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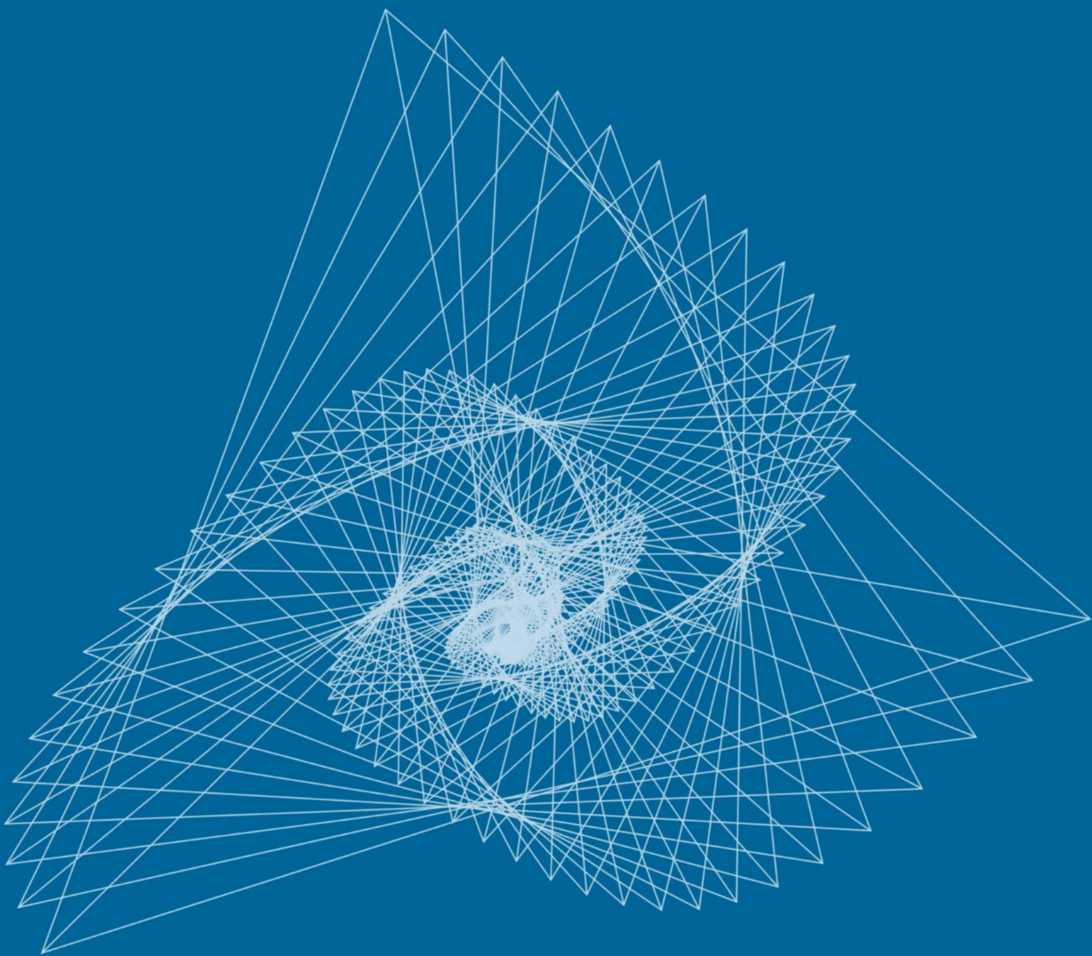

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TWO METHODS OF DESCRIBING 2-LOCAL DERIVATIONS AND AUTOMORPHISMS

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Abstract: In the present paper, we investigate 2-local linear operators on vector spaces. Sufficient conditions are obtained for the linearity of a 2-local linear operator on a finite-dimensional vector space. To do this, families of matrices of a certain type are selected and it is proved that every 2-local linear operator generated by these families is a linear operator. Based on these results we prove that each 2-local derivation of a finite-dimensional null-filiform Zinbiel algebra is a derivation. Also, we develop a method of construction of 2-local linear operators which are not linear operators. To this end, we select matrices of a certain type and using these matrices we construct a 2-local linear operator. If these matrices are distinct, then the 2-local linear operator constructed using these matrices is not a linear operator. Applying this method we prove that each finite-dimensional filiform Zinbiel algebra has a 2-local derivation that is not a derivation. We also prove that each finite-dimensional naturally graded quasi-filiform Leibniz algebras of type I has a 2-local automorphism that is not an automorphism.

Keywords: Linear operator, 2-Local linear operator, Leibniz algebra, Zinbiel algebra, Derivation, 2-Local derivations, Automorphism, 2-Local automorphism

1. Introduction

In 1997, P. Šemrl [20] introduced and investigated so-called 2-local derivations and 2-local automorphisms on operator algebras. He described such maps on the algebra $B(H)$ of all bounded linear operators on an infinite-dimensional separable Hilbert space H. Namely, he proved that every 2-local derivation (automorphism) on $B(H)$ is a derivation (respectively an automorphism).

A similar description of 2-local derivations for the finite-dimensional case appeared later in [17]. In the paper [19] 2-local derivations have been described on matrix algebras over finite-dimensional division rings. In [9] Sh. Ayupov and K. Kudaybergenov suggested a new technique and have generalized the above-mentioned results of [20] and [17] for arbitrary Hilbert spaces. Namely, they proved that every 2-local derivation on the algebra $B(H)$ of all linear bounded operators on an arbitrary Hilbert space H is a derivation. They obtained also a similar result for the automorphisms. In [4, 10] the authors extended the above results for 2-local derivations and gave a proof of the theorem for arbitrary von Neumann algebras.

Afterwards, 2-local derivations have been investigated by many authors on different algebras and many results have been obtained. In [15] it was established that every 2-local $*$-homomorphism from a von Neumann algebra into a C*-algebra is a linear $*$-homomorphism. These authors also proved that every 2-local Jordan $*$-homomorphism from a JBW*-algebra into a JB*-algebra is a Jordan $*$-homomorphism. Later, in [14] the authors prove that any 2-local automorphism on an arbitrary AW*-algebra without finite type I direct summands is an automorphism.

In the paper [11] 2-local derivations of finite-dimensional Lie algebras are described. The authors proved that every 2-local derivation on a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero is a derivation. They also showed that each finite-dimensional nilpotent Lie algebra L with $\dim L \geq 2$ admits a 2-local derivation which is not a derivation. At the same time, in [18] X. Lai and Z.X. Chen describe 2-local Lie derivations for the case of finite-dimensional simple Lie algebras.

In the paper [12] the authors proved that every 2-local automorphism on a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero is an automorphism and showed that each finite-dimensional nilpotent Lie algebra with dimension ≥ 2 admits a 2-local automorphism which is not an automorphism. Later, in [13] similar results were obtained in the case of finite-dimensional Leibniz algebras. Many papers were devoted to 2-local derivations and automorphisms on Lie and Leibniz algebras. In particular, in the paper [6]it was proven that every 2-local inner derivation on the Lie ring of skew-symmetric matrices over a commutative ring is an inner derivation. They also proved that every 2-local spatial derivation on various Lie algebras of infinite-dimensional skew-adjoint matrix-valued maps on a set is a spatial derivation. In [8] the previous results were extended of the Lie ring of skew-adjoint matrices over a commutative $*$-ring and various Lie algebras of skew-adjoint operator-valued maps on a set, respectively.

In [5] 2-local inner derivations on the Jordan ring $H_n(\Re)$ of symmetric $n \times n$ matrices over a commutative associative ring $\Re$ were investigated. It was proven that every such 2-local inner derivation is a derivation. In the paper [7], the authors introduced and investigated the notion of 2- local linear maps on vector spaces. A sufficient condition was obtained for the linearity of a 2-local linear map on a finite-dimensional vector space. Based on this result, the authors proved that every 2-local inner derivation on finite-dimensional semi-simple Jordan algebras over an algebraically closed field of characteristics different from 2 and a field of characteristics 0 is a derivation. Also, they showed that every 2-local 1-automorphism (i.e. implemented by single symmetries) of the mentioned Jordan algebra is an automorphism.

The present paper is devoted to 2-local linear operators, 2-local derivations and automorphisms on finite-dimensional vector spaces, Leibniz and Zinbiel algebras. This paper is organized as follows:

In Section 2, we investigate 2-local linear operators on vector spaces. Sufficient conditions are obtained for the linearity of a 2-local linear operator on a finite-dimensional vector space. To do this, families of matrices of a certain type are selected and it is proved that every 2-local linear operator generated by these families is a linear operator.

In Section 3, we develop a method of construction of 2-local linear operators which are not linear operators. For this purpose we select matrices of a certain type and using these matrices we construct a 2-local linear operator. If these matrices are distinct, then the 2-local linear operator constructed using these matrices is not a linear operator.

In Section 4, basing on the results of Section 2 we describe 2-local derivations of finite-dimensional null-filiform Zinbiel algebras. Namely, we prove that each 2-local derivation of a finite-dimensional null-filiform Zinbiel algebra is a derivation. Also, applying the method of Section 3 we prove that n-dimensional filiform Zinbiel algebras, $n \geq 5$, have 2-local derivations that are not derivations.

In Section 5, applying the method of Section 3 we prove that finite-dimensional naturally graded quasi-filiform Leibniz algebras of type I have 2-local automorphisms which are not automorphisms.

2. 2-Local liner operators of finite-dimensional vector spaces which are liner operators

Definition 1. *Let V be a vector space over a field $\mathbb{F}$, $\Delta : V \to V$ be a map such that for each pair v, w of elements in V there exists a linear operator $L_{v,w}$ of V satisfying the following conditions*

$$\Delta(v) = L_{v,w}(v), \quad \Delta(w) = L_{v,w}(w).$$

Then Δ is called a 2-local linear operator.

Definition 2. *Let V be a vector space of dimension n over a field $\mathbb{F}$, and let $\nu = \{e_1, e_2, \dots e_n\}$ be a basis of the vector space V. Let $\mathcal{M}$ be a set of $n \times n$ matrices. Then a mapping $\Delta : V \to V$ is called a 2-local linear operator generated by matrices in $\mathcal{M}$, if, for each pair v and w of elements in V, there exists a linear operator $L_{v,w}$ generated by a matrix in $\mathcal{M}$ with respect to ν such that*

$$\Delta(v) = L_{v,w}(v), \quad \Delta(w) = L_{v,w}(w).$$

Let n and m be natural numbers such that $m \leq n$. Let, for fixed k, p such that $1 \leq k \leq n$, $1 \leq p \leq m$,

$$f_{ij}(x_1, x_2, \dots, x_p), \quad i = 1, 2, \dots, m, \quad j \neq k, \quad j = 1, 2, \dots, n,$$

be functions with values in a field $\mathbb{F}$ (including the function $f_{ij}(x_1, x_2, \dots, x_p) \equiv 0$),

$$g_i(x_1, x_2, \dots, x_p), \quad i = 1, 2, \dots m,$$

be functions with values in the field $\mathbb{F}$ such that, for any nonzero elements $\{a_1, a_2, \dots, a_p\} \subset \mathbb{F}$, the following system of equations

$$g_i(x_1, x_2, \dots, x_p) = g_i(a_1, a_2, \dots, a_p), \quad i = 1, 2, \dots m,$$

has a unique solution $x_j = a_j$, $j = 1, 2, \dots, p$, and let $\mathcal{M}_{m,n}(k, p)$ be a set of $m \times n$ matrices A with components a_{ij} such that, there exist nonzero elements $a_i \in \mathbb{F}$, $i = 1, 2, \dots, p$, satisfying the following equalities

$$a_{ik} = g_i(a_1, a_2, \dots, a_p), \quad i = 1, 2, \dots m,$$
$$a_{ij} = f_{ij}(a_1, a_2, \dots, a_p), \quad i = 1, 2, \dots m, \quad j \neq k.$$

Remark 1. Note that, in the definition of the set $\mathcal{M}_{m,n}(k, p)$ components of every matrix A in $\mathcal{M}_{m,n}(k, p)$ are computed using some nonzero elements $a_i \in \mathbb{F}$, $i = 1, 2, \dots, p$.

Also, note that, by the definition of the set $\mathcal{M}_{m,n}(k, p)$, a matrix of this set may contain a row, all components of which are zeros, since $p \leq m$.

Theorem 1. *Let V be a vector space of dimension n over the field $\mathbb{F}$, and let $\nu = \{e_1, e_2, \ldots e_n\}$ be a basis of the vector space V. Let Δ be a 2-local linear operator on V generated by matrices in $\mathcal{M}_{n,n}(k,p)$ with respect to the basis ν. Then Δ is a linear operator generated by a matrix in $\mathcal{M}_{n,n}(k,p)$ with respect to the basis ν.*

P r o o f. Without loss of the generality, we suppose that $k = 1$. Indeed, matrices in $\mathcal{M}_{n,n}(k,p)$ depend on the basis $\nu = \{e_1, e_2, \ldots e_n\}$. If we swap the vectors e_1 and e_k, then we get the set of matrices $\mathcal{M}_{n,n}(1,p)$, i.e., $k = 1$. By the definition, for every element $x \in V$,

$$x = \sum_{i=1}^{n} x_i e_i,$$

there exists a matrix $A_{x,e_1} = (a_{ij}^{x,e_1})_{i,j=1}^{n}$ in $\mathcal{M}_{n,n}(1,p)$ such that

$$\Delta(x) = \widehat{A_{x,e_1}\bar{x}},$$

where $\bar{x} = (x_1, x_2, \ldots, x_n)^T$ is the vector corresponding to x, $\widehat{\bar{x}}$ is an operation on $\bar{x}$ such that $\widehat{\bar{x}} = x$, and

$$\overline{\Delta(e_1)} = A_{x,e_1}\overline{e_1} = (a_{11}^{x,e_1}, a_{21}^{x,e_1}, a_{31}^{x,e_1}, \ldots, a_{n1}^{x,e_1})^T.$$

Since $\Delta(e_1) = L_{x,e_1}(e_1) = L_{y,e_1}(e_1)$, we have

$$\overline{\Delta(e_1)} = (a_{11}^{x,e_1}, a_{21}^{x,e_1}, a_{31}^{x,e_1}, \ldots, a_{n1}^{x,e_1})^T = (a_{11}^{y,e_1}, a_{21}^{y,e_1}, a_{31}^{y,e_1}, \ldots, a_{n1}^{y,e_1})^T$$

for each pair x, y of elements in V. Hence, $a_{q1}^{x,e_1} = a_{q1}^{y,e_1}$, $q = 1, 2, \ldots n$. By the condition, there exist nonzero elements a_i^{x,e_1}, $a_i^{y,e_1} \in \mathbb{F}$, $i = 1, 2, \ldots, p$ such that

$$a_{q1}^{x,e_1} = g_i(a_1^{x,e_1}, a_2^{x,e_1}, \ldots, a_p^{x,e_1}), \quad i = 1, 2, \ldots n,$$
$$a_{q1}^{y,e_1} = g_i(a_1^{y,e_1}, a_2^{y,e_1}, \ldots, a_p^{y,e_1}), \quad i = 1, 2, \ldots n.$$

So, we have

$$g_i(a_1^{x,e_1}, a_2^{x,e_1}, \ldots, a_p^{x,e_1}) = g_i(a_1^{y,e_1}, a_2^{y,e_1}, \ldots, a_p^{y,e_1}), \quad i = 1, 2, \ldots n.$$

By the definition of g_i, $i = 1, 2, \ldots n$, we have

$$a_i^{x,e_1} = a_i^{y,e_1}, \quad i = 1, 2, \ldots p.$$

By the condition, for every component a_{ij}^{z,e_1}, $j \neq 1$, of A_{z,e_1} we have

$$a_{ij}^{z,e_1} = f_{ij}(a_1^{z,e_1}, a_2^{z,e_1}, \ldots, a_p^{z,e_1}), \quad i = 1, 2, \ldots n, j \neq 1.$$

where $z \in \{x, y\}$. Therefore $a_{ij}^{x,e_1} = a_{ij}^{y,e_1}$, $i, j = 1, 2, \ldots n$, i.e. $A_{x,e_1} = A_{y,e_1}$, and

$$\Delta(x) = \widehat{A_{y,e_1}\bar{x}}$$

for any $x \in V$, and the matrix of $\Delta(x)$ does not depend on x. Hence Δ is a linear operator, and the matrix A_{y,e_1} is the matrix of Δ. The proof is complete. $\square$

Let n be a natural number, and let $\{i_1, i_2, \dots i_p\}$ and $\{j_1, j_2, \dots j_q\}$ be subsets of $\{1, 2, \dots, n\}$ such that

$$p + q = n, \quad \{i_1, i_2, \dots i_p\} \cup \{j_1, j_2, \dots j_q\} = \{1, 2, \dots, n\}.$$

Let, for fixed k, m, l and s such that $1 \leq k, m, l, s \leq n$, $k \neq m$,

$$\mathcal{M}_n(k, m, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$$

be a set of $n \times n$ matrices $A = (a_{ij})_{i,j=1}^n$ such that the $p \times n$ submatrix

$$A_1 : a_{\alpha\beta}, \alpha \in \{i_1, i_2, \dots i_p\}, \quad \beta = 1, 2, \dots, n,$$

belongs to the set $\mathcal{M}_{p,n}(k, l)$ and the $q \times n$ submatrix

$$A_2 : a_{\alpha\beta}, \alpha \in \{j_1, j_2, \dots j_q\}, \quad \beta = 1, 2, \dots, n,$$

belongs to the set $\mathcal{M}_{q,n}(m, s)$. Then the following theorem takes place.

Theorem 2. *Let V be a vector space of dimension n over the field $\mathbb{F}$, and let $\nu = \{e_1, e_2, \dots e_n\}$ be a basis of the vector space V. Let Δ be a 2-local linear operator on V generated by matrices in $\mathcal{M}_n(k, m, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$ with respect to the basis ν. Then Δ is a linear operator generated by a matrix in*

$$\mathcal{M}_n(k, m, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$$

with respect to the basis ν.

P r o o f. Without loss of generality, we suppose that $k = 1$, $m = n$. Indeed, matrices in $\mathcal{M}_n(k, m, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$ depend on the basis $\nu = \{e_1, e_2, \dots e_n\}$. If we swap the vectors e_1 and e_k, e_m and e_n respectively then we get the set of matrices $\mathcal{M}_n(1, n, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$, i.e., $k = 1$, $m = n$. Then, by definition of Δ, for every element $x \in V$,

$$x = \sum_{i=1}^{n} x_i e_i,$$

there exists a matrix

$$A_{x,e_1} = (a_{ij}^{x,e_1})_{i,j=1}^n$$

in $\mathcal{M}_n(1, n, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$ such that

$$\Delta(x) = \widehat{A_{x,e_1}\bar{x}},$$

where $\bar{x} = (x_1, x_2, \dots, x_n)^T$ is the vector corresponding to x, $\hat{\bar{x}}$ is an operation on $\bar{x}$ such that $\hat{\bar{x}} = x$, and

$$\overline{\Delta(e_1)} = \overline{L_{x,e_1}(e_1)} = A_{x,e_1}\overline{e_1} = (a_{11}^{x,e_1}, a_{21}^{x,e_1}, a_{31}^{x,e_1}, \dots, a_{n1}^{x,e_1})^T,$$

where L_{x,e_1} is a linear operator, generated by A_{x,e_1}. Since $\Delta(e_1) = L_{x,e_1}(e_1) = L_{y_1,e_1}(e_1)$, we have

$$\overline{\Delta(e_1)} = (a_{11}^{x,e_1}, a_{21}^{x,e_1}, a_{31}^{x,e_1}, \dots, a_{n1}^{x,e_1})^T = (a_{11}^{y_1,e_1}, a_{21}^{y_1,e_1}, a_{31}^{y_1,e_1}, \dots, a_{n1}^{y_1,e_1})^T$$

for each pair, x, y_1 of elements in V. Hence,

$$a_{\alpha 1}^{x,e_1} = a_{\alpha 1}^{y_1,e_1}, \alpha \in \{i_1, i_2, \dots i_p\}. \tag{2.1}$$

By the definition of $\mathcal{M}_n(1, n, i_1, i_2, \dots i_p, j_1, j_2, \dots j_q, l, s)$ the submatrix

$$\{a^{x,e_1}_{\alpha j}\}_{\alpha\in\{i_1,i_2,\dots i_p\},\ j=1,2,\dots,n}$$

belongs to the set of matrices $\mathcal{M}_{p,n}(1, l)$. Hence, by definition of the set $\mathcal{M}_{p,n}(1, l)$ there exist mappings

$$g_i(x_1, x_2, \dots, x_l), \quad i = 1, 2, \dots p,$$

with values in the field $\mathbb{F}$ and nonzero elements $\{a^{x,e_1}_1, a^{x,e_1}_2, \dots, a^{x,e_1}_l\} \subset \mathbb{F}$ depending on x and e_1 such that

$$a^{x,e_1}_{i_\alpha 1} = g_\alpha(a^{x,e_1}_1, a^{x,e_1}_2, \dots, a^{x,e_1}_l), \quad \alpha \in \{1, 2, \dots p\}.$$

Also, there exist nonzero elements $\{a^{x,e_1}_1, a^{x,e_1}_2, \dots, a^{x,e_1}_l\} \subset \mathbb{F}$ depending on x and e_1 such that

$$a^{y_1,e_1}_{\alpha 1} = g_\alpha(a^{y_1,e_1}_1, a^{y_1,e_1}_2, \dots, a^{y_1,e_1}_l), \quad \alpha \in \{i_1, i_2, \dots i_p\}.$$

By the equalities (2.1), we have

$$g_\alpha(a^{x,e_1}_1, a^{x,e_1}_2, \dots, a^{x,e_1}_l) = g_\alpha(a^{y_1,e_1}_1, a^{y_1,e_1}_2, \dots, a^{y_1,e_1}_l), \quad \alpha \in \{1, 2, \dots p\}.$$

By the definition of the functions g_v, $v = 1, 2, \dots p$ in the definition of the set $\mathcal{M}_{p,n}(1, l)$, we have

$$a^{x,e_1}_i = a^{y_1,e_1}_i, \quad i = 1, 2, \dots l. \tag{2.2}$$

By the definition of the set $\mathcal{M}_{p,n}(1, l)$, there exist functions

$$f_{\alpha j}(x_1, x_2, \dots, x_p), \quad \alpha \in \{i_1, i_2, \dots i_p\}, \quad j = 2, \dots, n,$$

with values in the field $\mathbb{F}$ such that, for every component $a^{z,e_1}_{\alpha j}$, $\alpha \in \{i_1, i_2, \dots i_p\}$, $j = 2, 3, \dots, n$, of A_{z,e_1} we have

$$a^{z,e_1}_{\alpha j} = f_{\alpha,j}(a^{z,e_1}_1, a^{z,e_1}_2, \dots, a^{z,e_1}_p), \quad \alpha \in \{i_1, i_2, \dots i_p\}, \quad j = 2, 3, \dots, n.$$

where $z \in \{x, y_1\}$. Therefore, by (2.2), $a^{x,e_1}_{\alpha j} = a^{y_1,e_1}_{\alpha j}$, $\alpha \in \{i_1, i_2, \dots i_p\}$, $j = 1, 2, \dots n$. Hence, for the elements $v \in V_1$, where V_1 is the vector subspace, generated by the vectors $\{e_{i_1}, e_{i_2}, \dots, e_{i_p}\}$, i.e.,

$$V_1 = \langle e_{i_1}, e_{i_2}, \dots, e_{i_p}\rangle$$

and $w \in V_2$, where V_2 is the vector subspace, generated by the vectors $\{e_{j_1}, e_{j_2}, \dots, e_{j_p}\}$, i.e.,

$$V_2 = \langle e_{j_1}, e_{j_2}, \dots, e_{j_p}\rangle$$

such that

$$\widehat{A_{x,e_1}}\bar{x} = v + w,$$

the elements $t \in V_1$ and $r \in V_2$ such that

$$\widehat{A_{y_1,e_1}}\bar{x} = t + r$$

we have

$$v = t.$$

Similarly, from $L_{x,e_n}(e_n) = L_{y_2,e_n}(e_n)$ it follows that

$$a_{\alpha n}^{x,e_n} = a_{\alpha n}^{y_2,e_n}, \quad \alpha \in \{j_1, j_2, \dots j_q\}$$

and

$$a_{\alpha j}^{x,e_n} = a_{\alpha j}^{y_2,e_n}, \quad \alpha \in \{j_1, j_2, \dots j_q\}, \quad j = 1, 2, \dots n.$$

Hence, for the elements $a \in V_1$ and $b \in V_2$ such that

$$\widehat{A_{x,e_n}}\bar{x} = a + b,$$

the elements $c \in V_1$ and $d \in V_2$ such that

$$\widehat{A_{y_2,e_n}}\bar{x} = c + d$$

we have

$$b = d.$$

Therefore, if we take $y_1 = e_n$, $y_2 = e_1$, then, for the elements $f \in V_1$ and $g \in V_2$ such that

$$\widehat{A_{e_1,e_n}}\bar{x} = f + g,$$

we have

$$\widehat{A_{x,e_1}}\bar{x} = v + w = f + w = f + b = f + g = \widehat{A_{e_1,e_n}}\bar{x}$$

since $v = f$, $A_{x,e_1}\bar{x} = A_{x,e_n}\bar{x}$ and $b = g$. So,

$$L_{x,e_1}(x) = L_{x,e_n}(x) = L_{e_1,e_n}(x).$$

for any $x \in V$, and the matrix of $\Delta(x)$ does not depend on x. Hence Δ is a linear operator and the matrix A_{e_1,e_n} is the matrix of Δ. This ends the proof. $\square$

Example 1. Let $\mathcal{J}_{56}$ be the Jordan algebra with a basis $\{e_1, n_1, n_2, n_3\}$ such that

$$n_1^2 = n_2, \quad e_1 n_3 = \frac{1}{2} n_3, \quad e_1 n_i = n_i, \quad i = 1, 2$$

(see Table 3 in [16]). Then the matrix of its arbitrary derivation has the following form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & \beta & 2\alpha & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}.$$

If we take $k = 2$, $m = 4$, $i_1 = 2$, $i_2 = 3$, $j_1 = 4$, $l = 2$, $s = 1$, then the set of such matrices we can take as the set $\mathcal{M}_4(k, m, i_1, i_2, j_1, l, s)$.

Therefore, by Theorem 2, each 2-local automorphism of the Jordan algebra $\mathcal{J}_{56}$ is an automorphism. In this case, $\mathcal{M}_4(k, m, i_1, i_2, j_1, l, s)$ is a set of 4×4 matrices such that the 3×4 submatrix

$$A_1 : a_{\alpha,\beta}, \quad \alpha \in \{1, 2, 3\}, \quad \beta = 1, 2, 3, 4,$$

belongs to the set $\mathcal{M}_{3,4}(2, 2)$, and, the 1×4 submatrix

$$A_2 : a_{\alpha,\beta}, \quad \alpha = 4, \quad \beta = 1, 2, 3, 4,$$

belongs to the set $\mathcal{M}_{1,4}(4, 1)$.

3. 2-Local liner operators on finite-dimensional vector spaces which are not linear operators

Let n be a natural number, V be a vector space of dimension n over a field $\mathbb{F}$ with a basis $\{e_1, e_2, \ldots, e_n\}$. Let, for fixed k, m, α, β, γ, η such that

$$1 \le k, m, \alpha, \beta \le n, \quad 2 \le \eta \le n, \quad k \ne m, \quad \alpha \le \beta, \quad 0 \le \gamma \le (n-\beta)n + \beta(n-\eta)$$

and, for fixed subsets $\{i_1, i_2, \ldots, i_\beta\}$ and $\{j_1, j_2, \ldots, j_\eta\}$ of natural numbers from $\{1, 2, \ldots, n\}$ such that $k, m \in \{j_1, j_2, \ldots, j_\eta\}$,

$$f_{ij}(x_1, x_2, \ldots, x_\alpha), \quad i \in \{i_1, i_2, \ldots, i_\beta\}, \quad j \in \{j_1, j_2, \ldots, j_\eta\}, \quad j \ne k, \quad j \ne m,$$
$$f_{ij}(x_1, x_2, \ldots, x_\gamma), \quad i \in \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_\beta\}, \quad j \in \{1, 2, \ldots, n\} \quad \text{if} \quad \beta \ne n,$$
$$f_{ij}(x_1, x_2, \ldots, x_\gamma), \quad i \in \{1, 2, \ldots, n\}, \quad j \in \{1, 2, \ldots, n\} \setminus \{j_1, j_2, \ldots, j_\eta\} \quad \text{if} \quad \eta \ne n$$

be functions with values in the field $\mathbb{F}$ (including the function $f_{ij} \equiv 0$) and, for fixed nonzero elements a_1, a_2, $\ldots$, a_α, b_1, b_2, $\ldots$, b_β, z_1, z_2, $\ldots$, z_γ in $\mathbb{F}$,

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, \ldots, a_\alpha, b_1, b_2, \ldots, b_\beta, z_1, z_2, \ldots, z_\gamma)$$

be a $n \times n$ matrix with components a_{ij}, $i, j = 1, 2, \ldots, n$, such that

1) for $i \in \{i_1, i_2, \ldots, i_\beta\}$, $a_{ik} \in \{a_1, a_2, \ldots a_\alpha\}$ or $a_{ik} = 0$ and for any $a \in \{a_1, a_2, \ldots a_\alpha\}$ there exists $l \in \{i_1, i_2, \ldots, i_\beta\}$ such that $a_{lk} = a$;
2) for every component a_{ij}, $i \in \{i_1, i_2, \ldots, i_\beta\}$, $j \in \{j_1, j_2, \ldots, j_\eta\}$, $j \ne k$, $j \ne m$, of $\mathcal{M}_n^{k,m,\eta}(a_1, a_2, \ldots, a_\alpha, b_1, b_2, \ldots, b_\beta, z_1, z_2, \ldots, z_\gamma)$,
$$a_{ij} = f_{ij}(a_1, a_2, \ldots, a_\alpha);$$
3) $a_{i_s m} = b_s$, $s = 1, 2, \ldots, \beta$;
4) every component a_{ij} of the submatrices
$$B : a_{ij}, i \in \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_\beta\}, \quad j \in \{1, 2, \ldots, n\},$$
$$C : a_{ij}, i \in \{1, 2, \ldots, n\}, \quad j \in \{1, 2, \ldots, n\} \setminus \{j_1, j_2, \ldots, j_\eta\}$$
is equal to $f_{ij}(z_1, z_2, \ldots, z_\gamma)$;
5) if $\beta = n$ and $\eta = n$, then $\gamma = 0$ and we use the designation
$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, \ldots, a_\alpha, b_1, b_2, \ldots, b_n)$$
instead of $\mathcal{M}_n^{k,m,\eta}(a_1, a_2, \ldots, a_\alpha, b_1, b_2, \ldots, b_\beta, z_1, z_2, \ldots, z_\gamma)$.

Let V_1, V_2 be vector subspaces generated by the sets of vectors

$$\big\{e_j : j \ne m, \ j \in \{j_1, j_2, \ldots, j_\eta\}\big\}$$

and $\{e_m\}$ respectively, i.e.,

$$V_1 = \big\langle \big\{e_j : j \ne m, \ j \in \{j_1, j_2, \ldots, j_\eta\}\big\}\big\rangle, \quad V_2 = \langle e_m \rangle.$$

If $\eta \ne n$, then let V_3 be a vector subspace generated by the set of vectors

$$\big\{e_j : j \in \{1, 2, \ldots, n\} \setminus \{j_1, j_2, \ldots, j_\eta\}\big\},$$

i.e.,

$$V_3 = \big\langle \big\{e_j : j \in \{1, 2, \ldots, n\} \setminus \{j_1, j_2, \ldots, j_\eta\}\big\}\big\rangle.$$

Lemma 1. *If $\eta \neq n$, then, for any $v \in V_3$ and $x_1, x_2, \dots x_\alpha,\ y_1, y_2, \dots, y_\beta \in \mathbb{F}$,*

$$\mathcal{M}_n^{k,m,\eta}(x_1, x_2, \dots x_\alpha, y_1, y_2, \dots, y_\beta, z_1, z_2, \dots, z_\gamma)\bar{v}$$
$$= \mathcal{M}_n^{k,m,\eta}(a_1, a_2, \dots, a_\alpha, b_1, b_2, \dots, b_\beta, z_1, z_2, \dots, z_\gamma)\bar{v}.$$

P r o o f. We have

$$\mathcal{M}_n^{k,m,\eta}(x_1, x_2, \dots x_\alpha, y_1, y_2, \dots, y_\beta, z_1, z_2, \dots, z_\gamma)\bar{v} = \overline{\sum_{i=1}^{n} \sum_{j \in \{1,2,\dots,n\} \setminus \{j_1, j_2, \dots, j_\eta\}} a_{ij} v^j e_i} = C\bar{v},$$

where

$$v = \sum_{j \in \{1,2,\dots,n\} \setminus \{j_1, j_2, \dots, j_\eta\}} v^j e_j,$$

C is a matrix from item 4) of the definition of $\mathcal{M}_n^{k,m,\eta}(a_1, a_2, \dots, a_\alpha, b_1, b_2, \dots, b_\beta, z_1, z_2, \dots, z_\gamma)$ above. Since x_1, x_2, … x_α, y_1, y_2, …, y_β in $\mathbb{F}$ are chosen arbitrarily we have the statement of the lemma. □

Theorem 3. *Let V be a vector space of dimension n over a field $\mathbb{F}$ with a basis $\{e_1, e_2, \dots, e_n\}$. Then, for any nonzero elements $c_1, c_2, \dots, c_\alpha$ from the field $\mathbb{F}$, a mapping Δ on V defined as follows*

(I) *in the case $\eta \neq n$,*

1) *if $v = v_1 + v_3$ or $v = v_3$, $v_1 \in V_1$, $v_1 \neq 0$, $v_3 \in V_3$ then*

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, \dots a_\alpha, b_1, b_2, \dots b_\beta, z_1, z_2, \dots, z_\gamma)\bar{v},$$

2) *if $v = v_1 + v_2 + v_3$, $v_1 \in V_1$, $v_2 \in V_2$, $v_2 \neq 0$, $v_3 \in V_3$, then*

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(c_1, c_2, \dots, c_\alpha, b_1, b_2, \dots b_\beta, z_1, z_2, \dots, z_\gamma)\bar{v},$$

(II) *in the case $\eta = n$,*

1) *if $v = v_1$, $v_1 \in V_1$, $v_1 \neq 0$, then*

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, \dots a_\alpha, b_1, b_2, \dots b_\beta, z_1, z_2, \dots, z_\gamma)\bar{v},$$

2) *if $v = v_1 + v_2$, $v_1 \in V_1$, $v_2 \in V_2$, $v_2 \neq 0$, then*

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(c_1, c_2, \dots, c_\alpha, b_1, b_2, \dots b_\beta, z_1, z_2, \dots, z_\gamma)\bar{v}$$

is a 2-local linear operator, and Δ is a linear operator if and only if

$$a_i = c_i, \quad i = 1, 2, \dots, \alpha.$$

P r o o f. We will prove the theorem in the case (I). In the case (II), the theorem is proved similarly. We prove that the mapping Δ, defined in the theorem, is a 2-local linear operator on V. Take the subspace $V_1 \oplus V_3$ and arbitrary two elements v, w from $V_1 \oplus V_3$. Then, by the definition of Δ, item 1) of the theorem and by Lemma 1, for the linear operator $L_{v,w}$ with the matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma),$$

we have $\Delta(v) = L_{v,w}(v)$, $\Delta(w) = L_{v,w}(w)$.

Take the subspace $V_2 \oplus V_3$ and two elements v, w from $V_2 \oplus V_3$ such that

$$v = v_2 + v_3, \quad v_2 \in V_2, \quad v_2 \neq 0, \quad v_3 \in V_3, \quad w = w_2 + w_3, \quad w_2 \in V_2, \quad w_2 \neq 0, \quad w_3 \in V_3.$$

Then, by item 2) of the theorem, for the linear operator $L_{v,w}$ with the matrix

$$\mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma),$$

we have $\Delta(v) = L_{v,w}(v)$, $\Delta(w) = L_{v,w}(w)$.

Now, if we take elements $v \in V_1 \oplus V_3$ such that

$$v = v_1 + v_3, \quad v_1 \in V_1, \quad v_1 \neq 0, \quad v_3 \in V_3, \quad w \in V_2 \oplus V_3$$

such that

$$w = w_2 + w_3, \quad w_2 \in V_2, \quad w_2 \neq 0, \quad w_3 \in V_3,$$

then, by items 1) and 2) of the theorem

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v},$$

and

$$\overline{\Delta(w)} = \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{w}$$

respectively. In this case, by Lemma 1, for the linear operator $T_{v,w}$ with the matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma),$$

we have

$$\Delta(v) = T_{v,w}(v), \quad \Delta(w) = T_{v,w}(w).$$

Now, if $v \in V_1 \oplus V_2 \oplus V_3$ such that

$$v = v_1 + v_2 + v_3, \quad v_1 \in V_1, \quad v_2 \in V_2, \quad v_2 \neq 0, \quad v_3 \in V_3, \quad w \in V_1 \oplus V_3$$

such that

$$w = w_1 + w_3, \quad w_1 \in V_1, \quad w_1 \neq 0, \quad w_3 \in V_3,$$

then, by items 2) and 1) of the theorem,

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}$$

and

$$\overline{\Delta(w)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{w}$$

respectively. In this case, there exist elements $\lambda_1, \lambda_2, ..., \lambda_\beta$ in the field $\mathbb{F}$ such that for the linear operator $L_{v,w}$ with the matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, \lambda_1, \lambda_2, ..., \lambda_\beta, z_1, z_2, ..., z_\gamma),$$

we have

$$\Delta(v) = L_{v,w}(v), \quad \Delta(w) = L_{v,w}(w).$$

Indeed, the equality $\Delta(w) = L_{v,w}(w)$ is obviously true for any λ_1, λ_2, ... λ_β in $\mathbb{F}$ by Lemma 1. As for the equality $\Delta(v) = L_{v,w}(v)$, we rewrite it in the following form

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, \lambda_1, \lambda_2, ..., \lambda_\beta, z_1, z_2, ..., z_\gamma)\bar{v}$$

$$= \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}.$$

The last equality is a system of linear equations with respect to the variables λ_1, λ_2, ... λ_β. By Lemma 1, this system can be written in the following way

$$h_i + v_2^m \lambda_i = g_i + v_2^m b_i, \quad i \in \{i_1, i_2, ..., i_\beta\}, \quad h_j = h_j, \quad j \in \{1, 2, ..., n\} \setminus \{i_1, i_2, ..., i_\beta\},$$

for some elements h_i, $i = 1, 2, ..., n$ and g_j, $j \in \{i_1, i_2, ..., i_\beta\}$, from $\mathbb{F}$, where $v_2 = v_2^m e_m$. Since, $v_2^m \neq 0$, this system of linear equations has the solution

$$\lambda_i = \frac{1}{v_2^m}(g_i + v_2^m b_i - h_i), \quad i \in \{i_1, i_2, ..., i_\beta\}.$$

Hence,

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, \lambda_1, \lambda_2, ..., \lambda_\beta, z_1, z_2, ..., z_\gamma)$$

is a desired matrix.

The case

$$v = v_1 + v_2 + v_3, \quad v_1 \in V_1, \quad v_2 \in V_2, \quad v_2 \neq 0, \quad v_3 \in V_3,$$
$$w = w_1 + w_2 + w_3, \quad w_1 \in V_1, \quad w_2 \in V_2, \quad w_2 \neq 0, \quad w_3 \in V_3$$

is also trivial, i.e., by item 2) of the theorem, for the linear operator $L_{v,w}$ with the matrix

$$\mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma),$$

we have $\Delta(v) = L_{v,w}(v)$, $\Delta(w) = L_{v,w}(w)$.

The case $v \in V_3$ and $w \in V_1 \oplus V_2 \oplus V_3$ such that

$$w = w_1 + w_2 + w_3, \quad w_1 \in V_1, \quad w_1 \neq 0, \quad w_2 \in V_2, \quad w_2 \neq 0, \quad w_3 \in V_3$$

follows by Lemma 1. Indeed, we have

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}$$

by item 1 of the theorem, and,

$$\overline{\Delta(w)} = \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{w}$$

by item 2 of the theorem. At the same time,

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}$$

by Lemma 1. Hence,

$$\Delta(v) = L_{v,w}(v), \quad \Delta(w) = L_{v,w}(w)$$

for the linear operator $L_{v,w}$, generated by the matrix $\mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)$.

Thus, in all cases, for any pair v and w of elements from V, there exists a linear operator $L_{v,w}$ on V such that $\Delta(v) = L_{v,w}(v)$, $\Delta(w) = L_{v,w}(w)$, i.e., Δ is a 2-local linear operator.

Now, if $a_i = c_i$, $i = 1, 2, ..., \alpha$, then, by items 1) and 2) of the theorem, for any $v \in V$,

$$\overline{\Delta(v)} = \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}.$$

So Δ is linear.

Suppose that $(a_1, a_2, ..., a_\alpha) \neq (c_1, c_2, ..., c_\alpha)$. Then there exists a vector $v \in V_1$, $v \neq 0$, such that

$$\mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v} \neq \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}.$$

Then, for any $w \in V_2$, $w \neq 0$, we have

$$\begin{aligned}\overline{\Delta(v+w)} &= \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\overline{(v+w)},\\ \overline{\Delta(v)} &= \mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v},\\ \overline{\Delta(w)} &= \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{w}.\end{aligned}$$

So,

$$\begin{aligned}\overline{\Delta(v+w) - (\Delta(v) + \Delta(w))} &= \mathcal{M}_n^{k,m,\eta}(c_1, c_2, ..., c_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v}\\ &-\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)\bar{v} \neq 0,\end{aligned}$$

i.e., Δ is not additive. This ends the proof. □

4. 2-Local derivations of complex null-filiform and filiform Zinbiel algebras

An algebra $\mathcal{A}$ over a field $\mathbb{F}$ is called Zinbiel algebra if, for any $x, y, z \in \mathcal{A}$, the identity

$$(xy)z = x(yz) + x(zy)$$

holds. For a given Zinbiel algebra $\mathcal{A}$, we define the following sequence:

$$\mathcal{A}^1 = \mathcal{A}, \quad \mathcal{A}^{i+1} = \sum_{k=1}^{i} \mathcal{A}^k \mathcal{A}^{i+1-k}, \quad i \geq 1.$$

A Zinbiel algebra $\mathcal{A}$ is said to be nilpotent if $\mathcal{A}^i = 0$ for some $i \in \mathbb{N}$. The minimal number i satisfying $\mathcal{A}^i = 0$ is called index of nilpotency or nilindex of the algebra $\mathcal{A}$.

It is clear that the index of nilpotency of an arbitrary n-dimensional nilpotent Zinbiel algebra does not exceed the number $n+1$.

Definition 3. *An n-dimensional Zinbiel algebra $\mathcal{A}$ is said to be null-filiform if*

$$\dim \mathcal{A}^i = (n+1) - i,$$

where $\dim \mathcal{A}^i$ is the dimension of $\mathcal{A}^i$, $1 \leq i \leq n+1$.

It is evident that the last definition is equivalent to the fact that the Zinbiel algebra $\mathcal{A}$ has maximal index of nilpotency.

Theorem 4 [2]. *An arbitrary n-dimensional null-filiform Zinbiel algebra over the field $\mathbb{C}$ of complex numbers is isomorphic to the algebra*

$$F_n^0: \ e_i e_j = C_{i+j-1}^{j} e_{i+j}, \quad 2 \leq i + j \leq n,$$

where omitted products $e_k e_l$ are equal to zero and $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra, the symbols C_s^t are binomial coefficients defined as

$$C_s^t = \frac{s!}{t!(s-t!)}.$$

Definition 4. *An n-dimensional Zinbiel algebra $\mathcal{A}$ is said to be filiform if*

$$\dim \mathcal{A}^i = n - i, \quad 2 \leq i \leq n.$$

Theorem 5 [2]. *An arbitrary n-dimensional, $n \geq 5$, filiform Zinbiel algebra over the field $\mathbb{C}$ of complex numbers is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$F_n^1: \ e_ie_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq n-1,$$
$$F_n^2: \ e_ie_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq n-1, \quad e_ne_1 = e_{n-1},$$
$$F_n^3: \ e_ie_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq n-1, \quad e_ne_n = e_{n-1},$$

where omitted products e_ke_l are equal to zero and $\{e_1, e_2, \ldots, e_n\}$ is a basis of the appropriate algebra.

Theorem 6 [21]. *A linear map $\triangle : F_n^0 \to F_n^0$ is a derivation if and only if $\triangle$ is of the following form:*

$$\triangle(e_i) = \sum_{j=i}^{n} C_j^{i-1} \alpha_{j-i+1} e_j, \quad 1 \leq i \leq n,$$

where $\alpha_i \in \mathbb{C}$, $1 \leq i \leq n$.

Theorem 7 [21]. *A linear map $\triangle : F_n^1 \to F_n^1$ is a derivation if and only if $\triangle$ is of the following form*:

$$\triangle(e_1) = \sum_{j=1}^{n} \alpha_j e_j, \quad \triangle(e_i) = \sum_{j=i}^{n} C_j^{i-1} \alpha_{j-i+1} e_j, \quad 2 \leq i \leq n-1, \quad \triangle(e_n) = b_{n-1}e_{n-1} + b_ne_n,$$

where $\alpha_i \in \mathbb{C}$, $1 \leq i \leq n$.

Theorem 8 [21]. *A linear map $\triangle : F_n^2 \to F_n^2$ is a derivation if and only if $\triangle$ is of the following form:*

$$\triangle(e_1) = \sum_{j=1}^{n} \alpha_j e_j, \quad \triangle(e_2) = \sum_{j=2}^{n-1} C_j^1 \alpha_{j-1} e_j + \alpha_n e_{n-1},$$
$$\triangle(e_i) = \sum_{j=i}^{n-1} C_j^{i-1} \alpha_{j-i+1} e_j, \quad 3 \leq i \leq n-1, \quad \triangle(e_n) = b_{n-1}e_{n-1} + (n-2)\alpha_1 e_n,$$

where $\alpha_i \in \mathbb{C}$, $1 \leq i \leq n$.

Theorem 9 [21]. *A linear map $\triangle : F_n^3 \to F_n^3$ is a derivation if and only if $\triangle$ is of the following form:*

$$\triangle(e_1) = \sum_{j=1}^{n} \alpha_j e_j, \quad \triangle(e_i) = \sum_{j=i}^{n-1} C_j^{i-1} \alpha_{j-i+1} e_j, \quad 2 \leq i \leq n-1,$$
$$\triangle(e_n) = -\alpha_n e_{n-2} + b_{n-1}e_{n-1} + \frac{n-1}{2}\alpha_1 e_n,$$

where $\alpha_i \in \mathbb{C}$, $1 \leq i \leq n$.

The following theorems are the main theorems of the present section.

