and related association schemes. *J. Comb. Theory Ser. A.*, 2022. Vol. 191. Art. no. 105646. DOI: 10.1016/j.jcta.2022.105646
13. Tsiovkina L. Yu. On a class of edge-transitive distance-regular antipodal covers of complete graphs. *Ural Math. J.*, 2021. Vol. 7, No. 2. P. 136–158. DOI: 10.15826/umj.2021.2.010
14. Tsiovkina L. Yu. Arc-transitive groups of automorphisms of antipodal distance-regular graphs of diameter 3 in affine case. *Sib. Èlektron. Mat. Izv.*, 2020. Vol. 17. P. 445–495. DOI: 10.33048/semi.2020.17.029 (in Russian)
15. Tsiovkina L. Yu. On a class of vertex-transitive distance-regular covers of complete graphs. *Sib. Èlektron. Mat. Izv.*, 2021. Vol. 18, No. 2. P. 758–781. DOI: 10.33048/semi.2021.18.056 (in Russian)
16. Tsiovkina L. Yu. On a class of vertex-transitive distance-regular covers of complete graphs II. *Sib. Èlektron. Mat. Izv.*, 2022. Vol. 19, No. 1. P. 348–359. DOI: 10.33048/semi.2022.19.030 (in Russian)
17. Tsiovkina L. Yu. On some vertex-transitive distance-regular antipodal covers of complete graphs. *Ural Math. J.*, 2022. Vol. 8, No. 2. P. 162–176. DOI: 10.15826/umj.2022.2.014
18. Wilson R. A. *The Finite Simple Groups*. Grad. Texts in Math., vol. 251. London: Springer–Verlag, 2009. 298 p. DOI: 10.1007/978-1-84800-988-2

Editor: Tatiana F. Filippova

Managing Editor: Oxana G. Matviychuk

Design: Alexander R. Matviychuk

Contact Information

16 S. Kovalevskaya str., Ekaterinburg, Russia, 620990

Phone: +7 (343) 375-34-73

Fax: +7 (343) 374-25-81

Email: secretary@umjuran.ru

Web-site: https://umjuran.ru

N.N.Krasovskii Institute of Mathematics and Mechanics
of the Ural Branch of Russian Academy of Sciences

Ural Federal University named after the first President of Russia B.N.Yeltsin

Distributed for free