Theorem 10. *Each 2-local derivation on F_n^0 is a derivation.*

P r o o f. Let Δ be an arbitrary 2-local derivation on F_n^0. By the definition, for any x, $y \in F_n^0$ there exists a derivation $D_{x,y}$ on F_n^0 such that

$$\Delta(x) = D_{x,y}(x), \quad , \Delta(x) = D_{x,y}(x).$$

By Theorem 6, the matrix of the derivation $D_{x,y}$ has the following matrix form:

$$D_{x,y} = \begin{pmatrix} \alpha_1^{x,y} & 0 & 0 & \dots & 0 & 0 \\ \alpha_2^{x,y} & C_2^1\alpha_1^{x,y} & 0 & \dots & 0 & 0 \\ \alpha_3^{x,y} & C_3^1\alpha_2^{x,y} & C_3^2\alpha_1^{x,y} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-1}^{x,y} & C_{n-1}^1\alpha_{n-2}^{x,y} & C_{n-1}^2\alpha_{n-3}^{x,y} & \dots & C_{n-1}^{n-2}\alpha_1^{x,y} & 0 \\ \alpha_n^{x,y} & C_n^1\alpha_{n-1}^{x,y} & C_n^2\alpha_{n-2}^{x,y} & \dots & C_n^{n-2}\alpha_2^{x,y} & C_n^{n-1}\alpha_1^{x,y} \end{pmatrix}.$$

Clearly, the set of all $n \times n$ matrices of the form above we can set as a set $\mathcal{M}_{m,n}(k,p)$ defined in Section 2, where $m = n$, $k = 1$, $p = n$, i.e., $\mathcal{M}_{m,n}(k,p) = \mathcal{M}_{n,n}(1,n)$

Each 2-local derivation on F_n^0 is a 2-local linear operator on F_n^0 generated by matrices in $\mathcal{M}_{n,n}(1,n)$ with respect to the basis $\{e_1, e_2, ..., e_n\}$. Conversely, every 2-local linear operator on F_n^0 generated by matrices in $\mathcal{M}_{n,n}(1,n)$ is a 2-local derivation on F_n^0 by Theorem 6.

Therefore, by Theorem 1, each 2-local derivation on F_n^0 is a linear operator generated by a matrix from $\mathcal{M}_{n,n}(1,n)$. Hence, each 2-local derivation on F_n^0 is a derivation by Theorem 6. This ends the proof. □

Theorem 11. *The algebras F_n^1, F_n^2 and F_n^3 have 2-local derivations which are not derivations.*

P r o o f. Let D be an arbitrary derivation on F_n^1. By Theorem 7, the matrix of the derivation D has the following form:

$$\begin{pmatrix} \alpha_1 & 0 & 0 & \dots & 0 & 0 \\ \alpha_2 & C_2^1\alpha_1 & 0 & \dots & 0 & 0 \\ \alpha_3 & C_3^1\alpha_2 & C_3^2\alpha_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-1} & C_{n-1}^1\alpha_{n-2} & C_{n-1}^2\alpha_{n-3} & \dots & C_{n-1}^{n-2}\alpha_1 & \beta_{n-1} \\ \alpha_n & 0 & 0 & \dots & 0 & \beta_n \end{pmatrix}.$$

Let $a_1 = \alpha_{n-1}$, $a_2 = \alpha_n$, $b_1 = \beta_{n-1}$, $b_2 = \beta_n$ and

$$z_1 = \alpha_1, \quad z_2 = \alpha_2, \quad ..., \quad z_{n-2} = \alpha_{n-2}.$$

Then, if this matrix we denote by $\mathcal{M}_n^{1,n,n}(a_1, a_2, b_1, b_2, z_1, z_2, ..., z_{n-2})$, then $\mathcal{M}_n^{1,n,n}(a_1, a_2, b_1, b_2, z_1, z_2, ..., z_{n-2})$ satisfies the all conditions of the definition in Section 3 of a matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)$$

in the case of $k = 1$, $m = n$, $\eta = n$, $\alpha = 2$, $\beta = 2$ and $\gamma = n - 2$.

Therefore, by Theorem 3, we can find a 2-local derivation on F_n^1 which is not linear.

Now we take the algebra F_n^2 and a derivation D on F_n^2. By Theorem 8, the matrix of the derivation D has the following form:

$$\begin{pmatrix} \alpha_1 & 0 & 0 & \dots & 0 & 0 \\ \alpha_2 & C_2^1\alpha_1 & 0 & \dots & 0 & 0 \\ \alpha_3 & C_3^1\alpha_2 & C_3^2\alpha_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-1} & C_{n-1}^1\alpha_{n-2}+\alpha_n & C_{n-1}^2\alpha_{n-3} & \dots & C_{n-1}^{n-2}\alpha_1 & \beta_{n-1} \\ \alpha_n & 0 & 0 & \dots & 0 & (n-2)\alpha_1 \end{pmatrix}.$$

Similar to the previous case, we take $a_1 = \alpha_{n-1}$, $b_1 = \beta_{n-1}$ and

$$z_1 = \alpha_1, \quad z_2 = \alpha_2, \quad ..., \quad z_{n-2} = \alpha_{n-2}, \quad z_{n-1} = \alpha_n.$$

Then, if this matrix we denote by $\mathcal{M}_n^{1,n,n}(a_1, b_1, z_1, z_2, ..., z_{n-1})$, then $\mathcal{M}_n^{1,n,n}(a_1, b_1, z_1, z_2, ..., z_{n-1})$ satisfies the all conditions of the definition in Section 3 of a matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)$$

in the case of $k = 1, \;\; m = n, \;\; \eta = n, \;\; \alpha = 1, \;\; \beta = 1$ and $\gamma = n-1$.

Therefore, by Theorem 3, we can find a 2-local derivation on F_n^1 which is not linear.

Similarly we prove that F_n^3 has 2-local derivations which are not derivations. This ends the proof. □

5. 2-Local automorphisms of naturally graded quasi-filiform Leibniz algebras of type I

A vector space with a bilinear bracket $(\mathcal{L}, [\cdot,\cdot])$ is called a Leibniz algebra if, for any x, y, $z \in L$, the so-called Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

holds. For a given Leibniz algebra $(\mathcal{L}, [\cdot,\cdot])$, the sequence of two-sided ideals is defined recursively as follows:

$$\mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}], \quad k \geq 1.$$

This sequence is said to be the lower central series of $\mathcal{L}$.

A Leibniz algebra $\mathcal{L}$ is said to be nilpotent, if there exists $n \in \mathbb{N}$ such that $\mathcal{L}^n = \{0\}$.

It is easy to see that the sum of two nilpotent ideals of a Leibniz algebra is also nilpotent. Therefore, the maximal nilpotent ideal of a finite-dimensional Leibniz algebra always exists. The maximal nilpotent ideal of a Leibniz algebra is said to be the nilradical of the algebra.

Now we give the definitions of automorphisms and 2-local automorphisms.

Let $\mathcal{A}$ be an algebra. A linear bijective map $\varphi : \mathcal{A} \to \mathcal{A}$ is called an automorphism if it satisfies

$$\varphi([x,y]) = [\varphi(x), \varphi(y)] \quad \text{for all} \quad x, y \in \mathcal{A}.$$

Let $\mathcal{A}$ be an algebra. A (not necessarily linear) map $\Delta : \mathcal{A} \to \mathcal{A}$ is called a 2-local automorphism if, for any elements x, $y \in \mathcal{A}$, there exists an automorphism $\varphi_{x,y} : \mathcal{A} \to \mathcal{A}$ such that

$$\Delta(x) = \varphi_{x,y}(x), \quad \Delta(y) = \varphi_{x,y}(y).$$

Below we define the notion of a quasi-filiform Leibniz algebra.

An n-dimensional Leibniz algebra $\mathcal{L}$ is called quasi-filiform if $\mathcal{L}^{n-2} \neq \{0\}$ and $\mathcal{L}^{n-1} = \{0\}$.

Given an n-dimensional nilpotent Leibniz algebra $\mathcal{L}$ such that $\mathcal{L}^{s-1} \neq \{0\}$ and $\mathcal{L}^s = \{0\}$, put

$$\mathcal{L}_i = \mathcal{L}^i / \mathcal{L}^{i+1}, \quad 1 \leq i \leq s-1,$$

and

$$\mathrm{gr}(\mathcal{L}) = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_{s-1}.$$

Due to $[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}$ we obtain the graded algebra $\mathrm{gr}(\mathcal{L})$. If $\mathrm{gr}(\mathcal{L})$ and $\mathcal{L}$ are isomorphic, i.e., if $\mathrm{gr}(\mathcal{L}) \cong \mathcal{L}$, then we say that $\mathcal{L}$ is naturally graded.

Let x be a nilpotent element of the set $\mathcal{L} \backslash \mathcal{L}^2$. For the nilpotent operator of right multiplication $\mathcal{R}_x$ we define a decreasing sequence $C(x) = (n_1, n_2, \ldots, n_k)$, where $n = n_1 + n_2 + \cdots + n_k$, which consists of the dimensions of Jordan blocks of the operator $\mathcal{R}_x$. On the set of such sequences we consider the lexicographic order, that is,

$$C(x) = (n_1, n_2, \ldots, n_k) \leq C(y) = (m_1, m_2, \ldots, m_t)$$

iff there exists $i \in \mathbb{N}$ such that $n_j = m_j$ for any $j < i$ and $n_i < m_i$.

The sequence

$$C(\mathcal{L}) = \max_{x \in \mathcal{L} \backslash \mathcal{L}^2} C(x)$$

is called the characteristic sequence of the algebra $\mathcal{L}$.

A quasi-filiform non Lie Leibniz algebra $\mathcal{L}$ is called an algebra of the type I (respectively, type II) if there exists an element $x \in \mathcal{L} \backslash \mathcal{L}^2$ such that the operator $\mathcal{R}_x$ has the form

$$\begin{pmatrix} J_{n-2} & 0 \\ 0 & J_2 \end{pmatrix}, \quad \text{(respectively,} \quad \begin{pmatrix} J_2 & 0 \\ 0 & J_{n-2} \end{pmatrix}).$$

The following theorem obtained in [1] gives the classification of naturally graded quasifiliform Leibniz algebras of type I.

Theorem 12. *An arbitrary n-dimensional naturally graded quasi-filiform Leibniz algebra of type I is isomorphic to one of the pairwise non-isomorphic algebras of the following families*:

$$\mathcal{L}_n^{1,\lambda} : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_{n-1}, e_1] = e_n, & \\ [e_1, e_{n-1}] = \lambda e_n, & \lambda \in \mathbb{C}, \end{cases} \qquad \mathcal{L}_n^{2,\lambda} : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_{n-1}, e_1] = e_n, & \\ [e_1, e_{n-1}] = \lambda e_n, & \lambda \in \{0, 1\}, \\ [e_{n-1}, e_{n-1}] = e_n, & \end{cases}$$

$$\mathcal{L}_n^{3,\lambda} : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_{n-1}, e_1] = e_n + e_2, & \\ [e_1, e_{n-1}] = \lambda e_n, & \lambda \in \{-1, 0, 1\}, \end{cases} \qquad \mathcal{L}_n^{4,\mu} : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_{n-1}, e_1] = e_n + e_2, & \\ [e_{n-1}, e_{n-1}] = \mu e_n, & \mu \neq 0, \end{cases}$$

$$\mathcal{L}_n^{5,\lambda,\mu} : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_{n-1}, e_1] = e_n + e_2, & \\ [e_1, e_{n-1}] = \lambda e_n, & (\lambda, \mu) = (1,1) \ \mathit{or} \ (2,4), \\ [e_{n-1}, e_{n-1}] = \mu e_n, & \end{cases}$$

where $\{e_1, e_2, \ldots, e_n\}$ *is a basis of the algebra.*

In this section we use the following theorem from [3] concerning automorphisms of naturally graded quasi-filiform Leibniz algebras of type I.

Theorem 13. *A linear map $\varphi : \mathcal{L} \to \mathcal{L}$ is an automorphism if and only if φ has the following form*:

$$\varphi\left(\mathcal{L}_n^{1,\lambda}\right): \begin{cases} \varphi(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1\Big(\sum_{i=2}^{n-2} \alpha_{i-1} e_i + \alpha_{n-1}(1+\lambda) e_n\Big), \\ \varphi(e_j) = \alpha_1^{j-1} \sum_{i=j}^{n-2} \alpha_{i-j+1} e_i, \quad 3 \le j \le n-2, \\ \varphi(e_{n-1}) = \sum_{i=n-3}^{n} b_i e_i, \\ \varphi(e_n) = \alpha_1 (b_{n-3} e_{n-2} + b_{n-1} e_n), \end{cases}$$

where $\alpha_i \in \mathbb{C}$, $1 \le i \le n$, $\alpha_1 b_{n-1} \neq 0$;

$$\varphi\left(\mathcal{L}_n^{2,0}\right): \begin{cases} \varphi(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 \sum_{i=2}^{n-2} \alpha_{i-1} e_i + \alpha_{n-1}(\alpha_1 + \alpha_{n-1}) e_n, \\ \varphi(e_j) = \alpha_1^{j-1} \sum_{i=j}^{n-2} \alpha_{i-j+1} e_i, \quad 3 \le j \le n-2, \\ \varphi(e_{n-1}) = b_{n-2} e_{n-2} + b_{n-1} e_{n-1} + b_n e_n, \\ \varphi(e_n) = (\alpha_1 + \alpha_{n-1}) b_{n-1} e_n, \end{cases}$$

where $\alpha_i \in \mathbb{C}$, $1 \le i \le n$, $\alpha_1 b_{n-1} \neq 0$, $b_{n-1} = \alpha_1 + \alpha_{n-1}$;

$$\varphi\left(\mathcal{L}_n^{2,1}\right): \begin{cases} \varphi(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 \sum_{i=2}^{n-2} \alpha_{i-1} e_i + \alpha_{n-1}(2\alpha_1 + \alpha_{n-1}) e_n, \\ \varphi(e_j) = \alpha_1^{j-1} \sum_{i=j}^{n-2} \alpha_{i-j+1} e_i, \quad 3 \le j \le n-2, \\ \varphi(e_{n-1}) = b_{n-2} e_{n-2} + b_{n-1} e_{n-1} + b_n e_n, \\ \varphi(e_n) = (\alpha_1 + \alpha_{n-1}) b_{n-1} e_n, \end{cases}$$

where $\alpha_i \in \mathbb{C}$, $1 \le i \le n$, $\alpha_1 b_{n-1} \neq 0$, $b_{n-1} = \alpha_1 + \alpha_{n-1}$;

$$\varphi\left(\mathcal{L}_n^{3,-1}\right): \begin{cases} \varphi(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \\ \varphi(e_j) = \alpha_1^{j-1}(\alpha_1 + \alpha_{n-1}) e_j + \alpha_1^{n-1} \sum_{i=j+1}^{n-2} \alpha_{i-j+1} e_i, \quad 2 \le j \le n-2, \\ \varphi(e_{n-1}) = \sum_{i=2}^{n-3} \alpha_i e_i + b_{n-2} e_{n-2} + (\alpha_1 + \alpha_{n-1}) e_{n-1} + b_n e_n, \\ \varphi(e_n) = \alpha_1 (\alpha_1 + \alpha_{n-1}) e_n, \end{cases}$$

where $\alpha_i \in \mathbb{C}$, $1 \le i \le n$, $\alpha_1(\alpha_1 + \alpha_{n-1}) \neq 0$;

$$\varphi\left(\mathcal{L}_n^{3,0}\right): \begin{cases} \varphi(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1(\alpha_1 + \alpha_{n-1}) e_2 + \alpha_1 \sum_{i=3}^{n-2} \alpha_{i-1} e_i + \alpha_1 \alpha_{n-1} e_n, \\ \varphi(e_j) = \alpha_1^{j-1}(\alpha_1 + \alpha_{n-1}) e_j + \alpha_1^{j-1} \sum_{i=j+1}^{n-2} \alpha_{i-j+1} e_i, \quad 2 \le j \le n-2, \\ \varphi(e_{n-1}) = \sum_{i=2}^{n-4} \alpha_i e_i + b_{n-3} e_{n-3} + b_{n-2} e_{n-2} + (\alpha_1 + \alpha_{n-1}) e_{n-1} + b_n e_n, \\ \varphi(e_n) = (b_{n-3} - \alpha_{n-3}) \alpha_1 e_{n-2} + \alpha_1^2 e_n, \end{cases}$$

where $\alpha_i \in \mathbb{C}$, $1 \le i \le n$, $\alpha_1(\alpha_1 + \alpha_{n-1}) \neq 0$; *for the algebras* $\mathcal{L}_n^{3,1}, \mathcal{L}_n^{4,\mu}, \mathcal{L}_n^{5,\lambda,\mu}$

$$\begin{cases} \varphi(e_1) = \sum_{i=1}^{n-2} \alpha_i e_i + \alpha_n e_n, \\ \varphi(e_j) = \alpha_1^{i-1} \sum_{i=j}^{n-2} \alpha_{i-j+1} e_i, \quad 2 \le j \le n-2, \\ \varphi(e_{n-1}) = b_{n-2} e_{n-2} + \alpha_1 e_{n-1} + b_n e_n, \\ \varphi(e_n) = 2\alpha_1^2 e_n, \end{cases}$$

where $\alpha_i \in \mathbb{C}$, $1 \le i \le n-2$, $\alpha_n \in \mathbb{C}$, $\alpha_1 \neq 0$.

The following theorem is one of the main results of the present paper concerning 2-local automorphisms.

Theorem 14. *The algebras $\mathcal{L}_n^{1,\lambda}$, $\mathcal{L}_n^{2,\lambda}$, where $\lambda \in \{0,1\}$, $\mathcal{L}_n^{3,\lambda}$, where $\lambda \in \{-1,0,1\}$, $\mathcal{L}_n^{4,\mu}$ and $\mathcal{L}_n^{5,\lambda,\mu}$, where $(\lambda,\mu) = (1,1)$ or $(2,4)$, have 2-local automorphisms which are not automorphisms.*

P r o o f. Let φ be an arbitrary automorphism on $\mathcal{L}_n^{1,\lambda}$. By Theorem 13, the matrix of the automorphism φ has the following form:

$$\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\alpha_2 & \alpha_1^2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\alpha_3 & \alpha_1\alpha_2 & \alpha_1^3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{n-4} & \alpha_1\alpha_{n-5} & \alpha_1^2\alpha_{n-6} & \dots & \alpha_1^{n-6}\alpha_2 & \alpha_1^{n-4} & 0 & 0 & 0 & 0 \\
\alpha_{n-3} & \alpha_1\alpha_{n-4} & \alpha_1^2\alpha_{n-5} & \dots & a_1^{n-6}a_3 & \alpha_1^{n-5}\alpha_2 & \alpha_1^{n-3} & 0 & \beta_{n-3} & 0 \\
\alpha_{n-2} & \alpha_1\alpha_{n-3} & \alpha_1^2\alpha_{n-4} & \alpha_1^3\alpha_{n-5} & \dots & \alpha_1^{n-5}\alpha_3 & \alpha_1^{n-4}\alpha_2 & \alpha_1^{n-2} & \beta_{n-2} & \alpha_1\beta_{n-3} \\
\alpha_{n-1} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \beta_{n-1} & 0 \\
\alpha_n & \alpha_{n-1}(1+\lambda) & 0 & 0 & 0 & \dots & 0 & 0 & \beta_n & \alpha_1\beta_{n-1}
\end{pmatrix}.$$

Let $a_1 = \alpha_n,\ \alpha_{n-1} = 0,\ b_1 = \beta_n$ and

$$z_1 = \alpha_1, \quad z_2 = \alpha_2, \quad ..., \quad z_{n-2} = \alpha_{n-2}, \quad z_{n-1} = \beta_{n-1}, \quad z_n = \beta_{n-2}, \quad z_{n+1} = \beta_{n-3}.$$

Then, denoting this matrix by $\mathcal{M}_n^{1,n,n}(a_1, b_1, z_1, z_2, ..., z_{n+1})$, we see that $\mathcal{M}_n^{1,n,n}(a_1, b_1, z_1, z_2, ..., z_{n+1})$ satisfies all conditions of the definition in Section 3 of a matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)$$

in the case of $k = 1,\ m = n-1,\ \eta = n-1,\ \alpha = 1,\ \beta = 1$ and $\gamma = n+1$.

Therefore, by Theorem 3, we can find a 2-local automorphism on $\mathcal{L}_n^{1,\lambda}$ which is not linear.

Now we take the algebra $\mathcal{L}_n^{2,0}$ and an automorphism φ on $\mathcal{L}_n^{2,0}$. By Theorem 13, the matrix of the automorphism φ has the following form:

$$\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\alpha_2 & \alpha_1^2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\alpha_3 & \alpha_1\alpha_2 & \alpha_1^3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{n-4} & \alpha_1\alpha_{n-5} & \alpha_1^2\alpha_{n-6} & \dots & \alpha_1^{n-6}\alpha_2 & \alpha_1^{n-4} & 0 & 0 & 0 & 0 \\
\alpha_{n-3} & \alpha_1\alpha_{n-4} & \alpha_1^2\alpha_{n-5} & \dots & a_1^{n-6}a_3 & \alpha_1^{n-5}\alpha_2 & \alpha_1^{n-3} & 0 & 0 & 0 \\
\alpha_{n-2} & \alpha_1\alpha_{n-3} & \alpha_1^2\alpha_{n-4} & \alpha_1^3\alpha_{n-5} & \dots & \alpha_1^{n-5}\alpha_3 & \alpha_1^{n-4}\alpha_2 & \alpha_1^{n-2} & \beta_{n-2} & 0 \\
\alpha_{n-1} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \alpha_1+\alpha_{n-1} & 0 \\
\alpha_n & \alpha_{n-1}(\alpha_1+\alpha_{n-1}) & 0 & 0 & 0 & \dots & 0 & 0 & \beta_n & (\alpha_1+\alpha_{n-1})^2
\end{pmatrix}.$$

Similar to the previous case, we take $a_1 = \alpha_n,\ \alpha_{n-1} = 0,\ b_1 = \beta_n$ and

$$z_1 = \alpha_1, \quad z_2 = \alpha_2, \quad ..., \quad z_{n-2} = \alpha_{n-2}, \quad z_{n-1} = \beta_{n-2}.$$

Then, if this matrix we denote by $\mathcal{M}_n^{1,n,n}(a_1, b_1, z_1, z_2, ..., z_{n-1})$, then $\mathcal{M}_n^{1,n,n}(a_1, b_1, z_1, z_2, ..., z_{n-1})$ satisfies all conditions of definition in Section 3 of a matrix

$$\mathcal{M}_n^{k,m,\eta}(a_1, a_2, ..., a_\alpha, b_1, b_2, ..., b_\beta, z_1, z_2, ..., z_\gamma)$$

in the case of $k = 1$, $m = n-1$, $\eta = n-1$, $\alpha = 1$, $\beta = 1$ and $\gamma = n-1$. Therefore, by Theorem 3, we can find a 2-local automorphism on $\mathcal{L}_n^{2,\lambda}$ which is not linear.

Similarly we prove that $\mathcal{L}_n^{2,1}$ has 2-local automorphisms which are not automorphisms.

Now, we take $\mathcal{L}_n^{3,-1}$, $\mathcal{L}_n^{3,0}$, $\mathcal{L}_n^{3,1}$, $\mathcal{L}_n^{4,\mu}$ and $\mathcal{L}_n^{5,\lambda,\mu}$. By Theorem 13, the matrix of automorphisms of $\mathcal{L}_n^{3,-1}$ and $\mathcal{L}_n^{3,0}$ has the following forms respectively:

$$\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
\alpha_2 & \lambda_2 & 0 & 0 & 0 & 0 & \dots & 0 & \alpha_2 & 0\\
\alpha_3 & \alpha_1^{n-1}\alpha_2 & \lambda_3 & 0 & 0 & 0 & \dots & 0 & \alpha_3 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\alpha_{n-4} & \alpha_1^{n-1}\alpha_{n-5} & \alpha_1^{n-1}\alpha_{n-6} & \dots & \alpha_1^{n-1}\alpha_2 & \lambda_{n-4} & 0 & 0 & \alpha_{n-4} & 0\\
\alpha_{n-3} & \alpha_1^{n-1}\alpha_{n-4} & \alpha_1^{n-1}\alpha_{n-5} & \dots & \alpha_1^{n-1}\alpha_3 & \alpha_1^{n-1}\alpha_2 & \lambda_{n-3} & 0 & \alpha_{n-3} & 0\\
\alpha_{n-2} & \alpha_1^{n-1}\alpha_{n-3} & \alpha_1^{n-1}\alpha_{n-4} & \alpha_1^{n-1}\alpha_{n-5} & \dots & \alpha_1^{n-1}\alpha_3 & \alpha_1^{n-1}\alpha_2 & \lambda_{n-2} & \beta_{n-2} & \alpha_1\beta_{n-3}\\
\alpha_{n-1} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \alpha_1+\alpha_{n-1} & 0\\
\alpha_n & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \beta_n & \alpha_1(\alpha_1+\alpha_{n-1})
\end{pmatrix}$$

and

$$\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
\alpha_2 & \lambda_2 & 0 & 0 & \dots & 0 & \alpha_2 & 0\\
\alpha_3 & \alpha_1\alpha_2 & \lambda_3 & 0 & \dots & 0 & \alpha_3 & 0\\
\alpha_4 & \alpha_1\alpha_3 & \alpha_1^2\alpha_2 & \lambda_4 & \dots & 0 & \alpha_4 & 0\\
\alpha_5 & \alpha_1\alpha_4 & \alpha_1^2\alpha_3 & \alpha_1^3\alpha_2 & \dots & 0 & \alpha_5 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\alpha_{n-4} & \alpha_1\alpha_{n-5} & \alpha_1^2\alpha_{n-6} & \alpha_1^3\alpha_{n-7} & \dots & 0 & \alpha_{n-4} & 0\\
\alpha_{n-3} & \alpha_1\alpha_{n-4} & \alpha_1^2\alpha_{n-5} & \alpha_1^3\alpha_{n-6} & \dots & 0 & \beta_{n-3} & 0\\
\alpha_{n-2} & \alpha_1\alpha_{n-3} & \alpha_1^2\alpha_{n-4} & \alpha_1^3\alpha_{n-5} & \dots & \lambda_{n-2} & \beta_{n-2} & (\beta_{n-3}-\alpha_{n-3})\alpha_1\\
\alpha_{n-1} & 0 & 0 & 0 & \dots & 0 & \alpha_1+\alpha_{n-1} & 0\\
\alpha_n & \alpha_1\alpha_{n-1} & 0 & 0 & \dots & 0 & \beta_n & \alpha_1^2
\end{pmatrix},$$

where $\lambda_i = \alpha_1^{i-1}(\alpha_1+\alpha_{n-1})$, $i = 2, 3, \dots, n-2$.

For the algebras $\mathcal{L}_n^{3,1}$, $\mathcal{L}_n^{4,\mu}$ and $\mathcal{L}_n^{5,\lambda,\mu}$ the matrix of their automorphisms has the following form

$$\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
\alpha_2 & \alpha_1^2 & 0 & 0 & \dots & 0 & 0 & 0\\
\alpha_3 & \alpha_1^2\alpha_2 & \alpha_1^3 & 0 & \dots & 0 & 0 & 0\\
\alpha_4 & \alpha_1^3\alpha_3 & \alpha_1^3\alpha_2 & \alpha_1^4 & \dots & 0 & 0 & 0\\
\alpha_5 & \alpha_1^4\alpha_4 & \alpha_1^4\alpha_3 & \alpha_1^4\alpha_2 & \dots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\alpha_{n-2} & \alpha_1^{n-3}\alpha_{n-3} & \alpha_1^{n-3}\alpha_{n-4} & \alpha_1^{n-3}\alpha_{n-5} & \dots & \alpha_1^{n-2} & \beta_{n-2} & 0\\
0 & 0 & 0 & 0 & \dots & 0 & \alpha_1 & 0\\
\alpha_n & 0 & 0 & 0 & \dots & 0 & \beta_n & 2\alpha_1^2
\end{pmatrix}$$

By these forms and Theorem 3, similar to the cases of $\mathcal{L}_n^{1,\lambda}$ and $\mathcal{L}_n^{2,0}$ we can prove that the algebras $\mathcal{L}_n^{3,-1}$, $\mathcal{L}_n^{3,0}$, $\mathcal{L}_n^{3,1}$, $\mathcal{L}_n^{4,\mu}$ and $\mathcal{L}_n^{5,\lambda,\mu}$ also have 2-local automorphisms which are not automorphisms. This ends the proof. □

Conclusion

In conclusion, it can be said that the article generalizes the methods of studying 2-local derivations and automorphisms of algebras. The method proposed in the second section allows one to make a direct conclusion about whether all 2-local derivations (respectively, automorphisms) are derivations (respectively, automorphisms) based on the general matrix form of the matrix of a derivation (respectively, an automorphism) of an algebra. This method is useful since often the derivation (automorphism) of an algebra has the matrix form in the method under consideration. In the third section, a method is developed that allows one to obtain an entire subspace (an entire subgroup) of 2-local derivations (respectively, 2-local automorphisms) that are not derivations (respectively, automorphisms). As is known, the set of all 2-local derivations (2-local automorphisms) of an algebra forms a vector space (respectively, a group) and the description of this vector space (this group) is an open problem. We think that the method developed in the third section allows to solve this problem.

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ATTRACTION SETS IN ATTAINABILITY PROBLEMS WITH ASYMPTOTIC-TYPE CONSTRAINTS

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Abstract: In control theory, the problem of constructing and investigating attainability domains is very important. However, under perturbations of constraints, this problem lacks stability. It is useful to single out the case when the constraints are relaxed. In this case, greater opportunities arise in terms of attainability, and often a useful effect can be observed even under slight relaxation of the constraints. This situation is analogous to the duality gap in convex programming. Very often, it is not possible to specify in advance how much relaxation of the constraints will occur. Therefore, attention is focused on the limit of the attainability domains under unrestricted tightening of the relaxed conditions. As a result, a certain attainability problem with asymptotic-type constraints arises. This problem formulation can be significantly generalized. Namely, we do not consider any unperturbed conditions at all and instead pose asymptotic-type constraints directly by means of a nonempty family of sets in the space of ordinary controls. Moreover, not only the case of control problems can be considered. In this general formulation, an analogue of the limit of attainability domains naturally appears as the relaxed conditions are infinitely tightened. For asymptotic constraints of this kind, we introduce solutions which are, at the conceptual level, similar to the approximate solutions of J. Warga, but we use filters or directedness, and not just sequences of ordinary solutions (controls). We investigate the most general attainability problem, in which asymptotic-type constraints can be generated by any nonempty family of sets in the ordinary solution space. It is shown, however, that the most practically interesting case is realized by filters, and the role of ultrafilters is noted as well. The action of constraints is associated with sets and elements of attraction. Furthermore, some properties of the family of all attraction sets are investigated.

Keywords: Attraction set, Constraints, Filter, Topology, Ultrafilter.

1. Introduction

We consider attainability problems in topological spaces with asymptotic-type constraints. These asymptotic-type constraints may arise when standard constraints (such as inequalities in mathematical programming, phase constraints, or boundary conditions in control theory) are relaxed, but they can also be posed from the outset. In all cases, we deal with a nonempty family of sets in the space of ordinary (implementable) solutions. Thus, our concrete solutions must be essentially asymptotic; here we focus on the approximate solutions in the sense of Warga (see [17, Ch. III]), allowing, however, for nonsequential variants (i.e., directed sets or filters). In addition, for the family generating asymptotic-type constraints, we require that the solution direction eventually takes values in each set of this family (a similar requirement is imposed when using filters and, in particular, ultrafilters).

In addition, we have a certain target operator with values in a topological space. Using the solution direction, we obtain a directed set of its values (when using a filter, the filter base is realized). We consider those points in the topological space that are realized as generalized limits of such directed sets of values. The set of these generalized limits is called the attraction set for the given asymptotic-type constraints. Thus, for every nonempty family of sets in the space of ordinary solutions, the corresponding attraction set in the fixed topological space is defined. By varying these families, we obtain a family of attraction sets. The latter family is the main subject of our research. We strive to develop a kind of "calculus" of attraction sets. Filters and ultrafilters will play an important role in this construction.

We note that, for the investigation of extremal problems with weakened constraints, extension constructions are used very widely (see [17, Ch. III–V]). This approach motivated the development of the theory of generalized solutions (controls); in this connection, we would like to especially mention the monographs [9, 11, 17, 18]. In [11, 12], the fundamental alternative theorem was established; this theorem defined the current state of differential game theory. In the construction of the proof, the idea of observing phase constraints in the form of sections of the stable bridge of N.N. Krasovskii was employed. We also note the wide application of generalized controls in solving the performance problem; see [9].

For control problems involving impulses, N.N. Krasovskii suggested (see [13]) using the apparatus of generalized functions to represent (generalized) controls. This approach served as the basis for the development of impulse control theory (see [7, 10, 13, 15, 16, 19] and others). In [2, 3, 6], for abstract control problems with impulse-type and momentary-type constraints, and with discontinuous dependencies among the conditions, extension constructions in the class of finitely additive measures were proposed. Finally, we note the approach of [4, Ch. 8], which is connected with the use of ultrafilters as generalized elements in attainability problems with asymptotic-type constraints. The present article continues the investigations of [4, Ch. 8].

Now, we note essential differences between the present investigation and the constructions in the author's earlier works. Namely, here, not a single attraction set is considered, but rather the space of such objects is explored. In particular, we study the transformations of attraction sets when the asymptotic-type constraints are varied. Cases where attraction sets are generated by filters forming asymptotic-type constraints are particularly highlighted. The role of ultrafilters in the above-mentioned transformations is clarified. Namely, each ultrafilter on the set of ordinary solutions is associated with an element of attraction. As a consequence, an attraction operator is defined; by means of this operator, a new representation for attraction sets generated by filters is established.

2. General notions and definitions

We use standard set-theoretical notation, including quantifiers ($\forall$, $\exists$), logical connectives (&, $\vee$, $\Longrightarrow$, $\Longleftrightarrow$, and others), and special symbols: def (by definition), $\triangleq$ (equality by definition), and $\exists!$ (there exists a unique element). We assume that a family is a set whose elements are themselves sets. We also adopt the axiom of choice.

If a and b are objects, then by $\{a;b\}$ we denote the set such that $a \in \{a;b\}$, $b \in \{a;b\}$, and for any $z \in \{a;b\}$, $(z = a) \vee (z = b)$ holds; that is, $\{a;b\}$ is the unordered pair of these objects. For any object x, the set $\{x\} \triangleq \{x;x\}$ is the singleton corresponding to x. Sets are objects; therefore, for any objects x and y, the expression $(x,y) \triangleq \{\{x\};\{x;y\}\}$ defines the ordered pair with first element x and second element y (see [14, Ch. II, Sect. 3]). If h is an ordered pair, then $\mathrm{pr}_1(h)$ and $\mathrm{pr}_2(h)$ denote the first and second elements of h, respectively; by virtue of the equality $h = (\mathrm{pr}_1(h), \mathrm{pr}_2(h))$, these elements are uniquely defined.

If H is a set, then $\mathcal{P}(H)$ denotes the family of all subsets of H, and $\mathcal{P}'(H) \triangleq \mathcal{P}(H) \setminus \{\varnothing\}$. Moreover, let $\mathrm{Fin}(H)$ denote the family of all finite sets in $\mathcal{P}'(H)$, that is, the family of all nonempty finite subsets of H (any family can be used as H).

Functions. If A and B are nonempty sets, then B^A denotes the set of all functions from A

to B; for $g \in B^A$ (that is, for $g : A \to B$) and $a \in A$, the element $g(a) \in B$ is the value of g at the point a. If A and B are nonempty sets, $f \in B^A$, and $C \in \mathcal{P}(A)$, then [14, Ch. II, Sect. 7]

$$f^1(C) \triangleq \{f(x) :\ x \in C\} \in \mathcal{P}(B)$$

is the image of the set C under the action of f; if $D \in \mathcal{P}(B)$, then, as usual, $f^{-1}(D)$ denotes the preimage of the set D under f. For a nonempty family $\mathcal{M}$, we introduce the family

$$(\mathrm{Cen})[\mathcal{M}] \triangleq \big\{\mathcal{Z} \in \mathcal{P}'(\mathcal{M})|\ \bigcap_{Z \in \mathcal{K}} Z \neq \varnothing \quad \forall \mathcal{K} \in \mathrm{Fin}(\mathcal{Z})\big\} \in \mathcal{P}(\mathcal{P}'(\mathcal{M}))$$

of all nonempty centered subfamilies of $\mathcal{M}$. As usual, $\mathbb{R}$ is the real line, $\mathbb{N} \triangleq \{1; 2; \ldots\} \in \mathcal{P}'(\mathbb{R})$, and $\overline{1,n} \triangleq \{k \in \mathbb{N}|\ k \leq n\}$ under $n \in \mathbb{N}$. We suppose that the elements of $\mathbb{N}$ (the natural numbers) are not sets. Taking this into account, for every nonempty set H and $n \in \mathbb{N}$, we use the notation H^n instead of $H^{\overline{1,n}}$ for the set of all functions from $\overline{1,n}$ to H (these functions are called tuples). Of course, any nonempty family can be used as H. In denoting functions, we often use the index form (families with indices, see [17, Sect. 1.1]).

For every family $\mathcal{H}$ and set T, we define

$$\big([\mathcal{H}](T) \triangleq \{H \in \mathcal{H}|\ T \subset H\} \in \mathcal{P}(\mathcal{H})\big)\&\big(\mathcal{H}|_T \triangleq \{H \cap T :\ H \in \mathcal{H}\} \in \mathcal{P}(\mathcal{P}(T))\big).$$

If $\mathbb{M}$ is a set and $\mathcal{M} \in \mathcal{P}'(\mathcal{P}(\mathbb{M}))$, then

$$\mathbf{C}_{\mathbb{M}}[\mathcal{M}] \triangleq \{\mathbb{M} \setminus M :\ M \in \mathcal{M}\} \in \mathcal{P}'(\mathcal{P}(\mathbb{M}))$$

is the family of subsets of $\mathbb{M}$ dual to $\mathcal{M}$.

Special families. Fix a set $\mathbf{I}$ throughout this section. We consider families from $\mathcal{P}'(\mathcal{P}(\mathbf{I}))$, that is, nonempty families of subsets of $\mathbf{I}$. In particular,

$$\pi[\mathbf{I}] \triangleq \big\{\mathcal{I} \in \mathcal{P}'(\mathcal{P}(\mathbf{I}))|\ (\varnothing \in \mathcal{I})\&(\mathbf{I} \in \mathcal{I})\&(A \cap B \in \mathcal{I} \quad \forall A \in \mathcal{I}\ \forall B \in \mathcal{I})\big\} \tag{2.1}$$

is the family of all π-systems of subsets of $\mathbf{I}$ containing the "zero" $\varnothing$ and the "unit" $\mathbf{I}$. Define

$$(\mathrm{LAT})_0[\mathbf{I}] \triangleq \big\{\mathcal{I} \in \pi[E]|\ A \cup B \in \mathcal{I} \quad \forall A \in \mathcal{I}\ \forall B \in \mathcal{I}\big\}$$

as the family of all lattices of subsets of $\mathbf{I}$ containing the "zero" and "unit". Next,

$$\tilde{\pi}^0[\mathbf{I}] \triangleq \big\{\mathcal{I} \in \pi[\mathbf{I}]|\ \forall I \in \mathcal{I}\ \forall x \in \mathbf{I} \setminus I\ \exists J \in \mathcal{I} :\ (x \in J)\&(J \cap I = \varnothing)\big\} \tag{2.2}$$

is the family of all separable π-systems of (2.1). We also use the family

$$(\mathrm{top})[\mathbf{I}] \triangleq \big\{\tau \in \pi[\mathbf{I}]|\ \bigcup_{G \in \mathcal{G}} G \in \tau \quad \forall \mathcal{G} \in \mathcal{P}(\tau)\big\} = \big\{\tau \in (\mathrm{LAT})_0[\mathbf{I}]|\ \bigcup_{G \in \mathcal{G}} G \in \tau \quad \forall \mathcal{G} \in \mathcal{P}(\tau)\big\}$$

of all topologies on the set $\mathbf{I}$. If $\tau \in (\mathrm{top})[\mathbf{I}]$, then $(\mathbf{I}, \tau)$ is a topological space with unit $\mathbf{I}$, and $\mathbf{C}_{\mathbf{I}}[\tau] \in (\mathrm{LAT})_0[\mathbf{I}]$ is the family of all closed in $(\mathbf{I}, \tau)$ subsets of $\mathbf{I}$. Define

$$(\mathbf{c}-\mathrm{top})[\mathbf{I}] \triangleq \big\{\tau \in (\mathrm{top})[\mathbf{I}]|\ \bigcap_{F \in \mathcal{F}} F \neq \varnothing \quad \forall \mathcal{F} \in (\mathrm{Cen})[\mathbf{C}_{\mathbf{I}}[\tau]]\big\} \tag{2.3}$$

as the family of all compact topologies on $\mathbf{I}$. If $\tau \in (\mathbf{c}-\text{top})[\mathbf{I}]$, then $(\mathbf{I}, \tau)$ is a compact topological space. For $\tau \in (\text{top})[\mathbf{I}]$ and $x \in \mathbf{I}$, let $N^0_\tau(x) \triangleq \{G \in \tau |\ x \in G\}$ and

$$N_\tau(x) \triangleq \big\{H \in \mathcal{P}(\mathbf{I})|\ \exists G \in N^0_\tau(x):\ G \subset H\big\} \tag{2.4}$$

be the family of all neighborhoods of the point x in the topological space $(\mathbf{I}, \tau)$. Define

$$\begin{aligned}(\text{top})_0[\mathbf{I}] &\triangleq \big\{\tau \in (\text{top})[\mathbf{I}]|\ \forall y \in \mathbf{I}\ \forall z \in \mathbf{I}\setminus\{y\}\ \exists G_1 \in N^0_\tau(y)\ \exists G_2 \in N^0_\tau(z):\ G_1 \cap G_2 = \varnothing\big\}\\ &= \big\{\tau \in (\text{top})[\mathbf{I}]|\ \forall y \in \mathbf{I}\ \forall z \in \mathbf{I}\setminus\{y\}\ \exists H_1 \in N_\tau(y)\ \exists H_2 \in N_\tau(z):\ H_1 \cap H_2 = \varnothing\big\}\end{aligned}$$

as the family of all topologies that make $\mathbf{I}$ a T_2-space. Let

$$(\mathbf{c}-\text{top})_0[\mathbf{I}] \triangleq (\mathbf{c}-\text{top})[\mathbf{I}] \cap (\text{top})_0[\mathbf{I}];$$

if $\tau \in (\mathbf{c}-\text{top})_0[\mathbf{I}]$, then the topological space $(\mathbf{I}, \tau)$ is called a compactum.

If $\tau \in (\text{top})[\mathbf{I}]$ and $A \in \mathcal{P}(\mathbf{I})$, then $[\mathbf{C}_\mathbf{I}[\tau]](A) \in \mathcal{P}'(\mathbf{C}_\mathbf{I}[\tau])$ and

$$\text{cl}(A, \tau) \triangleq \bigcap_{F \in [\mathbf{C}_\mathbf{I}[\tau]](A)} F \in [\mathbf{C}_\mathbf{I}[\tau]](A)$$

is the closure of A in the topological space $(\mathbf{I}, \tau)$.

3. Some topological constructions

If (X, τ) is a topological space and $Y \in \mathcal{P}(X)$, then $\tau|_Y \in (\text{top})[Y]$; the resulting topological space $(Y, \tau|_Y)$ is called a subspace of (X, τ). For every topological space (X, τ), define

$$(\tau-\text{comp})[X] \triangleq \big\{K \in \mathcal{P}(X)|\ \tau|_K \in (\mathbf{c}-\text{top})[K]\big\}$$

as the family of all compact (in (X, τ)) subsets of X. Throughout this (brief) section, we fix topological spaces (U, τ_1) and (V, τ_2) with $U \neq \varnothing$ and $V \neq \varnothing$; that is, $\tau_1 \in (\text{top})[U]$ and $\tau_2 \in (\text{top})[V]$. Define

$$C(U, \tau_1, V, \tau_2) \triangleq \big\{f \in V^U|\ f^{-1}(G) \in \tau_1\ \forall G \in \tau_2\big\}, \tag{3.1}$$

$$\begin{aligned}C_{\text{cl}}(U, \tau_1, V, \tau_2) &\triangleq \big\{f \in C(U, \tau_1, V, \tau_2)|\ f^1(F) \in \mathbf{C}_V[\tau_2]\ \forall F \in \mathbf{C}_U[\tau_1]\big\}\\ &= \big\{f \in V^U|\ f^1(\text{cl}(A, \tau_1)) = \text{cl}(f^1(A), \tau_2)\ \forall A \in \mathcal{P}(U)\big\}.\end{aligned} \tag{3.2}$$

Note the following important special case:

$$\big((\tau_1 \in (\mathbf{c}-\text{top})[U])\&(\tau_2 \in (\text{top})_0[V])\big) \Longrightarrow \big(C(U, \tau_1, V, \tau_2) = C_{\text{cl}}(U, \tau_1, V, \tau_2)\big). \tag{3.3}$$

In (3.1), the set of all continuous functions from (U, τ_1) to (V, τ_2) is defined; (3.2) is the set of all closed (i.e., continuous and closed) functions between these spaces. By (3.3), every continuous function from a compact topological space to a T_2-space is closed. Of course, every constant function is continuous.

If $f \in V^U$ and $\mathcal{H} \in \mathcal{P}'(\mathcal{P}(U))$, then the family

$$f^1[\mathcal{H}] \triangleq \{f^1(H):\ H \in \mathcal{H}\} \in \mathcal{P}'(\mathcal{P}(V)) \tag{3.4}$$

is called the "image" of the initial nonempty family $\mathcal{H}$. If $\mathbb{H} \in \mathcal{P}(U)$ and $\mathcal{H} = \{\mathbb{H}\}$, then

$$f^1[\mathcal{H}] = f^1[\{\mathbb{H}\}] = \{f^1(\mathbb{H})\}.$$

The following important property holds:

$$f^1(K) \in (\tau_2-\text{comp})[V] \quad \forall f \in C(U, \tau_1, V, \tau_2) \quad \forall K \in (\tau_1-\text{comp})[U];$$

see [8, 3.1.10]. That is, the continuous image of a compact set is compact.

4. Directed families, filters, and filter bases

In this section, we fix a nonempty set J.

In what follows, this set may be realized in various ways. In essence, J serves as a parameter with specific realizations to be considered as needed. We consider various subfamilies of $\mathcal{P}(J)$. In particular,

$$\beta[J] \triangleq \{\mathcal{J} \in \mathcal{P}'(\mathcal{P}(J))|\ \forall J_1 \in \mathcal{J}\ \forall J_2 \in \mathcal{J}\ \exists J_3 \in \mathcal{J}:\ J_3 \subset J_1 \cap J_2\} \tag{4.1}$$

is the family of all nonempty directed subfamilies of $\mathcal{P}(J)$. In addition,

$$\{\cap\}_\sharp(\tilde{\mathcal{J}}) \triangleq \Big\{\bigcap_{\Sigma \in \mathcal{K}} \Sigma:\ \mathcal{K} \in \mathrm{Fin}(\tilde{\mathcal{J}})\Big\} \in \beta[J] \quad \forall \tilde{\mathcal{J}} \in \mathcal{P}'(\mathcal{P}(J)). \tag{4.2}$$

Now, we introduce filter bases; namely, we consider the family

$$\beta_0[J] \triangleq \{\mathcal{B} \in \beta[J] | \varnothing \notin \mathcal{B}\} = \{\mathcal{B} \in \mathcal{P}'(\mathcal{P}'(J))|\ \forall B_1 \in \mathcal{B}\ \forall B_2 \in \mathcal{B}\ \exists B_3 \in \mathcal{B}:\ B_3 \subset B_1 \cap B_2\} \tag{4.3}$$

of all filter bases on the set J. Moreover, note that (see [1, Ch. I])

$$\mathfrak{F}[J] \triangleq \{\mathcal{F} \in \mathcal{P}'(\mathcal{P}'(J))|\ \big(A \cap B \in \mathcal{F}\ \ \forall A \in \mathcal{F}\ \ \forall B \in \mathcal{F}\big) \& \big([\mathcal{P}(J)](F) \subset \mathcal{F}\ \ \forall F \in \mathcal{F}\big)\} \tag{4.4}$$

is the nonempty family (indeed, $\{J\} \in \mathfrak{F}[J]$) of all filters on J. In addition,

$$(J - \mathbf{fi})[\mathcal{B}] \triangleq \{F \in \mathcal{P}[J]|\ \exists B \in \mathcal{B}:\ B \subset F\} \in \mathfrak{F}[J] \quad \forall \mathcal{B} \in \beta_0[J]. \tag{4.5}$$

Thus (see (4.5)), filter bases from (4.3) generate filters of the family (4.4) via the simple rule (4.5). In connection with (2.4), note that for all $\tau \in (\mathrm{top})[J]$, for all $x \in J$,

$$N_\tau^0(x) \in \beta_0[J]: \quad N_\tau[x] = (J - \mathbf{fi})[N_\tau^0(x)] \in \mathfrak{F}[J]. \tag{4.6}$$

Using (4.6), recall the well-known convergence notion [1, Ch. I]: for all $\tau \in (\mathrm{top})[J]$, $\mathcal{B} \in \beta_0[J]$, and $x \in J$,

$$(\mathcal{B} \overset{\tau}{\Longrightarrow} x) \overset{\mathrm{def}}{\Longleftrightarrow} (N_\tau(x) \subset (J - \mathbf{fi})[\mathcal{B}]). \tag{4.7}$$

Using the inclusion $\mathfrak{F}[J] \subset \beta_0[J]$ and the evident property

$$(J - \mathbf{fi})[\mathcal{F}] = \mathcal{F} \quad \forall \mathcal{F} \in \mathfrak{F}[J],$$

from (4.7), we obtain the following natural corollary for filters: for all $\tau \in (\mathrm{top})[J]$, $\mathcal{F} \in \mathfrak{F}[J]$, and $x \in J$,

$$(\mathcal{F} \overset{\tau}{\Longrightarrow} x) \Longleftrightarrow (N_\tau(x) \subset \mathcal{F}). \tag{4.8}$$

We use ultrafilters, i.e., maximal filters; then,

$$\mathfrak{F}_{\mathbf{u}}[J] \triangleq \{\mathcal{U} \in \mathfrak{F}[J] |\ \forall \mathcal{F} \in \mathfrak{F}[J]\ \ (\mathcal{U} \subset \mathcal{F}) \Longrightarrow (\mathcal{U} = \mathcal{F})\} \tag{4.9}$$

is the nonempty family of all ultrafilters on J. As the simplest example of an ultrafilter, for $x \in J$, we set

$$(J - \mathrm{ult})[x] \triangleq \{F \in \mathcal{P}(J)|\ x \in F\} \in \mathfrak{F}_{\mathbf{u}}[J] \tag{4.10}$$

((4.10) is the trivial ultrafilter associated with x). Clearly, (4.10) realizes an embedding of the set J into the family (4.9):

$$(J - \mathrm{ult})[\cdot] \triangleq \big((J - \mathrm{ult})[x]\big)_{x \in J} \in \mathfrak{F}_{\mathbf{u}}[J]^J. \tag{4.11}$$

Along with (4.11), define the mapping $\mathbf{S}_J \in \mathcal{P}(\mathfrak{F}_{\mathbf{u}}[J])^{\mathcal{P}(J)}$ as follows:

$$\mathbf{S}_J(A) \triangleq \{\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[J] |\ A \in \mathcal{U}\} \quad \forall A \in \mathcal{P}(J). \tag{4.12}$$

By [4, Sect. 8.2], we define the Stone topology

$$\tau_{\mathbf{fi}}[J] \triangleq \big\{G \in \mathcal{P}(\mathfrak{F}_{\mathbf{u}}[J]) |\ \forall \mathcal{U} \in G\ \exists U \in \mathcal{U}:\ \mathbf{S}_J(U) \subset G\big\} \in (\mathbf{c}-\mathrm{top})_0[\mathfrak{F}_{\mathbf{u}}[J]]; \tag{4.13}$$

thus, we obtain a zero-dimensional compactum

$$(\mathfrak{F}_{\mathbf{u}}[J], \tau_{\mathbf{fi}}[J]). \tag{4.14}$$

5. Attraction sets in topological spaces

In this section, we fix a nonempty set E, whose elements are called usual solutions. We keep in mind that each element $e \in E$ admits immediate realization. We also fix a nonempty set X and a topology $\tilde{\tau} \in (\mathrm{top})[X]$; thus, $(X, \tilde{\tau})$, $X \neq \varnothing$, is a topological space. Finally, we fix $f \in X^E$ as a target operator. Recall that $f^1(\Sigma) = \{f(x):\ x \in \Sigma\}$ for $\Sigma \in \mathcal{P}(E)$. Then,

$$(\mathrm{AS})[E; X; \tilde{\tau}; f; \mathcal{E}] \triangleq \bigcap_{\Sigma \in \mathcal{E}} \mathrm{cl}(f^1(\Sigma), \tilde{\tau}) \in \mathbf{C}_X[\tilde{\tau}] \quad \forall \mathcal{E} \in \beta[E]; \tag{5.1}$$

where this definition is considered as a preliminary one. If $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, then (see (4.2)) we consider the following attraction set:

$$(\mathbf{as})[E; X; \tilde{\tau}; f; \mathcal{E}] \triangleq (\mathrm{AS})[E; X; \tilde{\tau}; f; \{\cap\}_\sharp(\mathcal{E})] \in \mathbf{C}_X[\tilde{\tau}]. \tag{5.2}$$

In connection with (5.1) and (5.2), we note (see [4, (8.3.10), Propositions 8.3.1 and 8.4.1] that a series of equivalent representations for attraction sets can be obtained. Now, recall that

$$(\mathbf{as})[E; X; \tilde{\tau}; f; \mathcal{E}] = (\mathrm{AS})[E; X; \tilde{\tau}; f; \mathcal{E}] \quad \forall \mathcal{E} \in \beta[E] \tag{5.3}$$

(see [4, Proposition 8.4.1]).

Now, let us consider the simplest example of an attraction set (5.1), (5.3). Here, we present the construction in a meaningful way, using a scalar controlled system:

$$\dot{x}(t) = u(t), \quad t \in [0, 1[, \tag{5.4}$$

with zero initial state: $x(0) = 0$. In (5.4), we allow nonnegative controls u of the following type: u is any piecewise constant, right-continuous, real-valued function on $[0, 1[$ satisfying

$$\int_0^1 u(t)\,\mathrm{dt} \leqslant 1. \tag{5.5}$$

Let $\mathbb{U}$ denote the set of all such functions (see (5.5)). Then,

$$\mathbf{x}_u(t) \triangleq \int_0^t u(\tau)\,d\tau \in [0, \infty[\quad \forall u \in \mathbb{U} \quad \forall t \in [0, 1].$$

In this example, we identify E with $\mathbb{U}$. For $u \in \mathbb{U}$, consider the following phase constraints:

$$\mathbf{x}_u(t) = 0 \quad \forall t \in [0, 1[. \tag{5.6}$$

Define the set

$$\mathbb{G} \triangleq \{\mathbf{x}_u(1) : u \in \mathbb{U},\ \mathbf{x}_u(t) = 0 \ \ \forall t \in [0,1[\}$$

as the reachability domain under these phase constraints. It is clear that $\mathbb{G} = \{0\}$. Now, let

$$\mathbf{B}_\theta \triangleq \{u \in \mathbb{U} \mid \mathbf{x}_u(t) = 0 \ \ \forall t \in [0, \theta\,[\} \quad \forall \theta \in [0,1[$$

and define

$$\mathfrak{B} \triangleq \{\mathbf{B}_\tau : \tau \in [0,1\,[\}.$$

Clearly, the intersection of all sets in $\mathfrak{B}$ is the set of all $u \in \mathbb{U}$ satisfying (5.6), which coincides with $\{\mathbf{O}\}$, where $\mathbf{O} \in \mathbb{U}$ and $\mathbf{O}(t) = 0$ for all $t \in [0,1\,[$. Moreover, we have $\mathfrak{B} \in \beta_0[\mathbb{U}]$, so we may use (5.1) with $\mathcal{E} = \mathfrak{B}$. Define $\tilde{h} \in \mathbb{R}^{\mathbb{U}}$ by

$$\tilde{h}(u) \triangleq \int_0^1 u(t)\,\mathrm{d}t \quad \forall\, u \in \mathbb{U}.$$

That is, in (5.1), we set $f = \tilde{h}$. Then,

$$\tilde{h}^1(\mathbf{B}_\theta) = \{\tilde{h}(u) : u \in \mathbf{B}_\theta\} = [0,1]$$

for each $\theta \in [0,1[$. Indeed, for $\theta \in [0,1\,[$ we can construct a function $\tilde{u}_\theta \in \mathbf{B}_\theta$, defined by

$$\big(\tilde{u}_\theta(\xi) \triangleq 0 \quad \forall\,\xi \in [0,\theta\,[\big) \& \Big(\tilde{u}_\theta(\xi) \triangleq \frac{1}{1-\theta} \quad \forall\,\xi \in [\theta, 1\,[\Big),$$

so that $\mathbf{x}_{\tilde{u}_\theta}(1) = 1$. Moreover, $\tilde{h}^1(\mathbf{B}_\theta)$ is a convex set. Therefore, in this example, the attraction set (5.1) coincides with $[0,1]$, while $[0,1] \neq \bar{\mathbb{G}}$, where the overline denotes the closure in $\mathbb{R}$ with respect to the usual $|\cdot|$-topology. Thus, there is a jump when (5.6) is weakened. Therefore, in this example, (5.1) is more interesting from a practical point of view.

Of course, we can use filter bases and filters as $\mathcal{E}$ in (5.1)–(5.3); in addition, $\mathfrak{F}[E] \subset \beta_0[E]$. In this connection, we note the following easily verifiable property:

$$(\mathrm{AS})[E; X; \tilde{\tau}; f; \mathcal{B}] = (\mathrm{AS})\big[E; X; \tilde{\tau}; f; (E - \mathbf{fi})[\mathcal{B}]\big] \quad \forall \mathcal{B} \in \beta_0[E]. \tag{5.7}$$

Recall that for any $\mathcal{B} \in \beta_0[E]$, the property $f^1[\mathcal{B}] \in \beta_0[X]$ holds and

$$\big((E - \mathbf{fi})[\mathcal{B}] \in \mathfrak{F}_{\mathbf{u}}[E]\big) \Longrightarrow \big((X - \mathbf{fi})[f^1[\mathcal{B}]] \in \mathfrak{F}_{\mathbf{u}}[X]\big) \tag{5.8}$$

(see [4, Proposition 8.2.1; 1, Ch. I]). Using (5.8), we obtain the following representation of the attraction set (see [4, Propositions 8.3.1, 8.4.1, and 8.4.2]): for any $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$

$$(\mathbf{as})[E; X; \tilde{\tau}; f; \mathcal{E}] = \big\{x \in X \mid \exists\, \mathcal{U} \in [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{E}) :\ f^1[\mathcal{U}] \stackrel{\tilde{\tau}}{\Longrightarrow} x\big\}; \tag{5.9}$$

so, by (5.9), ultrafilters can be used as analogs of Warga's approximate solutions (see [17, Ch. III]). Moreover, for any $\Sigma \in \mathcal{P}(E)$, we have the inclusion $\{\Sigma\} \in \beta[E]$, and by (5.1),

$$(\mathrm{AS})[E; X; \tilde{\tau}; f; \{\Sigma\}] = \mathrm{cl}(f^1(\Sigma), \tilde{\tau}). \tag{5.10}$$

In connection with (5.9) and (5.10), we note the equivalent representation [4, (8.3.10)] realized in the directed class. Now, we introduce two families of attraction sets. Set

$$(\tilde{\tau} - \mathbf{AS})[f] \triangleq \big\{(\mathbf{as})[E; X; \tilde{\tau}; f; \mathcal{E}] :\ \mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))\big\} \in \mathcal{P}'(\mathbf{C}_X[\tilde{\tau}]) \tag{5.11}$$

as the family of all attraction sets under fixed X, $\tilde{\tau}$, and f (recall that X is uniquely specified by $\tilde{\tau}$). Moreover,

$$\mathfrak{F}_{\mathbf{u}}[E] \subset \mathfrak{F}[E] \subset \beta_0[E] \subset \beta[E]. \tag{5.12}$$

Then, by (5.1), (5.3), and (5.12), for any $\mathcal{F} \in \mathfrak{F}[E]$,

$$(\mathbf{as})[E; X; \tilde{\tau}; f; \mathcal{F}] = (\mathrm{AS})[E; X; \tilde{\tau}; f; \mathcal{F}] = \bigcap_{F \in \mathcal{F}} \mathrm{cl}(f^1(F), \tilde{\tau}) \in \mathbf{C}_X[\tilde{\tau}]; \tag{5.13}$$

where, of course, ultrafilters can be used as $\mathcal{F}$. Using (5.13), we set

$$\begin{aligned} ((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f] &\triangleq \big\{(\mathbf{as})[E; X; \tilde{\tau}; f; \mathcal{F}] : \ \mathcal{F} \in \mathfrak{F}[E]\big\} \\ &= \big\{(\mathrm{AS})[E; X; \tilde{\tau}; f; \mathcal{F}] : \ \mathcal{F} \in \mathfrak{F}[E]\big\} \in \mathcal{P}'(\mathbf{C}_X[\tilde{\tau}]). \end{aligned} \tag{5.14}$$

Clearly,

$$((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f] \subset (\tilde{\tau} - \mathbf{AS})[f].$$

Proposition 1. *The following equality holds*:

$$(\tilde{\tau} - \mathbf{AS})[f] = ((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f] \cup \{\varnothing\}. \tag{5.15}$$

P r o o f. Let $M \in (\tilde{\tau} - \mathbf{AS})[f]$. Using (5.11), we choose $\mathcal{M} \in \mathcal{P}'\mathcal{P}(E))$ such that

$$M = (\mathbf{as})[E; X; \tilde{\tau}; f; \mathcal{M}].$$

Then, by (4.2), for

$$\mu \triangleq \{\cap\}_\sharp(\mathcal{M}) \in \beta[E]$$

we obtain (see (5.2))

$$M = (\mathrm{AS})[E; X; \tilde{\tau}; f; \mu]. \tag{5.16}$$

In addition, by (4.1) and (4.3), either $\mu \in \beta_0[E]$ or $\varnothing \in \mu$. We consider both cases separately.

Let $\mu \in \beta_0[E]$. Then, by (4.5), define $\mathfrak{M} \triangleq (E - \mathbf{fi})[\mu] \in \mathfrak{F}[E]$. Therefore, by (5.1), (5.7), and (5.14),

$$M = (\mathrm{AS})[E; X; \tilde{\tau}; f; \mathfrak{M}] \in ((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f].$$

Hence,

$$(\mu \in \beta_0[E]) \Longrightarrow (M \in ((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f]).$$

If $\varnothing \in \mu$, then by (5.1), $(\mathrm{AS})[E; X; \tilde{\tau}; f; \mu] = \varnothing$, and by (5.16), $M = \varnothing$. Thus, $(\varnothing \in \mu) \Longrightarrow (M = \varnothing)$. Consequently,

$$M \in ((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f] \cup \{\varnothing\}.$$

Therefore,

$$(\tilde{\tau} - \mathbf{AS})[f] \subset ((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f] \cup \{\varnothing\}. \tag{5.17}$$

Note that $\varnothing \in \mathcal{P}(E)$ and $\{\varnothing\} \in \beta[E]$. Then, by (5.1) and (5.3),

$$(\mathbf{as})[E; X; \tilde{\tau}; f; \{\varnothing\}] = (\mathrm{AS})[E; X; \tilde{\tau}; f; \{\varnothing\}] = \varnothing \in (\tilde{\tau} - \mathbf{AS})[f].$$

Therefore, $\{\varnothing\} \subset (\tilde{\tau} - \mathbf{AS})[f]$, and hence,

$$((\tilde{\tau}, \mathfrak{F}) - \mathbf{AS})[f] \cup \{\varnothing\} \subset (\tilde{\tau} - \mathbf{AS})[f].$$

Using (5.17), we obtain the required equality (5.15).

Let us recall the example of [4, Sect. 8.9]. In this example, attainability problems are presented in which the attraction set coincides with $\varnothing$. The asymptotic-type constraints are specified by filter bases. By using (5.7), we can interpret this example as an attainability problem where the asymptotic-type constraints are generated by a filter. Thus, in general, the families appearing in (5.15)–specifically, those on the right-hand side–need not be disjoint. In the following sections, we will introduce a natural condition that excludes this possibility.

6. Attainability problem with precompact target operator and some representations for attraction sets

In what follows, we fix a nonempty topological space (Y, τ), $Y \neq \varnothing$, as the main object. Thus, $\tau \in (\text{top})[Y]$. Let

$$\mathbb{F}^0_{\mathbf{c}}[E; Y; \tau] \stackrel{\triangle}{=} \{f \in Y^E | \ f^1(E) \in (\tau - \text{comp})^0[Y]\}, \tag{6.1}$$

where $(\tau - \text{comp})^0[Y] \stackrel{\triangle}{=} \{H \in \mathcal{P}(Y) \mid \exists\, K \in (\tau - \text{comp})[Y] : H \subset K\}$. We call functions from (6.1) precompact functions. It is easy to check that if $\tau \in (\text{top})_0[Y]$, $h \in \mathbb{F}^0_{\mathbf{c}}[E; Y; \tau]$, and $\mathcal{B} \in \beta_0[E]$, then

$$(\text{AS})[E; Y; \tau; h; \mathcal{B}] \in (\tau - \text{comp})[Y] \setminus \{\varnothing\}. \tag{6.2}$$

In this connection, recall that $\beta_0[E] \subset (\text{Cen})[\mathcal{P}(E)]$ (see (4.3) and [3, (3.3.16)]). From (6.2), it follows that if $\tau \in (\text{top})_0[Y]$, $h \in \mathbb{F}^0_{\mathbf{c}}[E; Y; \tau]$, and $\mathcal{F} \in \mathfrak{F}[E]$, then

$$(\text{AS})[E; Y; \tau; h; \mathcal{F}] \in (\tau - \text{comp})[Y] \setminus \{\varnothing\}. \tag{6.3}$$

Recall that for any topological space $(K, \mathbf{t})$, $K \neq \varnothing$, with $\mathbf{t} \in (\mathbf{c} - \text{top})[K]$ (i.e., any nonempty compact topological space $(K, \mathbf{t})$), $m \in K^E$, $\tau \in (\text{top})_0[Y]$, and $g \in C(K, \mathbf{t}, Y, \tau)$,

$$g \circ m \in \mathbb{F}^0_{\mathbf{c}}[E; Y; \tau]$$

(where, $\circ$ denotes composition). Furthermore; we have the following useful property (see [3, Proposition 5.2.1]):

$$(\text{AS})[E; Y; \tau; g \circ m; \mathcal{E}] = g^1\big((\text{AS})[E; K; \mathbf{t}; m; \mathcal{E}]\big) \quad \forall \mathcal{E} \in \beta[E]. \tag{6.4}$$

We note that (6.4) allows a number of generalizations (for example, see [6, Propositins 3.4.10 and 3.4.11], [5]). Of course, in (6.4), the compactum (4.14) can be taken as $(K, \mathbf{t})$. Moreover, recall that by [4, Proposition 8.3.1],

$$(\mathbf{as})[E; Y; \tau; f; \mathcal{E}] = \big\{y \in Y | \ \exists\, \mathcal{U} \in [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{E}) : \ f^1[\mathcal{U}] \stackrel{\tau}{\Longrightarrow} y\big\} \quad \forall f \in Y^E \quad \forall \mathcal{E} \in \mathcal{P}'(\mathcal{P}(E));$$

see also [4, (8.3.10)], where a representation of the attraction set in the directedness class is given.

7. Filters and attainability sets, 1

Recall some properties noted in [4, Ch. 9] and [1, Ch. I]. To this end, we assume that

$$\mathcal{E}_1\{\cap\}\mathcal{E}_2 \stackrel{\triangle}{=} \big\{\text{pr}_1(z) \cap \text{pr}_2(z) : \ z \in \mathcal{E}_1 \times \mathcal{E}_2\big\} \quad \forall \mathcal{E}_1 \in \mathcal{P}'(\mathcal{P}(E)) \quad \forall \mathcal{E}_2 \in \mathcal{P}'(\mathcal{P}(E)); \tag{7.1}$$

see [4, (9.3.6)]. We can use (7.1) for filters; furthermore, by [4, Proposition 9.3.1], for all $\mathcal{F}_1 \in \mathfrak{F}[E]$, $\mathcal{F}_2 \in \mathfrak{F}[E]$, and $\mathcal{F}_3 \in \mathfrak{F}[E]$,

$$\begin{gathered}\big((\mathcal{F}_1 \subset \mathcal{F}_3)\&(\mathcal{F}_2 \subset \mathcal{F}_3)\&(\forall \mathcal{F} \in \mathfrak{F}[E]\ ((\mathcal{F}_1 \subset \mathcal{F})\&(\mathcal{F}_2 \subset \mathcal{F})) \Longrightarrow (\mathcal{F}_3 \subset \mathcal{F}))\big) \\ \Longrightarrow (\mathcal{F}_3 = \mathcal{F}_1\{\cap\}\mathcal{F}_2).\end{gathered} \tag{7.2}$$

In connection with (7.2), we also recall the constructions of [1, Ch. I, § 6]. The following obvious corollary holds: in (7.2), the specified representation of the supremum for $\{\mathcal{F}_1; \mathcal{F}_2\}$ applies if this supremum exists. We also have the following consequence (see [4, Corollary 9.3.1]): for any $\mathcal{F}_1 \in \mathfrak{F}[E]$ and $\mathcal{F}_2 \in \mathfrak{F}[E]$,

$$\big(\exists \mathcal{F} \in \mathfrak{F}[E] : \ (\mathcal{F}_1 \subset \mathcal{F})\&(\mathcal{F}_2 \subset \mathcal{F})\big) \Longleftrightarrow (\mathcal{F}_1\{\cap\}\mathcal{F}_2 \in \mathfrak{F}[E]). \tag{7.3}$$

From (7.3), we obtain the following equivalence:

$$\big(A\cap B\neq\varnothing\quad\forall A\in\mathcal{F}_1\quad\forall B\in\mathcal{F}_2\big)\Longleftrightarrow\big(\mathcal{F}_1\{\cap\}\mathcal{F}_2\in\mathfrak{F}[E]\big).\tag{7.4}$$

(In connection with (7.4), we only note that $\mathcal{F}_1\{\cap\}\mathcal{F}_2\in\beta_0[E]$ under the property $A\cap B\neq\varnothing$ for all $A\in\mathcal{F}_1$ and $B\in\mathcal{F}_2$. Here, (4.5) and (7.3) should be used.) Note that in the general case, for $\mathcal{F}_1\in\mathfrak{F}[E]$ and $\mathcal{F}_2\in\mathfrak{F}[E]$,

$$\mathcal{F}_1\{\cap\}\mathcal{F}_2\in\beta[E]:\ (\mathcal{F}_1\subset\mathcal{F}_1\{\cap\}\mathcal{F}_2)\&(\mathcal{F}_2\subset\mathcal{F}_1\{\cap\}\mathcal{F}_2);\tag{7.5}$$

moreover, in this case, we obtain the following equality:

$$[\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_1\cup\mathcal{F}_2)=[\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_1\{\cap\}\mathcal{F}_2),\tag{7.6}$$

where the following natural representation holds for $\mathcal{F}_1\{\cap\}\mathcal{F}_2$:

$$\mathcal{F}_1\{\cap\}\mathcal{F}_2=\{\cap\}_\sharp(\mathcal{F}_1\cup\mathcal{F}_2).\tag{7.7}$$

From (7.7), by induction, we obtain: for any $n\in\mathbb{N}$ and $(\mathcal{F}_i)_{i\in\overline{1,n}}\in\mathfrak{F}[E]^n$, for the families

$$\Big(\bigcup_{i=1}^{n}\mathcal{F}_i\in\mathcal{P}'(\mathcal{P}'(E))\Big)\&\Big(\{\cap\}_{i=1}^{n}(\mathcal{F}_i)\triangleq\Big\{\bigcap_{i=1}^{n}F_i:\ (F_i)_{i\in\overline{1,n}}\in\prod_{i=1}^{n}\mathcal{F}_i\Big\}\in\beta[E]\Big),\tag{7.8}$$

the following equality holds:

$$\{\cap\}_{i=1}^{n}(\mathcal{F}_i)=\{\cap\}_\sharp\Big(\bigcup_{i=1}^{n}\mathcal{F}_i\Big).\tag{7.9}$$

(The verification of (7.9) follows straightforwardly from the definitions). As a consequence, if $\tau\in(\mathrm{top})[Y]$, $h\in Y^E$, $n\in\mathbb{N}$, and $(\mathcal{F}_i)_{i\in\overline{1,n}}\in\mathfrak{F}[E]^n$, we have

$$(\mathbf{as})\Big[E;X;\tau;h;\bigcup_{i=1}^{n}\mathcal{F}_i\Big]=(\mathrm{AS})\big[E;Y;\tau;h;\{\cap\}_{i=1}^{n}(\mathcal{F}_i)\big].\tag{7.10}$$

Now, consider the case of an arbitrary family of filters. That is, fix a nonempty set T and $(\mathcal{F}_t)_{t\in T}\in\mathfrak{F}[E]^T$; consider the family

$$\bigcup_{t\in T}\mathcal{F}_t\in\mathcal{P}'(\mathcal{P}'(E)).\tag{7.11}$$

To study the attraction set corresponding to asymptotic-type constraints generated by (7.11), we introduce the family

$$\{\cap\}_{t\in T}^{(\sharp)}(\mathcal{F}_t)\triangleq\bigcup_{K\in\mathrm{Fin}(T)}\Big\{\bigcap_{t\in K}F_t:\ (F_t)_{t\in K}\in\prod_{t\in K}\mathcal{F}_t\Big\}\in\mathcal{P}'(\mathcal{P}(E)).\tag{7.12}$$

Proposition 2. *If $A\in\{\cap\}_{t\in T}^{(\sharp)}(\mathcal{F}_t)$ and $B\in\{\cap\}_{t\in T}^{(\sharp)}(\mathcal{F}_t)$, then*

$$A\cap B\in\{\cap\}_{t\in T}^{(\sharp)}(\mathcal{F}_t).$$

The proof follows directly from (4.4) and (7.12); in this argument, standard properties of finite sets are used. From (4.1), (7.12), and Proposition 2, we obtain

$$\{\cap\}_{t\in T}^{(\sharp)}(\mathcal{F}_t)\in\beta[E].\tag{7.13}$$

Proposition 3. *The following equivalence is valid*:

$$\Big(\bigcap_{t\in K} F_t \neq \varnothing \quad \forall K\in \mathrm{Fin}(T) \quad \forall (F_t)_{t\in K}\in \prod_{t\in K}\mathcal{F}_t\Big) \iff \big(\{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t)\in \mathfrak{F}[E]\big). \tag{7.14}$$

P r o o f. Let

$$\bigcap_{t\in K} F_t \neq \varnothing \quad \forall K\in \mathrm{Fin}(T) \quad \forall (F_t)_{t\in K}\in \prod_{t\in K}\mathcal{F}_t. \tag{7.15}$$

Then, by (7.12) and (7.15),

$$\varnothing \notin \{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t).$$

Therefore,

$$\{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t)\in \mathcal{P}'(\mathcal{P}'(E)): \ A\cap B\in \{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t) \quad \forall A\in \{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t) \quad \forall B\in \{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t). \tag{7.16}$$

Let $\Phi\in \{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t)$. Using (7.12), we choose $\mathbb{K}\in \mathrm{Fin}(T)$ and

$$(\Phi_t)_{t\in\mathbb{K}}\in \prod_{t\in\mathbb{K}}\mathcal{F}_t$$

such that

$$\Phi = \bigcap_{t\in\mathbb{K}}\Phi_t. \tag{7.17}$$

Let $\mathbb{H}\in [\mathcal{P}(E)](\Phi)$. Then $\mathbb{H}\in\mathcal{P}(E)$ and $\Phi\subset\mathbb{H}$. By (4.4), we obtain

$$\tilde{\Phi}_t \triangleq \Phi_t\cup \mathbb{H} \ \in \mathcal{F}_t \tag{7.18}$$

for all $t\in\mathbb{K}$ (indeed, $\Phi_t\in\mathcal{F}_t$ and $\tilde{\Phi}_t\in[\mathcal{P}(E)](\Phi_t)$). From (7.18), we have

$$(\tilde{\Phi}_t)_{t\in\mathbb{K}}\in \prod_{t\in\mathbb{K}}\mathcal{F}_t: \quad \tilde{\Phi}\triangleq \bigcap_{t\in\mathbb{K}}\tilde{\Phi}_t\in \{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t). \tag{7.19}$$

By (7.18) and (7.19), $\mathbb{H}\subset\tilde{\Phi}$. Let $x_*\in\tilde{\Phi}$. Then, by (7.19), we have

$$x_*\in\tilde{\Phi}_t \quad \forall t\in\mathbb{K}. \tag{7.20}$$

In addition, $(x_*\notin\mathbb{H})\vee(x_*\in\mathbb{H})$. Suppose $x_*\notin\mathbb{H}$. Then, by (7.20), $x_*\in\tilde{\Phi}_t\setminus\mathbb{H}$ for all $t\in\mathbb{K}$, so by (7.18),

$$x_*\in\Phi_t \quad \forall t\in\mathbb{K}.$$

Therefore, $x_*\in\Phi$ (see (7.17)), and consequently $x_*\in\mathbb{H}$, which contradicts the assumption. Therefore, the property $x_*\notin\mathbb{H}$ is impossible, and so $x_*\in\mathbb{H}$. Since the choice of x_* was arbitrary, the inclusion $\tilde{\Phi}\subset\mathbb{H}$ is established. As a consequence (see (7.19)),

$$\mathbb{H}=\tilde{\Phi}\in \{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t).$$

Thus, we obtain the following important property:

$$[\mathcal{P}(E)](\Phi)\subset \{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t).$$

Since Φ was arbitrary, by (4.4) and (7.16), the inclusion

$$\{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t)\in\mathfrak{F}[E]$$

holds (under condition (7.15)). Thus, the implication

$$\Big(\bigcap_{t\in K} F_t \neq \varnothing \quad \forall K \in \mathrm{Fin}(T) \quad \forall (F_t)_{t\in K} \in \prod_{t\in K} \mathcal{F}_t\Big) \Longrightarrow \big(\{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t) \in \mathfrak{F}[E]\big)$$

is valid. From (4.4) and (7.12), the coverse implication follows directly. Accordingly, (7.14) is established.

Proposition 4. *The following equality holds*:

$$\{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t) = \{\cap\}_\sharp\Big(\bigcup_{t\in T}\mathcal{F}_t\Big). \tag{7.21}$$

P r o o f. Thus, we have two nonempty families. Let $\mathbb{P} \in \{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t)$. Then, by (7.12), for some $\mathbb{K} \in \mathrm{Fin}(T)$ and $(P_t)_{t\in\mathbb{K}} \in \prod_{t\in\mathbb{K}} \mathcal{F}_t$, the equality

$$\mathbb{P} = \bigcap_{t\in\mathbb{K}} P_t \tag{7.22}$$

holds. Set

$$\mathbf{P} \triangleq \{P_t : t \in \mathbb{K}\} \in \mathrm{Fin}\Big(\bigcup_{t\in T}\mathcal{F}_t\Big).$$

Then, $\mathbb{P}$ is the intersection of all sets of $\mathbf{P}$, and by (4.2) and (7.22),

$$\mathbb{P} \in \{\cap\}_\sharp\Big(\bigcup_{t\in T}\mathcal{F}_t\Big).$$

Therefore, we obtain the inclusion

$$\{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t) \subset \{\cap\}_\sharp\Big(\bigcup_{t\in T}\mathcal{F}_t\Big). \tag{7.23}$$

Now, choose any set

$$Q \in \{\cap\}_\sharp\Big(\bigcup_{t\in T}\mathcal{F}_t\Big).$$

Then, for some $r \in \mathbb{N}$ and tuple

$$(Q_l)_{l\in\overline{1,r}} \in \Big(\bigcup_{t\in T}\mathcal{F}_t\Big)^r, \tag{7.24}$$

we have

$$Q = \bigcap_{l=1}^{r} Q_l. \tag{7.25}$$

From (7.24), we have the following obvious property:

$$\mathbb{T}_l \triangleq \{t \in T \mid Q_l \in \mathcal{F}_t\} \in \mathcal{P}'(T) \quad \forall l \in \overline{1,r}. \tag{7.26}$$

Hence,

$$(\mathbb{T}_l)_{l\in\overline{1,r}} \in \mathcal{P}'(T)^r : \prod_{l=1}^{r}\mathbb{T}_l = \big\{(t_l)_{l\in\overline{1,r}} \in T^r \mid t_s \in \mathbb{T}_s \quad \forall s \in \overline{1,r}\big\} \in \mathcal{P}'(T^r); \tag{7.27}$$

Using (7.27), fix any tuple

$$(\theta_l)_{l\in\overline{1,r}} \in \prod_{l=1}^{r} \mathbb{T}_l. \tag{7.28}$$

Then, by (7.27) and (7.28), $(\theta_l)_{l\in\overline{1,r}} \in T^r$ and, as a result,

$$\Theta \triangleq \{\theta_l : \ l \in \overline{1,r}\} \in \mathrm{Fin}(T). \tag{7.29}$$

From (7.26) and (7.28), for each $l \in \overline{1,r}$, we have $Q_l \in \mathcal{F}_{\theta_l}$. For $t \in \Theta$, set

$$\mathcal{L}_t \triangleq \{l \in \overline{1,r}|\ \theta_l = t\} \in \mathcal{P}'(\overline{1,r}). \tag{7.30}$$

We note that, by (4.4) and reasoning by induction, the following property is established:

$$\bigcap_{i=1}^{m} F_i \in \mathcal{F} \quad \forall \mathcal{F} \in \mathfrak{F}[E] \quad \forall m \in \mathbb{N} \quad \forall (F_i)_{i\in\overline{1,m}} \in \mathcal{F}^m. \tag{7.31}$$

Using (7.30) and (7.31), for each $t \in \Theta$, set

$$\mathbb{Q}_t \triangleq \bigcap_{l\in\mathcal{L}_t} Q_l \in \mathcal{F}_t. \tag{7.32}$$

Thus,

$$(\mathbb{Q}_t)_{t\in\Theta} \in \prod_{t\in\Theta} \mathcal{F}_t.$$

From (7.12) and (7.29),

$$\mathbf{Q} \triangleq \bigcap_{t\in\Theta} \mathbb{Q}_t \in \{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t). \tag{7.33}$$

Consider two sets: Q and $\mathbf{Q}$. Let $y_* \in Q$. Then $y_* \in Q_l$ for all $l \in \overline{1,r}$. By (7.30) and (7.32),

$$y_* \in \mathbb{Q}_t \quad \forall t \in \Theta,$$

so, by (7.33), $y_* \in \mathbf{Q}$. Thus,

$$Q \subset \mathbf{Q}. \tag{7.34}$$

Let $y^* \in \mathbf{Q}$. Then, for $y^* \in E$, we have

$$y^* \in \mathbb{Q}_t \quad \forall t \in \Theta. \tag{7.35}$$

Now, let $\nu \in \overline{1,r}$. Then $\mathbb{T}_\nu = \{t \in T|\ Q_\nu \in \mathcal{F}_t\}$ (see (7.26)). By (7.28), $\theta_\nu \in \mathbb{T}_\nu$, so

$$Q_\nu \in \mathcal{F}_{\theta_\nu},$$

where $\theta_\nu \in \Theta$ by (7.29). From (7.35), $y^* \in \mathbb{Q}_{\theta_\nu}$. Therefore, by (7.32),

$$y^* \in Q_l \quad \forall l \in \mathcal{L}_{\theta_\nu}. \tag{7.36}$$

By (7.30), $\nu \in \mathcal{L}_{\theta_\nu}$, so by (7.36), $y^* \in Q_\nu$. Since ν was arbitrary, it follows that

$$y^* \in Q_l \quad \forall l \in \overline{1,r}.$$

By (7.25), $y^* \in Q$. Thus, $\mathbf{Q} \subset Q$. Using (7.34), we obtain the equality $Q = \mathbf{Q}$ and, by (7.33), $Q \in \{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t)$. Therefore, the inclusion

$$\{\cap\}_\sharp\Big(\bigcup_{t\in T} \mathcal{F}_t\Big) \subset \{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t)$$

holds. Using (7.23), we obtain the required equality (7.21).

Corollary 1. *If* $h \in Y^E$, *then*

$$(\mathbf{as})\Big[E;Y;\tau;h;\bigcup_{t\in T}\mathcal{F}_t\Big] = (\mathrm{AS})\big[E;Y;\tau;h;\{\cap\}^{(\sharp)}_{t\in T}(\mathcal{F}_t)\big].$$

The corresponding proof uses (5.1), (7.13), and Proposition 4. In Corollary 1, an essential generalization is obtained in comparison with (7.10). By Proposition 3 and (5.14), we have

$$\Big(\bigcap_{t\in K} F_t \neq \varnothing \quad \forall K \in \mathrm{Fin}(T) \quad \forall (F_t)_{t\in K} \in \prod_{t\in K}\mathcal{F}_t\Big)$$
$$\Longrightarrow \Big((\mathbf{as})\Big[E;Y;\tau;h;\bigcup_{t\in T}\mathcal{F}_t\Big] \in ((\tau,\mathfrak{F}) - \mathbf{AS})[h] \quad \forall h \in Y^E\Big).$$

8. Filters and attainability sets, 2

In what follows, we suppose that $\tau \in (\mathrm{top})_0[Y]$. Thus, we consider the T_2-space (Y,τ), $Y \neq \varnothing$. Moreover, we fix a precompact function $\mathbf{h} \in \mathbb{F}^0_{\mathbf{c}}[E;Y;\tau]$. By (6.2), we have the following important property:

$$(\mathrm{AS})[E;Y;\tau;\mathbf{h};\mathcal{F}] \in (\tau - \mathrm{comp})[Y] \setminus \{\varnothing\} \quad \forall \mathcal{F} \in \mathfrak{F}[E]. \tag{8.1}$$

Returning to Proposition 1, we note that, by (5.14) and (8.1),

$$((\tau,\mathfrak{F}) - \mathbf{AS})[\mathbf{h}] \subset (\tau - \mathbf{AS})[\mathbf{h}] \setminus \{\varnothing\}$$

and, as, a consequence (see Proposition 1),

$$((\tau,\mathfrak{F}) - \mathbf{AS})[\mathbf{h}] = (\tau - \mathbf{AS})[\mathbf{h}] \setminus \{\varnothing\}. \tag{8.2}$$

Thus, in our case, the attraction set is nonempty if and only if it can be generated by a filter. Therefore, in this case, we avoid pathologies such as those in the example of [4, Sect. 8.9]. It is useful to note that both the precompactness condition for $\mathbf{h}$ and the T_2-separability of (Y,τ) are typical in control problems. Thus, (8.2) holds for an important class of practical problems.

Now, we note that in (8.1) we can take ultrafilters as $\mathcal{F}$; that is:

$$(\mathrm{AS})[E;Y;\tau;\mathbf{h};\mathcal{U}] \in (\tau - \mathrm{comp})[Y] \setminus \{\varnothing\} \quad \forall \mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E]. \tag{8.3}$$

We will now consider certain constructions related to (8.3). For this, we first introduce some auxiliary statements regarding filter convergence. If $\mathcal{F} \in \mathfrak{F}[Y]$, we define the sets

$$\Big((\tau - \mathrm{LIM})[\mathcal{F}] \triangleq \{y \in Y \mid \mathcal{F} \overset{\tau}{\Longrightarrow} y\} \in \mathcal{P}(Y)\Big)\&\Big((\tau - \mathrm{CL})[\mathcal{F}] \triangleq \bigcap_{F\in\mathcal{F}} \mathrm{cl}(F,\tau) \in \mathcal{P}(Y)\Big) \tag{8.4}$$

which satisfy

$$(\tau - \mathrm{LIM})[\mathcal{F}] \subset (\tau - \mathrm{CL})[\mathcal{F}]$$

(see [4, (8.3.37)]), and

$$\big((\tau - \mathrm{LIM})[\mathcal{F}] = \varnothing\big) \vee \big(\exists y \in (\tau - \mathrm{LIM})[\mathcal{F}] :\ (\tau - \mathrm{CL})[\mathcal{F}] = \{y\}\big) \tag{8.5}$$

(see [4, Proposition 8.3.3]). Moreover, by [4, Proposition 8.3.2], we always have

$$(\tau - \mathrm{LIM})[\mathcal{U}] = (\tau - \mathrm{CL})[\mathcal{U}] \quad \forall \mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E]. \tag{8.6}$$

We will use statements (8.4)–(8.6) in the investigation of the properties of the attraction set (8.3). In more general form, these statements are presented in [4, Sect. 8.3].

Proposition 5. *If* $\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E]$, *then* $\exists!\, \mathbf{y} \in Y:\ (\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] = \{\mathbf{y}\}$.

P r o o f. Fix $\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E]$. Then, in particular, $\mathcal{U} \in \beta_0[E]$ and $\mathbf{h}^1[\mathcal{U}] \in \beta_0[Y]$. By [4, Proposition 8.2.1],

$$(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]] \in \mathfrak{F}_{\mathbf{u}}]Y]. \tag{8.7}$$

By (5.1), (5.7), and (8.4), we have

$$(\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] = (\tau - \mathrm{CL})\big[(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]]\big]. \tag{8.8}$$

Using (8.6) and (8.7), we obtain the following equality:

$$(\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] = (\tau - \mathrm{LIM})\big[(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]]\big].$$

From (8.3), we have

$$(\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] \neq \varnothing.$$

Therefore, by (8.5), (8.7), and (8.8),

$$(\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] = \{\mathbf{y}\}, \tag{8.9}$$

where

$$\mathbf{y} \in (\tau - \mathrm{LIM})\big[(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]]\big].$$

From (8.4), it follows that $\mathbf{y} \in Y$. The element $\mathbf{y} \in Y$ satisfying (8.9) is, of course, unique.

In connection with Proposition 5, we recall (5.9). Using this proposition, we introduce the operator

$$\Psi[E; Y; \tau; \mathbf{h}] \in Y^{\mathfrak{F}_{\mathbf{u}}[E]} \tag{8.10}$$

by the following natural rule: for any $\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E]$, the value $\Psi[E; Y; \tau; \mathbf{h}](\mathcal{U}) \in Y$ is defined by the equality

$$(\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] = \big\{\Psi[E; Y; \tau; \mathbf{h}](\mathcal{U})\big\}; \tag{8.11}$$

we call $\Psi[E; Y; \tau; \mathbf{h}](\mathcal{U})$ the attraction element corresponding to the ultrafilter $\mathcal{U}$.

Proposition 6. *If* $\mathcal{F} \in \mathfrak{F}[E]$, *then*

$$(\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}] = \Psi[E; Y; \tau; \mathbf{h}]^1\big([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})\big). \tag{8.12}$$

P r o o f. Fix $\mathcal{F} \in \mathfrak{F}[E]$. Let $y_0 \in \Psi[E; Y; \tau; \mathbf{h}]^1\big([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})\big)$. Then $y_0 \in Y$, and for some ultrafilter $\mathcal{U}_0 \in [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})$, the equality $y_0 = \Psi[E; Y; \tau; \mathbf{h}](\mathcal{U}_0)$ holds. Using (8.11), we have

$$(\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}_0] = \{y_0\}. \tag{8.13}$$

By (5.1), we have the inclusion

$$(\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}_0] \subset (\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}]$$

(since, by the choice of $\mathcal{U}_0$, we have $\mathcal{F} \subset \mathcal{U}_0$). Then, by (8.13), $y_0 \in (\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}]$. Since y_0 was arbitrary, the inclusion

$$\Psi[E; Y; \tau; \mathbf{h}]^1\big([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})\big) \subset (\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}] \tag{8.14}$$

is established. Let $y_* \in (\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}]$. Now, we use (5.9) and [4, (8.2.6) and Proposition 8.3.1]. Then, for some $\mathcal{U}_* \in [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})$,

$$\mathbf{h}^1[\mathcal{U}_*] \overset{\tau}{\Longrightarrow} y_*.$$

As a consequence, by (4.7) and (4.8) we obtain

$$(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}_*]] \stackrel{\tau}{\Longrightarrow} y_*.$$

Therefore,

$$y_* \in (\tau - \mathrm{LIM})\big[(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}_*]]\big],$$

where

$$(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}_*]] \in \mathfrak{F}_{\mathbf{u}}[Y]$$

(see (5.8)). By (8.6),

$$y_* \in (\tau - \mathrm{CL})\big[(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}_*]]\big].$$

Using (3.4), (4.5), and (8.4), we obtain the following chain of equalities:

$$(\tau - \mathrm{CL})[(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}_*]]] = \bigcap_{\Sigma \in (Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}_*]]} \mathrm{cl}(\Sigma, \tau)$$

$$= \bigcap_{\Sigma \in \mathbf{h}^1[\mathcal{U}_*]} \mathrm{cl}(\Sigma, \tau) = \bigcap_{U \in \mathcal{U}_*} \mathrm{cl}(\mathbf{h}^1(U), \tau) = (\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}_*].$$

Thus,

$$y_* \in (\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}_*].$$

By (8.11),

$$y_* = \Psi[E; Y; \tau; \mathbf{h}](\mathcal{U}_*).$$

Since $\mathcal{U}_* \in [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})$, we conclude (by the definition of the image) that

$$y_* \in \Psi[E; Y; \tau; \mathbf{h}]^1\big([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F})\big).$$

Therefore, the inclusion

$$(\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}] \subset \Psi[E; Y; \tau; \mathbf{h}]^1([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}))$$

is established. Using (8.14), we obtain the required equality (8.12).

So, the attraction set for asymptotic-type constraints generated by a filter is exhausted by the attraction elements corresponding to ultrafilters that majorize this filter. We note the following obvious property of the attraction element for trivial ultrafilters:

$$\Psi[E; Y; \tau; \mathbf{h}]\big((E - \mathrm{ult})[x]\big) = \mathbf{h}(x) \quad \forall x \in E. \tag{8.15}$$

Next, we state two simple facts regarding the nonemptiness of the attraction set. For $\mathcal{E} \in \beta[E]$,

$$\big((\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{E}] \neq \varnothing)\big) \Longleftrightarrow (\mathcal{E} \in \beta_0[E]).$$

Moreover, for $\mathcal{E} \in \mathcal{P}'(\mathcal{P}(E))$, the following equivalence holds:

$$\big((\mathbf{as})[E; Y; \tau; \mathbf{h}; \mathcal{E}] \neq \varnothing\big) \Longleftrightarrow (\mathcal{E} \in (\mathrm{Cen})[E]).$$

Proposition 7. *The following equality is valid*:

$$((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}] = (\tau - \mathbf{AS})[\mathbf{h}] \setminus \{\varnothing\}. \tag{8.16}$$

P r o o f. By Proposition 1, we obtain

$$(\tau - \mathbf{AS})[\mathbf{h}] \setminus \{\varnothing\} \subset ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}]. \tag{8.17}$$

On the other hand, from (5.14) and (6.3), we have (in our case)

$$((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}] \subset (\tau - \mathbf{AS})[\mathbf{h}] \setminus \{\varnothing\}.$$

Using (8.17), we obtain (8.16).

Recall Proposition 6. Now, we will use some properties of ultrafilters. We have that for all $\mathcal{F}_1 \in \mathfrak{F}[E]$, $\mathcal{F}_2 \in \mathfrak{F}[E]$, and $\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E]$,

$$(\mathcal{F}_1 \cap \mathcal{F}_2 \subset \mathcal{U}) \Longrightarrow \big((\mathcal{F}_1 \subset \mathcal{U}) \vee (\mathcal{F}_2 \subset \mathcal{U})\big); \tag{8.18}$$

here we use [4, Proposition 9.4.3 and (1.5.1)]; in addition, $\mathcal{F}_1 \cap \mathcal{F}_2 \in \mathfrak{F}[E]$. Given $\mathcal{E}_1 \in \mathcal{P}'(\mathcal{P}(E))$ and $\mathcal{E}_2 \in \mathcal{P}'(\mathcal{P}(E))$, we define

$$\mathcal{E}_1\{\cup\}\mathcal{E}_2 \triangleq \big\{\mathrm{pr}_1(z) \cup \mathrm{pr}_2(z) : \; z \in \mathcal{E}_1 \times \mathcal{E}_2\big\} \in \mathcal{P}'(\mathcal{P}(E)).$$

If $\mathcal{F}_1 \in \mathfrak{F}[E]$ and $\mathcal{F}_2 \in \mathfrak{F}[E]$, the following obvious equality holds:

$$\mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}_1\{\cup\}\mathcal{F}_2 \in \mathfrak{F}[E]. \tag{8.19}$$

From (8.18) and (8.19), we obtain the following chain of equalities:

$$[\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_1 \cap \mathcal{F}_2) = [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_1\{\cup\}\mathcal{F}_2) = [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_1) \cup [\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_2.) \tag{8.20}$$

Now, recall Proposition 6. Then, by (8.19) and (8.20),

$$\begin{aligned}(\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}_1 \cap \mathcal{F}_2] &= \Psi[E; Y; \tau; \mathbf{h}]^1\big([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_1 \cap \mathcal{F}_2)\big) \\ &= \Psi[E; Y; \tau; \mathbf{h}]^1([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_1)) \cup \Psi[E; Y; \tau; \mathbf{h}]^1([\mathfrak{F}_{\mathbf{u}}[E]](\mathcal{F}_2)) \\ &= (\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}_1] \cup (\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}_2] \quad \forall \mathcal{F}_1 \in \mathfrak{F}[E] \quad \forall \mathcal{F}_2 \in \mathfrak{F}[E].\end{aligned} \tag{8.21}$$

From (4.4), it follows that for $m \in \mathbb{N}$ and $(\mathcal{F}_i)_{i \in \overline{1,m}} \in \mathfrak{F}[E]^m$,

$$\bigcap_{i=1}^{m} \mathcal{F}_i \in \mathfrak{F}[E]. \tag{8.22}$$

In connection with (8.22), we introduce

$$\{\cup\}_{i=1}^{m}(\mathcal{E}_i) \triangleq \Big\{ \bigcup_{i=1}^{m} \Sigma_i : \; (\Sigma_i)_{i \in \overline{1,m}} \in \prod_{i=1}^{m} \mathcal{E}_i \Big\} \quad \forall m \in \mathbb{N} \quad \forall (\mathcal{E}_i)_{i \in \overline{1,m}} \in \mathcal{P}'(\mathcal{P}(E))^m.$$

It is easy to see that for $m \in \mathbb{N}$ and $(\mathcal{F}_i)_{i \in \overline{1,m}} \in \mathfrak{F}[E]^m$,

$$\bigcap_{i=1}^{m} \mathcal{F}_i = \{\cup\}_{i=1}^{m}(\mathcal{F}_i). \tag{8.23}$$

Remark 1. In fact, (8.22) and (8.23) can be generalized as follows: if T is a nonempty set and $(\mathcal{F}_t)_{t \in T} \in \mathfrak{F}[E]^T$, then

$$\bigcap_{t \in T} \mathcal{F}_t = \Big\{ \bigcup_{t \in T} F_t : \; (F_t)_{t \in T} \in \prod_{t \in T} \mathcal{F}_t \Big\} \in \mathfrak{F}[E].$$

By (8.21) and reasoning by induction, the following general statement is established.

Proposition 8. *If* $n \in \mathbb{N}$ *and* $(\mathcal{F}_i)_{i \in \overline{1,n}} \in \mathfrak{F}[E]^n$, *then*

$$\bigcup_{i=1}^{n} (\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{F}_i] = (\mathrm{AS})\Big[E; Y; \tau; \mathbf{h}; \bigcap_{i=1}^{n} \mathcal{F}_i\Big].$$

Corollary 2. *If* $n \in \mathbb{N}$ *and* $(\mathcal{B}_i)_{i \in \overline{1,n}} \in \beta_0[E]^n$, *then*

$$\bigcup_{i=1}^{n} (\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{B}_i] = (\mathrm{AS})\Big[E; Y; \tau; \mathbf{h}; \bigcap_{i=1}^{n} (E - \mathbf{fi})[\mathcal{B}_i]\Big].$$

The corresponding proof follows immediately from (5.7) and Proposition 8. As a consequence, from (5.14), we obtain

$$\bigcup_{i=1}^{n} (\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{B}_i] \in \big((\tau, \mathfrak{F}) - \mathbf{AS}\big)[\mathbf{h}] \quad \forall n \in \mathbb{N} \quad \forall (\mathcal{B}_i)_{i \in \overline{1,n}} \in \beta_0[E]^n.$$

9. Some topological properties

Now, we consider the question of the continuity property of the operator $\Psi[E; Y; \tau; \mathbf{h}]$ and some its consequences. Since $\mathcal{P}(E) \in \tilde{\pi}^0[E]$, by [4, (1.5.8), (2.4.4)] we use

$$\mathbb{F}_{\lim}[E; Y; \mathcal{P}(E); \tau] \triangleq \big\{g \in Y^E | \ \forall \mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E] \ \exists y \in Y : \ g^1[\mathcal{U}] \overset{\tau}{\Longrightarrow} y\big\} \in \mathcal{P}'(Y^E) \tag{9.1}$$

for which

$$\mathbb{F}^0_{\mathbf{c}}[E; Y; \tau] \subset \mathbb{F}_{\lim}[E; Y; \mathcal{P}(E); \tau]$$

(the corresponding proof is obvious). Thus, $\mathbf{h} \in \mathbb{F}_{\lim}[E; Y; \mathcal{P}(E); \tau]$, and (see [4, p. 58]) we define

$$\varphi_{\lim}[E; Y; \mathcal{P}(E); \tau; \mathbf{h}] \in Y^{\mathfrak{F}_{\mathbf{u}}[E]};$$

moreover, in our case, the following equality holds:

$$\varphi_{\lim}[E; Y; \mathcal{P}(E); \tau; \mathbf{h}] = \Psi[E; Y; \tau; \mathbf{h}]. \tag{9.2}$$

Remark 2. In connection with (9.2), we note (8.6). Indeed, let $\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E]$. Then, $\mathcal{U} \in \beta[E]$, and by (3.4), (5.1), and (8.6),

$$\begin{gathered}(\mathrm{AS})[E; Y; \tau; \mathbf{h}; \mathcal{U}] = \bigcap_{U \in \mathcal{U}} \mathrm{cl}(\mathbf{h}^1(U), \tau) = \bigcap_{\Sigma \in \mathbf{h}^1[\mathcal{U}]} \mathrm{cl}(\Sigma, \tau) \\ \bigcap_{\Sigma \in (Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]]} \mathrm{cl}(\Sigma, \tau) = (\tau - \mathrm{CL})\big[(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]]\big] = (\tau - \mathrm{LIM})\big[(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]]\big],\end{gathered} \tag{9.3}$$

where

$$(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]] \in \mathfrak{F}_{\mathbf{u}}[Y]$$

(see (5.8)). As a consequence, by (8.4), (8.11), and (9.3),

$$(Y - \mathbf{fi})[\mathbf{h}^1[\mathcal{U}]] \overset{\tau}{\Longrightarrow} \Psi[E; Y; \tau; \mathbf{h}](\mathcal{U}). \tag{9.4}$$

From (4.7), (4.8), and (9.4), we obtain the following convergence:

$$\mathbf{h}^1[\mathcal{U}] \overset{\tau}{\Longrightarrow} \Psi[E; Y; \tau; \mathbf{h}](\mathcal{U}).$$

Now, by (9.1) and [4, (1.5.8),(2.4.5),(2.4.6)], the obvious equality holds:

$$\varphi_{\lim}[E;Y;\mathcal{P}(E);\tau;\mathbf{h}](\mathcal{U}) = \Psi[E;Y;\tau;\mathbf{h}](\mathcal{U}).$$

Since the choice of $\mathcal{U}$ was arbitrary, equality (9.2) is established.

Until the end of this section, we suppose that (Y,τ) is a regular topological space; that is, (Y,τ) is both a T_1-space and a T_3-space. Furthermore, the separability property holds in our case, that is, $\tau \in (\mathrm{top})_0[Y]$. From (9.2) and [4, Proposition 2.4.2], we obtain the following statement.

Proposition 9. *The mapping* $\Psi[E;Y;\tau;\mathbf{h}]$ *has the continuity property*:

$$\Psi[E;Y;\tau;\mathbf{h}] \in C(\mathfrak{F}_{\mathbf{u}}[E], \tau_{\mathbf{fi}}[E], Y, \tau).$$

Using (3.3) and (4.13), we obtain

$$\Psi[E;Y;\tau;\mathbf{h}] \in C_{\mathrm{cl}}(\mathfrak{F}_{\mathbf{u}}[E], \tau_{\mathbf{fi}}[E], Y, \tau);$$

therefore, by (3.2),

$$\Psi[E;Y;\tau;\mathbf{h}]^1(\mathrm{cl}(A, \tau_{\mathbf{fi}}[E])) = \mathrm{cl}(\Psi[E;Y;\tau;\mathbf{h}]^1(A), \tau) \quad \forall A \in \mathcal{P}(\mathfrak{F}_{\mathbf{u}}[E]). \tag{9.5}$$

Now, we use (9.5) and the natural variant of [4, (9.7.18)]:

$$\mathfrak{F}_{\mathbf{u}}[E] = \mathrm{cl}\big(\big\{(E-\mathrm{ult})[x] :\ x \in E\big\}, \tau_{\mathbf{fi}}[E]\big); \tag{9.6}$$

in connection with (9.6), we also recall [4, (1.5.8), (1.5.9), (8.2.4)]. Using (8.15), (9.5), and (9.6), we obtain

$$\begin{aligned}\Psi[E;Y;\tau;\mathbf{h}]^1(\mathfrak{F}_{\mathbf{u}}[E]) &= \mathrm{cl}\big(\big\{\Psi[E;Y;\tau;\mathbf{h}]((E-\mathrm{ult})[x]) :\ x \in E\big\}, \tau\big)\\ &= \mathrm{cl}(\{\mathbf{h}(x) :\ x \in E\}, \tau) = \mathrm{cl}(\mathbf{h}^1(E), \tau).\end{aligned} \tag{9.7}$$

Proposition 10. *Nonempty finite subsets of* $\mathrm{cl}(\mathbf{h}^1(E), \tau)$ *are attraction sets generated by filters*:

$$\mathrm{Fin}(\mathrm{cl}(\mathbf{h}^1(E), \tau)) \subset ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}]. \tag{9.8}$$

P r o o f. We use (9.7) to verify (9.8). Let

$$V \in \mathrm{Fin}(\mathrm{cl}(\mathbf{h}^1(E), \tau)). \tag{9.9}$$

Then for some $n \in \mathbb{N}$ and some tuple $(v_i)_{i\in\overline{1,n}} \in V^n$, we have

$$V = \{v_i :\ i \in \overline{1,n}\}.$$

In particular, $(v_i)_{i\in\overline{1,n}} \in Y^n$. Moreover, by (9.9),

$$v_j \in \mathrm{cl}(\mathbf{h}^1(E), \tau) \quad \forall j \in \overline{1,n}. \tag{9.10}$$

By (9.7) and (9.10), we obtain

$$\mathfrak{V}_j \triangleq \{\mathcal{U} \in \mathfrak{F}_{\mathbf{u}}[E] |\ v_j = \Psi[E;Y;\tau;\mathbf{h}](\mathcal{U})\} \in \mathcal{P}'(\mathfrak{F}_{\mathbf{u}}[E]) \quad \forall j \in \overline{1,n}.$$

It follows that

$$\prod_{i=1}^{n} \mathfrak{V}_i = \big\{(\mathcal{U}_i)_{i\in\overline{1,n}} \in \mathfrak{F}_{\mathbf{u}}[E]^n |\ \mathcal{U}_j \in \mathfrak{V}_j \quad \forall j \in \overline{1,n}\big\} \in \mathcal{P}'(\mathfrak{F}_{\mathbf{u}}[E]^n). \tag{9.11}$$

Since the set (9.11) is nonempty, we can choose

$$(\mathcal{V}_i)_{i\in\overline{1,n}} \in \prod_{i=1}^{n} \mathfrak{V}_i. \tag{9.12}$$

From (9.11) and (9.12), for each $j \in \overline{1,n}$, the ultrafilter $\mathcal{V}_j \in \mathfrak{F}_{\mathbf{u}}[E]$ satisfies the equality

$$v_j = \Psi[E; Y; \tau; \mathbf{h}](\mathcal{V}_j).$$

Of course, $\mathcal{V}_j \in \mathfrak{F}[E]$ for all $j \in \overline{1,n}$. Therefore, by (8.22),

$$\bigcap_{i=1}^{n} \mathcal{V}_i \in \mathfrak{F}[E].$$

Then, by (5.14),

$$(\mathrm{AS})\big[E; Y; \tau; \mathbf{h}; \bigcap_{i=1}^{n} \mathcal{V}_i\big] \in ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}].$$

Using Proposition 8, we obtain

$$\bigcup_{i=1}^{n} (\mathrm{AS})\big[E; Y; \tau; \mathbf{h}; \mathcal{V}_i\big] \in ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}]. \tag{9.13}$$

By (8.11) and (9.12),

$$(\mathrm{AS})\big[E; Y; \tau; \mathbf{h}; \mathcal{V}_j\big] = \{v_j\}$$

for all $j \in \overline{1,n}$. Thus, by (9.13),

$$\bigcup_{i=1}^{n} \{v_i\} \in ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}],$$

where the union $\{v_i\}$, $i \in \overline{1,n}$, coincides with V. As a consequence,

$$V \in ((\tau, \mathfrak{F}) - \mathbf{AS})[\mathbf{h}].$$

Since the choice of V in (9.9) was arbitrary, (9.8) holds.

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ASYMPTOTIC BEHAVIOR FOR THE LOTKA–VOLTERRA EQUATION WITH DISPLACEMENTS AND DIFFUSION

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Abstract: In this paper, we consider the Lotka–Volterra equation with displacements and diffusion, that is transport-diffusion system describing the evolution of prey and predator populations with their displacements and their diffusion, in a periodic domain in $\mathbb{R}$. It is shown that the solution to this equation and its logarithm are globally bounded, and that, when the solution converges to the stationary solution in mean value, it converges uniformly with respect to the time variable as well as the space variable. These results are obtained by using L^2-estimate of the well-known Lyapunov functional, and, in particular, an estimate of the point-wise growth of the solution by means of the application of the fundamental solution of the heat equation.

Keywords: Lotka–Volterra equation, Asymptotic behavior, Diffusion, Transport/Displacement, Numerical example.

1. Introduction

As is well-known, the system of equations called *Lotka–Volterra equation*,

$$\begin{cases} \dfrac{d}{dt}u_1 = \alpha u_1 - \beta u_1 u_2, \\ \dfrac{d}{dt}u_2 = -\gamma u_2 + \delta u_1 u_2, \end{cases}$$

$(\alpha, \beta, \gamma, \delta > 0)$ was proposed to model the evolution of prey and predator populations (represented by u_1 and u_2, respevtively). This system of equations has the particularity that all its (positive) solutions are periodic, as illustrated in [16]. In [16], we also find a detailed analysis of the behavior of the solution and various versions of the equation.

As for the Lotka–Volterra equation with diffusion, Rothe [15] considered the Lotka–Volterra equation with diffusion (with the same diffusion coefficient for both species) in one-dimensional domain $0 < x < 1$ with periodic boundary conditions in x (or homogeneous Neumann conditions) and proved the uniform convergence to the time-periodic solution of the Lotka–Volterra equation (independent of x) (see also [14], which had made similar reasoning). On the other hand, Gabbuti and Negro [8] proved the convergence of the solution of the Lotka–Volterra equation with diffusion in a bounded domain of $\mathbb{R}^2$ with the homogeneous Neumann condition to the time-periodic solution of the Lotka–Volterra equation (independent of x); in the article [8], the diffusion coefficients are not the same for both species and the convergence is in an integral sense, but sufficiently strong. Successively, the asymptotic behavior of the solution of the Lotka–Volterra equation with diffusion with the Dirichlet condition was studied in [18], while the aspects of spatial propagation of the solution to the Lotka–Volterra equation continue to attract the interest of researchers (see for example [4, 5]).

As far as concerns the Lotka–Volterra equation with diffusion in one spatial dimension, the question concerning the travelling waves has attracted the interest of many researchers. However, the results of [14] and [15] exclude the existence of a travelling wave for the classical Lotka–Volterra equation with simple diffusion. For this reason, several researchers have sought some aspects of travelling wave for slightly modified equations (see for example [2, 3, 6, 10, 17]).

In the context of stochastic equations, the Lotka–Volterra equation with logistic effect and diffusion has been studied in [7] and [9]. In [7] the existence and uniqueness theorem of the solution has been proved, and in [9] the existence of an invariant measure has been shown.

In [13] the author has considered the Lotka–Volterra equation with diffusion and population displacements. The results of this article are essentially numerical. However, the question of population displacement/immigration has attracted the attention of many researchers, as evidenced by several recent publications (see for example [1, 11, 12]).

In this article, we consider the Lotka–Volterra equation for the population density $u_1(t,x)$ and $u_2(t,x)$ with diffusion and population displacements on the periodic domain of $\mathbb{R}$ and prove the uniform boundedness of $u_1(t,x)$, $u_2(t,x)$, $\log u_1(t,x)$, $\log u_2(t,x)$. We also prove that in the case where the solution (u_1,u_2) tends to the stationary solution in mean value, (u_1,u_2) converges uniformly to the stationary solution. In order to obtain this result, we use the function

$$U = -\alpha\log(u_2) - \gamma\log(u_1) + \beta u_2 + \delta u_1,$$

but due to the population displacements we cannot directly deduce a conclusion from the equation for U, as the authors of [14] and [15] did. In order to overcome this difficulty, we estimate not only U in $L^2(0,2\pi)$ but also point-wise growth of $u_1(t,x)$ and $u_2(t,x)$.

Our study is motivated not only by the general interest of the effect of displacement/immigration for population dynamics but also by the specific behavior that arises from the numerical calculation of the solution of the Lotka–Volterra equation with population displacement in opposite directions for prey and predator populations. This will be illustrated in the following section.

2. Motivation and some numerical examples

As we mentioned in Introduction, the evolution of prey and predator populations is described, in its basic form, by Lotka–Volterra equation

$$\frac{d}{dt}u_1(t) = \alpha u_1(t) - \beta u_1(t)u_2(t), \tag{2.1}$$

$$\frac{d}{dt}u_2(t) = -\gamma u_2(t) + \delta u_1(t)u_2(t), \tag{2.2}$$

where $u_1(t)$ and $u_2(t)$ denote the prey and predator populations, respectively, while the coefficients α, β, γ and δ are assumed to be constants and strictly positive. We consider the system of equations (2.1)–(2.2) with the initial conditions

$$u_1(0) = u_{1,0} > 0, \quad u_2(0) = u_{2,0} > 0. \tag{2.3}$$

We first recall the well-known fundamental properties of the solution of the system of equations (2.1)–(2.2). For this, we define the function $U_0(\cdot,\cdot)$ as

$$U_0(u_1,u_2) = -\alpha\log u_2 - \gamma\log u_1 + \beta u_2 + \delta u_1, \quad u_1>0, \quad u_2>0.$$

Remark 1. For any initial data $u_{1,0}>0$, $u_{2,0}>0$, the solution $(u_1(t),u_2(t))$ of the Cauchy problem (2.1)–(2.3) exists for all $t>0$ and it is periodic in t. Furthermore, the function $U_0(u_1(t),u_2(t))$ remains constant, i.e.

$$U_0(u_1(t),u_2(t)) = U_0(u_1(0),u_2(0)) = -\alpha\log(u_2(0)) - \gamma\log(u_1(0)) + \beta u_2(0) + \delta u_1(0)$$

for all $t \geq 0$ and the solution $(u_1(t), u_2(t))$ moves along the closed curve

$$\gamma = \{\, (u_1, u_2) \,|\, u_1 > 0,\ u_2 > 0,\ U_0(u_1, u_2) = U_0(u_1(0), u_2(0)) \,\}$$

with a constant period.

The proof of this fact can be found in [16] (and in many other manuals on population dynamics).

The model (2.1)–(2.2) is constructed for the prey and predators populations homogeneously distributed in a territory. But, if the populations are not homogeneously distributed and if there are population displacements, the relations mentioned in Remark 1 will not be guaranteed. Let us see an example of changing the behavior of the solution.

Consider the equation system

$$\begin{cases} \partial_t u_1(t,x) = -v_1(t)\partial_x u_1(t,x) + \alpha u_1(t,x) - \beta u_1(t,x)u_2(t,x), \\ \partial_t u_2(t,x) = -v_2(t)\partial_x u_2(t,x) - \gamma u_2(t,x) + \delta u_1(t,x)u_2(t,x), \end{cases} \quad t > 0, \quad x \in \mathbb{R}, \tag{2.4}$$

with the initial condition

$$u_1(0,x) = \overline{u}_1(x), \quad u_2(0,x) = \overline{u}_2(x).$$

Let us choose a particular initial data $(\overline{u}_1(x), \overline{u}_2(x))$ defined as follows: consider the equation system (2.1)–(2.2) and write x instead of t in the solution $(\overline{u}_1(\cdot), \overline{u}_2(\cdot))$ to these equations. It is clear that the thus defined functions $\overline{u}_1(x)$ and $\overline{u}_2(x)$ can be defined on $\mathbb{R}$ and are periodic in x. Let us further assume that

$$v_1(t) = -v_2(t) \quad \forall t \geq 0$$

and that they are periodic in t with the same period as the solution of the equation system (2.1)–(2.2). Then, for a certain choice of functions $(v_1(t), v_2(t))$ we find the amplification of the oscillation of the solution in certain points x and the contraction in certain points x, as illustrated in the graphs obtained by numerical calculation (see Fig. 1–2).

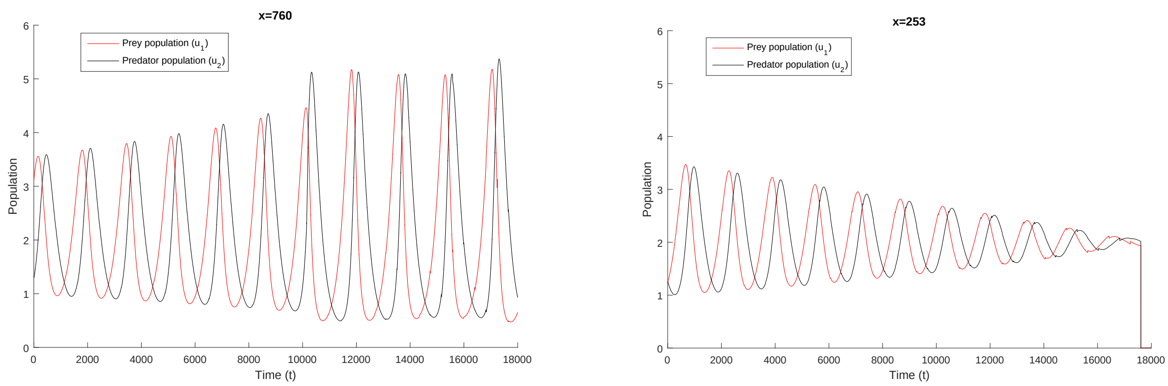

Figure 1. Solution of the equation system (2.4) at a point where amplification occurs and at a point where contraction occurs.

However, even with displacements, the equation system (2.4) in a periodic domain $x \in \mathbb{R}/\mathrm{mod}\,\mathrm{L}$ has a globally similar behavior to what we have seen in Remark 1.

Remark 2. Let L be a strictly positive number. Let $u_{1,0}(x)$ and $u_{2,0}(x)$ be two functions with strictly positive values and periodic in $x \in \mathbb{R}$ with period L. If the solution $(u_1(t,x), u_2(t,x))$ to the equation system (2.4) with the initial condition

$$u_1(0,x) = u_{1,0}(x), \quad u_2(0,x) = u_{2,0}(x),$$

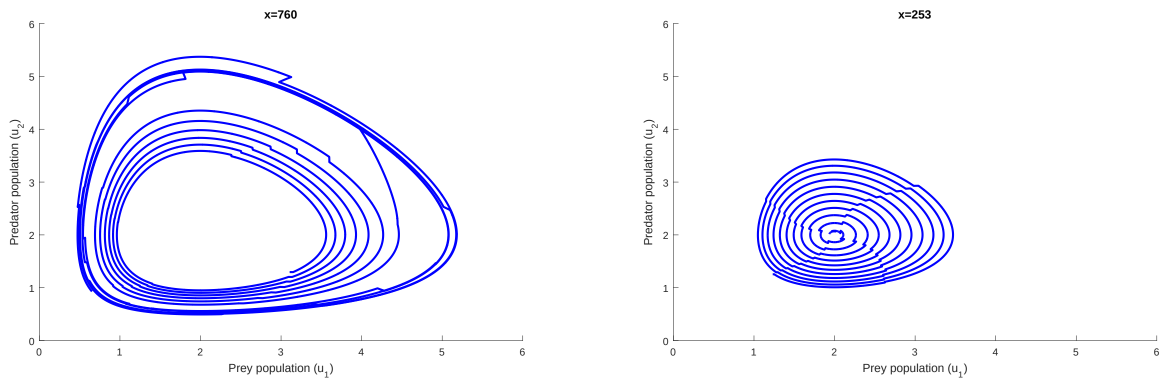

Figure 2. Trajectories of the solution of the equation system (2.4) on the phase plane at a point where amplification occurs and at a point where contraction occurs in the space (u_1, u_2).

exists and is periodic in $x \in \mathbb{R}$ with period L, then we have

$$\int_0^L U_0(u_1(t,x), u_2(t,x))dx = \text{Const} = \int_0^L U_0(u_{1,0}(x), u_{2,0}(x))dx. \tag{2.5}$$

Indeed, it follows immediately from (2.4) that

$$\partial_t \log u_1 = -v_1 \partial_x \log u_1 + \alpha - \beta u_2, \tag{2.6}$$

$$\partial_t \log u_2 = -v_2 \partial_x \log u_2 - \gamma + \delta u_1, \tag{2.7}$$

from (2.4), (2.6) and (2.7), by direct calculations, we obtain

$$\partial_t U_0(u_1(t,x), u_2(t,x)) = -v_1 \partial_x(-\gamma \log u_1 + \delta u_1) - v_2 \partial_x(-\alpha \log u_2 + \beta u_2). \tag{2.8}$$

Given the assumption that $u_1(t,x)$ and $u_2(t,x)$ are periodic in x with period L, we have

$$\int_0^L \partial_x(-\gamma \log u_1 + \delta u_1)dx = \int_0^L \partial_x(-\alpha \log u_2 + \beta u_2)dx = 0.$$

Thus

$$\frac{d}{dt}\int_0^L U_0(u_1(t,x), u_2(t,x))dx = 0,$$

which implies (2.5). But, we cannot deduce that $\sup_{0 \le x \le 2\pi} U_0(u_1(t,x), u_2(t,x))$ is bounded at t.

Given these circumstances, we are interested in the asymptotic behavior of the solution $(u_1(t,x), u_2(t,x))$ of the Lotka–Volterra equation with displacements and diffusion (see (3.1)–(3.2) in the next section).

3. Position of problem and preliminary result

We consider in this article the following equation system

$$\partial_t u_1(t,x) = -v_1(t)\partial_x u_1(t,x) + \kappa \partial_x^2 u_1(t,x) + \alpha u_1(t,x) - \beta u_1(t,x) u_2(t,x), \tag{3.1}$$

$$\partial_t u_2(t,x) = -v_2(t)\partial_x u_2(t,x) + \kappa \partial_x^2 u_2(t,x) - \gamma u_2(t,x) + \delta u_1(t,x) u_2(t,x), \tag{3.2}$$

for $t \geq 0$ and $x \in \mathbb{R}$, where $\alpha, \beta, \gamma, \delta$ and κ are strictly positive constants and $v_1(t)$ and $v_2(t)$ are functions of t. The system (3.1)–(3.2) will be considered with the initial condition

$$u_i(t,x) = u_{i,0}(x), \quad i = 1, 2. \tag{3.3}$$

For the functions $u_{1,0}(x)$ and $u_{2,0}(x)$, it is assumed that

$$u_{i,0}(x) > 0, \quad u_{i,0}(x) = u_{i,0}(x+2\pi) \quad \forall x \in \mathbb{R}, \quad u_{i,0}(\cdot) \in L^\infty(\mathbb{R}), \quad i = 1, 2. \tag{3.4}$$

Since the equations (3.1)–(3.2) are parabolic equations subject to the conditions (3.3)–(3.4), the existence and uniqueness of the local solution follow from the classical theory of parabolic equations. Furthermore, considering the equations (3.1)–(3.2) on $\mathbb{R}_+ \times \mathbb{T}$ with the torus $\mathbb{T} = \mathbb{R}/\mathrm{mod}\, 2\pi$, the periodicity in x of the solution $(u_1(t,x), u_2(t,x))$ follows automatically. As far as concerns the global solution, we will first prove the inequality (4.3) on the interval of the existence of the solution $(u_1(t,x), u_2(t,x))$ and then deduce from the inequality (4.3) and the theorem of the existence and the uniqueness of the local solution the existence and the uniqueness of the global solution.

We now define the functions $U_1(u_1)$, $U_2(u_2)$ and $U(u_1, u_2)$:

$$U_1(u_1) = -\gamma\Big(\log u_1 - \log\frac{\gamma}{\delta}\Big) + \delta\left(u_1 - \frac{\gamma}{\delta}\right), \tag{3.5}$$

$$U_2(u_2) = -\alpha\Big(\log u_2 - \log\frac{\alpha}{\beta}\Big) + \beta\Big(u_2 - \frac{\alpha}{\beta}\Big), \tag{3.6}$$

$$U(u_1, u_2) = U_1(u_1) + U_2(u_2). \tag{3.7}$$

Since

$$\min_{u_1>0}(-\gamma \log u_1 + \delta u_1) = -\gamma \log\left(\frac{\gamma}{\delta}\right) + \gamma, \tag{3.8}$$

$$\min_{u_2>0}(-\alpha \log u_2 + \beta u_2) = -\alpha \log\left(\frac{\alpha}{\beta}\right) + \alpha, \tag{3.9}$$

it follows that $U_1(u_1) \geq 0$, $U_2(u_2) \geq 0$ and $U(u_1, u_2) \geq 0$ for any $u_1 > 0$ and $u_2 > 0$. Thus

$$\min_{u_1>0} U_1(u_1) = \min_{u_2>0} U_2(u_2) = \min_{u_1>0,\ u_2>0} U(u_1, u_2) = 0, \tag{3.10}$$

$$U(u_1, u_2) = 0 \iff u_1 = \frac{\gamma}{\delta} \quad \text{and} \quad u_2 = \frac{\alpha}{\beta}. \tag{3.11}$$

Let us set

$$\tilde{U}(t) = \frac{1}{2\pi}\int_0^{2\pi} U(u_1(t,x), u_2(t,x))dx. \tag{3.12}$$

Let us first note the following fact, which can be proved in a manner similar to the raisoning presented in [14] and [15].

Proposition 1. *Assume that*

$$\sup_{0\leq x\leq 2\pi} U(u_{1,0}(x), u_{2,0}(x)) < \infty$$

and that the problem (3.1)–(3.3) *with* (3.4) *admits the unique solution* $(u_1(t,x), u_2(t,x))$ *in the time interval* $[0, \tau[\ \ (0 < \tau \leq \infty)$. *Then, the function* $\tilde{U}(t)$ *is decreasing on the interval* $[0, \tau[$.

P r o o f. In a manner similar to deriving (2.8), but adding the terms that result from the diffusion termes, we obtain

$$\partial_t U = \kappa\partial_x^2 U - \kappa\sigma - v_1\partial_x U_1 - v_2\partial_x U_2, \tag{3.13}$$

where

$$\sigma = \sigma(t,x) = \gamma\left(\frac{\partial_x u_1(t,x)}{u_1(t,x)}\right)^2 + \alpha\left(\frac{\partial_x u_2(t,x)}{u_2(t,x)}\right)^2.$$

By integrating both sides of the equality (3.13) with respect to x from 0 to 2π, we obtain

$$\int_0^{2\pi} \partial_t U dx = \int_0^{2\pi} \left(\kappa\partial_x^2 U - \kappa\sigma - v_1\partial_x U_1 - v_2\partial_x U_2\right) dx.$$

Since the functions $U(u_1(t,x), u_2(t,x))$, $U_1(u_1(t,x))$ and $U_2(u_2(t,x))$ are 2π-periodic in x, we have

$$\frac{d}{dt}\int_0^{2\pi} U(u_1(t,x), u_2(t,x))dx = -\kappa \int_0^{2\pi} \sigma(t,x)dx.$$

This, together with the relation $\sigma \geq 0$, implies that the function $\tilde{U}(t)$ is decreasing. □

Corollary 1. *If the solution $(u_1(t,x), u_2(t,x))$ of the problem (3.1)–(3.3) (with (3.4)) exists for all $t > 0$, then the function $\tilde{U}(t)$ converges to a value $\tilde{U}_\infty$ for $t \to \infty$.*

P r o o f. It immediately follows from Proposition 1 and the relation (3.10). □

4. Main result

Our main result is the following.

Theorem 1. *Assume that*

$$\sup_{t\geq 0} |v_1(t) - v_2(t)| \equiv C_v < \infty, \tag{4.1}$$

$$\sup_{0\leq x\leq 2\pi} U(u_{1,0}(x), u_{2,0}(x)) < \infty. \tag{4.2}$$

Then, the problem (3.1)–(3.3) with (3.4) admits a unique solution $(u_1(t,x), u_2(t,x))$ for all $t > 0$ and we have

$$\sup_{t\geq 0,\ 0\leq x\leq 2\pi} U(u_1(t,x), u_2(t,x)) < \infty. \tag{4.3}$$

More precisely,

i) *there exists a continuous and increasing function $\Lambda_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\limsup_{t\to\infty} \sup_{0\leq x\leq 2\pi} U(u_1(t,x), u_2(t,x)) \leq \Lambda_1(\tilde{U}_\infty),$$

ii) *if $\tilde{U}_\infty = 0$, then we have*

$$\lim_{t\to\infty} \sup_{0\leq x\leq 2\pi} U(u_1(t,x), u_2(t,x)) = 0,$$

where $\tilde{U}_\infty = \lim_{t\to\infty} \tilde{U}(t)$ with $\tilde{U}(t)$ defined in (3.12).

For the proof of Theorem 1 we use the proposition.

Proposition 2. *Assume that the conditions* (4.1)–(4.2) *and* (3.4) *are satisfied and that the problem* (3.1)–(3.3) *admits a unique solution* $(u_1(t,x), u_2(t,x))$ *for all* $t > 0$*. Then, there exists an increasing and continuous function* $\Lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\limsup_{t\to\infty} \|U(u_1(t,\cdot), u_2(t,\cdot))\|^2_{L^2(0,2\pi)} \le \Lambda_2(\tilde{U}_\infty), \tag{4.4}$$
$$\Lambda_2(0) = 0.$$

The function $\Lambda_2(\cdot)$ can be given for example by the formula (5.13).

In the following section, we will prove Proposition 2. Theorem 1 will be proved in the successive section.

5. Proof of Proposition 2

In order to prove Proposition 2, we begin with the following lemma.

Lemma 1. *Let* $U = U(x)$ *be a positive and* 2π*-periodic function such that*

$$\left\| \frac{d}{dx} U \right\|_{L^2(0,2\pi)} < \infty.$$

If

$$\|U\|_{L^2(0,2\pi)} > \sqrt{8\pi}\,\overline{U}, \tag{5.1}$$

then we have

$$\left\| \frac{d}{dx} U \right\|^2_{L^2(0,2\pi)} \ge \frac{1}{256\pi^3 \overline{U}^2} \left(1 - \frac{4\sqrt{2\pi}\,\overline{U}}{3\|U\|_{L^2(0,2\pi)}} \right) \|U\|^4_{L^2(0,2\pi)}, \tag{5.2}$$

where

$$\overline{U} = \frac{1}{2\pi} \int_0^{2\pi} U(x)dx.$$

P r o o f. Set

$$\mu = \frac{\|U\|_{L^2(0,2\pi)}}{2\sqrt{2\pi}}, \qquad D_\mu = \left\{ x \in [0,2\pi] \,|\, U(x) > \mu \right\}, \tag{5.3}$$

and denote by $|D_\mu|$ the measure of the set D_μ. Since $U(x) > \mu$ on D_μ, it follows from the definition of $\overline{U}$ and μ that

$$\mu |D_\mu| \le 2\pi \overline{U}. \tag{5.4}$$

Since

$$U(x)^2 = (U(x) - \mu)^2 + 2\mu(U(x) - \mu) + \mu^2,$$

it follows that

$$\int_{D_\mu} |U(x)|^2 dx = \int_{D_\mu} (U(x) - \mu)^2 dx + 2\int_{D_\mu} \mu(U(x) - \mu)dx + \int_{D_\mu} \mu^2 dx.$$

Hence

$$\int_{D_\mu} (U(x) - \mu)^2 dx = \int_{D_\mu} |U(x)|^2 dx - 2\int_{D_\mu} \mu(U(x) - \mu)dx - |D_\mu|\mu^2$$
$$\ge \int_{D_\mu} |U(x)|^2 dx - 3|D_\mu|\mu^2 - \frac{1}{2}\int_{D_\mu} (U(x) - \mu)^2 dx.$$

Thus, taking into account (5.3), we have

$$\int_{D_\mu} (U(x)-\mu)^2 dx \geq \frac{2}{3}\int_{D_\mu} |U(x)|^2 dx - 2|D_\mu|\mu^2 = \frac{2}{3}\int_{D_\mu} |U(x)|^2 dx - \frac{|D_\mu| \|U\|^2_{L^2(0,2\pi)}}{4\pi}. \tag{5.5}$$

On the other hand, we have

$$\int_{D_\mu^c} |U(x)|^2 dx \leq (2\pi - |D_\mu|)\mu^2.$$

Hence, taking into account (5.3), we have

$$\int_{D_\mu} |U(x)|^2 dx \geq \|U\|^2_{L^2(0,2\pi)} - (2\pi - |D_\mu|)\mu^2 = \Big(\frac{3}{4} + \frac{|D_\mu|}{8\pi}\Big)\|U\|^2_{L^2(0,2\pi)}. \tag{5.6}$$

From (5.5) and (5.6) we obtain

$$\int_{D_\mu} (U(x)-\mu)^2 dx \geq \left(\frac{1}{2} - \frac{|D_\mu|}{6\pi}\right) \|U\|^2_{L^2(0,2\pi)}. \tag{5.7}$$

Recall that under the condition (5.1) the relation (5.4) implies that $|D_\mu| < 2\pi$, and thus there exists at least one $\bar{x} \in \mathbb{R}$ such that $U(\bar{x}) \leq \mu$. Since $U(x)$ is 2π-periodic, it is not restrictive to assume that $\bar{x} = 0$ (and thus $U(\bar{x}+2\pi) \leq \mu$).

We first consider the case

$$D_\mu =]x_0, x_0 + |D_\mu|[\,.$$

In this case, since we have

$$\int_{D_\mu} (U(x)-\mu)^2 dx = \int_{D_\mu} 2\int_{x_0}^{x} (U(x')-\mu)\frac{d}{dx'}U(x')dx'dx,$$

and thus

$$\int_{D_\mu} (U(x)-\mu)^2 dx \leq 2|D_\mu| \Big(\int_{D_\mu} (U(x)-\mu)^2 dx\Big)^{1/2} \Big(\int_{D_\mu} \Big(\frac{d}{dx}U(x)\Big)^2 dx\Big)^{1/2},$$

we obtain

$$\int_{D_\mu} (U(x)-\mu)^2 dx \leq 4|D_\mu|^2 \int_{D_\mu} \left(\frac{d}{dx}U(x)\right)^2 dx. \tag{5.8}$$

Even in the general case with

$$D_\mu = \bigcup_{k=0}^{N}]x_k, x_k'[, \quad |D_\mu| = \sum_{k=1}^{N} (x_k' - x_k), \quad N \in \mathbb{N}, \quad N \geq 2 \quad \text{or} \quad N = +\infty,$$

on every interval $]x_k, x_k'[$ we have

$$\int_{x_k}^{x_k'} (U(x)-\mu)^2 dx \leq 4|D_\mu|^2 \int_{x_k}^{x_k'} \left(\frac{d}{dx}U(x)\right)^2 dx.$$

By summing these inequalities, we obtain (5.8).

From (5.7) and (5.8) we have

$$\int_{D_\mu} \left(\frac{d}{dx}U(x)\right)^2 dx \geq \frac{1}{4|D_\mu|^2}\Big(\frac{1}{2} - \frac{|D_\mu|}{6\pi}\Big)\|U\|^2_{L^2(0,2\pi)}. \tag{5.9}$$

Since, according to (5.4), we have

$$|D_\mu| \le \frac{4\pi\sqrt{2\pi}\,\overline{U}}{\|U\|_{L^2(0,2\pi)}},$$

from (5.9) we obtain

$$\int_{D_\mu}\left(\frac{d}{dx}U(x)\right)^2 dx \ge \frac{1}{256\pi^3\overline{U}^2}\left(1-\frac{4\sqrt{2\pi}\,\overline{U}}{3\|U\|_{L^2(0,2\pi)}}\right)\|U\|^4_{L^2(0,2\pi)}.$$

Since

$$\int_0^{2\pi}\left(\frac{d}{dx}U(x)\right)^2 dx \ge \int_{D_\mu}\left(\frac{d}{dx}U(x)\right)^2 dx,$$

we deduce the inequality (5.2). This completes the proof of the lemma. □

Lemma 1 leads to the following property.

Lemma 2. *Assume that the conditions (4.1)–(4.2) and (3.4) are satisfied and that the problem (3.1)–(3.3) admits a unique solution $(u_1(t,x), u_2(t,x))$ for all $t > 0$. Let $U(\cdot,\cdot)$ and $\tilde{U}(t)$ be the functions defined in (3.7) and (3.12), respectively. If*

$$\|U(u_1(t,\cdot), u_2(t,\cdot))\|_{L^2(0,2\pi)} > \sqrt{8\pi}\tilde{U}(t),$$

then we have

$$\frac{d}{dt}\|U\|^2_{L^2} \le \left(\frac{C_v^2}{\kappa} - \frac{\kappa}{256\pi^3\tilde{U}^2}\left(1-\frac{4\sqrt{2\pi}}{3\|U\|_{L^2}}\tilde{U}\right)\|U\|^2_{L^2}\right)\|U\|^2_{L^2}, \tag{5.10}$$

where $\tilde{U} = \tilde{U}(t)$ and

$$\|U\|_{L^2} = \|U(u_1(t,\cdot), u_2(t,\cdot))\|_{L^2(0,2\pi)}.$$

P r o o f. By writing $v_1(t) - v_2(t) + v_2(t)$ instead of $v_1(t)$ in (3.13), we have

$$\partial_t U = \kappa\partial_x^2 U - \kappa\sigma(t,x) - v_2(t)\partial_x U - (v_1(t) - v_2(t))\partial_x U_1. \tag{5.11}$$

If we multiply both sides of (5.11) by U and integrate them on $[0, 2\pi]$, then, using integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}|U|^2dx = -\kappa\int_0^{2\pi}|\partial_x U|^2dx - \kappa\int_0^{2\pi}\sigma U dx + (v_1(t)-v_2(t))\int_0^{2\pi}U_1\partial_x U dx.$$

Note that due to relations $U = U_1 + U_2$, $U_1 \ge 0$, $U_2 \ge 0$ (see (3.5)–(3.9)), we have

$$\int_0^{2\pi}U_1\partial_x U dx \le \frac{1}{2\kappa}\int_0^{2\pi}|U_1|^2dx + \frac{\kappa}{2}\int_0^{2\pi}|\partial_x U|^2dx \le \frac{1}{2\kappa}\int_0^{2\pi}|U|^2dx + \frac{\kappa}{2}\int_0^{2\pi}|\partial_x U|^2dx.$$

Thus, taking into account the relation $\sigma U \ge 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}|U|^2dx \le -\frac{\kappa}{2}\int_0^{2\pi}|\partial_x U|^2dx + \frac{|v_1(t)-v_2(t)|^2}{2\kappa}\int_0^{2\pi}|U|^2dx. \tag{5.12}$$

Applying the inequality (5.2) to the first term on the right-hand side of the inequality (5.12) and taking into account the condition (4.1), we obtain (5.10). This completes the proof of the lemma. □

P r o o f (of Proposition 2). Note that if $\|U\|_{L^2} > \sqrt{8\pi}\tilde{U}$, then we have

$$1 - \frac{4\sqrt{2\pi}}{3\|U\|_{L^2}}\tilde{U} \geq \frac{1}{3}.$$

Thus, in this case, the right-hand side of the inequality (5.10) is bounded from above by

$$\left(\frac{C_v^2}{\kappa} - \frac{\kappa}{256\pi^3\tilde{U}^2}\frac{\|U\|_{L^2}^2}{3}\right)\|U\|_{L^2}^2.$$

Therefore, from Lemma 2 it follows that

$$\limsup_{t\to\infty}\int_0^{2\pi}|U(u_1(t,x),u_2(t,x))|^2dx \leq \Lambda_2(\tilde{U}_\infty),$$

where $\Lambda_2(\cdot)$ is defined by

$$\Lambda_2(a) = \max\left(8\pi,\ \frac{768\pi^3C_v^2}{\kappa^2}\right)a^2, \tag{5.13}$$

which completes the proof of Proposition 2. □

6. Proof of Theorem 1

In order to prove Theorem 1, we begin with an estimate of the $\|\partial_x U(u_1(t,\cdot),u_2(t,\cdot))\|_{L^2(0,2\pi)}$. We have the following lemma (in Lemmas 3–9, we assume that the hypothesis of Proposition 2 is satisfied).

Lemma 3. *For all* $t_2 > t_1 \geq 0$, *we have*

$$\begin{aligned}&\int_{t_1}^{t_2}\|\partial_x U(u_1(t,\cdot),u_2(t,\cdot))\|_{L^2(0,2\pi)}^2dt\\ &\leq \frac{C_v}{\kappa^2}\int_{t_1}^{t_2}\|U(u_1(t,\cdot),u_2(t,\cdot))\|_{L^2_{(0,2\pi)}}^2dt + \frac{1}{\kappa}\|U(u_1(t_1,\cdot),u_2(t_1,\cdot))\|_{L^2(0,2\pi)}^2.\end{aligned} \tag{6.1}$$

P r o o f. From (5.12) we deduce that

$$\int_0^{2\pi}|\partial_x U(t,x)|^2dx \leq \frac{|v_1(t)-v_2(t)|^2}{\kappa^2}\int_0^{2\pi}|U(t,x)|^2dx - \frac{1}{\kappa}\frac{d}{dt}\int_0^{2\pi}|U(t,x)|^2dx,$$

where $U(t,x) = U(u_1(t,x),u_2(t,x))$. Integrating both sides of this inequality with respect to t from t_1 to t_2, we obtain

$$\int_{t_1}^{t_2}\|\partial_x U(t,\cdot)\|_{L^2(0,2\pi)}^2dt \leq \frac{C_v}{\kappa^2}\int_{t_1}^{t_2}\|U(t,\cdot)\|_{L^2(0,2\pi)}^2dt - \frac{1}{\kappa}\left(\|U(t_2,\cdot)\|_{L^2(0,2\pi)}^2 - \|U(t_1,\cdot)\|_{L^2(0,2\pi)}^2\right). \tag{6.2}$$

Eliminating the negative terme of the right-hand side of the inequality (6.2), we obtain (6.1). □

We deduce from Lemma 3 the following relation.

Lemma 4. *We have*

$$\begin{aligned}&\int_t^{t+1}\sup_{0\leq x\leq 2\pi}U(u_1(t',x),u_2(t',x))dt'\\ &\leq \tilde{U}(t) + \sqrt{2\pi}\Big(\frac{C_v}{\kappa^2}\int_t^{t+1}\|U(u_1(t',\cdot),u_2(t',\cdot))\|_{L(0,2\pi)^2}^2dt' + \frac{1}{\kappa}\|U(u_1(t,\cdot),u_2(t,\cdot))\|_{L^2(0,2\pi)}^2\Big)^{1/2},\end{aligned} \tag{6.3}$$

where $\tilde{U}(t)$ *is the notation introduced in* (3.12).

P r o o f. We use the notation $U(t,x) = U(u_1(t,x), u_2(t,x))$ as in the proof of Lemma 3. Since

$$\|\varphi\|_{L^1(0,2\pi)} \leq \sqrt{2\pi}\|\varphi\|_{L^2(0,2\pi)}$$

for all $\varphi \in L^2(0,2\pi)$, from the relation

$$\sup_{0\leq x\leq 2\pi} U(t,x) \leq \tilde{U}(t) + \|\partial_x U(t,\cdot)\|_{L^1(0,2\pi)},$$

we obtain

$$\sup_{0\leq x\leq 2\pi} U(t,x) \leq \tilde{U}(t) + \sqrt{2\pi}\|\partial_x U(t,\cdot)\|_{L^2(0,2\pi)}. \tag{6.4}$$

Taking into account the decreasing of $\tilde{U}(t)$, the inequality (6.3) follows immediatly from (6.1) and (6.4). □

We will now estimate the growth of

$$\sup_{0\leq x\leq 2\pi} u_1(t,x), \quad \sup_{0\leq x\leq 2\pi} u_2(t,x), \quad \sup_{0\leq x\leq 2\pi} (-\log u_1(t,x)), \quad \sup_{0\leq x\leq 2\pi} (-\log u_2(t,x)).$$

To this end, we return to the equations (3.1) and (3.2). Note that, if we introduce the function

$$\xi_1(t,x) = x + \int_0^t v_1(t')dt',$$

and if we consider the variables (t,ξ_1) instead of (t,x), then the equation (3.1) is rewritten as

$$\partial_t u_1(t,\xi_1) = \kappa\partial^2_{\xi_1} u_1(t,\xi_1) + \alpha u_1(t,\xi_1) - \beta u_1(t,\xi_1)u_2(t,\xi_1). \tag{6.5}$$

Analogously, if we introduce the function

$$\xi_2(t,x) = x + \int_0^t v_2(t')dt',$$

and if we consider the variables (t,ξ_2) instead of (t,x), then the equation (3.2) is rewritten as

$$\partial_t u_2(t,\xi_2) = \kappa\partial^2_{\xi_2} u_2(t,\xi_2) - \gamma u_2(t,\xi_2) + \delta u_1(t,\xi_2)u_2(t,\xi_2). \tag{6.6}$$

Using (6.5) and (6.6), we will prove the four following lemmas.

Lemma 5. *Set*

$$u_1^+(t) = \sup_{0\leq x\leq 2\pi} u_1(t,x) = \sup_{\xi_1\in\mathbb{R}} u_1(t,\xi_1). \tag{6.7}$$

Then, for $0 \leq t_0 \leq t$, *we have*

$$u_1^+(t) \leq u_1^+(t_0)e^{\alpha(t-t_0)} \equiv \Phi_1(u_1^+(t_0), t-t_0). \tag{6.8}$$

P r o o f. By formally applying the fundamental solution of the heat equation to (6.5), we have

$$u_1(t,\xi_1) = \int_{-\infty}^{+\infty} \Theta(t-t_0,\xi'-\xi_1)u_1(t_0,\xi')d\xi'$$
$$+ \int_{t_0}^{t}\int_{-\infty}^{+\infty} \Theta(t-t',\xi'-\xi_1)\left(\alpha u_1(t',\xi') - \beta u_1(t',\xi')u_2(t',\xi')\right)d\xi' dt',$$

where

$$\Theta(\tau,\eta)=\frac{1}{\sqrt{(4\pi\tau\kappa)}}\exp(-\frac{|\eta|^2}{4\tau\kappa}).$$

Since

$$\int_{-\infty}^{+\infty}\Theta(\tau,\eta)d\eta=1$$

for all $\tau>0$, we have

$$u_1^+(t)\le u_1^+(t_0)+\alpha\int_{t_0}^{t}u_1^+(t')dt',$$

so that we obtain (6.8). □

Lemma 6. *Set*

$$w_2^+(t)=\sup_{0\le x\le 2\pi}(-\log u_2(t,x))=\sup_{\xi_2\in\mathbb{R}}(-\log u_2(t,\xi_2)).$$

Then, for $0\le t_0\le t$, *we have*

$$w_2^+(t)\le w_2^+(t_0)+\gamma(t-t_0)\equiv\Psi_2(w_2^+(t_0),t-t_0). \tag{6.9}$$

P r o o f. If we divide both sides of (6.6) by $-u_2(t,\xi_2)$, we have

$$\partial_t(-\log(u_2(t,\xi_2)))=\kappa\partial_{\xi_2}^2(-\log(u_2(t,\xi_2)))-(\partial_{\xi_2}\log(u_2(t,\xi_2)))^2+\gamma-\delta u_1(t,\xi_2). \tag{6.10}$$

By formally applying the fundamental solution of the heat equation to (6.10), we have

$$-\log(u_2(t,\xi_2))\le\int_{-\infty}^{+\infty}\Theta(t-t_0,\xi'-\xi_2)(-\log(u_2(t_0,\xi')))d\xi'+\gamma(t-t_0),$$

and this inequality implies (6.9). □

Lemma 7. *Set*

$$u_2^+(t)=\sup_{0\le x\le 2\pi}u_2(t,x)=\sup_{\xi_2\in\mathbb{R}}u_2(t,\xi_2).$$

Then, for $0\le t_0\le t$, *we have*

$$\begin{aligned}u_2^+(t)\le u_2^+(t_0)\left(1+\delta u_1^+(t_0)\int_{t_0}^{t}e^{\alpha(t'-t_0)}e^{\delta/\alpha\cdot u_1^+(t_0)(e^{\alpha(t-t_0)}-e^{\alpha(t'-t_0)})}dt'\right)\\ \equiv\Phi_2(u_1^+(t_0),u_2^+(t_0),t-t_0).\end{aligned} \tag{6.11}$$

P r o o f. We formally apply the fundamental solution of the heat equation to (6.6), so that we have

$$u_2(t,\xi_2)\le\int_{-\infty}^{+\infty}\Theta(t-t_0,\xi'-\xi_2)u_2(t_0,\xi')d\xi'+\delta\int_{t_0}^{t}\int_{-\infty}^{+\infty}\Theta(t-t',\xi'-\xi_2)u_1(t',\xi')u_2(t',\xi')d\xi'dt'.$$

Hence, using the inequality (6.8), we have

$$u_2^+(t)\le u_2^+(t_0)+\delta u_1^+(t_0)\int_{t_0}^{t}e^{\alpha(t'-t_0)}u_2^+(t')dt',$$

or

$$Y'(t) \le e^{\alpha(t-t_0)} u_2^+(t_0) + \delta u_1^+(t_0) e^{\alpha(t-t_0)} Y(t), \quad Y(t) = \int_{t_0}^{t} e^{\alpha(t'-t_0)} u_2^+(t')dt'.$$

From this inequality follows (6.11). □

Lemma 8. *Set*

$$w_1^+(t) = \sup_{0\le x\le 2\pi} (-\log u_1(t,x)) = \sup_{\xi_1\in\mathbb{R}} (-\log u_1(t,\xi_1)).$$

Then, for $0 \le t_0 \le t$*, we have*

$$w_1^+(t) \le w_1^+(t_0) + \beta \int_{t_0}^{t} \Phi_2(t_0, u_2^+(t_0), t')dt' \equiv \Psi_1(u_1^+(t_0), u_2^+(t_0), w_1^+(t_0), t-t_0). \tag{6.12}$$

P r o o f. From the equation

$$\partial_t(-\log(u_1(t,\xi_1))) = \kappa \partial^2_{\xi_1}(-\log(u_1(t,\xi_1))) - \kappa(\partial_{\xi_1}\log(u_1(t,\xi_1)))^2 - \alpha + \beta u_2(t,\xi_1),$$

we deduce (in a similar way to the previous case)

$$-\log(u_1(t,\xi_1)) \le w_1^+(t_0) + \beta\int_{t_0}^{t} u_2^+(t')dt'.$$

Hence, using (6.11) we obtain (6.12). □

Let us define $w_1^+(U)$, $u_1^+(U)$, $w_2^+(U)$ and $u_2^+(U)$, for all $U \ge 0$, as follows:

$$w_1^+(U) = -\log(\bar u_1), \quad U_1(\bar u_1) = U, \quad 0 < \bar u_1 \le \frac{\gamma}{\delta},$$
$$u_1^+(U) = \bar{\bar u}_1, \quad U_1(\bar{\bar u}_1) = U, \quad \bar{\bar u}_1 \ge \frac{\gamma}{\delta},$$
$$w_2^+(U) = -\log(\bar u_2), \quad U_2(\bar u_2) = U, \quad 0 < \bar u_2 \le \frac{\alpha}{\beta},$$
$$u_2^+(U) = \bar{\bar u}_2, \quad U_2(\bar{\bar u}_2) = U, \quad \bar{\bar u}_2 \ge \frac{\alpha}{\beta}.$$

It is clear that

$$U = U_1(e^{-w_1^+(U)}) = U_1(u_1^+(U)) = U_2(e^{-w_2^+(U)}) = U_2(u_2^+(U)).$$

These definitions are justified due to the definition (3.5)–(3.6) of $U_1(u_1)$ and $U_2(u_2)$.

Lemma 9. *If we set*

$$U^+(t) = \sup_{0\le x\le 2\pi} U(u_1(t,x), u_2(t,x)),$$

we have

$$U^+(t) \le \tilde M(U^+(t_0), t-t_0), \quad t \ge t_0,$$

where

$$\tilde M(U^+(t_0), t-t_0) = U_1^{\max}(U^+(t_0), t-t_0) + U_2^{\max}(U^+(t_0), t-t_0), \tag{6.13}$$

$$U_1^{\max}(U^+(t_0), t-t_0)$$
$$= \max(U_1(\Phi_1(u_1^+(U^+(t_0)), t-t_0)), U_1(e^{-\Psi_1(u_1^+(U^+(t_0)), u_2^+(U^+(t_0)), w_1^+(U^+(t_0)), t-t_0)})),$$

$$U_2^{\max}(U^+(t_0), t-t_0) = \max(U_2(\Phi_2(u_1^+(U^+(t_0)), u_2^+(U^+(t_0)), t-t_0)), U_2(e^{-\Psi_2(w_2^+(U^+(t_0)), t-t_0)})).$$

P r o o f. The lemma follows immediatly from the definition of $\tilde{M}(U^+(t_0), t-t_0)$ and Lemmas 5–8. □

Remark 3. The function $\tilde{M}(a, b)$ can be defined for any values $a \geq 0$ and $b \geq 0$ (independently of the solution $(u_1(t, x), u_2(t, x))$ of the problem (3.1)–(3.3)). Furthermore, the function $\tilde{M}(a, b)$ is continuous and increasing with respect to either $a \geq 0$ or $b \geq 0$.

Indeed, this remark follows immediately from the definition (6.13).

We are now able to prove the main result.

P r o o f (of Theorem 1). In this proof we use the notations $\tilde{U}(t)$ introduced in (3.12) and $U(t, x) = U(u_1(t, x), u_1(t, x))$. Lemma 2 (see also (5.13)) implies that, if

$$\|U(t, \cdot)\|^2_{L^2(0,2\pi)} > \Lambda_2(\tilde{U}(t)),$$

then $\|U(t, \cdot)\|^2_{L^2(0,2\pi)}$ decreases. Taking into account that $\tilde{U}(t)$ is decreasing, we have

$$\|U(t, \cdot)\|^2_{L^2(0,2\pi)} \leq \max\Big(\|U(0, \cdot)\|^2_{L^2(0,2\pi)}, \Lambda_2(\tilde{U}(0))\Big) \equiv B_U, \quad \forall t \geq 0.$$

This inequality, together with (6.3) and Proposition 1, allows us to conclude the existence of τ in each interval $[t, t+1]$ such that

$$\sup_{0 \leq x \leq 2\pi} U(\tau, x) \leq \tilde{U}(0) + \sqrt{2\pi}\Big(\frac{C_v}{\kappa^2} + \frac{1}{\kappa}\Big)^{1/2}\sqrt{B_U} \equiv A_U.$$

On the other hand, it follows from Lemma 9 (see also Remark 3) that

$$\sup_{0 \leq x \leq 2\pi} U(t, x) \leq \tilde{M}(A_U, t-\tau),$$

for $t \geq \tau$. Thus, from these relations it follows that, for all $t \geq 0$, we have

$$\sup_{0 \leq x \leq 2\pi} U(t', x) \leq \tilde{M}(A_U, 1), \quad \forall t' \in [t, t+1],$$

in other words, we have

$$\sup_{0 \leq x \leq 2\pi} U(t, x) \leq \tilde{M}(A_U, 1), \quad \forall t \geq 0,$$

with $\tilde{M}(A_U, 1) < \infty$ (see (6.13)), which completes the proof of (4.3).

We now set

$$\Lambda_1(\tilde{U}_\infty) = \tilde{M}(A^*_U(\tilde{U}_\infty), 1),$$

where

$$A^*_U(\tilde{U}_\infty) = \tilde{U}_\infty + \sqrt{2\pi}\Big(\frac{C_v}{\kappa^2} + \frac{1}{\kappa}\Big)^{1/2}\sqrt{\Lambda_2(\tilde{U}_\infty)}. \tag{6.14}$$

We note that the right-hand side of (6.14) does not depend on t and we can deduce from the definition of $\tilde{M}$ that the function $\Lambda_1(\tilde{U}_\infty)$ is continuous and increasing. From the reasoning of the proof of (4.3), taking into account (4.4), we deduce that

$$\limsup_{t \to \infty} \sup_{0 \leq x \leq 2\pi} U(t, x) \leq \Lambda_1(\tilde{U}_\infty),$$

which completes the proof of the statement *i*) of Theorem 1.

We now assume that $\tilde{U}_\infty = 0$. Then, according to Lemma 4, we have

$$\int_{t-1}^{t} \sup_{0\le x\le 2\pi} U(\tau,x)d\tau \le \tilde{U}(t-1) + \sqrt{2\pi}\Big(\frac{C_v}{\kappa^2}\int_{t-1}^{t} \|U(\tau,\cdot)\|^2_{L(0,2\pi)^2}d\tau + \frac{1}{\kappa}\|U(t-1,\cdot)\|^2_{L^2(0,2\pi)}\Big)^{1/2}.$$

According to Proposition 2 the upper limit of the right-hand side of this inequality is $A^*_U(\tilde{U}_\infty)$, as given in (6.14). Thus, we have

$$\lim_{t\to\infty}\int_{t-1}^{t} \sup_{0\le x\le 2\pi} U(\tau,x)d\tau = 0. \tag{6.15}$$

In order to prove the statement ii) of Theorem 1, we argue by contradiction by assuming that

$$\lim_{t\to\infty} \sup_{0\le x\le 2\pi} U(t,x) \neq 0,$$

in other words, suppose that there exists $\varepsilon > 0$ such that, for each $t > 0$, there exists $t' \ge t$ such that

$$\sup_{0\le x\le 2\pi} U(t',x) \ge \varepsilon. \tag{6.16}$$

Let us define the function $U^{(\varepsilon)}(s)$, for each $s > 0$, as

$$\tilde{M}(U^{(\varepsilon)}(s), s) = \varepsilon. \tag{6.17}$$

Then, from the definition of $\tilde{M}$ it follows that, for t' satisfaying (6.16), we have for $\tau < t'$

$$U^{(\varepsilon)}(t'-\tau) \le \sup_{0\le x\le 2\pi} U(\tau,x).$$

Thus

$$\int_{t'-1}^{t'} U^{(\varepsilon)}(t'-\tau)d\tau \le \int_{t'-1}^{t'} \sup_{0\le x\le 2\pi} U(\tau,x)d\tau. \tag{6.18}$$

Recall that the definition of $\tilde{M}$ (and also of $U_1^{\max}$ and $U_2^{\max}$; see (6.13)) implies that for any $t_0 > 0$, we have

$$\lim_{t\to t_0^+} U_1^{\max}(U^+(t_0), t-t_0) = \max(U_1(u_1^+(U^+(t_0))), U_1(e^{-w_1^+(U^+(t_0))})) = U^+(t_0),$$
$$\lim_{t\to t_0^+} U_2^{\max}(U^+(t_0), t-t_0) = \max(U_2(u_2^+(U^+(t_0))), U_2(e^{-w_2^+(U^+(t_0))})) = U^+(t_0),$$

and thus

$$\lim_{t\to t_0^+} \tilde{M}(U^+(t_0), t-t_0) = 2U^+(t_0).$$

This relation also implies that

$$\lim_{\tau\to t'^-} U^{(\varepsilon)}(t'-\tau) = \frac{1}{2}\varepsilon > 0. \tag{6.19}$$

From the continuity of $\tilde{M}(a,b)$ we can deduce that $U^{(\varepsilon)}(s)$ is continuous (see (6.17)). Thus, from (6.19) it follows that there exists some $s_\varepsilon > 0$ such that $U^{(\varepsilon)}(s) > 0$ for $0 < s < s_\varepsilon$, and we have

$$\int_{t'-s_\varepsilon}^{t'} U^{(\varepsilon)}(t'-\tau)d\tau = \int_0^{s_\varepsilon} U^{(\varepsilon)}(s)ds \equiv c_\varepsilon > 0.$$

Thus, it follows from (6.18) that

$$\int_{t'-1}^{t'} \sup_{0\le x\le 2\pi} U(\tau,x)d\tau \ge c_\varepsilon > 0,$$

where c_ε is independant of t'. This result contradicts (6.15). Therefore we have

$$\lim_{t\to\infty} \sup_{0\le x\le 2\pi} U(t,x) = 0.$$

This completes the proof of the theorem. □

7. Conclusion

In this article, we have analyzed the asymptotic behavior of the solution to the Lotka–Volterra equation with diffusion and population displacements in a periodic domain of $\mathbb{R}$. From this analysis we have obtained the global boundedness of the solution and its logarithm and also its uniform convergence to the stationary solution in the case in which the solution converges in mean-value to the stationary solution. This result guarantees that, even if there can be the growth of oscillation of the solution in certain points as we have seen in the example of numerical calculation in the Section 2, these phenomena cannot develop infinitely, and the growth of oscillation is limited.

Moreover we have developed some particular techniques of estimate of the solution. Even if the conditions we have set for the equation are relatively simple, the techniques we have developed here can, with possible adaptation, be used also for analogous problem with more complex conditions.

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A PAIR OF FOUR-ELEMENT HORIZONTAL GENERATING SETS OF A PARTITION LATTICE[1][2]

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Abstract: Let $\lfloor x\rfloor$ and $\lceil x\rceil$ denote the lower integer part and the upper integer part of a real number x, respectively. Our main goal is to construct four partitions of a finite set A with $n\geq 7$ elements such that each of the four partitions has exactly $\lceil n/2\rceil$ blocks and any other partition of A can be obtained from the given four by forming joins and meets in a finite number of steps. We do the same with $\lceil n/2\rceil-1$ instead of $\lceil n/2\rceil$, too. To situate the paper within lattice theory, recall that the *partition lattice* $\mathrm{Eq}(A)$ of a set A consists of all partitions (equivalently, of all equivalence relations) of A. For a natural number n, $[n]$ and $\mathrm{Eq}(n)$ will stand for $\{1,2,\dots,n\}$ and $\mathrm{Eq}([n])$, respectively. In 1975, Heinrich Strietz proved that, for any natural number $n\geq 3$, $\mathrm{Eq}(n)$ has a four-element generating set; half a dozen papers have been devoted to four-element generating sets of partition lattices since then. We give a simple proof of his just-mentioned result. We call a generating set X of $\mathrm{Eq}(n)$ *horizontal* if each member of X has the same height, denoted by $h(X)$, in $\mathrm{Eq}(n)$; no such generating sets have been known previously. We prove that for each natural number $n\geq 4$, $\mathrm{Eq}(n)$ has two four-element horizontal generating sets X and Y such that $h(Y)=h(X)+1$; for $n\geq 7$, $h(X)=\lfloor n/2\rfloor$.

Keywords: Partition lattice, Equivalence lattice, Minimum-sized generating set, Horizontal generating set, Four-element generating set.

1. Notes on the dedication

Árpád Kurusa, 1961–2024, was an excellent geometer. The present paper is dedicated to his memory. In addition to his high reputation in geometry, his editorial and technical editorial work for several mathematical journals as well as his textbooks (in Hungarian) were also deeply acknowledged. From 2000 to 2018, he led the Department of Geometry at the Bolyai (Mathematical) Institute of the University of Szeged. As the title of [5] shows, our collaboration has added a piece to the traditionally strong interrelation between geometry and lattice theory. At the motivational level, the present paper has some (but very slight) connection to the just-mentioned joint paper. Indeed, partition lattices form a specific subclass of *geometric* lattices, and the term "horizontal" is rooted in a *geometric* perspective of these lattices.

2. Introduction and our theorem

Given a set A, the collection of *equivalences*, that is, the collection of reflexive, symmetric, transitive relations of A forms a lattice $\mathrm{Eq}(A)$, the *equivalence lattice* of A. In this lattice, the meet and the join are the intersection and the transitive hull of the union, respectively. By the well-known bijective correspondence between the equivalences of A and the partitions of A, $\mathrm{Eq}(A)$

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[2]*Dedicated to the memory of my local colleague and co-author Árpád Kurusa.*

is isomorphic to the *partition lattice* of A, which consists of all partitions of A. By the just-mentioned correspondence, we make no sharp distinction between equivalences and partitions in our terminology and notations. To explain that we use the notation $\mathrm{Eq}(A)$ rather than something like $\mathrm{Part}(A)$, note that equivalences are more appropriate for performing the lattice operations and forming restrictions. For a natural number n, we let $[n] := \{1, 2, \dots, n\}$, and we usually abbreviate $\mathrm{Eq}([n])$ to $\mathrm{Eq}(n)$.

Partition lattices play an important role in lattice theory since congruence lattices, which play a central role in universal algebra, are naturally embedded in partition lattices. In fact, every lattice is embeddable into a partition lattice by Whitman [12] and each finite lattice into a finite partition lattice by Pudlák and Tůma [9]; note that these facts can be exploited in some proofs, for example, in [1]. Furthermore, every partition lattice $\mathrm{Eq}(A)$ is known to be a *geometric lattice*, that is, an atomistic semimodular lattice; see, e.g., Grätzer [7, Section IV.4] or [8, Section V.3]. Being *atomistic* means that each element x of $\mathrm{Eq}(A)$ is the join of all atoms below x. *Semimodularity* is understood as upper semimodularity, that is, for any $x, y, z \in \mathrm{Eq}(A)$, $x \preceq y$ implies that $x \vee z \preceq y \vee z$, where $\preceq$ is the "is covered by or equal to" relation.

A subset X of $\mathrm{Eq}(A)$ is a *generating set* of $\mathrm{Eq}(A)$ if X extends to no proper subset S of $\mathrm{Eq}(A)$ such that S is closed with respect to joins and meets. In the seventies, Strietz [10] and [11] proved that, for any natural number $n \geq 3$, $\mathrm{Eq}(n)$ has a four-element generating set. His result is optimal, since $\mathrm{Eq}(n)$ does not have a three-element generating set provided that $n \geq 4$. Since Strietz's pioneering work was published in [10] and [11], five additional papers have already been devoted to the four-element generating sets of equivalence lattices; see [6], the 2nd-, the 3rd-, and the 4th-item in the "References" section of [6], and Zádori [13].

For $n \geq 3$, which is always assumed, each permutation of $[n]$ extends to an automorphism of $\mathrm{Eq}(n)$, and such an automorphism sends generating sets to generating sets. We say that two generating sets of $\mathrm{Eq}(n)$ are *essentially different* if no such automorphism sends one of them to the other one. We know even from Strietz [10] and [11] that, for n large enough, $\mathrm{Eq}(n)$ has several essentially different four-element generating sets. Many more (essentially different) four-element generating sets have been given in [6]. However, it is very likely by the computer-assisted section of [6] that only an infinitesimally small percentage of the four-element generating sets of $\mathrm{Eq}(n)$ are known for n large. Exploring more such generating sets seems to be a reasonable target in its own right, and there is an additional motivation: Namely, the more small generating sets of $\mathrm{Eq}(n)$ are available, the more the cryptographic ideas of [2] can benefit from equivalence lattices. (If there are and *we know* many four-element generating sets, then we can extend them to small generating sets in very many ways.)

Before explaining what sort of new four-element generating sets of $\mathrm{Eq}(n)$ we are going to present, note that even at the very beginning of this type of research in the seventies, Strietz himself paid attention to some lattice theoretical properties of his four-element generating sets. For $n \geq 4$, he showed that a four-element generating set is either an *antichain* (that is, a subset with no comparable elements) or it is of order type $1+1+2$, that is, exactly two out of the four generators are comparable. He managed to prove that $\mathrm{Eq}(n)$ has a four-element generating set of order type $1+1+2$ for every integer $n \geq 10$. Briefly saying, $\mathrm{Eq}(n)$ is $(1+1+2)$-*generated* for $n \geq 10$. With ingenious constructions, Zádori [13] improved "$n \geq 10$" to $n \geq 7$, and he gave a visual proof of Strietz's result that $\mathrm{Eq}(n)$ has a four-element generating set; his proofs are simpler than Strietz's ones. Zádori [13] left open the problem whether $\mathrm{Eq}(5)$ and $\mathrm{Eq}(6)$ are $(1+1+2)$-generated. This problem was solved as recently as 2020 in [6], where an affirmative answer for $\mathrm{Eq}(6)$ was given but a computer-assisted negative answer for $\mathrm{Eq}(5)$ was provided.

As $\mathrm{Eq}(n)$ is a geometric lattice, there is a natural property of a subset, which is more restrictive than being an antichain. To introduce it, recall that the *length* of an n-element chain is $n-1$. The least element and the largest element of $\mathrm{Eq}(n)$ or $\mathrm{Eq}(A)$ will be denoted by Δ and ∇, respectively.

If confusion threatens, we write Δ_n, ∇_A, etc. The height of an element $\mu \in \mathrm{Eq}(n)$ is the length of a maximal chain in the interval $[\Delta, \mu]$; we know from the Jordan-Hölder Chain Condition for semimodular lattices, see, e.g., Grätzer [7, Theorem IV.2.1, p. 226] or [8, Theorem 377], that no matter which maximal chain is taken. We denote the *height* of μ by $h(\mu)$. A subset X of $\mathrm{Eq}(n)$ is *horizontal* if its elements are of the same height; in this case, the common height of the elements of X is denoted by $h(X)$. A horizontal subset of $\mathrm{Eq}(n)$ is necessarily an antichain. Clearly, $\mathrm{Eq}(n)$ for $n \geq 3$ has a *horizontal generating set*, since the set of atoms is such. To get a better insight into the four-element generating sets of partition lattices, it is reasonable to determine those natural numbers n for which $\mathrm{Eq}(n)$ has a *four-element horizontal generating set*. In fact, we are going to do more by showing that whenever $\mathrm{Eq}(n)$ has a four-element antichain at all, that is, whenever $n \geq 4$, then it has two four-element horizontal generating sets of neighboring heights. To smooth our terminology, let us introduce the notation

$$\mathrm{HFHGS}(n) := \{h(X) : X \text{ is a four-element horizontal generating set of } \mathrm{Eq}(n)\};$$

the acronym above comes from the **h**eights of **f**our-element **h**orizontal **g**enerating **s**ets. For a real number r, we denote by $\lfloor r \rfloor$ and $\lceil r \rceil$ the *lower integer part* and the *upper integer part* of r; for example, $\lfloor \sqrt{2} \rfloor = 1$ and $\lceil \sqrt{2} \rceil = 2$. Let $\mathbb{N}^+$ denote the set of positive integers.

Theorem 1. *For every natural number $n \geq 4$, the partition lattice $\mathrm{Eq}(n)$ has two four-element horizontal generating sets X and Y such that $h(Y) = h(X)+1$ holds for their heights. Furthermore,*

$$\mathrm{HFHGS}(n) \supseteq \{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1\} \text{ for all integers } n \geq 7 \text{ and also for } n = 5, \text{ and} \tag{2.1}$$
$$\mathrm{HFHGS}(n) \subseteq \{k \in \mathbb{N}^+ : \lfloor (n-1)/4 \rfloor + 1 \leq k \leq n - \lceil \sqrt[4]{n} \rceil\} \text{ for all integers } n \geq 4. \tag{2.2}$$

Based on the following statement, we conjecture that "$\supseteq$" in (2.1) is never an equality for $n \geq 7$. We do not know whether $\lim_{n\to\infty} |\mathrm{HFHGS}(n)| = \infty$ and $\mathrm{HFHGS}(n)$ is always a convex subset of $\mathbb{N}^+$. We know $\mathrm{HFHGS}(n)$ only for $n \in \{4, 5, 6, 7, 8\}$. In the proposition below, each occurrence of the relation symbol $\overset{\text{comp}}{=}$ denotes an equality that we could prove only with the assistance of the brute force of a computer.

Proposition 1. *We have the following equalities and inclusions:*

$$\mathrm{HFHGS}(4) = \{1, 2\}, \tag{2.3}$$
$$\mathrm{HFHGS}(5) = \{2, 3\}, \tag{2.4}$$
$$\{2, 3\} \subseteq \mathrm{HFHGS}(6) \subseteq \{2, 3, 4\}, \quad \textit{in fact,} \quad \mathrm{HFHGS}(6) \overset{\text{comp}}{=} \{2, 3\}, \tag{2.5}$$
$$\{2, 3, 4\} \subseteq \mathrm{HFHGS}(7) \subseteq \{2, 3, 4, 5\}, \quad \textit{in fact,} \quad \mathrm{HFHGS}(7) \overset{\text{comp}}{=} \{2, 3, 4\}, \quad \textit{and} \tag{2.6}$$
$$\{3, 4, 5\} \subseteq \mathrm{HFHGS}(8) \subseteq \{2, 3, 4, 5, 6\}, \quad \textit{in fact,} \quad \mathrm{HFHGS}(8) \overset{\text{comp}}{=} \{3, 4, 5\}. \tag{2.7}$$

Remark 1. (2.3) and (2.5) witness that (2.1) fails for $n \in \{4, 6\}$. Note also that concrete four-element horizontal generating sets witnessing (2.1) and (2.3)–(2.7) are defined by Lemma 5 combined with Assertion 1, by Lemmas 6, 7 and 8 combined with both (the Key) Lemma 4 and Assertion 1, and in the rest of the lemmas presented in Section 5. For n large, the just-mentioned four-element horizontal generating sets are given only inductively; the inductive feature could be eliminated but we do not strive for non-inductive definitions of these generating sets.

The rest of the paper is devoted to proving Theorem 1 and Proposition 1. Unless explicitly stated otherwise, we assume that $4 \leq n \in \mathbb{N}^+$ for the remainder of the paper.

3. Some lemmas, the Key Lemma, and a new proof of one of Strietz's results

For a finite nonempty set A, if $\{a_{1,1},\dots,a_{1,t_1}\}$, …, $\{a_{k,1},\dots,a_{k,t_k}\}$ is a repetition-free list of the blocks of a partition $\mu\in\mathrm{Eq}(A)$, then we denote both μ and the corresponding equivalence by

$$\mathrm{eq}(a_{1,1},\dots,a_{1,t_1};\dots;a_{k,1},\dots,a_{k,t_k})\quad\text{or}\quad \mathrm{eq}(a_{1,1}\dots a_{1,t_1};\dots;a_{k,1}\dots a_{k,t_k}).$$

That is, we omit the commas when no confusion threatens but not the block-separating semicolons. Usually, the elements in a block and the blocks are listed in lexicographic order. For example,

$$\Delta_4=\mathrm{eq}(1;2;3;4),\quad \nabla_4=\mathrm{eq}(1234),\quad\text{and}\quad \nabla_{11}=\mathrm{eq}(1,2,3,4,5,6,7,8,9,10,11);$$

for more involved examples, see Lemmas 5–15. For $u,v\in A$, the least equivalence of A collapsing u and v will be denoted by $\mathrm{at}(u,v)$ or, if confusion threatens, by $\mathrm{at}_A(u,v)$. For example, in $\mathrm{Eq}(6)$, $\mathrm{at}(2,5)=\mathrm{eq}(1;25;3;4;6)$. Note that $\mathrm{at}(u,v)$ is an ***at**om* of $\mathrm{Eq}(A)$ (that is, a cover of Δ), and every atom of $\mathrm{Eq}(A)$ is of this form.

We define the *graph* $G(S)$ of a sublattice S of $\mathrm{Eq}(A)$ by letting A be the *vertex set* of $G(S)$ and letting $\{(a,b) : a\neq b \text{ and } \mathrm{at}(a,b)\in S\}$ be the *edge set* of $G(S)$. (No matter if we consider (a,b) and (b,a) equal or different.) A *Hamiltonian circle* of $G(S)$ is a permutation $a_1,a_2,\dots,a_n$ of the elements of A such that $\mathrm{at}(a_{i-1},a_i)\in S$ for $i\in[n]-\{1\}$ and $\mathrm{at}(a_n,a_1)\in S$. Of course, $G(S)$ need not have a Hamiltonian circle. The following lemma occurs, explicitly or implicitly, in several papers dealing with generating sets of equivalence lattices; see, for example, Czédli and Oluoch [6, Lemma 2.5]. For the reader's convenience, we are going to outline its trivial proof.

Lemma 1 (Hamiltonian Cycle Lemma). *For a finite set A with at least three elements and a sublattice S of $\mathrm{Eq}(A)$, we have that $S=\mathrm{Eq}(A)$ if and only if $G(S)$ has a Hamiltonian circle.*

P r o o f. The "only if" part is trivial. To prove the "if" part, let $a_1,\dots,a_n$ be a Hamiltonian circle of $G(S)$. As each element of the atomistic lattice $\mathrm{Eq}(A)$ is the join of some atoms, it suffices to show that for all $i\neq j$, $i,j\in[n]$, we have that $\mathrm{at}(a_i,a_j)\in S$. This membership follows from

$$\begin{aligned}\mathrm{at}(a_i,a_j)&=\big(\mathrm{at}(a_i,a_{i+1})\vee\mathrm{at}(a_{i+1},a_{i+2})\vee\dots\vee\mathrm{at}(a_{j-1},a_j)\big)\\&\wedge\big(\mathrm{at}(a_i,a_{i-1})\vee\mathrm{at}(a_{i-1},a_{i-2})\vee\dots\vee\mathrm{at}(a_2,a_1)\\&\vee\mathrm{at}(a_1,a_n)\vee\mathrm{at}(a_n,a_{n-1})\vee\mathrm{at}(a_{n-1},a_{n-2})\vee\dots\vee\mathrm{at}(a_{j+1},a_j)\big)\end{aligned}$$

and the "commutativity" $\mathrm{at}(x,y)=\mathrm{at}(y,x)$. □

Let $\mathbb{Z}_4:=(\{0,1,2,3\},+)$ denote the cyclic group of order 4; the addition in it is performed modulo 4. To give the lion's share of the proof of (2.3) and also to present an easy consequence of Lemma 1, we present the following lemma, in which the addition is understood in $\mathbb{Z}_4$.

Lemma 2. *Both*

$$X:=\{\mathrm{at}(i,i+1) : i\in\mathbb{Z}_4\}$$

and

$$Y:=\{\mathrm{at}(i,i+1)\vee\mathrm{at}(i+1,i+2) : i\in\mathbb{Z}_4\}$$

are four-element horizontal generating sets of $\mathrm{Eq}(\mathbb{Z}_4)\cong\mathrm{Eq}(4)$.

P r o o f. By Lemma 1, X generates $\mathrm{Eq}(\mathbb{Z}_4)$. Since

$$\mathrm{at}(i,i+1)=\big(\mathrm{at}(i,i+1)\vee\mathrm{at}(i+1,i+2)\big)\wedge\big(\mathrm{at}(i-1,i)\vee\mathrm{at}(i,i+1)\big)\quad\text{for}\quad i\in\mathbb{Z}_4,$$

it follows that X is contained in the sublattice of $\mathrm{Eq}(\mathbb{Z}_4)$ generated by Y, whence Y also generates $\mathrm{Eq}(\mathbb{Z}_4)$. □

Next, we introduce a concept that is crucial in the proof of Theorem 1. By an n-element *eligible structure* we mean a 7-tuple

$$\mathcal{A} = (A, \alpha, \beta, \gamma, \delta, u, v) \tag{3.1}$$

such that A is an n-element finite set, u and v are distinct elements of A, $\{\alpha, \beta, \gamma, \delta\}$ is a four-element generating set of $\mathrm{Eq}(A)$, and

$$\alpha \vee \delta = \nabla, \quad \alpha \wedge \delta = \Delta, \tag{3.2}$$

$$\beta \wedge \bigl(\gamma \vee \mathrm{at}(u,v)\bigr) = \Delta, \quad \gamma \wedge \bigl(\beta \vee \mathrm{at}(u,v)\bigr) = \Delta, \tag{3.3}$$

$$\text{and} \;\; \beta \vee \gamma \vee \mathrm{at}(u,v) = \nabla. \tag{3.4}$$

To present an example and also for a later reference, we formulate the following statement.

Lemma 3. *With* $\alpha = \mathrm{eq}(123;4)$, $\beta = \mathrm{eq}(14;2;3)$, $\gamma = \mathrm{eq}(1;2;34)$, *and* $\delta = \mathrm{eq}(1;24;3)$,

$$\mathcal{A} := ([4], \alpha, \beta, \gamma, \delta, 1, 2) \tag{3.5}$$

is an eligible structure.

P r o o f. Let S be the sublattice of $\mathrm{Eq}(4)$ generated by $\{\alpha, \beta, \gamma, \delta\}$. Since

$$\mathrm{at}(1,2) = \mathrm{eq}(12;3;4) = \alpha \wedge (\beta \vee \delta) \in S, \quad \mathrm{at}(2,3) = \alpha \wedge (\gamma \vee \delta) \in S, \quad \mathrm{at}(3,4) = \gamma \in S,$$

and $\mathrm{at}(4,1) = \beta \in S$, the sequence 1, 2, 3, 4 is a Hamiltonian cycle in $G(S)$. Thus, Lemma 1 implies that $\{\alpha, \beta, \gamma, \delta\}$ generates $\mathrm{Eq}(4)$. Since (3.2), (3.3), and (3.4) are trivially satisfied, the proof of Lemma 3 is complete. □

For $A \subseteq B$ and $\mu \in \mathrm{Eq}(A)$, the smallest equivalence of B that includes μ will be denoted by μ_B^{ext}. The superscript in the notation comes from "<u>**ext**</u>ension". As a partition, μ_B^{ext} consists of the blocks of μ and the singleton blocks $\{b\}$ for $b \in B - A$.

Lemma 4 (Key Lemma). *Assume that* $(A, \alpha, \beta, \gamma, \delta, u, v)$ *is an eligible structure,* $|A| \geq 4$, $w \notin A$, *and* $B = A \cup \{w\}$. *Let*

$$\begin{gathered} \alpha' := \beta_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w), \quad \beta' := \alpha_B^{\mathrm{ext}}, \quad \gamma' := \delta_B^{\mathrm{ext}}, \quad \delta' := \gamma_B^{\mathrm{ext}} \vee \mathrm{at}_B(v,w), \\ u' := u, \quad v' := w. \end{gathered} \tag{3.6}$$

Then the <u>**e**</u>xtended <u>**s**</u>tructure

$$\mathrm{ES}(\mathcal{A}) := \mathcal{B} = \bigl(B, \alpha', \beta', \gamma', \delta', u', v'\bigr) \tag{3.7}$$

is also an eligible structure. The heights of the partitions occurring in (3.6)–(3.7) *satisfy that*

$$h(\alpha') = h(\beta) + 1, \quad h(\beta') = h(\alpha), \quad h(\gamma') = h(\delta), \quad h(\delta') = h(\gamma) + 1. \tag{3.8}$$

P r o o f. Assume that $\mathcal{A}$ is an eligible structure and $\mathcal{B} = \mathrm{ES}(\mathcal{A})$ is as in (3.7). We will frequently but mostly implicitly use the obvious fact that the function $f\colon \mathrm{Eq}(A) \to \mathrm{Eq}(B)$ defined by $\mu \mapsto \mu_B^{\mathrm{ext}}$ is a lattice embedding and, for any $\mu \in \mathrm{Eq}(A)$, $h(f(\mu)) = h(\mu)$. Denote by S the sublattice generated by $\{\alpha', \beta', \gamma', \delta'\}$ in $\mathrm{Eq}(B)$. For $\mu \in \mathrm{Eq}(B)$, let $\mu\restriction_A$ denote the *restriction* of μ to A. That is, as an equivalence, $\mu\restriction_A = \mu \cap (A \times A)$. E.g.,

$$\big((\Delta_A)^{\mathrm{ext}}_B\big)\restriction_A = \Delta_A.$$

Note the obvious rule:

$$(\rho^{\mathrm{ext}}_B)\restriction_A = \rho \quad \text{and} \quad (\mu\restriction_A)^{\mathrm{ext}}_B = \mu \wedge (\nabla_A)^{\mathrm{ext}}_B \quad \text{for every} \quad \rho \in \mathrm{Eq}(A) \quad \text{and} \quad \mu \in \mathrm{Eq}(B). \tag{3.9}$$

Let us agree that, for $x, y \in B$, $\mathrm{at}(x, y)$ is understood as $\mathrm{at}_B(x, y)$ even when $x, y \in A$. We claim that for any $\mu \in \mathrm{Eq}(A)$ and for any $d \in A$,

$$\big(\mu^{\mathrm{ext}}_B \vee \mathrm{at}_B(d, w)\big)\restriction_A = \mu; \tag{3.10}$$

and, in particular,

$$\alpha'\restriction_A = \beta \quad \text{and} \quad \delta'\restriction_A = \gamma. \tag{3.11}$$

The inequality

$$\big(\mu^{\mathrm{ext}}_B \vee \mathrm{at}_B(d, w)\big)\restriction_A \geq \mu$$

is clear. To show the converse inequality, assume that $a \neq b$ and (a, b) belongs to $\big(\mu^{\mathrm{ext}}_B \vee \mathrm{at}_B(d, w)\big)\restriction_A$. Then $a, b \in A$ and, by the description of the join in equivalence lattices, there exists a *shortest* sequence $x_0 = a$, x_1, …, x_{t-1}, $x_t = b$ of elements of B such that, for each $i \in [t]$,

$$\text{either } (x_{i-1}, x_i) \in \mu^{\mathrm{ext}}_B \quad \text{or} \quad (x_{i-1}, x_i) \in \{(d, w), (w, d)\}. \tag{3.12}$$

Since this sequence is repetition-free, the first alternative in (3.12) means that $(x_{i-1}, x_i) \in \mu$. By way of contradiction, suppose that not all elements of the sequence are in A. Let j be the smallest subscript such that $x_j \notin A$. As $x_0 = a \in A$ and $x_t = b \in A$, we have that $0 < j < t$. By the choice of j, $x_{j-1} \in A$. This rules out that $(x_{j-1}, x_j) = (w, d)$. Since $x_j \notin A$, $(x_{j-1}, x_j) \in \mu$ cannot occur either. Hence, $(x_{j-1}, x_j) = (d, w)$. However, then the only possibility to continue the sequence is that $(x_j, x_{j+1}) = (w, d)$. So d occurs in the sequence at least twice, which contradicts the fact that our sequence is repetition-free. Therefore, all elements of the sequence are in A, whereby the first alternative of (3.12) holds for all i. Thus, $(x_{i-1}, x_i) \in \mu$ for $i \in [t]$, and we obtain the required membership $(a, b) = (x_0, x_t) \in \mu$ by transitivity. We have shown (3.10). Letting $(\mu, d) := (\beta, u)$ and $(\mu, d) := (\gamma, v)$, (3.10) implies (3.11).

Next, using the first half of (3.2) (and the fact that f is an embedding), we obtain that

$$(\nabla_A)^{\mathrm{ext}}_B = (\alpha \vee \delta)^{\mathrm{ext}}_B = \alpha^{\mathrm{ext}}_B \vee \delta^{\mathrm{ext}}_B = \beta' \vee \gamma'$$

belongs to S. Hence, so does $\alpha' \wedge (\nabla_A)^{\mathrm{ext}}_B$. By the second half of (3.9) applied to $\mu := \alpha'$, this equivalence is $(\alpha'\restriction_A)^{\mathrm{ext}}_B$, whence $(\alpha'\restriction_A)^{\mathrm{ext}}_B \in S$. Therefore, applying (3.11), $\beta^{\mathrm{ext}}_B \in S$. As β and γ play a symmetric role, γ^{ext}_B is also in S. By (3.6), S contains $\alpha^{\mathrm{ext}}_B = \beta'$ and $\delta^{\mathrm{ext}}_B = \gamma'$. So $f(\mu) = \mu^{\mathrm{ext}}_B \in S$ for every $\mu \in \{\alpha, \beta, \gamma, \delta\}$. Since f is an embedding and $\{\alpha, \beta, \gamma, \delta\}$ generates $\mathrm{Eq}(A)$, we conclude that $f(\mathrm{Eq}(A)) \subseteq S$. In particular, $\mathrm{at}_B(u, v) = f(\mathrm{at}_A(u, v)) \in S$. Based on this containment, we claim that

$$\mathrm{at}_B(u, w) = \alpha' \wedge \big(\mathrm{at}_B(u, v) \vee \delta'\big) \in S. \tag{3.13}$$

As $\mathrm{at}_B(u, v), \alpha', \delta' \in S$, it suffices to show the equality in (3.13). The inequality "$\leq$" in place of the equality is clear by the definition of α' given in (3.6). To show the converse inequality, assume that $a \neq b$ and (a, b) belongs to the right-hand side of the equality in (3.13). Let

$$\nu := \mathrm{at}_A(u, v) \vee \gamma.$$

Observe that

$$(a,b) \in \alpha' \wedge \big(\nu_B^{\text{ext}} \vee \text{at}_B(v,w)\big), \tag{3.14}$$

since

$$\begin{aligned}\alpha' \wedge \big(\nu_B^{\text{ext}} \vee \text{at}_B(v,w)\big) &= \alpha' \wedge \big((\text{at}_A(u,v) \vee \gamma)_B^{\text{ext}} \vee \text{at}_B(v,w)\big)\\ &= \alpha' \wedge \big((\text{at}_A(u,v))_B^{\text{ext}} \vee \gamma_B^{\text{ext}} \vee \text{at}_B(v,w)\big)\end{aligned} \tag{3.15}$$

$$= \alpha' \wedge \big(\text{at}_B(u,v) \vee \gamma_B^{\text{ext}} \vee \text{at}_B(v,w)\big) \overset{(3.6)}{=} \alpha' \wedge \big(\text{at}_B(u,v) \vee \delta'\big). \tag{3.16}$$

As $a \neq b$ and $|B - A| = |\{w\}| = 1$, at least one of a and b is in A. By symmetry, we can assume that $a \in A$. Depending on the position of b, there are two cases.

First, assume that b is also in A. Then $(a,b) \in \alpha'$ and (3.11) give that $(a,b) \in \beta$. As (a,b) is in the second meetand in (3.14) and $a, b \in A$, we have that

$$(a,b) \in \big(\nu_B^{\text{ext}} \vee \text{at}_B(v,w)\big)\rceil_A.$$

Hence, (3.10) applied to $(\mu, d) := (\nu, v)$ yields that $(a,b) \in \nu$. Thus, (a,b) belongs to

$$\beta \wedge \nu = \beta \wedge \big(\text{at}_A(u,v) \vee \gamma\big),$$

which is Δ_A by (3.3). Since $(a,b) \in \Delta_A$ contradicts the assumption $a \neq b$, the first case cannot occur.

Second, assume that $b \notin A$. Then

$$(a,w) = (a,b) \in \alpha' \wedge \big(\text{at}_B(u,v) \vee \delta'\big)$$

and $a \in A$. By (3.6), $(w,u) \in \alpha'$. As both (w,v) and (v,u) belong to the second meetand of (3.15), (w,u) belongs to this meetand, too. These facts, (3.15), and (3.16) give that $\alpha' \wedge \big(\text{at}_B(u,v) \vee \delta'\big)$ contains (w,u). By transitivity, it contains (a,u), too. If we had that $a \neq u$, then (a,u) (with u playing the role of b) would be a contradiction by the first case. Thus, $a = u$, that is, $(a,b) = (u,w) \in \text{at}_B(u,w)$, as required. We have shown the validity of (3.13).

We obtain the following fact analogously; we can derive it also from (3.13) by symmetry, since $(A, \delta, \gamma, \beta, \alpha, v, u)$ is also an eligible structure:

$$\text{at}_B(v,w) = \delta' \wedge \big(\text{at}_B(u,v) \vee \alpha'\big) \in S. \tag{3.17}$$

With $n := |A|$, list the elements of B as follows:

$$c_1 := u, \quad c_2, \ \ldots, \ c_{n-1}, \ c_n := v, \quad c_{n+1} := w.$$

Since $f(\text{Eq}(A)) \subseteq S$ and $c_1, \ldots, c_n \in A$, we have that

$$\text{at}_B(c_i, c_{i+1}) = f\big(\text{at}_A(c_i, c_{i+1})\big) \in S,$$

that is, (c_i, c_{i+1}) is an edge of $G(S)$ for $i \in [n-1]$. So are $(c_n, c_{n+1}) = (v,w)$ and $(c_{n+1}, c_1) = (w,u)$ by (3.17) and by (3.13), respectively. Therefore, our list is a Hamiltonian cycle, and Lemma 1 implies that $\{\alpha', \beta', \gamma', \delta'\}$ is a generating set of $\text{Eq}(B)$. This set is four-element since $|B| \geq 4$ and so we know from Strietz [10] or [11] that $\text{Eq}(B)$ cannot be generated by less than four elements.

Clearly, $u' = u \in A$ is distinct from $v' = w \in B - A$. Since

$$\begin{aligned}\alpha' \vee \delta' &\overset{(3.6)}{=} \beta_B^{\text{ext}} \vee \text{at}_B(u,w) \vee \gamma_B^{\text{ext}} \vee \text{at}_B(v,w) = \beta_B^{\text{ext}} \vee \gamma_B^{\text{ext}} \vee \text{at}_B(u,v) \vee \text{at}_B(v,w)\\ &= \big(\beta \vee \gamma \vee \text{at}_A(u,v)\big)_B^{\text{ext}} \vee \text{at}_B(v,w) \overset{(3.4)}{=} (\nabla_A)_B^{\text{ext}} \vee \text{at}_B(v,w) = \nabla_B,\end{aligned}$$

$\mathcal{B}$ satisfies the first half of (3.2). To show by way of contradiction that $\mathcal{B}$ fulfills the second half, suppose that $a \neq b$ and $(a,b) \in \alpha' \wedge \delta'$. If $a, b \in A$, then (3.11) leads to $(a,b) \in \beta \wedge \gamma = \Delta_A$, contradicting that $a \neq b$. So one of a and b is w, and we can assume that $a \in A$ and $b = w$. As $(a,w) = (a,b) \in \alpha'$ and $(w,u) \in \alpha'$, we have that $(a,u) \in \alpha'$. Hence, $(a,u) \in \beta$ by (3.11). Similarly, $(a,w),(w,v) \in \delta'$ and (3.11) imply that $(a,v) \in \gamma$. The just-obtained memberships and relations give that

$$(a,u) \in \beta \wedge \big(\gamma \vee \mathrm{at}_A(u,v)\big) \quad \text{and} \quad (a,v) \in \gamma \wedge \big(\beta \vee \mathrm{at}_A(u,v)\big).$$

Combining this with (3.3), we obtain that $a = u$ and $a = v$, contradicting $u \neq v$. So we have proved that $\mathcal{B}$ fulfills (3.2).

By symmetry, to show that $\mathcal{B}$ satisfies (3.3), it suffices to deal with its first half. For the sake of contradiction, suppose that

$$\beta' \wedge \big(\gamma' \vee \mathrm{at}_B(u',v')\big) \neq \Delta_B.$$

Then we can pick $a, b \in B$ such that $a \neq b$ and

$$(a,b) \in \beta' \wedge \big(\gamma' \vee \mathrm{at}_B(u',v')\big) \overset{(3.6)}{=} \alpha_B^{\mathrm{ext}} \wedge \big(\delta_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w)\big). \tag{3.18}$$

The containment $(a,b) \in \alpha_B^{\mathrm{ext}}$ gives that $a, b \in A$. The meet in $\mathrm{Eq}(B)$ is the set-theoretic intersection, so it commutes with the restriction map. Hence, applying the first equality of (3.9) with $\rho := \alpha$ and (3.10) with $(\mu, d) := (\delta, u)$ at $\overset{*}{=}$, (3.18) leads to

$$(a,b) \in \big(\alpha_B^{\mathrm{ext}} \wedge (\delta_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w))\big)\restriction_A = \alpha_B^{\mathrm{ext}}\restriction_A \wedge \big(\delta_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w)\big)\restriction_A \overset{*}{=} \alpha \wedge \delta \overset{(3.2)}{=} \Delta_A \subseteq \Delta_B,$$

which contradicts the assumption $a \neq b$ and proves that $\mathcal{B}$ satisfies (3.3). Since

$$\beta' \vee \gamma' \vee \mathrm{at}_B(u',v') \overset{(3.6)}{=} \alpha_B^{\mathrm{ext}} \vee \delta_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w) = (\alpha \vee \delta)_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w)$$
$$\overset{(3.2)}{=} (\nabla_A)_B^{\mathrm{ext}} \vee \mathrm{at}_B(u,w) = \nabla_B,$$

$\mathcal{B}$ satisfies (3.4), too. We have proved that $\mathcal{B}$ is an eligible structure, as required.

For a finite nonempty set H and μ in $\mathrm{Eq}(H)$, let $\mathrm{NumB}(\mu)$ denote the ***num**ber of **b**locks* of μ. For example, if $\mu = \mathrm{eq}(14; 25; 3) \in \mathrm{Eq}(5)$, then $\mathrm{NumB}(\mu) = 3$. The following folkloric fact is trivial:

$$\text{For any } \mu \in \mathrm{Eq}(H), \quad h(\mu) + \mathrm{NumB}(\mu) = |H|. \tag{3.19}$$

Clearly, (3.6) leads to

$$\mathrm{NumB}(\alpha') = \mathrm{NumB}(\beta), \quad \mathrm{NumB}(\beta') = \mathrm{NumB}(\alpha) + 1,$$
$$\mathrm{NumB}(\gamma') = \mathrm{NumB}(\delta) + 1, \quad \mathrm{NumB}(\delta') = \mathrm{NumB}(\gamma).$$

These equalities and (3.19) imply (3.8), completing the proof of the Key Lemma. □

Now we are in the position to give a new proof of Strietz's result stating that $\mathrm{Eq}(n)$ is four-generated. For those who prefer theoretical arguments rather than long and tedious computations with concrete partitions, the proof below is presumably simpler than the earlier ones.

Corollary 1 (Strietz [10] and [11]). *For any natural number $n \geq 3$, $\mathrm{Eq}(n)$ has a four-element generating set.*

P r o o f. As the case $n = 3$ is trivial, we assume that $n \geq 4$. Let $\mathcal{A}_4$ be the eligible structure given in (3.5); see (3.1). For $n > 4$, define $\mathcal{A}_n$ as $\mathrm{ES}(\mathcal{A}_{n-1})$. Then, for each $n \geq 4$, $\mathcal{A}_n$ is an n-element eligible structure by Lemmas 3 and (the Key) Lemma 4. Thus, by the definition of eligible structures, $\mathrm{Eq}(n)$ is four-generated, completing the proof of Corollary 1. □

4. A tediously provable lemma

The n-th *Bell number* $B(n)$ is defined to be the number of elements of $\mathrm{Eq}(n)$, that is, $B(n) := |\mathrm{Eq}(n)|$. As n grows, $B(n)$ grows very fast; see https://oeis.org/A000110 of N. J. A. Sloan's Online Encyclopedia of Integer Sequences. For example,

$$|\mathrm{Eq}(6)| = B(6) = 203, \quad |\mathrm{Eq}(8)| = 4\,140, \quad |\mathrm{Eq}(9)| = 21\,147, \quad \text{and}$$
$$|\mathrm{Eq}(20)| = 51\,724\,158\,235\,372 \approx 5.17 \cdot 10^{13}.$$

These large numbers explain our experience that even when it is feasible to prove that a four-element subset X of $\mathrm{Eq}(n)$ generates $\mathrm{Eq}(n)$, this task requires straightforward but tedious computations in general. Each of Lemmas 5–15 belongs to this category by stating that a subset X of $\mathrm{Eq}(n)$ generates $\mathrm{Eq}(n)$; some of these lemmas state slightly more, but these surpluses are trivial to verify. We offer two ways to verify these lemmas.

First, one can read their proofs based on Lemma 1. One of these proofs is given in this section. As the rest of these proofs are long without containing a single new idea, the proofs of Lemmas 6–15 are given only in Appendix 1 of the extended version of the paper. At the time of writing, this extended version is at https://tinyurl.com/czg-h4ge (and also at the author's website[3] http://tinyurl.com/g-czedli/), and it will be available at www.arxiv.org soon.

Second, the author has developed three closely related computer programs in Dev-Pascal 1.9.2 under Windows 10. These programs, available at https://tinyurl.com/czg-equ2024p or at the author's website given in the previous paragraph, form a mini-package. The main program and its auxiliary program are also given in Appendices 2 and 3 of the extended version of the paper. The third program performs the same tasks as the first one and also uses the auxiliary program. Despite being slower, it is more cross-platform because it requires less computer memory. For $n \le 9$, the auxiliary program lists the elements of $\mathrm{Eq}(n)$; the other two programs rely on this list. In what follows, by a program, we mean the main program. The program can "prove" Lemmas 5–15, and it can also "prove" the $\overset{\text{comp}}{=}$ parts of (2.5)–(2.7). In fact, the program has been designed to perform the following two tasks.

First, the program can take an $n \in \{4, 5, \dots, 9\}$ and a four-element subset X of $\mathrm{Eq}(n)$ as inputs. After enlarging X by adding the join and the meet of any two of its elements as long as the enlargement is proper, the program computes the sublattice S generated by X. Then the program displays the size $|S|$ of S on the screen and tells whether X generates $\mathrm{Eq}(n)$. The program can prove Lemma 8, where $n = 9$, in about fifteen minutes. For Lemma 14, where $n = 8$, 25 seconds suffice. Note that for just one four-element subset X of $\mathrm{Eq}(n)$, it is not worthwhile to create and the program does not create the operation tables of $\mathrm{Eq}(n)$. For this (the first) task, there is no difference between the main program and its slower variant.

Second, for a given $n \in \{4, 5, \dots, 9\}$ and a $k \in [n-1]$ as inputs, the program decides whether $\mathrm{Eq}(n)$ has a four-element horizontal generating set of height k. For $(n, k) = (8, 2)$, this takes about three and a half minutes, provided the program runs on a desktop computer with AMD Ryzen 7 2700X Eight-Core Processor and 3.70 GHz with 16 GB memory. For $(n, k) = (9, 3)$, if $\mathrm{Eq}(9)$ has no four-element horizontal generating set of height 3, which we do not know, the program would need about a month; partially because there is not enough computer memory to store the operation tables of $\mathrm{Eq}(9)$ and also because there are significantly more cases.

The quotation marks around "proved" in a paragraph above indicate that the author believes but cannot prove that the program itself is error-free. The source code of the program and that of its auxiliary program are 24 and 8 kilobytes, respectively, totaling 32 kilobytes. Proving *exactly* that the program is perfect would probably be harder than verifying all proofs in Appendix 1.

[3]This standard "tiny" short link redirects us to the real URL https://www.math.u-szeged.hu/~czedli/ .

Lemma 5. *With*

$$\alpha := \mathrm{eq}(123;4;5), \tag{4.1}$$
$$\beta := \mathrm{eq}(1;23;45), \tag{4.2}$$
$$\gamma := \mathrm{eq}(13;25;4), \text{ and} \tag{4.3}$$
$$\delta := \mathrm{eq}(15;2;34), \tag{4.4}$$

$([5], \alpha, \beta, \gamma, \delta, 1, 4)$ *is an eligible structure and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 2$.

P r o o f. Let S denote the sublattice of $\mathrm{Eq}(5)$ generated by $\{\alpha, \beta, \gamma, \delta\}$. We will list some members of S; each of them belongs to S by earlier containments as indicated.

$$\mathrm{eq}(1;23;4;5) = \mathrm{eq}(123;4;5) \wedge \mathrm{eq}(1;23;45) \in S \quad \text{by (4.1) and (4.2)}, \tag{4.5}$$
$$\mathrm{eq}(13;2;4;5) = \mathrm{eq}(123;4;5) \wedge \mathrm{eq}(13;25;4) \in S \quad \text{by (4.1) and (4.3)}, \tag{4.6}$$
$$\mathrm{eq}(1235;4) = \mathrm{eq}(123;4;5) \vee \mathrm{eq}(13;25;4) \in S \quad \text{by (4.1) and (4.3)}, \tag{4.7}$$
$$\mathrm{eq}(15;234) = \mathrm{eq}(15;2;34) \vee \mathrm{eq}(1;23;4;5) \in S \quad \text{by (4.4) and (4.5)}, \tag{4.8}$$
$$\mathrm{eq}(1345;2) = \mathrm{eq}(15;2;34) \vee \mathrm{eq}(13;2;4;5) \in S \quad \text{by (4.4) and (4.6)}, \tag{4.9}$$
$$\mathrm{eq}(15;2;3;4) = \mathrm{eq}(15;2;34) \wedge \mathrm{eq}(1235;4) \in S \quad \text{by (4.4) and (4.7)}, \tag{4.10}$$
$$\mathrm{eq}(1;2;3;45) = \mathrm{eq}(1;23;45) \wedge \mathrm{eq}(1345;2) \in S \quad \text{by (4.2) and (4.9)}, \tag{4.11}$$
$$\mathrm{eq}(13;245) = \mathrm{eq}(13;25;4) \vee \mathrm{eq}(1;2;3;45) \in S \quad \text{by (4.3) and (4.11)}, \tag{4.12}$$
$$\mathrm{eq}(1;24;3;5) = \mathrm{eq}(15;234) \wedge \mathrm{eq}(13;245) \in S \quad \text{by (4.8) and (4.12)}. \tag{4.13}$$

Let $E(S)$ denote the edge set of the graph $G(S)$; it is defined in the paragraph preceding Lemma 1. Since $(1,3) \in E(S)$ by (4.6), $(3,2) \in E(S)$ by (4.5), $(2,4) \in E(S)$ by (4.13), $(4,5) \in E(S)$ by (4.11), and $(5,1) \in E(S)$ by (4.10), the sequence $1,3,2,4,5$ is a Hamiltonian cycle of $G(S)$. Hence, $\{\alpha, \beta, \gamma, \delta\}$ is a generating set of $\mathrm{Eq}(5)$ by Lemma 1. Armed with this fact, now it is a trivial task to verify that $([5], \alpha, \beta, \gamma, \delta, 1, 4)$ satisfies (3.2), (3.3), and (3.4), whereby it is an eligible structure. Thus, (3.19) completes the proof Lemma 5. □

5. The rest of tediously provable lemmas

We need the following ten lemmas, too. As indicated in the second paragraph of Section 4, their proofs are given only in Appendix 1 of the extended version of the paper.

Lemma 6. *With*

$$\alpha := \mathrm{eq}(134;256;7), \quad \beta := \mathrm{eq}(146;27;3;5), \quad \gamma := \mathrm{eq}(135;2;4;67), \quad \text{and} \quad \delta := \mathrm{eq}(12;357;46),$$

$([7], \alpha, \beta, \gamma, \delta, 2, 3)$ *is an eligible structure,* $h(\alpha) = h(\delta) = 4$, *and* $h(\beta) = h(\gamma) = 3$.

Lemma 7. *With*

$$\alpha := \mathrm{eq}(134;258;67), \quad \beta := \mathrm{eq}(14;2;36;578),$$
$$\gamma := \mathrm{eq}(17;25;348;6), \quad \text{and} \quad \delta := \mathrm{eq}(12;378;456),$$

$([8], \alpha, \beta, \gamma, \delta, 2, 6)$ *is an eligible structure,* $h(\alpha) = h(\delta) = 5$, *and* $h(\beta) = h(\gamma) = 4$.

Lemma 8. *With*

$$\alpha := \mathrm{eq}(178; 249; 356), \quad \beta := \mathrm{eq}(19; 26; 378; 45),$$
$$\gamma := \mathrm{eq}(1; 28; 359; 467), \quad \textit{and} \quad \delta := \mathrm{eq}(169; 258; 347),$$

$([9], \alpha, \beta, \gamma, \delta, 1, 2)$ *is an eligible structure,* $h(\alpha) = h(\delta) = 6$*, and* $h(\beta) = h(\gamma) = 5$.

Lemma 9. *With*

$$\alpha := \mathrm{eq}(134; 25), \quad \beta := \mathrm{eq}(13; 245), \quad \gamma := \mathrm{eq}(12; 345), \quad \textit{and} \quad \delta := \mathrm{eq}(124; 35),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(5)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 3$.

Lemma 10. *With*

$$\alpha := \mathrm{eq}(12; 34; 5; 6), \quad \beta := \mathrm{eq}(1; 2; 35; 46), \quad \gamma := \mathrm{eq}(1; 25; 36; 4), \quad \textit{and} \quad \delta := \mathrm{eq}(15; 24; 3; 6),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(6)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 2$.

Lemma 11. *With*

$$\alpha := \mathrm{eq}(13; 256; 4), \quad \beta := \mathrm{eq}(156; 2; 34), \quad \gamma := \mathrm{eq}(12; 35; 46), \quad \textit{and} \quad \delta := \mathrm{eq}(13; 246; 5),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(6)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 3$.

Lemma 12. *With*

$$\alpha := \mathrm{eq}(1; 24; 35; 6; 7), \quad \beta := \mathrm{eq}(14; 26; 3; 5; 7),$$
$$\gamma := \mathrm{eq}(1; 2; 34; 5; 67), \quad \textit{and} \quad \delta := \mathrm{eq}(17; 2; 3; 4; 56),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(7)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 2$.

Lemma 13. *With*

$$\alpha := \mathrm{eq}(13; 24; 567), \quad \beta := \mathrm{eq}(125; 3; 467) \quad \gamma := \mathrm{eq}(1357; 26; 4), \quad \textit{and} \quad \delta := \mathrm{eq}(126; 35; 47),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(7)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 4$.

Lemma 14. *With*

$$\alpha := \mathrm{eq}(18; 2; 35; 4; 67), \quad \beta := \mathrm{eq}(1; 24; 37; 5; 68),$$
$$\gamma := \mathrm{eq}(16; 2; 34; 57; 8), \quad \textit{and} \quad \delta := \mathrm{eq}(12; 3; 45; 6; 78),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(8)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 3$.

Lemma 15. *With*

$$\alpha := \mathrm{eq}(137; 246; 58), \quad \beta := \mathrm{eq}(146; 257; 38), \quad \gamma := \mathrm{eq}(136; 2; 4578), \quad \textit{and} \quad \delta := \mathrm{eq}(1245; 37; 68),$$

$\{\alpha, \beta, \gamma, \delta\}$ *generates* $\mathrm{Eq}(8)$ *and* $h(\alpha) = h(\beta) = h(\gamma) = h(\delta) = 5$.

6. Proving Theorem 1 and Proposition 1 with our lemmas

Since the proof of Theorem 1 relies on parts of Proposition 1 and the proof of Proposition 1 uses (2.2) from Theorem 1, we present a combined proof of both the theorem and the proposition.

P r o o f (Proving Theorem 1 and Proposition 1). First, we deal with (2.2). Assume that $\{\alpha_1, \dots, \alpha_4\}$ is a four-element horizontal generating set of $\mathrm{Eq}(n)$ with height k. That is, $k = h(\alpha_i)$ for $i \in [4]$. We need to prove that

$$\lfloor (n-1)/4 \rfloor + 1 \le k \le n - \lceil \sqrt[4]{n}\, \rceil. \tag{6.1}$$

By semimodularity, see Grätzer [7, Theorem IV.2.2, p. 226], the height of $\alpha_1 \vee \dots \vee \alpha_4$ is at most $h(\alpha_1) + \dots + h(\alpha_4) = 4k$. The just-mentioned join is the largest element of the sublattice S generated by $\{\alpha_1, \dots, \alpha_4\}$. But this sublattice is $\mathrm{Eq}(n)$, so this join is ∇_n, whereby $h(\nabla_n) \le 4k$. We know from, say, (3.19) that $h(\nabla_n) = n - 1$. Thus, the previous inequality turns into $(n-1)/4 \le k$. If $(n-1)/4 < k$, then $\lfloor (n-1)/4 \rfloor < k$ and we obtain the first inequality of (6.1) since k is an integer. Hence, it suffices to exclude that $(n-1)/4 = k$. To obtain a contradiction, suppose that $(n-1)/4 = k$, that is, $n-1 = h(\nabla_n) = 4k$. Let $i \in [4]$. As $h(\alpha_i) = k$, we can find k atoms $\beta_{k(i-1)+1}$, $\beta_{k(i-1)+2}$, $\dots$, β_{ki} in $\mathrm{Eq}(n)$ such that α_i is the join of these atoms; the existence of such atoms is clear in $\mathrm{Eq}(n)$ and it is true even in any geometric lattice by Grätzer [7, Theorems IV.2.4–IV.2.5, p. 228–229] or [8, Theorems 380–381]. As $\{\alpha_1, \dots, \alpha_4\}$ generates $\mathrm{Eq}(n)$, $\alpha_1 \vee \dots \vee \alpha_4 = \nabla_n$. Hence,

$$h\Big(\bigvee_{j=1}^{4k} \beta_j\Big) = h(\alpha_1 \vee \dots \vee \alpha_4) = h(\nabla_n) = n - 1 = 4k.$$

Therefore, Grätzer [7, Theorem IV.2.4, p. 228] or [8, Theorem 380] yields that $\{\beta_1, \dots, \beta_{4k}\}$ is an independent set of atoms; this means that $\{\beta_1, \dots, \beta_{4k}\}$ generates a Boolean sublattice T of $\mathrm{Eq}(n)$. In particular, T is a distributive. As $\alpha_1, \dots, \alpha_4$ are in T, they generate a sublattice of T, which is distributive, too. This means that $\mathrm{Eq}(n)$ is distributive, which contradicts the assumption that $n \ge 4$. Therefore, $(n-1)/4 = k$ cannot occur and we have proved the first inequality in (6.1).

Clearly, $\alpha_1 \wedge \dots \wedge \alpha_4$, which is the smallest element of S, is Δ_n. Let $b := \mathrm{NumB}(\alpha_i)$; by (3.19), $b = n - k$ does not depend on $i \in [4]$. The largest block C_1 of α_1 has at least n/b elements. When we form the meet $\alpha_1 \wedge \alpha_2$, then C_1 splits into at most b blocks of $\alpha_1 \wedge \alpha_2$ and the largest one of these blocks has at least $(n/b)/b$ elements. So $\alpha_1 \wedge \alpha_2$ has a block C_2 with at least n/b^2 elements. And so on; finally, $\Delta_n = \alpha_1 \wedge \dots \wedge \alpha_4$ has a block with at least n/b^4 elements. But Δ_n has only one-element blocks, whereby $n/b^4 \le 1$, that is, $b \ge \sqrt[4]{n}$. Thus $b \ge \lceil \sqrt[4]{n}\, \rceil$, since $b \in \mathbb{N}^+$. Therefore, as we know from (3.19) that $b = n - k$, we obtain that $k \le n - \lceil \sqrt[4]{n}\, \rceil$. This completes the proof of (6.1) and that of (2.2).

Next, assume that $\mathcal{A} = (A, \alpha, \beta, \gamma, \delta, u, v)$. With the "extended structure operator" introduced in (3.7), we use the notation $(C, \alpha'', \beta'', \gamma'', \delta'', u'', v'')$ for $\mathrm{ES}^2(\mathcal{A}) := \mathrm{ES}(\mathrm{ES}(\mathcal{A}))$. Clearly, (the Key) Lemma 4 implies the following assertion.

Assertion 1. *If $\mathcal{A} = (A, \alpha, \beta, \gamma, \delta, u, v)$ is an eligible structure and $\mathcal{C} = (C, \alpha'', \beta'', \gamma'', \delta'', u'', v'')$ is $\mathrm{ES}^2(\mathcal{A})$, then $\mathcal{C}$ is also an eligible structure,*

$$h(\alpha'') = h(\alpha) + 1, \quad h(\beta'') = h(\beta) + 1, \quad h(\gamma'') = h(\gamma) + 1, \quad \textit{and} \quad h(\delta'') = h(\delta) + 1.$$

Resuming the proof, let us agree that, for any meaningful x, $\mathcal{A}_{\mathrm{L}x}$ denotes the eligible structure defined in Lemma x. For example, $\mathcal{A}_{\mathrm{L}5}$ is defined in Lemma 5. We call an eligible structure *horizontal* if its four partitions have the same height; this common height is the *height* of the structure.

By Lemma 5, $\mathcal{A}_{L5}$ is a 5-element horizontal eligible structure of height 2. Applying Assertion 1 repeatedly, we obtain a 7-element horizontal eligible structure, a 9-element horizontal eligible structure, etc. of heights 3, 4, ..., respectively. Thus,

$$\text{for } n \geq 5 \text{ odd, } \mathrm{Eq}(n) \text{ has a four-element horizontal generating set of height } \lfloor n/2 \rfloor. \tag{6.2}$$

By Lemma 7 and (the Key) Lemma 4, $\mathrm{ES}(\mathcal{A}_{L7})$ is a 9-element horizontal eligible structure of height 5. Applying Assertion 1 repeatedly, we obtain an 11-element horizontal eligible structure, a 13-element horizontal eligible structure, etc. of heights 6, 7, ..., respectively. Hence,

$$\text{for } n \geq 9 \text{ odd, } \mathrm{Eq}(n) \text{ has a four-element horizontal generating set of height } \lfloor n/2 \rfloor + 1. \tag{6.3}$$

By Lemma 6 and (the Key) Lemma 4, $\mathrm{ES}(\mathcal{A}_{L6})$ is an 8-element horizontal eligible structure of height 4. Hence, the repeated use of Assertion 1 yields that

$$\text{for } n \geq 8 \text{ even, } \mathrm{Eq}(n) \text{ has a four-element horizontal generating set of height } \lfloor n/2 \rfloor. \tag{6.4}$$

By Lemma 8 and (the Key) Lemma 4, $\mathrm{ES}(\mathcal{A}_{L8})$ is a 10-element horizontal eligible structure of height 6. Hence, the repeated use of Assertion 1 yields that

$$\text{for } n \geq 10 \text{ even, } \mathrm{Eq}(n) \text{ has a four-element horizontal generating set of height } \lfloor n/2 \rfloor + 1. \tag{6.5}$$

We know from Lemma 9 that Eq(5) is generated by a four-element horizontal generating set of height $\lceil 5/2 \rceil + 1$. By Lemma 13, Eq(7) has four-element horizontal generating set of height $(\lfloor 7/2 \rfloor + 1)$. For Eq(8), a four-element horizontal generating set of height $(\lfloor 8/2 \rfloor + 1)$ is provided by Lemma 15. These three facts, (6.2), (6.3), (6.4), and (6.5) imply (2.1).

In what follows, we will implicitly use that $\mathrm{Eq}(n)$ has no four-element horizontal subset of height 0 or $n-1$. Since there is no four-element subset of height 0 or 3 in Eq(4), Lemma 2 implies (2.3).

Since $\{2,3\} \subseteq \mathrm{HFHGS}(5)$ by (2.2), (2.1) implies (2.4).

We obtain from (2.2) and Lemmas 10–11 that $\{2,3\} \subseteq \mathrm{HFHGS}(6) \subseteq \{2,3,4\}$. As the already mentioned computer program yields that $4 \notin \mathrm{HFHGS}(6)$ in less than a second[4], (2.5) holds.

Lemma 12, (2.1), and (2.2) imply that $\{2,3,4\} \subseteq \mathrm{HFHGS}(7) \subseteq \{2,3,4,5\}$. In 2 seconds, the program excludes that $5 \in \mathrm{HFHGS}(7)$. Thus, we have shown (2.6).

Lemma 14, (2.1) and (2.2) yield that $\{3,4,5\} \subseteq \mathrm{HFHGS}(8) \subseteq \{2,3,4,5,6\}$, as required. The program excludes 2 and 6 from HFHGS(8) in three and a half minutes and in one minute, respectively. Thus, we proved the validity of (2.7) and that of Proposition 1.

Finally, the first sentence of Theorem 1 follows from (2.3), (2.4) or (2.1), the first inclusion in (2.5), and from (2.1). The combined proof of Theorem 1 and Proposition 1 is complete. □

7. Conclusion

Motivated by earlier results on four-element generating sets of finite equivalence lattices and their link to cryptography, we have proved the existence of two four-element horizontal generating sets of consecutive heights in these lattices. After the first submission of the paper, this result—and the method behind it—motivated two subsequent papers on four-element generating sets of equivalence lattices with other special properties (see [3] and [4]). We anticipate similar results in the future.

[4]The auxiliary program creates the auxiliary files containing the lists of partitions of $[n]$ for $n \leq 9$ in 4 seconds, but this has to be done only once. Thus, here and later, even though the program needs these files, the just-mentioned 4 seconds are not counted. The time for entering n and k are not counted either.

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TOPOLOGIES ON THE FUNCTION SPACE Y^X WITH VALUES IN A TOPOLOGICAL GROUP

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Abstract: Let Y^X denote the set of all functions from X to Y. When Y is a topological space, various topologies can be defined on Y^X. In this paper, we study these topologies within the framework of function spaces. To characterize different topologies and their properties, we employ generalized open sets in the topological space Y. This approach also applies to the set of all continuous functions from X to Y, denoted by $C(X,Y)$, particularly when Y is a topological group. In investigating various topologies on both Y^X and $C(X,Y)$, the concept of limit points plays a crucial role. The notion of a topological ideal provides a useful tool for defining limit points in such spaces. Thus, we utilize topological ideals to study the properties and consequences for function spaces and topological groups.

Keywords: Topological group, Topological ideal, Function space Y^X.

1. Introduction

For any topological space Z and topological group H [6, 26], let $C(Z,H)$ denote the group of all continuous functions from Z to H, equipped with the "pointwise group operations". That is, the product of $f \in C(Z,H)$ and $g \in C(Z,H)$ is the function $fg \in C(Z,H)$ defined by

$$fg(z) = f(z)g(z)$$

for all $z \in Z$, and the inverse of f is the function $h \in C(Z,H)$ defined by

$$h(z) = (f(z))^{-1}$$

for all $z \in Z$. The space $C(Z,H)$ with the point-open topology was studied by Shakhmatov and Spěvák [25]. A set of the form

$$[z,V]^+ = \big\{f \in C(Z,H)|\ f(z) \in V\big\},$$

where $z \in Z$ and V is an open subset of H, is a subbase of the point-open topology on $C(Z,H)$. The space $C(Z,H)$ with the open-point topology has a subbase consisting of sets of the form

$$[U,r]^- = \big\{f \in C(Z,H)|\ f^{-1}(r) \cap U \neq \emptyset\big\},$$

where $r \in H$ and U is an open subset of Z.

The space $C(Z,H)$ with the bi-point-open topology has a subbase consisting of sets of both kinds: $[z,V]^+$ and $[U,r]^-$, where $z \in Z$ and V is an open subset of H, U is an open subset of Z, and $r \in H$.

The following three propositions serve as necessary tools for the development of this paper.

Proposition 1 [5]. *Let β be a basis of a topological group H. The collection*

$$\big\{[z_1, B_1]^+ \cap \cdots \cap [z_n, B_n]^+ |\ n \in \mathbb{N},\ z_i \in Z,\ B_i \in \beta\big\}$$

is a basis for the space $C(Z, H)$ equipped with the point-open topology.

Proposition 2 [26]. *Let β be a basis of a topological space X. The collection*

$$\big\{[B_1, r_1]^- \cap \cdots \cap [B_n, r_n]^- |\ n \in \mathbb{N},\ r_i \in H,\ B_i \in \beta\big\}$$

is a basis for the space $C(Z, H)$ equipped with the open-point topology.

Proposition 3 [26]. *Let β_Z and β_H be bases of a topological space Z and a topological group H, respectively. The collection*

$$\big\{[z_1, B_1]^+ \cap \cdots \cap [z_n, B_n]^+ \cap [V_1, r_1]^- \cap \cdots \cap [V_m, r_m]^- |$$
$$z_i \in Z,\ r_j \in H,\ r_i \in H,\ B_i \in \beta_H,\ \text{and } V_j \in \beta_Z,\ 1 \le i \le n,\ 1 \le j \le m\big\}$$

is a basis for the space $C(Z, H)$ equipped with the bi-point-open topology.

General definition of the point-open topology on Y^X:

Definition 1 [21]. *Given a point $x \in X$ and an open set U in a topological space Y, define*

$$S(x, U) = \big\{f \in Y^X |\ f(x) \in U\big\}.$$

The collection of all such sets $S(x, U)$ forms a subbasis for a topology on Y^X. This topology is called the ***point-open topology on*** *Y^X.*

To obtain a topology on Y^X, it is not necessary that Y be a topological space. That is, for any set Y, the following construction defines a topology on Y^X.

Let x be a point of the set X and A be any subset of Y. Consider

$$S(x, A) = \big\{f \in Y^X |\ f(x) \in A\big\}.$$

The sets $S(x, A)$ form a subbasis for a topology on Y^X. Suppose $\mathfrak{F} \subseteq Y^X$.

The question is: Is $\mathfrak{F}$ open in the topology on Y^X generated by the subbasis elements above?

Let $g \in \mathfrak{F}$. For any $x \in X$, we have $g(x) \in Y$. If X is finite, then $g \in S(x, \{g(x)\}) \subseteq \mathfrak{F}$. Thus, the subbasis

$$\{S(x, A)|\ x \in X,\ A \in \wp(Y)\}$$

generates the discrete topology on Y^X when X is finite. If we take $A = Y$, then the subbasis

$$\big\{\emptyset\big\} \cup \big\{S(x, Y)|\ x \in X\big\}$$

generates the indiscrete topology on Y^X. If we restrict the subsets of Y used in the subbasis, we obtain a weaker topology on Y^X. Therefore, we conclude that "Y being a topological space" is not essential for defining a topology on Y^X. In particular, starting with the discrete topology on Y yields the discrete topology on Y^X, while starting with the indiscrete topology on Y yields the indiscrete topology on Y^X.

In this paper, we will discuss various topologies on Y^X. For this purpose, the following generalized open sets are important tools.

Definition 2. *A subset A of a topological space Y is said to be*

– *semi-open* [15] *if* $A \subseteq \mathrm{Co}(\mathrm{Io}(A))$;
– *preopen* [16] *if* $A \subseteq \mathrm{Io}(\mathrm{Co}(A))$;
– *β-open* [10] *or semi-preopen* [3] *if* $A \subseteq \mathrm{Co}(\mathrm{Io}(\mathrm{Co}(A)))$;
– *b-open* [4] *if* $A \subseteq \mathrm{Io}(\mathrm{Co}(A)) \cup \mathrm{Co}(\mathrm{Io}(A))$;
– *h-open* [1] *if, for every nonempty open set* $U \neq Y$, $A \subseteq \mathrm{Io}(A \cup U)$,

where Io *and* Co *denote the interior and closure operators, respectively.*

We denote the collection of all semi-open sets, preopen sets, β-open sets, and b-open sets in a topological space Y by $SO(Y)$, $PO(Y)$, $\beta O(Y)$, and $BO(Y)$, respectively. These collections satisfy the following inclusion relations: the collection of open sets $\subseteq PO(Y) \subseteq BO(Y) \subseteq \beta O(Y)$ and the collection of open sets $\subseteq SO(Y) \subseteq BO(Y) \subseteq \beta O(Y)$.

The following is one way to obtain weaker and stronger topologies on Y^X; it serves as an introductory result of the paper.

Lemma 1. *Suppose σ and σ' are two topologies on the set Y such that $\sigma \subseteq \sigma'$. Then, the point-open topology induced by σ' is finer than the point-open topology induced by σ.*

P r o o f. Let β_τ and $\beta_{\tau'}$ be bases for the point-open topologies τ and τ' induced by σ and σ', respectively, on Y^X. Let

$$B = S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n)$$

be a member of β_τ, and suppose $f \in B$. Then $f \in S(x_i, U_i)$ for all $i = 1, 2, \ldots, n$. This implies that $f \in S(x_i, U_i')$, where $U_i = U_i'$ for all $i = 1, 2, \ldots, n$. So,

$$f \in S(x_1, U_1') \cap S(x_2, U_2') \cap \cdots \cap S(x_n, U_n') = B' \in \beta_{\tau'}$$

as $U_1', U_2', \ldots, U_n'$ are open subsets of (Y, σ'). Thus, for every $f \in B$, there exists $B' \in \beta_{\tau'}$ such that $B' \subseteq B$. This completes the proof. $\square$

Note that if σ' is strictly finer than σ, then the point-open topology induced by σ' is strictly finer than the point-open topology induced by σ.

Our aim is to discuss different point-open topologies for various operators in topological spaces. Thus, for various operators, we consider a topological ideal [2, 14].

An ideal $\mathbb{I}$ on a topological space (Y, σ) is a collection of subsets of Y satisfying:

(i) If $A \subseteq B \in \mathbb{I}$, then $A \in \mathbb{I}$;
(ii) If $A, B \in \mathbb{I}$, then $A \cup B \in \mathbb{I}$.

This concept of an ideal on a topological space was first introduced by Kuratowski [14] in 1933. The study of the local function (or the generalization of limit points) is an important aspect of the theory of topological ideals. It is defined as follows:

$$A^* = \{y \in Y \mid U_y \cap A \notin \mathbb{I},\ U_y \in \sigma(y)\},$$

where $\sigma(y)$ is the collection of all open sets of (Y, σ) containing y. The set-valued set function [20] associated with the operator $()^*$ is the operator ψ [18, 22], which is defined by the relation $\psi(A) = Y \setminus (Y \setminus A)^*$.

Throughout this paper, $(Y, \sigma, \mathbb{I})$ denotes an ideal topological space. Furthermore, an ideal $\mathbb{I}$ on the topological space (Y, σ) is called a codense ideal [9] (or, equivalently, the ideal topological space $(Y, \sigma, \mathbb{I})$ is called an H-S space [8]) if $\mathbb{I} \cap \sigma = \{\emptyset\}$.

2. Topologies on Y^X

In this section, we consider X as a set and Y as a topological space (or simply, a space).

Lemma 2. *Given a point $x \in X$ and a subset A of the topological space Y, define*

$$S(x, \mathrm{Io}(A)) = \{f \in Y^X \mid f(x) \in \mathrm{Io}(A)\}.$$

The sets $S(x, \mathrm{Io}(A))$ form a subbasis for a topology on Y^X.

P r o o f. Let $f \in Y^X$. Then

$$f \in S(x, Y) = S(x, \mathrm{Io}(Y)) \subseteq \bigcup_i S(x_i, \mathrm{Io}(A_i)),$$

where $x_i \in X$ and A_i are subsets of Y. So,

$$f \in \bigcup_i S(x_i, \mathrm{Io}(A_i)).$$

Thus,

$$Y^X \subseteq \bigcup_i S(x_i, \mathrm{Io}(A_i)).$$

Hence, the sets $S(x_i, \mathrm{Io}(A_i))$ form a subbasis for a topology on Y^X. □

The topology generated by the above subbasis is called the ***point-interior topology*** on Y^X.

As is well known, the operator Co is the set-valued set function [20] associated with Io. Thus, if we define the sets $S(x, \mathrm{Io}(A))$ by

$$\{f \in Y^X \mid f(x) \in X \setminus \mathrm{Co}(X \setminus A)\}$$

or

$$\{f \in Y^X \mid f(x) \notin \mathrm{Co}(X \setminus A)\},$$

then we obtain the same topology.

Now we state that the operator Co independently generates a topology on Y^X as follows.

Lemma 3. *Given a point $x \in X$ and a subset A of the topological space Y, define*

$$S(x, \mathrm{Co}(A)) = \{f \in Y^X \mid f(x) \in \mathrm{Co}(A)\}.$$

The sets $S(x, \mathrm{Co}(A))$ form a subbasis for a topology on Y^X.

The topology generated by the above subbasis is called the ***point-closure topology*** on Y^X.

As Io $\sim^Y$ Co [20], one can rewrite the above Lemma using the Io operator. The point-open topology and the point-interior topology on Y^X coincide. However, the point-interior topology and the point-closure topology are not comparable.

Example 1. Let $X = \{a, b\}$ and (Y, σ) be a topological space, where $Y = \{1, 2, 3\}$ and $\sigma = \{\emptyset, Y, \{2\}, \{3\}, \{2, 3\}\}$. All possible functions from X to Y are defined by

$$\begin{array}{llllll} f_1(a) = 1, & f_1(b) = 2; & f_2(a) = 1, & f_2(b) = 3; & f_3(a) = 2, & f_3(b) = 3; \\ f_4(a) = 2, & f_4(b) = 1; & f_5(a) = 3, & f_5(b) = 1; & f_6(a) = 3, & f_6(b) = 2; \\ f_7(a) = 1, & f_7(b) = 1; & f_8(a) = 2, & f_8(b) = 2; & f_9(a) = 3, & f_9(b) = 3. \end{array}$$

Then, a basis of the point-interior topology τ on Y^X is

$$\beta_\tau = \big\{\emptyset, Y^X, \{f_3\}, \{f_6\}, \{f_8\}, \{f_9\}, \{f_3, f_8\}, \{f_6, f_9\}, \{f_6, f_8\}, \{f_3, f_9\}, \{f_3, f_4, f_8\}, \{f_5, f_6, f_9\}, \{f_1, f_6, f_8\}, \{f_2, f_3, f_9\}, \{f_3, f_6, f_8, f_9\}, \{f_3, f_4, f_5, f_6, f_8, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}\big\}.$$

A basis of the point-closure topology τ' on Y^X is

$$\beta_{\tau'} = \big\{\emptyset, Y^X, \{f_7\}, \{f_1, f_7\}, \{f_2, f_7\}, \{f_4, f_7\}, \{f_5, f_7\}, \{f_1, f_2, f_7\}, \{f_4, f_5, f_7\}, \{f_1, f_4, f_7, f_8\}, \{f_2, f_3, f_4, f_7\}, \{f_1, f_5, f_6, f_7\}, \{f_2, f_5, f_7, f_9\}, \{f_1, f_2, f_3, f_4, f_7, f_8\}, \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_1, f_4, f_5, f_6, f_7, f_8\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}\big\}.$$

Here, $f_6 \in \{f_6\} \in \beta_\tau$ but there does not exist any $B' \in \beta_{\tau'}$ such that $f_6 \in B' \subseteq \{f_6\}$. Thus, τ' is not finer than τ.

Similarly, $f_7 \in \{f_7\} \in \beta_{\tau'}$ but there does not exist any $B \in \beta_\tau$ such that $f_7 \in B \subseteq \{f_7\}$. Thus, τ is not finer than τ'.

Hence, the point-interior topology and the point-closure topology on Y^X are not comparable.

Lemma 4. *Given a point $x \in X$ and a subset A of the topological space Y, define*

$$S(x, \mathrm{Io}(\mathrm{Co}(A))) = \big\{f \in Y^X \mid f(x) \in \mathrm{Io}(\mathrm{Co}(A))\big\}.$$

The sets $S(x, \mathrm{Io}(\mathrm{Co}(A)))$ form a subbasis for a topology on Y^X.

P r o o f. Let $f \in Y^X$. Then

$$f \in S(x, Y) = S(x, \mathrm{Io}(\mathrm{Co}(Y))) \subseteq \bigcup_i S(x_i, \mathrm{Io}(\mathrm{Co}(A_i))),$$

where $x_i \in X$ and A_i are subsets of Y. Therefore,

$$f \in \bigcup_i S(x_i, \mathrm{Io}(\mathrm{Co}(A_i))).$$

Thus,

$$Y^X \subseteq \bigcup_i S(x_i, \mathrm{Io}(\mathrm{Co}(A_i))).$$

Hence, the sets $S(x_i, \mathrm{Io}(\mathrm{Co}(A_i)))$ form a subbasis for a topology on Y^X. □

The topology generated by the above subbasis is called the ***point-interior-closure topology*** on Y^X. Since $\mathrm{Io}\,\mathrm{Co} \sim^Y \mathrm{Co}\,\mathrm{Io}$ [20], we may rewrite the subbasis of the point-interior-closure topology on Y^X using the $\mathrm{Co}\,\mathrm{Io}$ operator.

Proposition 4. *Suppose Y is a topological space. Then, the point-open topology on Y^X is finer than the point-interior-closure topology on Y^X.*

P r o o f. Let β_τ and $\beta_{\tau'}$ be bases for the point-interior-closure topology and the point-open topology on Y^X, respectively. Let

$$B = S(x_1, \mathrm{Io}(\mathrm{Co}(A_1))) \cap S(x_2, \mathrm{Io}(\mathrm{Co}(A_2))) \cap \cdots \cap S(x_n, \mathrm{Io}(\mathrm{Co}(A_n)))$$

be a member of β_τ, and let $f \in B$. Then

$$f \in S(x_i, \mathrm{Io}(\mathrm{Co}(A_i))) \quad \forall i = 1, 2, \dots, n.$$

This implies that $f \in S(x_i, U_i)$, where

$$U_i = \mathrm{Io}(\mathrm{Co}(A_i)) \quad \forall i = 1, 2, \dots, n.$$

Therefore,

$$f \in S(x_1, U_1) \cap S(x_2, U_2) \cap \dots \cap S(x_n, U_n) = B' \in \beta_{\tau'},$$

as $U_1, U_2, \dots, U_n$ are open subsets of Y. Thus, for every $f \in B$, there exists $B' \in \beta_{\tau'}$ such that $B' \subseteq B$. □

For the converse of this proposition, we have the following.

Let

$$B_1' = S(x_1, U_1) \cap S(x_2, U_2) \cap \dots \cap S(x_n, U_n)$$

be a member of $\beta_{\tau'}$, and let $g \in B_1'$. Then

$$g \in S(x_i, U_i) \Rightarrow g \in S(x_i, \mathrm{Io}(\mathrm{Co}(U_i))) \quad (\text{as} \quad U_i \subseteq \mathrm{Co}(U_i) \Rightarrow U_i \subseteq \mathrm{Io}(\mathrm{Co}(U_i))), \quad \forall i = 1, 2, \dots, n.$$

So,

$$g \in S(x_1, \mathrm{Io}(\mathrm{Co}(U_1))) \cap S(x_2, \mathrm{Io}(\mathrm{Co}(U_2))) \cap \dots \cap S(x_n, \mathrm{Io}(\mathrm{Co}(U_n))) = B_1 \in \beta_\tau.$$

Thus, for each $B_1' \in \beta_{\tau'}$, there exists $B_1 \in \beta_\tau$. However, $B_1 \subseteq B_1'$ does not hold in general. To justify this statement, we give the following example.

Example 2. Let (Y, σ) be a topological space, where $Y = \{a, b, c\}$ and $\sigma = \{\emptyset, Y, \{c\}\}$. Then

$$\{\mathrm{Io}(\mathrm{Co}(A)) |\ A \subseteq Y\} = \{\emptyset, Y\}.$$

Thus,

$$\{\mathrm{Io}(\mathrm{Co}(A)) |\ A \subseteq Y\}$$

is not equal to σ.

Lemma 5. *Given a point $x \in X$ and a subset A of the topological space Y, define*

$$S(x, \mathrm{Co}(\mathrm{Io}(A))) = \{f \in Y^X |\ f(x) \in \mathrm{Co}(\mathrm{Io}(A))\}.$$

The sets $S(x, \mathrm{Co}(\mathrm{Io}(A)))$ form a subbasis for a topology on Y^X.

P r o o f. Let $f \in Y^X$. Then

$$f \in S(x, Y) = S(x, \mathrm{Co}(\mathrm{Io}(Y))) \subseteq \bigcup_i S(x_i, \mathrm{Co}(\mathrm{Io}(A_i))),$$

where $x_i \in X$ and A_i are subsets of Y. So,

$$f \in \bigcup_i S(x_i, \mathrm{Co}(\mathrm{Io}(A_i))).$$

Thus,

$$Y^X \subseteq \bigcup_i S(x_i, \mathrm{Co}(\mathrm{Io}(A_i))).$$

Hence, the sets $S(x_i, \mathrm{Co}(\mathrm{Io}(A_i)))$ form a subbasis for a topology on Y^X. □

The topology generated by the above subbasis is called the ***point-closure-interior topology*** on Y^X.

The following example shows that the point-interior-closure topology and the point-closure-interior topology on Y^X are not comparable.

Example 3. In Example 1, a basis of the point-interior-closure topology τ on Y^X is

$$\beta_\tau = \big\{\emptyset, Y^X, \{f_3\}, \{f_6\}, \{f_8\}, \{f_9\}, \{f_3, f_4, f_8\}, \{f_5, f_6, f_9\}, \{f_1, f_6, f_8\}, \{f_2, f_3, f_9\}\big\}.$$

A basis of the point-closure-interior topology τ' on Y^X is

$$\begin{gathered}\beta_{\tau'} = \big\{\emptyset, Y^X, \{f_7\}, \{f_1, f_7\}, \{f_2, f_7\}, \{f_4, f_7\}, \{f_5, f_7\},\\ \{f_1, f_2, f_7\}, \{f_4, f_5, f_7\}, \{f_1, f_4, f_7, f_8\}, \{f_2, f_3, f_4, f_7\}, \{f_1, f_5, f_6, f_7\}, \{f_2, f_5, f_7, f_9\},\\ \{f_1, f_2, f_3, f_4, f_7, f_8\}, \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_1, f_4, f_5, f_6, f_7, f_8\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}\big\}.\end{gathered}$$

Here, $f_3 \in \{f_3\} \in \beta_\tau$, but there does not exist any $B' \in \beta_{\tau'}$ such that $f_3 \in B' \subseteq \{f_3\}$. Thus, τ' is not finer than τ.

Similarly, $f_7 \in \{f_7\} \in \beta_{\tau'}$, but there does not exist any $B_1 \in \beta_\tau$ such that $f_7 \in B_1 \subseteq \{f_7\}$. Thus, τ is not finer than τ'.

Hence, the point-interior-closure topology and the point-closure-interior topology of Y^X are not comparable.

Lemma 6. *Let Y be a topological space. Given a point $x \in X$ and a subset $A \in SO(Y)$ (resp. $PO(Y)$, $\beta O(Y)$, $BO(Y)$), define*

$$S(x, A) = \{f \in Y^X |\ f(x) \in A\}.$$

The sets $S(x, A)$ form a subbasis for a topology on Y^X.

The topology generated by the above subbasis is called the ***point-semi-open*** (resp. ***point-preopen, point-β-open, point-b-open***) topology on Y^X.

Theorem 1. *Suppose Y is a topological space. Then, the point-preopen topology on Y^X is finer than the point-open topology on Y^X.*

P r o o f. Let β_τ and $\beta_{\tau'}$ be bases for the point-open topology and the point-preopen topology on Y^X, respectively. Let

$$B = S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n)$$

be a member of β_τ, and let $f \in B$. Then $f \in S(x_i, U_i)$ for all $i = 1, 2, \ldots, n$. This implies that $f \in S(x_i, U_i)$, where $U_i \in PO(Y)$ for all $i = 1, 2, \ldots, n$ (since U_i are open in Y). So,

$$f \in S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n) = B' \in \beta_{\tau'}$$

as $U_1, U_2, \ldots, U_n \in PO(Y)$ are open subsets of Y. Thus, for every $f \in B$, there exists $B' \in \beta_{\tau'}$ such that $B' \subseteq B$. Hence, the proof is complete. □

For the converse of Theorem 1, we always obtain a set $B_1 \in \beta_\tau$ for any $B_1' \in \beta_{\tau'}$, but it is not necessarily the case that $B_1 \subseteq B_1'$. To illustrate this, we present the following example.

Example 4. Let (Y,σ) be a topological space, where $Y=\{a,b,c\}$ and $\sigma=\{\emptyset, Y, \{c\}\}$. Then

$$\{A\subseteq Y|\ A\in PO(Y)\}=\big\{\emptyset, Y, \{c\}, \{a,c\}, \{b,c\}\big\}.$$

Thus,

$$\{A\subseteq Y|\ A\in PO(Y)\}\neq\sigma.$$

However, the two topologies will be equal when $PO(Y)=\sigma$.

Theorem 2. *Suppose Y is a topological space. Then, the point-semi-open (resp. point-β-open, point-b-open) topology on Y^X is finer than the point-open topology on Y^X.*

The proof of this theorem follows from the fact that open sets in Y are contained in $SO(Y)$ (resp. $\beta O(Y)$, $BO(Y)$). The reader should not conclude that, for any collection $\mathcal{A}$ containing the collection of open sets of Y, the point-open topology with respect to $\mathcal{A}$ is necessarily finer than the point-open topology on Y^X. However, the result of Theorem 2 holds because every open set is a preopen (resp. semi-open, b-open, β-open) set.

Therefore, a common generalization is discussed in the following lemma.

Lemma 7. *Suppose a collection $\mathcal{G}\subseteq\wp(Y)$ (the power set of Y) satisfies the following conditions:*

1) $\emptyset$, $Y\in\mathcal{G}$;
2) *$\mathcal{G}$ is closed under arbitrary unions.*

Let $h:\mathcal{G}\to\mathcal{G}$ and $k:\mathcal{G}\to\mathcal{G}$ be two set-valued set functions [20] *such that $h(A)=Y\setminus k(Y\setminus A)$ for all $A\in\wp(Y)$ and $h(\emptyset)=\emptyset$, $h(Y)=Y$.*

Given a point $x\in X$ and a subset $A\subseteq h\circ k(A)$, define

$$S(x,A)=\{f\in Y^X\ |\ \ f(x)\in A\}.$$

The sets $S(x,A)$ form a subbasis for a topology on Y^X.

P r o o f.

$$h\circ k(Y)=h(Y\setminus h(Y\setminus Y))=h(Y\setminus h(\emptyset))=h(Y)\quad(\text{as}\quad h(\emptyset)=\emptyset)=Y.$$

Thus, $Y\subseteq h\circ k(Y)$.

Let $f\in Y^X$. Then

$$f\in S(x,Y)\subseteq\bigcup_i S(x_i,(A_i)),$$

where $x_i\in X$ and $A_i\subseteq h\circ k(A_i)$. So,

$$f\in\bigcup_i S(x_i,(A_i)).$$

Thus,

$$Y^X\subseteq\bigcup_i S(x_i,(A_i)).$$

Hence, the sets $S(x_i,(A_i))$ form a subbasis for a topology on Y^X. □

The topology generated by the above subbasis is called the ***point-associated topology*** on Y^X. The following is an example of this topology.

Example 5. By taking h and k to be the Io and Co operators, respectively, we see that Lemma 7 coincides with Lemma 4.

Lemma 8. *Let Y be a topological space. Given a point $x \in X$ and a subset $A \in \mathcal{D}(Y)$ (the set of all dense sets in Y), define*

$$S(x, A) = \{f \in Y^X | \ f(x) \in A\}.$$

The sets $S(x, A)$ form a subbasis for a topology on Y^X.

P r o o f. Let $f \in Y^X$. Then

$$f \in S(x, Y) \subseteq \bigcup_i S(x_i, (A_i)),$$

where $x_i \in X$ and $A_i \in \mathcal{D}(Y)$ (as $Y \in \mathcal{D}(Y)$). So, $f \in \bigcup_i S(x_i, (A_i))$. Thus,

$$Y^X \subseteq \bigcup_i S(x_i, (A_i)).$$

Hence, the sets $S(x_i, (A_i))$ form a subbasis for a topology on Y^X. □

The topology generated by the above subbasis is called the **point-dense topology** on Y^X.

Example 6.

1. In Example 1, a basis of the point-open topology τ on Y^X is

$$\beta_\tau = \big\{\emptyset, Y^X, \{f_3\}, \{f_6\}, \{f_8\}, \{f_9\}, \{f_3, f_8\}, \{f_3, f_9\}, \{f_6, f_8\}, \{f_6, f_9\}, \{f_3, f_4, f_8\}, \{f_5, f_6, f_9\}, \{f_1, f_6, f_8\}, \{f_2, f_3, f_9\}, \{f_3, f_6, f_8, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}, \{f_3, f_4, f_5, f_6, f_8, f_9\}\big\}.$$

   A basis of the point-dense topology τ' on Y^X is

$$\beta_{\tau'} = \big\{Y^X, \{f_3, f_6, f_8, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}, \{f_3, f_4, f_5, f_6, f_8, f_9\}\big\}.$$

   In this case, the point-open topology is strictly finer than the point-dense topology.

2. In Example 1 with $\sigma = \{\emptyset, Y, \{3\}\}$, a basis of the point-open topology τ on Y^X is

$$\beta_\tau = \big\{\emptyset, Y^X, \{f_9\}, \{f_5, f_6, f_9\}, \{f_2, f_3, f_9\}\big\}.$$

   A basis of the point-dense topology τ' on Y^X is

$$\beta_{\tau'} = \big\{Y^X, \{f_9\}, \{f_2, f_9\}, \{f_3, f_9\}, \{f_5, f_9\}, \{f_6, f_9\}, \{f_2, f_3, f_9\}, \{f_5, f_6, f_9\}, \{f_1, f_2, f_6, f_9\}, \{f_2, f_5, f_7, f_9\}, \{f_3, f_4, f_5, f_9\}, \{f_3, f_6, f_8, f_9\}, \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}, \{f_3, f_4, f_5, f_6, f_8, f_9\}\big\}.$$

   In this case, the point-dense topology is strictly finer than the point-open topology.

Hence, we conclude that the point-open topology and the point-dense topology on Y^X are not comparable.

3. Topologies on Y^X due to ideal

It is known from [7, 11, 17, 18] that ψ is not an interior operator. The following lemma shows that a noninterior operator may also serve as an essential tool in obtaining a topology on Y^X.

Lemma 9. *Let $\mathbb{I}$ be an ideal on the topological space Y. Given a point $x \in X$ and a subset A of the topological space Y, define*

$$S_{\mathbb{I}}(x, \psi(A)) = \{f \in Y^X \mid f(x) \in \psi(A)\}.$$

The sets $S_{\mathbb{I}}(x, \psi(A))$ form a subbasis for a topology on Y^X.

P r o o f. Let $f \in Y^X$. Then

$$f \in S_{\mathbb{I}}(x, Y) = S_{\mathbb{I}}(x, \psi(Y)) \subseteq \bigcup_i S_{\mathbb{I}}(x_i, \psi(A_i)),$$

where $x_i \in X$ and A_i are subsets of Y. So,

$$f \in \bigcup_i S_{\mathbb{I}}(x_i, \psi(A_i)).$$

Thus,

$$Y^X \subseteq \bigcup_i S_{\mathbb{I}}(x_i, \psi(A_i)).$$

Hence, the sets $S_{\mathbb{I}}(x_i, \psi(A_i))$ form a subbasis for a topology on Y^X. □

The topology generated by the above subbasis is called the ***point-ψ topology*** on Y^X.

Since $\psi \sim^Y *$ [20], the subbasis for the point-ψ topology on Y^X can be equivalently rewritten in terms of the $*$-operator.

Comparison of the point-ψ topology with other topologies on Y^X are as follows.

Proposition 5. *Suppose $\mathbb{I}$ is an ideal on the topological space Y. Then, the point-open topology on Y^X is finer than the point-ψ topology on Y^X.*

P r o o f. Let β_τ and $\beta_{\tau'}$ be bases for the point-ψ topology and the point-open topology on Y^X, respectively. Let

$$B = S_{\mathbb{I}}(x_1, \psi(A_1)) \cap S_{\mathbb{I}}(x_2, \psi(A_2)) \cap \cdots \cap S_{\mathbb{I}}(x_n, \psi(A_n))$$

be a member of β_τ, and let $f \in B$. Then $f \in S_{\mathbb{I}}(x_i, \psi(A_i))$ for all $i = 1, 2, \ldots, n$. This implies that $f \in S(x_i, U_i)$, where $U_i = \psi(A_i)$ (since for each i, $\psi(A_i)$ is open by [11, 18]), for all $i = 1, 2, \ldots, n$. Hence,

$$f \in S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n) = B' \in \beta_{\tau'}$$

since $U_1,\ U_2, \ldots,\ U_n$ are open subsets of Y. Thus, for each $f \in B$, there exists $B' \in \beta_{\tau'}$ such that $B' \subseteq B$. □

For the converse relation of this proposition, we give the following example.

Example 7. Consider Example 1 with $\sigma = \{\emptyset, Y, \{3\}, \{1,3\}, \{2,3\}\}$ and $\mathbb{I} = \{\emptyset, \{1\}\}$. Then, a basis of the point-ψ topology τ on Y^X is

$$\beta_\tau = \{\emptyset, Y^X, \{f_2, f_5, f_7, f_9\}, \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}\}.$$

A basis of the point-open topology τ' on Y^X is

$$\beta_{\tau'} = \big\{\emptyset, Y^X, \{f_9\}, \{f_2, f_9\}, \{f_3, f_9\}, \{f_5, f_9\}, \{f_6, f_9\}, \\ \{f_2, f_3, f_9\}, \{f_5, f_6, f_9\}, \{f_1, f_2, f_6, f_9\}, \{f_2, f_5, f_7, f_9\}, \{f_3, f_4, f_5, f_9\}, \{f_3, f_6, f_8, f_9\}, \\ \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}, \{f_3, f_4, f_5, f_6, f_8, f_9\}\big\}.$$

Here, $f_9 \in \{f_9\} \in \beta_{\tau'}$, but there does not exist any $B_1 \in \beta_\tau$ such that $f_9 \in B_1 \subseteq \{f_9\}$. Thus, τ is not finer than τ'.

However, the set $\{\psi(A) :\ A \subseteq Y\}$ does not form a topology on Y.

Example 8. Let $(Y, \sigma, \mathbb{I})$ be an ideal topological space, where $Y = \{a, b, c\}$, $\sigma = \big\{\emptyset, Y, \{c\}, \{a, c\}, \{b, c\}\big\}$, and $\mathbb{I} = \{\emptyset, \{a\}\}$. Then $\{\psi(A)|\ A \subseteq Y\} = \{\emptyset, Y, \{a, c\}\}$. In this example, it is clear that

$$\{\psi(A)|\ A \subseteq Y\} \neq \sigma$$

on Y.

As a consequences of the above results and Theorem 46.7 of [21], we have the following.

Theorem 3. *Suppose $\mathbb{I}$ is an ideal on the metric space (Y, d) and Y is a topological space. For the function space Y^X, the following inclusions of topologies hold*:

$$(\textit{uniform}) \supset (\textit{compact convergence}) \supset (\textit{point-open}) = (\textit{point-interior}) \supseteq (\textit{point-}\psi).$$

Proposition 6. *Suppose $\mathbb{I}$ is a codense ideal on the topological space Y. Given a point $x \in X$ and a subset A of the topological space Y, define*

$$S_{\mathbb{I}}(x, A^*) = \{f \in Y^X \ |\ f(x) \in A^*\}.$$

The sets $S_{\mathbb{I}}(x, A^*)$ form a subbasis for a topology on Y^X.

P r o o f. Let $f \in Y^X$. Then

$$f \in S_{\mathbb{I}}(x, Y) = S_{\mathbb{I}}(x, Y^*) \quad (\text{since } \mathbb{I} \text{ is a codense ideal}) \subseteq \bigcup_i S_{\mathbb{I}}(x_i, A_i^*),$$

where $x_i \in X$ and A_i are subsets of Y. Thus,

$$f \in \bigcup_i S_{\mathbb{I}}(x_i, A_i^*).$$

Thus,

$$Y^X \subseteq \bigcup_i S_{\mathbb{I}}(x_i, A_i^*).$$

Hence, the sets $S_{\mathbb{I}}(x_i, A_i^*)$ form a subbasis for a topology on Y^X. □

The topology generated by the above subbasis is called the ***point-$*$ topology*** on Y^X.

Example 9. Consider Example 1 with $\sigma = \big\{\emptyset, Y, \{3\}, \{1, 3\}, \{2, 3\}\big\}$ and $\mathbb{I} = \{\emptyset, \{1\}\}$. Then, a basis of the point-ψ topology τ on Y^X is

$$\beta_\tau = \big\{\emptyset, Y^X, \{f_2, f_5, f_7, f_9\}, \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}\big\}.$$

A basis of the point-$*$ topology τ' on Y^X is

$$\beta_{\tau'} = \big\{\emptyset, Y^X, \{f_8\}, \{f_3, f_4, f_8\}, \{f_1, f_6, f_8\}\big\}.$$

Here, $f_2 \in \{f_2, f_5, f_7, f_9\} \in \beta_\tau$, but there does not exist any $B' \in \beta_{\tau'}$ such that $f_2 \in B' \subseteq B$. Thus, τ' is not finer than τ.

Similarly, $f_8 \in \{f_8\} \in \beta_{\tau'}$, but there does not exist any $B_1 \in \beta_\tau$ such that $f_8 \in B_1 \subseteq \{f_8\}$. Thus, τ is not finer than τ'.

Hence, the point-ψ topology and point-$*$ topology of Y^X are not comparable.

To discuss further topologies on Y^X, we make use of the notion of ψ-sets in an ideal topological space. This concept was introduced by Modak and Bandyopadhyay in [7], whose definition is as follows.

Let $\mathbb{I}$ be an ideal on a topological space Y. A subset A of Y is called a ψ-set if $A \subset \mathrm{Io}(\mathrm{Co}(\psi(A)))$. The collection of all ψ-sets in the ideal topological space Y is denoted by $\psi^Y(Y)$.

Theorem 4. *Let $\mathbb{I}$ be an ideal on the topological space Y. Given a point $x \in X$ and $A \in \psi^Y(Y)$, define*

$$S_{\mathbb{I}}(x, A) = \{f \in Y^X \mid f(x) \in A\}.$$

The sets $S_{\mathbb{I}}(x, A)$ form a subbasis for a topology on Y^X.

Before proceeding to the proof of this theorem, we make a few remarks on ψ-sets. The collection $\psi^Y(Y)$ forms a topology on Y whenever the ideal $\mathbb{I}$ is a codense ideal or a σ-boundary ideal [23] on Y. Modak and Bandyopadhyay studied this topology in [7] and showed that this topology coincides with the α-topology [24] of the $*$-topology [12] generated by σ. Thus, we say that the topology obtained in Theorem 4 is the point-open topology for $\psi^Y(Y)$ (forms a topology on Y). If we denote the σ^*-topology generated by σ by $*$-topology, the topology constructed in Theorem 4 is the point-open topology of $(\sigma^*)^\alpha$. We also note that codenseness is not essential for the proof of Theorem 4. However, if we consider the point-open topology of Y^X arising from $(\sigma^*)^\alpha$, then codenseness is required. We omit the proof of this theorem, leaving it as an exercise for the reader.

For our next discussion, we will refer to the topology obtained in Theorem 4 as the ***point-*Co_ψ ***topology*** on Y^X.

The following gives a comparison of the point-Co_ψ topology on Y^X.

Corollary 1. *Suppose $\mathbb{I}$ is an ideal on the topological space Y. Then, the point-Co_ψ topology on Y^X is finer than the point-open topology on Y^X.*

P r o o f The proof of this corollary is only meaningful when $\mathbb{I}$ is not a codense ideal on Y; otherwise, the result follows immediately from Lemma 1. □

Theorem 5. *Suppose $\mathbb{I}$ is an ideal on the topological space Y. Then, the point-Co_ψ topology on Y^X is finer than the point-ψ topology on Y^X.*

P r o o f. Let β_τ and $\beta_{\tau'}$ be bases for the point-ψ topology and the point-Co_ψ topology on Y^X, respectively. Let

$$B = S_{\mathbb{I}}(x_1, \psi(A_1)) \cap S_{\mathbb{I}}(x_2, \psi(A_2)) \cap \cdots \cap S_{\mathbb{I}}(x_n, \psi(A_n))$$

be a member of β_τ, and let $f \in B$. Then

$$f \in S_{\mathbb{I}}(x_i, \psi(A_i)), \quad \forall i = 1, 2, \ldots, n.$$

This implies that $f \in S_{\mathbb{I}}(x_i, U_i)$, where $U_i = \psi(A_i)$ for all $i = 1, 2, \ldots, n$. Therefore,

$$f \in S_{\mathbb{I}}(x_1, U_1) \cap S_{\mathbb{I}}(x_2, U_2) \cap \cdots \cap S_{\mathbb{I}}(x_n, U_n) = B' \in \beta_{\tau'}$$

(as $U_1, U_2, \ldots, U_n$ are open subsets of Y and $U_i \in \psi^Y(Y)$). Thus, for every $f \in B$, there exists $B' \in \beta_{\tau'}$ such that $B' \subseteq B$. This completes the proof. □

The converse of this theorem does not necessarily hold in general.

If we replace the Co operator with $()^*$ operator, we obtain another topology on Y^X. To this end, we introduce Modak's $\dot{\psi}^*$-set [17]. Its formal definition is as follows.

Let $\mathbb{I}$ be an ideal on a space Y. A subset A of Y is called a $\dot{\psi}^*$-set if $A \subseteq \mathrm{Io}((\psi(A))^*)$. The collection of all $\dot{\psi}^*$-sets in an ideal topological space Y is denoted by $\dot{\psi}^*(Y)$.

Theorem 6. *Let $\mathbb{I}$ be a codense ideal on the topological space Y. Given a point $x \in X$ and a subset $A \in \dot{\psi}^*(Y)$, define*

$$S_{\mathbb{I}}(x, A) = \{f \in Y^X \mid f(x) \in A\}.$$

The sets $S(x, A)$ form a subbasis for a topology on Y^X.

P r o o f. Since Y is open, $Y \subseteq \psi(Y)$. Then $Y = Y^*$ (as $\mathbb{I}$ is codense)$\subseteq (\psi(Y))^*$. This implies $Y = \mathrm{Io}(Y) \subseteq \mathrm{Io}((\psi(Y))^*)$, and hence, $Y \in \dot{\psi}^*(Y)$.

Let $f \in Y^X$. Then

$$f \in S_{\mathbb{I}}(x, Y) \subseteq \bigcup_i S_{\mathbb{I}}(x_i, (A_i)),$$

where $x_i \in X$ and $A_i \in \dot{\psi}^*(Y)$. Therefore,

$$f \in \bigcup_i S(x_i, (A_i)).$$

Hence,

$$Y^X \subseteq \bigcup_i S_{\mathbb{I}}(x_i, (A_i)).$$

Thus, the sets $S_{\mathbb{I}}(x_i, (A_i))$ form a subbasis for a topology on Y^X. □

The topology generated by the above subbasis is called the ***point-$\dot{\psi}^*$ topology*** on Y^X.

Moreover, if $\mathbb{I}$ is a codense ideal on Y, then the collections $\psi^Y(Y)$ and $\dot{\psi}^*(Y)$ both represent the α-sets of the $*$-topology of σ (see [7]). Thus, the point-open topologies induced by $\psi^Y(Y)$ and $\dot{\psi}^*(Y)$ coincide.

Definition 3 [19]. *Let $(Y, \sigma, \mathbb{I})$ be an ideal topological space, and $A \subseteq Y$. Then A is called h^ψ-open if, for every nonempty open set $U \neq Y$, it holds $A \subseteq \psi(A \cup U)$.*

Theorem 7. *Let $(Y, \sigma, \mathbb{I})$ be an ideal topological space. Given a point $x \in X$ and a h^ψ-open set A of the topological space Y, define*

$$S_{\mathbb{I}}(x, A) = \{f \in Y^X \mid f(x) \in A\}.$$

The sets $S_{\mathbb{I}}(x, A)$ form a subbasis for a topology on Y^X.

P r o o f . This follows from the fact that the collection of h^ψ-open sets forms a topology on Y. □

The topology generated by the above subbasis is called the ***point-h^ψ-open topology*** on Y^X.

Theorem 8. *Suppose $\mathbb{I}$ is an ideal on the topological space Y. Then, the point-h^ψ-open topology on Y^X is finer than the point-open topology on Y^X.*

P r o o f . This follows directly from the fact that the topology generated by the h^ψ-open sets is finer than the topology σ on Y. □

Theorem 9. *Let Y be a topological space. Given a point $x \in X$ and an h-open set A of the space Y, define*

$$S(x, A) = \{f \in Y^X \mid f(x) \in A\}.$$

The sets $S(x, A)$ form a subbasis for a topology on Y^X.

The topology generated by the above subbasis is called the ***point-h-open topology*** on Y^X.

Theorem 10. *The point-h-open topology on Y^X is finer than the point-open topology on Y^X.*

From the above theorems, we conclude the following common phenomenon.

Corollary 2. *Let $\mathbb{I}$ be an ideal on a topological space Y. Then, the point-open topology on Y^X is contained in the point-h-open topology on Y^X, which in turn is contained in the point-h^ψ-topology on Y^X.*

4. Topologies on Y^X induced by continuous functions

In this section, we discuss the interrelation among the open-point topology, the point-open topology, and the bi-point topology [13, 26].

Theorem 11. *Let $C_{op}(Z, H)$ be the group of all continuous open functions from Z to H. Then, the open-point topology on $C_{op}(Z, H)$ is finer than the point-open topology on $C_{op}(Z, H)$.*

P r o o f. Let β_τ and $\beta_{\tau'}$ be bases for the open-point and point-open topologies on $C_{op}(Z, H)$, respectively. Let $B' = [z_1, V_1]^+ \cap \cdots \cap [z_n, V_n]^+$, where $n \in \mathbb{N}$, $z_i \in Z$, and each V_i is an open subset of H, be a member of $\beta_{\tau'}$, and let $f \in B'$. Then $f \in [z_i, V_i]^+$ for all $i = 1, 2, \dots, n$, and $f : Z \to H$ is continuous. Hence, $z_i \in B_i$, where $B_i = f^{-1}(V_i)$ for all $i = 1, 2, \dots, n$ (as $f^{-1}(V_i)$ are open in Z). Let $r_i \in V_i$ be such that $f(z_i) = r_i$. Then $z_i \in f^{-1}(r_i)$. Therefore,

$$z_i \in f^{-1}(r_i) \cap B_i \quad \forall i = 1, 2, \dots, n.$$

Thus,

$$f \in [B_1, r_1]^- \cap \cdots \cap [B_n, r_n]^- = B \in \beta_\tau.$$

Therefore, for every $f \in B'$, there exists $B \in \beta_\tau$.

It remains to show that $B \subseteq B'$. Let $f \in B$. Then

$$f^{-1}(r_i) \cap B_i \neq \emptyset.$$

Let

$$z_i \in f^{-1}(r_i) \cap B_i.$$

Then

$$f(z_i) \in f[f^{-1}(r_i) \cap B_i] \subseteq f(f^{-1}(r_i)) \cap f(B_i) \subseteq f(B_i) = V_i.$$

Hence, $f(z_i) \in V_i$, which implies $f \in B'$. This show that $B \subseteq B'$. This completes the proof. $\square$

Openness of a function is a necessary condition for Theorem 11. To illustrate this, we give the following example.

Example 10. Let (Z, τ) and (Y, σ) be two topological spaces, where $Z = \{a, b\}$, $\tau = \{\emptyset, Z, \{a\}\}$, $Y = \{1, 2, 3\}$, and $\sigma = \big\{\emptyset, Y, \{2\}, \{3\}, \{2, 3\}\big\}$. All possible functions from Z to Y are given by

$$\begin{array}{llllll} f_1(a) = 1, & f_1(b) = 2; & f_2(a) = 1, & f_2(b) = 3; & f_3(a) = 2, & f_3(b) = 3; \\ f_4(a) = 2, & f_4(b) = 1; & f_5(a) = 3, & f_5(b) = 1; & f_6(a) = 3, & f_6(b) = 2; \\ f_7(a) = 1, & f_7(b) = 1; & f_8(a) = 2, & f_8(b) = 2; & f_9(a) = 3, & f_9(b) = 3. \end{array}$$

Now,

$$C(Z, Y) = \{f_4, f_5, f_7, f_8, f_9\}.$$

Here, f_7 is not an open map since $f_7(\{a\}) = \{1\} \notin \sigma$. We have

$$[a, \{2\}]^+ = \{f_4, f_8\}, \quad [a, \{3\}]^+ = \{f_5, f_9\}, \quad [a, \{2, 3\}]^+ = \{f_4, f_5, f_8, f_9\},$$
$$[b, \{2\}]^+ = \{f_8\}, \quad [b, \{3\}]^+ = \{f_9\}, \quad [b, \{2, 3\}]^+ = \{f_8, f_9\},$$
$$[a, Y]^+ = [b, Y]^+ = \{f_4, f_5, f_7, f_8, f_9\}.$$

Then, a basis for the point-open topology on $C(Z, Y)$ is

$$\beta' = \big\{\emptyset, \{f_4, f_8\}, \{f_5, f_9\}, \{f_8\}, \{f_9\}, \{f_8, f_9\}, \{f_4, f_5, f_8, f_9\}, \{f_4, f_5, f_7, f_8, f_9\}\big\}.$$

Also,

$$[\{a\}, 1]^- = \{f_7\}, \quad [\{a\}, 2]^- = \{f_4, f_8\}, \quad [\{a\}, 3]^- = \{f_5, f_9\},$$
$$[Z, 1]^- = [Z, 2]^- = [Z, 3]^- = \{f_4, f_5, f_7, f_8, f_9\}.$$

Then, a basis for the open-point topology on $C(Z, Y)$ is

$$\beta = \big\{\emptyset, \{f_7\}, \{f_4, f_8\}, \{f_5, f_9\}, \{f_4, f_5, f_7, f_8, f_9\}\big\}.$$

In this example, we see that the open-point topology on $C(Z, Y)$ is not finer than the point-open topology on $C(Z, Y)$.

Theorem 12. *Let $C_{op}(Z, H)$ be the group of all continuous open functions from Z to H. Then, the bi-point-open topology on $C_{op}(Z, H)$ is finer than the point-open topology on $C_{op}(Z, H)$.*

P r o o f. Let β_τ, $\beta_{\tau'}$, and $\beta_{\tau''}$ be bases for the open-point topology on $C_{op}(Z, H)$, the point-open topology on $C_{op}(Z, H)$, and the bi-point-open topology on $C_{op}(Z, H)$, respectively. Let

$$B' = [z_1, V_1]^+ \cap \cdots \cap [z_n, V_n]^+,$$

where $n \in \mathbb{N}$, $z_i \in Z$, and each V_i is an open subset of H, be a member of $\beta_{\tau'}$, and let $f \in B'$. Then, from Theorem 11, there exists

$$B = [B_1, r_1]^- \cap \cdots \cap [B_n, r_n]^- \in \beta_\tau,$$

where $n \in \mathbb{N}$, $r_i \in H$, and B_i are open subsets of Z such that $f \in B$. Thus,

$$f \in [z_1, V_1]^+ \cap \cdots \cap [z_n, V_n]^+ \cap [B_1, r_1]^- \cap \cdots \cap [B_n, r_n]^- = B'' \in \beta_{\tau''}.$$

Clearly, $B'' \subseteq B'$. This completes the proof. $\square$

Theorem 13. *Let $C_{op}(Z,H)$ be the group of all continuous open functions from Z to H. Then, the bi-point-open topology on $C_{op}(Z,H)$ is finer than the open-point topology on $C_{op}(Z,H)$.*

P r o o f. Let β_τ, $\beta_{\tau'}$ and $\beta_{\tau''}$ be bases for the open-point topology on $C_{op}(Z,H)$, the point-open topology on $C_{op}(Z,H)$, and the bi-point-open topology on $C_{op}(Z,H)$, respectively. Let

$$B = [B_1, r_1]^- \cap \cdots \cap [B_n, r_n]^- \in \beta_\tau,$$

where $n \in \mathbb{N}$, each $r_i \in H$, and each B_i is an open subset of Z, be a member of β_τ and $f \in B$. Then

$$f^{-1}(r_i) \cap B_i \neq \emptyset.$$

Let $z_i \in f^{-1}(r_i) \cap B_i$. Then

$$f(z_i) \in f[f^{-1}(r_i) \cap B_i] \subseteq ff^{-1}(r_i) \cap f(B_i) \subseteq f(B_i) = V_i,$$

where each V_i is open in H. Therefore, $f(z_i) \in V_i$. This implies that

$$f \in B' = [z_1, V_1]^+ \cap \cdots \cap [z_n, V_n]^+ \in \beta_{\tau'}.$$

Thus,

$$f \in [z_1, V_1]^+ \cap \cdots \cap [z_n, V_n]^+ \cap [B_1, r_1]^- \cap \cdots \cap [B_n, r_n]^- = B'' \in \beta_{\tau''}.$$

Clearly, $B'' \subseteq B'$. This completes the proof. □

5. Conclusion

In this paper, the role of generated open sets in defining topologies on Y^X has been discussed. The interrelations among these topologies were also explored. We have shown that the concept of a topological ideal provides a useful framework for studying such topologies on Y^X. Furthermore, for a topological group H and a space Z, the relationship between the point-open topology and the bi-point-open topology on $C(Z,H)$ was also examined.

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ON λ-WEAK CONVERGENCE OF SEQUENCES DEFINED BY AN ORLICZ FUNCTION

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Abstract: In this article, we introduce and rigorously analyze the concept of difference λ-weak convergence for sequences defined by an Orlicz function. This notion generalizes the classical weak convergence by incorporating a λ-density framework and an Orlicz function, providing a more flexible tool for analyzing convergence behavior in sequence spaces. We systematically investigate the algebraic and topological properties of these newly defined sequence spaces, establishing that they form linear and normed spaces under suitable conditions. Our results include demonstrating the convexity of these spaces and identifying several important inclusion relationships among them, such as strict inclusions between spaces involving different orders of difference operators and Orlicz functions satisfying the Δ_2-condition.

Keywords: Weak convergence, Orlicz function, λ convergence.

1. Introduction and preliminaries

The concept of weak convergence, first introduced by Banach [1], is central to functional analysis, providing a foundation for evaluating how sequences converge in infinite-dimensional spaces. While important, weak convergence has its limitations, especially when applied to complex sequence structures or when more precise convergence criteria are required.

Recently, Mahanta and Tripathy [21] made important advances in the study of vector-valued sequence spaces by investigating novel types of convergence and their repercussions. Their contributions have improved our understanding of the algebraic and topological properties of these spaces, enabling the development of new tools and approaches for investigating convergence in broader contexts. This growing field of study emphasizes the continual growth and improvement of sequence space theory, overcoming the limitations of traditional weak convergence while responding to the demands of more complex mathematical analysis.

The concept of natural density for subsets of $\mathbb{N}$ was extended by Mursaleen [13] to what is known as λ-density, which enabled a further generalization of the statistical convergence of real sequences, leading to the concept of λ-statistical convergence. If $\lambda = \{\lambda_s\}_{s\in\mathbb{N}}$ represents a nondecreasing sequence of positive real numbers tending to infinity, satisfying $\lambda_1 = 1$ and $\lambda_{s+1} \leq \lambda_s + 1$, $s \in \mathbb{N}$, then for any subset $T \subset \mathbb{N}$, the λ-density $d_\lambda(T)$ is defined as

$$d_\lambda (T) = \lim_{s\to\infty} \frac{|\{k \in I_s : k \in T\}|}{\lambda_s},$$

where $I_s = [s - \lambda_s + 1, s]$.

A sequence $t = \{t_\alpha\}_{\alpha \in \mathbb{N}}$ of real numbers is called λ-statistically convergent or S_λ-convergent to $t_0 \in \mathbb{R}$ if, for every $\epsilon > 0$, $d_\lambda(T(\epsilon)) = 0$, where

$$T(\epsilon) = \{\alpha \in \mathbb{N} : |t_\alpha - t_0| \geq \epsilon\}.$$

The generalized de la Vallée-Poussin mean is defined by

$$q_s(t) = \frac{1}{\lambda_s} \sum_{\alpha \in I_s} t_\alpha$$

where $I_s = [s - \lambda_s + 1, s]$. A sequence is called (V, λ)-summable to a number t_0 if $q_s(t) \to t_0$ as $s \to \infty$.

If $\lambda_s = s$ for all $s \in \mathbb{N}$, then the notions of λ-density and λ-statistical convergence coincide with the notions of natural density and statistical convergence, respectively. Some discussions and applications related to λ-statistical convergence can be found in [2, 4, 5, 12, 14, 15, 17–20].

Let X be a normed space. The concept of the difference sequence space $Z(\Delta)$ was first introduced by Kizmaz [10] and is defined as follows:

$$Z(\Delta) = \{t = (t_\alpha) : (\Delta t_\alpha) \in X\},$$

where $\Delta t = (\Delta t_\alpha) = (t_\alpha - t_{\alpha+1})$ for all $\alpha \in \mathbb{N}$. Later, Et and Çolak [3] extended this idea by defining generalized difference sequence spaces, expressed as

$$Z(\Delta^p) = \{t = (t_\alpha) : (\Delta^p t_\alpha) \in X\}$$

for $Z = \ell_\infty, c$, and c_0, where $\Delta^p t_\alpha = \Delta^{p-1} t_\alpha - \Delta^{p-1} t_{\alpha+1}$ and $\Delta^0 t_\alpha = t_\alpha$ for all $\alpha \in \mathbb{N}$.

The binomial expansion for this generalized difference operator is given by

$$\Delta^p t_\alpha = \sum_{d=0}^{p} (-1)^d \binom{p}{d} t_{\alpha+d}, \quad \text{for all } \alpha \in \mathbb{N}. \tag{1.1}$$

These generalized difference sequence spaces have been further studied by researchers such as Tripathy [22, 23], Tripathy and Esi [24], among others.

Definition 1. *Let V be a real vector space and let $u, v \in V$. Then, the set of all convex combinations of u and v is the set of points*

$$\{w_\varrho \in V : w_\varrho = (1 - \varrho) u + \varrho v, \ 0 \leq \varrho \leq 1\}. \tag{1.2}$$

In, say, $\mathbb{R}^2$, this set is exactly the line segment joining the two points u and v. We now introduce the concept of a convex set.

Definition 2. *Let $M \subset V$. Then the set M is said to be convex if, for any two points $u, v \in M$, the set defined in (1.2) is a subset of M.*

An Orlicz function $\mathcal{U} : [0, \infty) \to [0, \infty)$ is defined such that $\mathcal{U}(0) = 0$, $\mathcal{U}(t) > 0$ for $t > 0$, and $\mathcal{U}(t) \to \infty$ as $t \to \infty$. This function is continuous, nondecreasing, and convex.

Lindenstrauss and Tzafriri [11] introduced the concept of an Orlicz function to define the sequence space

$$\ell_{\mathcal{U}} = \left\{ (t_i) \in \omega : \sum_{i=1}^{\infty} \mathcal{U}\left(\frac{|t_i|}{v}\right) < \infty \text{ for some } v > 0 \right\},$$

where ω denotes the class of all sequences. The norm on the sequence space $\ell_{\mathcal{U}}$ is defined by

$$\|t\| = \inf\left\{v > 0 : \sum_{i=1}^{\infty} \mathcal{U}\left(\frac{|t_i|}{v}\right) \leq 1\right\},$$

which turns $\ell_{\mathcal{U}}$ into a Banach space, commonly referred to as an Orlicz sequence space. Various researchers, including Khan [6], Khan et al. [7–9], Parashar and Choudhury [16], and Tripathy and Mahanta [21], have explored different forms of Orlicz sequence spaces.

Definition 3. *A sequence (t_i) in a normed linear space X is called weakly convergent to an element $t_0 \in X$ if*

$$\lim_{i\to\infty} f\left(t_i - t_0\right) = 0 \quad \text{for all } f \in X',$$

where X' denotes the continuous dual space of X.

Definition 4. *A sequence (t_i) in a normed linear space X is said to be λ-weakly convergent to $t_0 \in X$ if*

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{k\in I_s} f\left(t_k - t_0\right) = 0$$

for every $f \in X'$, where X' is the continuous dual space of X. In this context, the notation $\mathcal{D}^w_\lambda$ is used to denote the set of all λ-weakly convergent sequences.

Definition 5. *A sequence space E is called solid if, for any scalar sequence (β_i) with $|\beta_i| \leq 1$ for all $i \in \mathbb{N}$, the condition $(t_i) \in E$ implies that $(\beta_i t_i) \in E$.*

Definition 6. *A sequence space $E \subset \omega$ is called monotone if it contains all preimages of its step spaces.*

Definition 7. *A sequence space $E \subset \omega$ is called symmetric if, whenever $(t_i) \in E$, the permuted sequence $\left(t_{\pi(i)}\right)$ also belongs to E, where π is a permutation of $\mathbb{N}$.*

Lemma 1. *A sequence space E being solid does not necessarily mean that E is monotone.*

Definition 8. *An Orlicz function $\mathcal{U}$ satisfies the Δ_2-condition if there exists a constant $T > 0$ such that for all $u \geq 0$,*

$$\mathcal{U}(2u) \leq T\mathcal{U}(u).$$

2. Main result

This section presents the following classes of sequences and establishes results related to them:

$$[\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_0 = \left\{t = (t_\alpha) : \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) = 0 \text{ for some } v > 0\right\},$$

$$[\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_1 = \left\{t = (t_\alpha) : \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\frac{|f(\Delta^p t_\alpha - t_0)|}{v} \text{ for some } t_0 \text{ and } v > 0\right\},$$

$$[\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_\infty = \left\{t = (t_\alpha) : \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) < \infty \text{ for some } v > 0\right\}.$$

The following result is presented here with a sketch of the proof.

Theorem 1. *The classes of sequences* $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$, $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_1$, *and* $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\infty$ *are linear spaces.*

P r o o f. The proof is provided only for the class $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$; the other cases can be established using a similar approach. Let $(t_\alpha), (q_\alpha) \in [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$, and let $\mathfrak{y}, \mathfrak{z} \in \mathbb{C}$. To prove the result, we need to find some $v_3 > 0$ such that

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\mathfrak{y}\Delta^p t_\alpha + \mathfrak{z}\Delta^p q_\alpha)|}{v_3}\right) = 0.$$

Since $(t_\alpha), (q_\alpha) \in [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$, there exist $v_1, v_2 > 0$ such that

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v_1}\right) = 0$$

and

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p q_\alpha)|}{v_2}\right) = 0.$$

We set $v_3 = \max(2|\mathfrak{y}|v_1, 2|\mathfrak{z}|v_2)$. Suppose that $\mathcal{U}$ is both convex and nondecreasing; it follows that

$$\begin{aligned}\frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\mathfrak{y}\Delta^p t_\alpha + \mathfrak{z}\Delta^p q_\alpha)|}{v_3}\right) &\le \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\mathfrak{y}\Delta^p t_\alpha)|}{v_3} + \frac{|f(\mathfrak{z}\Delta^p q_\alpha)|}{v_3}\right)\\ &\le \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \frac{1}{2}\left[\mathcal{U}\left(\frac{f(\mathfrak{y}\Delta^p t_\alpha)}{v_1} + \frac{f(\mathfrak{z}\Delta^p q_\alpha)}{v_2}\right)\right] \to 0 \quad \text{as} \quad s \to \infty.\end{aligned}$$

This proves that $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$ is a linear space over the field $\mathbb{C}$ of complex numbers.

Theorem 2. *For any Orlicz function* $\mathcal{U}$, *the space* $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\infty$ *forms a normed linear space with respect to the norm*

$$\varkappa_{\Delta^p}(t) = \sum_{i=1}^{p} |f(x_i)| + \inf\left\{v > 0 : \sup_s \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) \le 1\right\}.$$

P r o o f. To prove the theorem, we begin by examining the implications of $\varkappa_{\Delta^p}(t) = \varkappa_{\Delta^p}(-t)$ and $t = \theta$, which leads to $\Delta^p t_\alpha = 0$. As a result, we find $\mathcal{U}(\theta) = 0$, which consequently yields $\varkappa_{\Delta^p}(\theta) = 0$. Conversely, suppose $\varkappa_{\Delta^p}(t) = 0$, which implies that

$$\sum_{i=1}^{p} |f(t_i)| + \inf\left\{v > 0 : \sup_s \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) \le 1\right\} = 0.$$

Thus, we conclude that

$$\sum_{i=1}^{p} |f(t_i)| = 0 \quad \text{and} \quad \inf\left\{v > 0 : \sup_s \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) \le 1\right\} = 0.$$

From the first part, it follows that

$$t_i = \bar{\theta} \quad \text{for} \quad i = 1, 2, 3, \dots, m, \tag{2.1}$$

where $\bar{\theta}$ denotes the zero element. For the second part, for any $\sigma > 0$, there exists some v_σ with $0 < v_\sigma < \sigma$ such that

$$\sup_s \frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v_\sigma}\right) \leq 1 \Rightarrow \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v_\sigma}\right) \leq 1.$$

Therefore,

$$\sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{\sigma}\right) \leq \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v_\sigma}\right) \leq 1.$$

Suppose that $\Delta^p t_{q_i} \neq \bar{\theta}$ for each $i \in \mathbb{N}$. As $\sigma \to 0$, it follows that

$$\frac{|f(\Delta^p t_{q_i})|}{\sigma} \to \infty.$$

Thus,

$$\frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{\sigma}\right) \to \infty$$

as $\sigma \to 0$, where $q_i \in I_s$, which leads to a contradiction. Hence, $\Delta^p t_{q_i} = \bar{\theta}$ for each $i \in \mathbb{N}$, and consequently $\Delta t_\alpha = \bar{\theta}$ for all $\alpha \in \mathbb{N}$. Therefore, it follows from (1.1) and (2.1) that $t_\alpha = \bar{\theta}$ for all $\alpha \in \mathbb{N}$, implying that $t = \theta$.

Next, let $v_1, v_2 > 0$ be such that

$$\sup_s \frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{v_1}\right) \leq 1$$

and

$$\sup_s \frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p \varpi_\alpha)|}{v_2}\right) \leq 1.$$

Let $v = v_1 + v_2$, then we have

$$\sup_s \frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p (t_\alpha + \varpi_\alpha))|}{v}\right) \leq 1.$$

Since v is nonnegative, we have

$$\varkappa_{\Delta^p} f(t+\varpi) = \sum_{i=1}^{p} |f(t_i+\varpi_i)| + \inf\left\{v>0 : \sup_s \frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p (t_\alpha+\varpi_\alpha))|}{v}\right) \leq 1\right\}$$

$$\Rightarrow \varkappa_{\Delta^p} f(t+\varpi) \leq \varkappa_{\Delta^p} f(t) + \varkappa_{\Delta^p} f(\varpi).$$

Let $\vartheta \neq 0$ and $\vartheta \in \mathbb{C}$. Then

$$\varkappa_{\Delta^p}(\vartheta t) = \sum_{i=1}^{p} |\vartheta t_i| + \inf\left\{v>0 : \sup_s \frac{1}{\lambda_s} \sum_{\alpha \in I_s} \mathcal{U}\left(\frac{|f(\Delta^p (\vartheta t_\alpha))|}{v}\right) \leq 1\right\} \leq |\vartheta| \varkappa_{\Delta^p} f(t).$$

This completes the proof. □

Every normed space is convex. In fact, we will show that the space $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\infty$, defined in this work, is convex, as stated in the following result.

Corollary 1. *The sequence space* $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\infty$ *is convex.*

P r o o f. Let (t_α), $(\varpi_\alpha) \in [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\infty$. Then, from the definition of the space, we can write

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p(t_\alpha))|}{v_t}\right) < \infty \quad \text{for some} \quad v_t > 0,$$

and

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p(\varpi_\alpha))|}{v_\varpi}\right) < \infty \quad \text{for some} \quad v_\varpi > 0.$$

For $\varrho = \mu t + (1-\mu)\varpi$, we have to show that

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p(\mu t_\alpha + (1-\mu)\varpi_\alpha))|}{v_\varrho}\right) < \infty \quad \text{for some} \quad v_\varrho > 0.$$

Since $\mathcal{U}$ is a convex function, we have

$$\mathcal{U}\left(\frac{|f(\Delta^p(\mu t_\alpha + (1-\mu)\varpi_\alpha))|}{v_\varrho}\right) \le \mu\mathcal{U}\left(\frac{|f(\Delta^p(t_\alpha))|}{v_t}\right) + (1-\mu)\mathcal{U}\left(\frac{|f(\Delta^p(\varpi_\alpha))|}{v_\varpi}\right),$$

where $v_\varrho = \mu v_t + (1-\mu)v_\varpi$.

Now, taking the limit as $s \to \infty$, we have

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p \varrho_\alpha)|}{v_\varrho}\right) \le \mu \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p(t_\alpha))|}{v_t}\right) + (1-\mu) \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}\left(\frac{|f(\Delta^p \varpi_\alpha)|}{v_\varpi}\right).$$

Therefore,

$$\varrho = \mu t + (1-\mu)\varpi \in [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\infty.$$

Hence, the space $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\infty$ is convex. □

Theorem 3. *Let* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *be Orlicz functions satisfying the* Δ_2*-condition. Then the following strict inclusions hold*:

(i) $[\mathcal{D}_\lambda^w, \mathcal{U}_1, \Delta^p]_\mathcal{K} \subseteq [\mathcal{D}_\lambda^w, \mathcal{U}_2 \cdot \mathcal{U}_1, \Delta^p]_\mathcal{K}$;

(ii) $[\mathcal{D}_\lambda^w, \mathcal{U}_1, \Delta^p]_\mathcal{K} \cap [\mathcal{D}_\lambda^w, \mathcal{U}_2, \Delta^p]_\mathcal{K} \subseteq [\mathcal{D}_\lambda^w, \mathcal{U}_1 + \mathcal{U}_2, \Delta^p]_\mathcal{K}$, *where* $\mathcal{K} = 0, 1,$ *and* ∞.

P r o o f. We first prove the statement in the case $\mathcal{K} = 0$. The same methods can then be applied to the remaining cases.

(i) Let $(t_\alpha) \in [\mathcal{D}_\lambda^w, \mathcal{U}_1, \Delta^p]_0$. Then there exists $v > 0$ such that

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} \mathcal{U}_1\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) = 0.$$

Let $0 < \sigma < 1$ and $0 < \beta < 1$ be such that $\mathcal{U}_2(m) < \sigma$ for $0 \le m < \beta$.

Define

$$\varpi_\alpha = \mathcal{U}_1\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right).$$

Consider the expression

$$\sum_{\alpha\in I_s} \mathcal{U}_2(\varpi_\alpha) = \sum_1 \mathcal{U}_2(\varpi_\alpha) + \sum_2 \mathcal{U}_2(\varpi_\alpha),$$

where the first summation runs over terms with $\varpi_\alpha > \beta$ and the second summation includes terms with $\varpi_\alpha \le \beta$. Since

$$\frac{1}{\lambda_s}\sum_{1}\mathcal{U}_2\left(\varpi_\alpha\right) < \mathcal{U}_2(2)\frac{1}{\lambda_s}\sum_{1}\left(\varpi_\alpha\right) \tag{2.2}$$

for $\varpi_\alpha > \beta$, we have

$$\varpi_\alpha < 1 + \frac{\varpi_\alpha}{\beta}.$$

Since $\mathcal{U}_2$ is nondecreasing and convex, it follows that

$$\mathcal{U}_2\left(\varpi_\alpha\right) < \frac{1}{2}\mathcal{U}_2(2) + \frac{1}{2}\mathcal{U}_2\left(\frac{2\varpi_\alpha}{\beta}\right).$$

Since $\mathcal{U}_2$ satisfies the Δ_2-conditions, we have

$$\mathcal{U}_2\left(\varpi_\alpha\right) = T\frac{\varpi_\alpha}{\beta}\mathcal{U}_2(2).$$

Hence,

$$\frac{1}{\lambda_s}\sum_{2}\mathcal{U}_2\left(\varpi_\alpha\right) \le \max\left(1, T\beta^{-1}\mathcal{U}_2(2)\right)\frac{1}{\lambda_s}\sum_{2}\varpi_\alpha. \tag{2.3}$$

Taking the limit as $s \to \infty$, from (2.2) and (2.3), we obtain

$$(t_\alpha) \in \left[\mathcal{D}_\lambda^w, \mathcal{U}_2 \cdot \mathcal{U}_1, \Delta^p\right]_0.$$

A similar approach can be applied to demonstrate the result for the remaining cases.

(ii) The proof is standard and is omitted. □

By taking $\mathcal{U}_1(t) = t$ and $\mathcal{U}_2 = \mathcal{U}(t)$ in Theorem 3 (i), we obtain the following particular case.

Corollary 2. *The inclusion* $\left[\mathcal{D}_\lambda^w, \Delta^p\right]_0 \subseteq \left[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p\right]_0$ *is strict.*

Here, the space $\left[\mathcal{D}_\lambda^w, \Delta^p\right]_0$ is defined by

$$\left[\mathcal{D}_\lambda^w, \Delta^p\right]_0 = \left\{t = (t_\alpha) : \lim_{s\to\infty}\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\left(\frac{|f(\Delta^p t_\alpha)|}{v}\right) = 0 \text{ for some } v > 0\right\}.$$

Theorem 4. *Let* $p \ge 1$ *and* $\mathcal{K} = 1, 2, \infty$. *Then, the inclusion* $\left[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^{p-1}\right]_{\mathcal{K}} \subset \left[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p\right]_{\mathcal{K}}$ *is strict. In general,* $\left[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^i\right]_{\mathcal{K}} \subset \left[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p\right]_{\mathcal{K}}$ *for* $i = 0, 1, 2, \ldots, p-1$.

Proof. Let $(t_\alpha) \in \left[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^{p-1}\right]_0$. Then we have

$$\lim_{s\to\infty}\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\Delta^{p-1}t_\alpha)|}{v}\right) = 0 \quad \text{for some } v > 0. \tag{2.4}$$

Since $\mathcal{U}$ is both convex and nondecreasing, we can deduce that

$$\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{2v}\right) = \frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\Delta^{p-1}t_\alpha - \Delta^{p-1}t_{\alpha+1})|}{2v}\right)$$

$$\le \left(\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\Delta^{p-1}t_\alpha)|}{v}\right) - \frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\Delta^{p-1}t_{\alpha+1})|}{v}\right)\right).$$

As $s \to \infty$, we have

$$\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\Delta^p t_\alpha)|}{2v}\right) = 0$$

by (2.4), which implies $(t_\alpha) \in \left[\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^{p-1}\right]_0$.

The other cases will follow by a similar approach. Using induction, we can establish that

$$\left[\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^i\right]_\mathcal{K} \subset [\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_\mathcal{K}$$

and $i = 0, 1, \ldots, p-1$. □

The following example directly illustrates this inclusion.

Example 1. Let $\lambda_s = (s)$ be a sequence and $\mathcal{U}(t) = t$. Consider the sequence $(t_\alpha) = \left(\alpha^{p-1}\right)$. Then

$$\Delta^p t_\alpha = 0, \quad \Delta^{p-1} t_\alpha = (-1)^{p-1}(p-1)!$$

for all $\alpha \in \mathbb{N}$. Therefore, $(t_\alpha) \in [\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_0$ but $(t_\alpha) \notin \left[\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^{p-1}\right]_0$.

Theorem 5. *The space $[\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_\mathcal{K}$, where $\mathcal{K} = 0, 1, \infty$, is generally not solid. The spaces $[\mathcal{D}^w_\lambda, \mathcal{U}]_0$ and $[\mathcal{D}^w_\lambda, \mathcal{U}]_\infty$ are solid.*

P r o o f. Let $(t_\alpha) \in [\mathcal{D}^w_\lambda, \mathcal{U}]_0$. Then there exists $v > 0$ such that

$$\lim_{s\to\infty}\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f\,(t_\alpha)|}{v}\right) = 0.$$

Let (δ_α) be a sequence of scalars such that $|\delta_\alpha| \leq 1$. Then, for each s, we can write

$$\begin{aligned}\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(\delta_\alpha t_\alpha)|}{v}\right) &\leq \frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f(t_\alpha)|}{v}\right)\\ \Rightarrow \lim_{s\to\infty}\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\mathcal{U}\left(\frac{|f\,(\delta_\alpha t_\alpha)|}{v}\right) &= 0\\ \Rightarrow (\delta_\alpha t_\alpha) &\in [\mathcal{D}^w_\lambda, \mathcal{U}]_0\,.\end{aligned} \tag{2.5}$$

From the inequality presented in (2.5), it follows that $[\mathcal{D}^w_\lambda, \mathcal{U}]_\infty$ is solid. □

To demonstrate that the spaces $[\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_1$ and $[\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_\infty$ are generally not solid, we provide the following example.

Example 2. Consider the function $f(t) = t$ for all $t \in \mathbb{R}$. Let $X = \mathbb{R}$ with $p = 1$. Let the sequence (t_α) be defined by $t_\alpha = \alpha$ for all $\alpha \in \mathbb{N}$. Let $\mathcal{U}(t) = t^r$ with $r \geq 1$, and $\lambda_s = (s)$. Then $(t_\alpha) \in [\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_1$ and $(t_\alpha) \in [\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_\infty$. Let $(\gamma_\alpha) = ((-1)^\alpha)$. Then $(\gamma_\alpha t_\alpha) \notin [\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_1$ and $(\gamma_\alpha t_\alpha) \notin [\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_\infty$.

The following example illustrates that $[\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_0$ is generally not solid.

Example 3. Let $X = \mathbb{R}$ and consider the function $f(t) = t$ for all $t \in \mathbb{R}$. Let $p = 1$. Consider the sequence (t_α) defined by $t_\alpha = 1$ for all $\alpha \in \mathbb{N}$. Assume $\mathcal{U}(t) = t^r$ with $r = 2$ and $\lambda_s = (s)$. Let $(\gamma_\alpha) = ((-1)^\alpha)$ for all $\alpha \in \mathbb{N}$. Then $(\gamma_\alpha t_\alpha) \notin [\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_0$. Thus, the set $[\mathcal{D}^w_\lambda, \mathcal{U}, \Delta^p]_0$ is not solid.

The following result is a consequence of Lemma 1 and Theorem 5.

Corollary 3. *The spaces* $[\mathcal{D}_\lambda^w, \mathcal{U}]_0$ *and* $[\mathcal{D}_\lambda^w, \mathcal{U}]_\infty$ *are monotone.*

Remark 1. The space $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$ is not convergence free.

P r o o f. The following example clearly illustrates this point. □

Example 4. Let $p = 1$, $\mathcal{U} = t$ and consider the sequence $\lambda_s = (s)$. Consider the sequence (t_α) defined by

$$t_\alpha = \frac{1}{2}\left(1 - (-1)^\alpha\right).$$

Then $(t_\alpha) \in [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$.

Now consider the sequence (ϖ_α) defined by

$$\varpi_\alpha = \begin{cases} \alpha & \text{if} \quad \alpha \text{ is odd,} \\ \bar{\theta} & \text{if} \quad \alpha \text{ is even.} \end{cases}$$

Then $(\varpi_\alpha) \notin [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$.

Remark 2. The spaces $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\mathcal{K}$, where $\mathcal{K} = 0, 1, \infty$, are generally not symmetric. The following example illustrates this fact.

Example 5. Let $p = 1$, $X = \mathbb{R}$, and consider the function $f(t) = t$ for all $t \in \mathbb{R}$. Let $\mathcal{U}(t) = t$ and $\lambda_s = (s)$. Consider the sequence (t_α) defined by $t_\alpha = \alpha$ for all $\alpha \in \mathbb{N}$. Then $(t_\alpha) \in [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_0$. Now, rearrange the sequence (t_α) to obtain the sequence (ϖ_α) defined by

$$\varpi_\alpha = (t_1, t_2, t_4, t_3, t_9, \ldots).$$

Then $(\varpi_\alpha) \notin [\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\mathcal{K}$, where $\mathcal{K} = 0, 1, \infty$. Hence, the spaces $[\mathcal{D}_\lambda^w, \mathcal{U}, \Delta^p]_\mathcal{K}$, where $\mathcal{K} = 0, 1, \infty$, are generally not symmetric.

3. Conclusion

In this paper, we introduced and analyzed the concept of difference λ-weak convergence for sequences defined by an Orlicz function. Our study provided an in-depth examination of the algebraic and topological properties of these sequences, offering a foundational perspective on their structure and behavior. We also established key inclusion relationships between these newly defined spaces and existing sequence spaces, thereby enhancing the overall framework of sequence space theory. Our results contribute to the broader field of functional analysis, particularly in the context of sequence spaces and Orlicz functions, and open new avenues for future research.

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A STUDY ON PERFECT ITALIAN DOMINATION OF GRAPHS AND THEIR COMPLEMENTS

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Abstract: Perfect Italian Domination is a type of vertex domination which can also be viewed as a graph labelling problem. The vertices of a graph G are labelled by $0, 1$ or 2 in such a way that a vertex labelled 0 should have a neighbourhood with exactly two vertices in it labelled 1 each or with exactly one vertex labelled 2. The remaining vertices in the neighbourhood of the vertex labelled 0 should be all 0's. The minimum sum of all labels of the graph G satisfying these conditions is called its Perfect Italian domination number. We study the behaviour of graph complements and how the Perfect Italian Domination number varies between a graph and its complement. The *Nordhaus–Gaddum type* inequalities in the Perfect Italian Domination number are also discussed.

Keywords: Perfect Italian domination, Graph complement, Nordhaus–Gaddum type inequalities.

1. Introduction

Analysing how graph properties vary across each graph family is always fascinating. That is the manner in which a graph's structural characteristics, such as its number of vertices, edges, connectivity, symmetry, etc., affect graph parameters such as its chromatic number, clique number, domination number, etc. The variation of a graph parameter between a graph and its complement has also been researched since the seminal work of Nordhaus and Gaddum [7]. On n-vertex graphs, they determined an upper and lower bound for the sum (and product) of chromatic numbers of a graph and its complement. The problems that include determining the upper and lower bounds of the sum or product of certain graph properties are referred to as *Nordhaus–Gaddum type* studies.

Perfect Italian Domination is a domination concept defined by T.W. Haynes and M.A. Henning. It can be viewed as a vertex labelling problem, where vertices are labelled by $0, 1$ or by 2. A vertex in a Perfect Italian Dominated (PID) graph is labelled 0 if and only if it is adjacent to two vertices labelled 1 each or one vertex labelled 2, and the remaining vertices in its neighbourhood are labelled 0. The sum of the vertex labels on a graph G that satisfies the PID condition is determined and the term *PID number* of G denoted as $\gamma_I^p(G)$ refers to the smallest sum that may be computed for a graph G [5].

The graph $\overline{G}$ is called the complement of a graph G, when two vertices are neighbours in G if and only if they are not neighbours in $\overline{G}$. In this paper, we examine the variation in the Perfect Italian Domination (PID) number of a graph and its complement. We find some *Nordhaus–Gaddum type* inequalities of Perfect Italian Domination number and, also characterise some graph classes which attain the upper bound and lower bound. We have also considered a few graph classes whose PID numbers are found and are compared with the PID numbers of their complements.

2. PID on graph complements and Nordhaus–Gaddum inequalities

The Perfect Italian domination number of any graph G is at least two and is at most its order. Hence, for a graph G of order n,

$$4 \leq \gamma_I^p(G) + \gamma_I^p(\overline{G}) \leq 2n.$$

In this paper, we prove that these bounds are tight by constructing classes of graphs. The gap between the bounds is shortened when a few restrictions are made to the graphs considered. We consider a few cases where the upper bound is small. We arrive at a conclusion that if G is any graph such that $\gamma_I^p(G) = n$, then $\gamma_I^p(\overline{G}) \geq 5$ or equal to 2. If G is a connected graph, then $\gamma_I^p(\overline{G}) \geq 5$. We have also determined the PID number of certain graph cases and their complements. This helps in the study of determining the criteria that the graph must satisfy in order to maximise or reduce a graph PID value. This study can help us find extremal graphs which is an important area of study in graph theory. Some of these will also would lead to optimal solutions.

We examine graphs that correspond to a specific PID number and analyze the PID number of its complement. We will start by considering graphs G with $\gamma_I^p(G) = 2, 3, 4$ and later $\gamma_I^p(G) \geq 5$.

The only possible graphs of order $n = 2$ are $2K_1$ and K_2. We know that PID number of each of them is 2 and they are complement to each other. When $n \geq 3$, $\gamma_I^p(G) = 2$ if and only if there is a universal vertex or if there exist two non adjacent vertices adjacent to all the remaining vertices of G. A universal vertex of G forms an isolated vertex in $\overline{G}$. Similarly, the non adjacent vertices adjacent to all the remaining vertices in G form a K_2 component. Hence when $n \geq 3$ if $\gamma_I^p(G) = 2$, then $\gamma_I^p(\overline{G})$ is always greater than or equal to 3.

Let G be any graph of order n and $\gamma_I^p(G) = 2$. Then $\overline{G}$ is a disconnected graph with

$$2 \leq \gamma_I^p(G) \leq n.$$

The following realization problem shows that for any integer $2 \leq a \leq n$, we can find a graph such that its PID number is 2 whereas the PID number of its complement is a.

Theorem 1. *For any $a \in \mathbb{N} - \{1\}$, there exists a graph G such that $\gamma_I^p(G) = 2$ and $\gamma_I^p(\overline{G}) = a$.*

Proof. Let G be a graph obtained from the join of a path complement graph- $\overline{P}_{2a-3}$ and K_1, $(\overline{P}_{2a-3} + K_1)$, where (see [8])

$$\gamma_I^p(\overline{P}_{2a-3} + K_1) = 2.$$

Then $\overline{G}$ will be $P_{2a-3} \cup K_1$. For any path P_n, (see [6])

$$\gamma_I^p(P_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Hence,

$$\gamma_I^p(\overline{G}) = \gamma_I^p(P_{2a-1} \cup K_1) = \left\lceil \frac{2a-3+1}{2} + 1 \right\rceil = a.$$

□

Proposition 1. *Let G be a graph such that $\gamma_I^p(G) = 3$. Then $\gamma_I^p(\overline{G}) \leq 6$.*

P r o o f. A graph G with $\gamma_I^p(G) > 2$ has $\gamma_I^p(G) = 3$ if and only if $\overline{G}$ has a perfect dominating set of size 3 [6]. This implies that $\gamma_I^p(\overline{G}) \leq 6$. □

From the above results it is clear that $\gamma_I^p(G) = 3$ and $\gamma_I^p(\overline{G}) = 2$ if and only if G is a disconnected graph.

Corollary 1. *Let G be a connected graph such that $\gamma_I^p(G) = 3$. Then $3 \leq \gamma_I^p(\overline{G}) \leq 6$.*

Proposition 2. *Let G be a graph such that $\gamma_I^p(G) = 4$. Then $\gamma_I^p(\overline{G}) \leq 4$.*

P r o o f. If G is a graph such that $\gamma_I^p(G) = 4$, then either of the following is true.

1) There exists a vertex set S in G consisting of four vertices $\{u_i\}$ for $i = 1, 2, 3, 4$ such that the remaining vertices in G are adjacent to exactly any two vertices of the set S.
2) There exists a set S in G consisting of two vertices, u_1, u_2 such that the remaining vertices in G are adjacent to exactly any one vertex of the set S.
3) There exists a set S in G consisting of three vertices, u_1, u_2, u_3 such that any other vertex, v belonging to G satisfies one of the following:
   (a) $N(v) \cap S = \{u_1\}$
   (b) $N(v) \cap S = \{u_2, u_3\}$.

If G satisfies 1), then the vertices belonging to $N(u_i) \cap N(u_j)$ in G will not be adjacent to u_i, u_j in $\overline{G}$, but will be adjacent to u_k where $k \neq i, j$. Hence labelling all the u_i's by 1 and the remaining vertices by 0 satisfies the PID condition. Thus, $\gamma_I^p(\overline{G}) \leq 4$.

If the graph G satisfies 2), then the vertices adjacent to $u_1 \in G$ are not adjacent to $u_1 \in \overline{G}$ but will be adjacent to u_2. Similar is the case of neighbours of u_2. Hence labelling u_1, u_2 by 2 and the remaining vertices by 0 satisfies the PID condition, i.e., $\gamma_I^p(\overline{G}) \leq 4$.

If G satisfies 3), then the vertices belonging to $N(u_1)$ in G are not adjacent to u_1 but are adjacent to u_2, u_3 in $\overline{G}$. Similarly the vertices belonging to $N(u_2) \cup N(u_3)$ are not adjacent to u_2, u_3 but are adjacent to u_1. Hence labelling u_1 by 2 and u_2, u_3 by 1 gives a PID labelling, i.e., $\gamma_I^p(\overline{G}) \leq 4$. □

Corollary 2. *Let G be a connected graph such that $\gamma_I^p(G) = 4$. Then $\gamma_I^p(\overline{G}) = 3$ or 4.*

If G is a connected graph with a PID number greater than or equal to 7, then from the above results, PID number of $\overline{G}$ cannot be $2, 3$ or 4. This implies that PID number of $\overline{G}$ is greater than or equal to 5 but less than or equal to the order of G.

The following realisation problem shows that the upper bound is tight.

Theorem 2. *For any $k \geq 5$, there exists a graph G of order n such that $\gamma_I^p(G) = k$ and $\gamma_I^p(\overline{G}) = n$.*

P r o o f. Let G be a graph constructed by the following steps:

Take k copies of P_4 where k is any integer greater than or equal to 5. Label each path as $Q_1, Q_2, ..., Q_k$. Let us consider a K_k whose vertices are $u_1, u_2, ..., u_k$. Then make each vertex of the path Q_i adjacent to u_i, u_{i+1} where $i = 1, 2, ..., (k-1)$. The vertices of Q_k are adjacent to u_1 and u_k. An illustration of the construction when $k = 5$ is given in Figure 1. This is a connected graph of order $5k$.

Since each vertex of the path P_i is adjacent to exactly two vertices among the $u_i's$, labelling all the $u_i's$ 1 and the vertices belonging to the paths 0 gives a PID labelling where

$$\gamma_I^p(G) \leq k \longrightarrow (a).$$

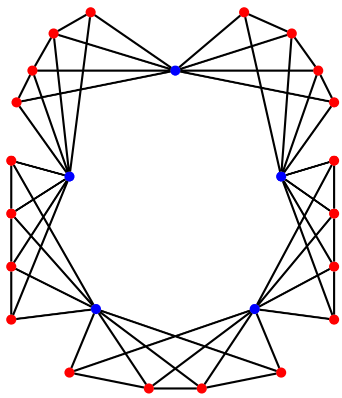

Figure 1. An illustration of construction of Graph G, where $k = 5$.

Obviously, degree of u_i is 8 which coincides with $\Delta(G)$. But from [3], we have

$$\gamma_I^p(G) \geq \gamma_I(G) \geq \frac{2(5k)}{\Delta(G)+2}, \quad \text{i. e.,} \quad \gamma_I^p(G) \geq k \longrightarrow (b).$$

From (a) and (b), $\gamma_I^p(G) = k$.

Since $\{u_1, u_2 ... u_k\}$ is a set of independent vertices in G, they induce a clique K_k in $\overline{G}$. As P_4 is a self-complementary graph, each Q_i remains the same in $\overline{G}$. Each vertex u_i is adjacent to the vertices of all the paths except P_{i-1}, P_i $j \neq i-1, i$ and $i, j = 2, 3, \ldots k$. The vertex u_1 is adjacent to the vertices of all the paths except P_k and P_1. Each vertex of the path P_i will be adjacent to all the vertices of the paths P_j where $j \neq i$ and $i, j = 1, 2, 3 \ldots k$.

Since G and $\overline{G}$ are connected graphs, $\gamma_I^p(\overline{G}) > 2$. Let us consider the following cases of possible labellings for $\overline{G}$:

1. Let a vertex v_i belonging to a path Q_s be labelled 0. Then, at most two vertices in its neighbourhood, say x, y, are non-zero labelled and the remaining vertices in its neighbourhood are zero labelled. Since each vertex in a path is of degree at least $5k-5$, there exist two vertices among the u_i's and at most two vertices in the path Q_s that are non-adjacent to the vertex v_i. If any one among this, say z is non zero labelled, then there exists at least one vertex on a path Q_i labelled 0 adjacent to x, y and z. This violates the perfect Italian domination condition. This implies that no vertex among the non adjacent vertices of v_i can be non-zero labelled. Hence, all remaining vertices in the graph are labelled 0. This contradicts $\gamma_I^p(\overline{G}) > 2$. Hence, no vertex on the path Q_i can be labelled 0 and its non adjacent vertices can be non-zero labelled. The remaining vertices in the graph are labelled 0. Since each vertex in a path is of degree of at least $5k-5$, there exist two vertices among the $u_i's$ and at most 2 vertices in the path Q_s that are non adjacent to the vertex v_i. If any one among this is non zero labelled, then there exists at least one vertex labelled 0 among the paths P_j where $j \neq k$ adjacent to all the vertices not labelled zero. This is a contradiction to the PID condition. Hence no vertex on an induced path P_i of the G can be labelled 0.

2. Each vertex u_i is adjacent to all the vertices of $k-2$ induced paths. From the above case we know that no vertex on an induced path of the graph G is labelled 0. Since $k \geq 5$, this implies that no vertex u_i can be labelled 0.

This shows that no vertex in $\overline{G}$ can be labelled 0. i.e., $\gamma_I^p(\overline{G}) = 5k$, the order of graph G. □

The following is a summary of the results mentioned above.

Remark 1. Let G be a connected graph of order n,

1. If $\gamma_I^p(G) = 3$, then $\gamma_I^p(\overline{G}) \in \{3, 4, 5, 6\}$.
2. If $\gamma_I^p(G) = 4$, then $\gamma_I^p(\overline{G}) \in \{3, 4\}$.
3. If $\gamma_I^p(G) \in \{5, 6\}$, then $\gamma_I^p(\overline{G}) \in \mathbb{N} - \{1, 2, 4\}$.
4. If $\gamma_I^p(G) \geq 7$, then $5 \leq \gamma_I^p(\overline{G}) \leq n$.

Based on the results above, we can deduce the following *Nordhaus–Gaddum type inequalities.*

Remark 2. Let G be a connected graph of order $n \geq 3$ and $\gamma_I^p(G) = 3$. Then,

$$6 \leq \gamma_I^p(G) + \gamma_I^p(\overline{G}) \leq 9, \quad 9 \leq \gamma_I^p(G) \cdot \gamma_I^p(\overline{G}) \leq 18.$$

Remark 3. Let G be a connected graph of order $n \geq 3$ and $\gamma_I^p(G) = 4$. Then,

$$7 \leq \gamma_I^p(G) + \gamma_I^p(\overline{G}) \leq 8, \quad 12 \leq \gamma_I^p(G) \cdot \gamma_I^p(\overline{G}) \leq 16.$$

Remark 4. Let G be a connected graph of order $n \geq 3$ and $7 \leq \gamma_I^p(G) \leq n$. Then,

$$12 \leq \gamma_I^p(G) + \gamma_I^p(\overline{G}) \leq 2n, \quad 35 \leq \gamma_I^p(G) \cdot \gamma_I^p(\overline{G}) \leq n^2.$$

Remark 5. Let G and $\overline{G}$ be connected graphs of order n. Then

$$6 \leq \gamma_I^p(G) + \gamma_I^p(\overline{G}) \leq 2n, \quad 6 \leq \gamma_I^p(G) \cdot \gamma_I^p(\overline{G}) \leq n^2.$$

3. PID of some graph classes and their complements

A vertex in a graph G is said to be dominated if it is either belonging to or is adjacent to a vertex belonging to the Dominating set S of G. A Perfect Dominating set, S_p of a graph G is a set of vertices such that any vertex of G not belonging to this set is dominated by exactly one vertex from S_p. The least number of vertices that can exist in such a set S_p is called Perfect Domination number $\gamma_p(G)$. [4].

Theorem 3 [2]. *For a path P_n, the perfect domination number,*

$$\gamma_p(P_n) = \begin{cases} \frac{n}{3}, & n \equiv 0 \pmod 3, \\ \frac{n+1}{3}, & n \equiv 2 \pmod 3, \\ \frac{n+2}{3}, & n \equiv 1 \pmod 3. \end{cases}$$

Theorem 4 [1]. *For a cycle C_n, the perfect domination number,*

$$\gamma_p(C_n) = \begin{cases} \frac{n}{3}, & n \equiv 0 \pmod 3, \\ \left\lceil \frac{n}{3} \right\rceil, & n \equiv 1 \pmod 3, \\ \left\lfloor \frac{n}{3} \right\rfloor + 2, & n \equiv 2 \pmod 3. \end{cases}$$

Theorem 5 [6]. *Let G be a connected graph with $\gamma_I^p(G) > 2$. Then $\gamma_I^p(G) = 3$ if and only if $\overline{G}$ has a perfect dominating set of size 3.*

Theorem 6. *For a path P_n, $\gamma_I^p(P_n) = \lceil (n+1)/2 \rceil$ and*

$$\gamma_I^p(\overline{P}_n) = \begin{cases} 1, & n = 1, \\ 2, & n = 2, \\ 3, & 3 \leq n \leq 9, \\ n, & \textit{otherwise.} \end{cases}$$

Proof. For a path P_n, $\gamma_I^p(P_n) = \lceil (n+1)/2 \rceil$ [6].

1. For $n \geq 10$: The two end vertices of P_n are adjacent vertices of degree $(n-2)$ in $\overline{P}_n$ and the remaining vertices which are of degree 2 in P_n are of degree $n-3$ in $\overline{P}_n$. This implies that $\gamma_I^p(\overline{P}_n) > 2$.
   (a) If a vertex of degree $(n-2)$, say u_i, is labelled 0, then u_{i+1} can be non-zero labelled and a vertex x in the neighbourhood of u_i is labelled 2 (or two vertices x, y in its neighbourhood are labelled 1 each). This implies that all the remaining vertices are labelled 0. Since $n \geq 10$, and vertices are of degree at least $n-3$ there exists a zero labelled vertex adjacent to the vertices x, y, u_{i+1}. This is a contradiction to the PID condition. Hence u_{i+1} is not labelled zero but then this is a contradiction to $\gamma_I^p(\overline{P}_n) > 2$.
   (b) If a vertex of degree $(n-3)$, say u_i, is labelled 0, then at most two of its adjacent vertices say a, b are non zero labelled and at least $n-5$ vertices are labelled 0. In the previous case we proved that the vertices of degree $(n-2)$ cannot be labelled 0, since $n \geq 10$ there exists at least one vertex of degree $(n-2)$ in the neighbourhood of u_i. This implies that at least one among a, b say a is of degree $(n-2)$. Let u_{i-1}, u_{i+1} be the vertices not adjacent to u_i and if one among them say u_{i-1} is non zero labelled, then u_{i-1} is not adjacent to u_i and at most one more vertex. a is not adjacent to one vertex and b is not adjacent to at most two vertices. This implies that there exists at least $n-5-(1+1+2) = n-9$ vertices labelled 0 adjacent to a, b and u_{i-1}. This is a contradiction to the perfect Italian domination condition. This implies that neither u_{i-1} nor u_{i+1} can be non-zero labelled.
   This is a contradiction to $\gamma_I^p(\overline{P}_n) > 2$. Hence no vertex of degree $(n-3)$ can be labelled 0.

   Thus no vertex in $\overline{P}_n$ where $n \geq 10$ can be labelled by 0. This implies that $\gamma_I^p(\overline{P}_n) = n$.
2. For $n = 1$, the complement is a K_1. Hence $\gamma_I^p(\overline{P}_1) = 1$.
3. For $n = 2$, $\overline{P}_2$ is two isolated vertices and $\gamma_I^p(\overline{P}_2) = 2$.
4. Assume $3 \leq n \leq 9$. The graph $\overline{P}_3$ is $K_1 \cup K_2$ and the PID number is 3. The graph $\overline{P}_4$ is P_4 and the PID number is 3. Let $u_1u_2...u_5$ be a P_5. Then $\{u_1, u_4, u_5\}$ is a perfect dominating set of size 3 and from the Theorem 5 we can conclude that $\gamma_I^p(\overline{P}_5) = 3$. Similarly the vertices $\{u_2, u_4, u_5\}$ is a perfect dominating set of a P_6, $u_1, u_2...u_6$. This implies that $\gamma_I^p(\overline{P}_6) = 3$ (from Theorem 5). For $n = 7, 8, 9$, $\gamma_p(P_n) = 3$ (from Theorem: 3), this implies that $\gamma_I^p(\overline{P}_n) = 3$ (from Theorem 5). Hence for $3 \leq n \leq 9$, $\gamma_I^p(\overline{P}_n) = 3$.

□

Theorem 7. *For a cycle C_n, $\gamma_I^p(C_n) = \lceil n/2 \rceil$ and*

$$\gamma_I^p(\overline{C}_n) = \begin{cases} 3, & n = 3, 5, 7, 9, \\ 4, & n = 4, 6, 8, \\ n, & \textit{otherwise.} \end{cases}$$

P r o o f. For a cycle C_n, $\gamma_I^p(C_n) = \lceil n/2 \rceil$ [6]. Since each vertex in C_n is of degree 2, the vertices of $\overline{C}_n$ are of degree $n-3$. This implies $\overline{C}_n$ is a $(n-3)$ regular graph and $\gamma_I^p(\overline{C}_n) > 2$.

1. Assume $n \geq 10$. If a vertex, v is labelled 0, then v is adjacent to $n-3$ vertices, say $u_1, u_2, u_3 ... u_{n-3}$, and is not adjacent to w_1, w_2. Among the $u_i's$ two vertices are labelled 1, say u_1, u_2 (or one vertex u_1 is labelled 2) and the remaining $(n-5)$ (or $(n-4)$) $u_i's$ are labelled 0. The vertex v is not adjacent to w_1, w_2, as $\gamma_I^p(\overline{C}_n) > 2$, at least one of them, say w_1, should be non-zero labelled.
   (a) If both w_1, w_2 are non-zero labelled, then at least $(n-6)$ zero labelled vertices are adjacent to each of them. Vertices u_1, u_2 are adjacent to at least $n-7$ vertices. Since $n \geq 10$, there exists at least one vertex adjacent to three non-zero labelled vertices. This is a contradiction to the PID condition.
   (b) If w_1 is non zero labelled and w_2 is zero labelled, then w_2 is adjacent to at least $n-5$ zero labelled vertices (as w_1 should be adjacent to w_2, it cannot be adjacent to one of the u_1, u_2, say u_2.) This implies that w_1 is adjacent to at least $n-6$ zero labelled vertices, u_1 is adjacent to $n-7$ vertices labelled 0 and u_2 is adjacent to $n-6$ zero labelled vertices. This means that there exists at least one zero labelled vertex adjacent to all the three non-zero labelled vertices. This is a contradiction to the PID condition.

   Thus no vertex in $\overline{C}_n$ can be labelled 0.
2. Assume $n = 3, 5, 7, 9$. The graph $\overline{C}_3$ is $3K_1$ and the PID number is 3. Perfect domination number of cycles C_n, where $n = 5, 7, 9$ is 3 (from the Theorem 4). This implies that $\gamma_I^p(\overline{C}_n) = 3$ (from the Theorem 5).
3. Assume $n = 4, 6, 8$. The graph $\overline{C}_4$ is $2K_2$ and the PID number is 4. When $\gamma_p(C_6) = 2$, it cannot have a perfect dominating set of size 3. This implies that $\gamma_I^p(\overline{C}_6) \neq 3$. Hence, $\gamma_p(C_8) = 4 \implies \gamma_I^p(\overline{C}_8) \neq 3$ (from the Theorems 4, 5). The Fig. 2 shows a PID labelling with γ_I^p value equals to 4. Hence, for $n = 4, 6, 8$, $\gamma_I^p(\overline{C}_n) = 4$.

□

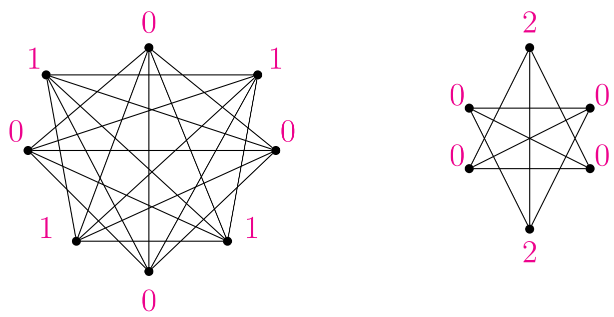

Figure 2. PID labelling of $\overline{C}_8, \overline{C}_6$.

Theorem 8. *Let G be a connected graph of order $n/2$. Then,*

$$\gamma_I^p(\overline{G \circ K_1}) = \begin{cases} 3, & G \cong C_3 \text{ or } P_3, \\ n, & \text{otherwise.} \end{cases}$$

P r o o f. Let the vertices of G be $u_1, u_2 ... u_{n/2}$ and the corresponding $K_1's$ be $v_1, v_2 ... v_{n/2}$. The $v_i's$ form a clique $K_{n/2}$ and each of these $v_i's$ will be adjacent to all the $u_j's$ such that $j \neq i$ for $i, j = 1, 2, 3, ..., n/2$.

Since G is a connected graph, $G \circ K_1$ has neither an isolated vertex nor a K_2. This implies that there exists neither a universal vertex nor two non-adjacent vertices adjacent to all the remaining vertices in $\overline{G \circ K_1}$. Thus, $\gamma_I^p(\overline{G \circ K_1}) > 2$ and degree of each vertex v_i belonging to the clique $K_{n/2}$ is $(n-1)$.

1. Assume any connected graph $G \ncong C_3$ or P_3, i.e., $n/2 \geq 4$.
   (a) If any vertex belonging to the clique $K_{n/2}$, say v_1, is labelled 0, then u_1 which is not adjacent to v_1 can be non-zero labelled and two vertices belonging to the neighbourhood of v_1 are labelled 1 each (or a vertex is labelled 2). This implies that all the remaining vertices of the graph is labelled 0. Since $n/2 \geq 4$, there exists a vertex belonging to the clique adjacent to all the three non-zero labelled vertices. This violates the PID condition, i.e., u_1 cannot be non-zero labelled. But this is a contradiction to $\gamma_I^p(\overline{G \circ K_1}) > 2$.
   (b) If a vertex u_i belonging to G is labelled 0, then it is adjacent to at least $n/2-1$ vertices belonging to the clique. From the above case it is clear that no vertex of K_k can be labelled 0, i.e., they are all non-zero labelled. A vertex u_i belonging to G is adjacent to at least $n/2-1$ vertices belonging to K_k. Hence, no vertex u_i belonging to G can be labelled 0.

   This implies that no vertex in $\overline{G \circ K_1}$ can be labelled 0. Hence, $\gamma_I^p(\overline{G \circ K_1}) = 2 \times n/2 = n$.
2. Assume $G \cong C_3$ or P_3. Labelling all the three vertices $v_i's$ 1 and all the $u_i's$ 0 gives a PID labelling, i.e., $\gamma_I^p(\overline{G \circ K_1}) \leq 3$. Since $\gamma_I^p(\overline{G \circ K_1}) > 2$, we can conclude that $\gamma_I^p(\overline{G \circ K_1}) = 3$.

□

Remark 6. Let G be a graph with an isolated vertex v. Then $\gamma_I^p(\overline{G \circ K_1}) = 2$ since $v \in G$ and its corresponding pendant vertices in $G \circ K_1$ are non-adjacent vertices of degree $n-2$ in $\overline{G \circ K_1}$.

Remark 7. Let G be a complete bipartite graph. Then $\gamma_I^p(G) = \gamma_I^p(\overline{G}) = 4$.

4. A unique family $\mathcal{G}$ of graphs G

Theorem 9. *For any positive integer $n \geq 20$ there exists a graph G of order n such that G, $\overline{G}$ are both connected and $\gamma_I^p(G) = \gamma_I^p(\overline{G}) = n$.*

P r o o f. Let $\mathcal{G}$ be a collection of graphs G each of order n. Then each graph G in $\mathcal{G}$ is constructed as follows.

Construction of the graph G in $\mathcal{G}$. Let $\{v_1, v_2, ...v_{n/2}\}$, $\{u_1, u_2, ...u_{n/2}\}$ be the vertices of two paths $P_{n/2}$ each of order $n/2$ and $P_{n/2} + P_{n/2}$ be the graph obtained by taking join of these two paths. Then G is a graph of order n obtained by removing the edge v_1u_1 from $P_{n/2} + P_{n/2}$.

Any vertex in G is of degree $n/2+2$, $n/2+1$ or $n/2$. This implies that there exists no universal vertex or two non-adjacent vertices of degree $n-2$. Hence $\gamma_I^p(G) > 2$. Let $A = \{u_1, u_2, ...u_{n/2}\}$ and $B = \{v_1, v_2, ...v_{n/2}\}$. Then the following are the possible labellings for the vertices of the graph G.

1. If two vertices belonging to the set A are labelled 1 each or one vertex in the set A is labelled 2, then labelling a vertex belonging to the set A makes all the vertices belonging to the set B labelled 0. (If the vertex labelled 0 is u_1, then all the vertices in B except v_1.) Since there exist vertices in B which are PI dominated by the non-zero labelled vertices in

A, all the remaining vertices in A should be labelled 0. (Since v_1 is adjacent to v_2 which is zero labelled and is PI dominated by the vertices of A, v_1 is also labelled 0). Similarly, if a vertex in B is labelled 0, then all the remaining vertices in A are labelled 0. (If v_1 is the vertex labelled zero, then all the remaining vertices except u_1 is labelled 0.) There exists at least one vertex x belonging to B adjacent to the zero labelled vertex which implies that x also should be labelled 0 and is PI dominated by the vertices of the set A. Since B is a connected graph, this continues and all the vertices of B are labelled 0. This forces u_1 also is to be labelled 0.

2. Let a vertex x from set A and a vertex y from a set B be labelled 1 each. Then a vertex in the neighbourhood of x and y belonging to the set A or B, is labelled zero forces all the remaining vertices in the other set are to be labelled 0. There exists at least one zero labelled vertex adjacent to the y in B. This implies that all the remaining vertices in A should be labelled 0.

Both the cases are contradictions to $\gamma_I^p(G) > 2$. This implies that no vertex in G is labelled 0. Hence

$$\gamma_I^p(G) = \frac{n}{2} + \frac{n}{2} = n.$$

The complement $\overline{G}$ is $\overline{P}_{n/2} \cup \overline{P}_{n/2}$ with an edge between v_1 and u_1. The vertex v_1 belonging to a path complement is adjacent to vertex u_1 belonging to another path complement. Hence, the adjacency between any two vertices of $\overline{G}$ other than $\{v_1, u_1\}$ is same as its adjacency in $\overline{P}_{n/2}$. This implies that as given in the proof of Theorem 6, if any vertex in the graph is labelled 0, then at most two vertices can only be non-zero labelled and they are labelled 1 each. Since $n \geq 20$ and v_1, u_1 are of degree $n/2 - 1 + 1 = n/2$ each, $\gamma_I^p(\overline{G}) > 2$. This implies that no vertex can be labelled 0 and

$$\gamma_I^p(\overline{G}) = \frac{n}{2} + \frac{n}{2} = n.$$

□

This theorem proves that there exists a family of graphs in which each of them and its corresponding complement are connected as well as have their PID number same as its order. This shows that the upper bound of *Nordhaus–Gaddum inequalities* for the Perfect Italian Domination is tight.

Thus, $\gamma_I^p(G) + \gamma_I^p(\overline{G}) = 2n$ if and only if $\gamma_I^p(G) = \gamma_I^p(\overline{G}) = n$. Since there is no complete characterization of graphs satisfying $\gamma_I^p(G) = n$, characterizing the graphs such that

$$\gamma_I^p(G) + \gamma_I^p(\overline{G}) = 2n$$

remains an open problem.

5. Conclusion

The lower and upper bounds in the Nordhaus–Gaddum type inequalities for the Perfect Italian domination number of an arbitrary graph G are way apart. Hence, particular cases of the graphs are considered to find the Nordhaus–Gaddum type inequalities. We have constructed different graph classes to show that the bounds are tight since there is no complete characterization of graphs satisfying $\gamma_I^p(G) = n$. Thus characterizing the graphs such that $\gamma_I^p(G) + \gamma_I^p(\overline{G}) = 2n$ remains an open problem.

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STABILITY OF GENERAL QUADRATIC EULER–LAGRANGE FUNCTIONAL EQUATIONS IN MODULAR SPACES: A FIXED POINT APPROACH

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Abstract: In this paper, we establish a result on the Hyers–Ulam–Rassias stability of the Euler–Lagrange functional equation. The work presented here is in the framework of modular spaces. We obtain our results by applying a fixed point theorem. Moreover, we do not use the Δ_α-condition of modular spaces in the proofs of our theorems, which introduces additional complications in establishing stability. We also provide some corollaries and an illustrative example. Apart from its main objective of obtaining a stability result, the present paper also demonstrates how fixed point methods are applicable in modular spaces.

Keywords: Hyers–Ulam–Rassias stability, Euler–Lagrange functional equation, Modular spaces, Convexity, Fixed point method.

1. Introduction

In this paper, our main result concerns the stability property of a type of Euler–Lagrange functional equation. This type of equations was introduced by Rassias [18] in 1992. The name is derived from the Euler–Lagrange identity [19] and has several variants [12, 20, 26, 30], but our study is conducted within the framework of modular spaces.

The kind of stability investigated for the functional equation considered here is well-known as Hyers–Ulam–Rassias stability, which is very general and applicable to diverse branches of mathematics [4, 7, 25]. The concept originates from a mathematical question posed by Ulam [27] in 1940, along with its extensions and partial answers provided by Hyers [6] and Rassias [21]. In the most general terms, following Gruber [5], Hyers–Ulam–Rassias stability holds for a mathematical equation if, whenever it approximately satisfies an equation from a certain class, it admits an exact solution close to that approximate one. It involves questions such as whether a given approximately linear equation has an exact linear approximation.

Our framework of study is modular spaces [13, 16, 17, 28]. A modular space is a linear space equipped with a modular function possessing specific properties. Such a function introduces an additional structure on the linear space, thereby broadening its scope. Several studies from different domains of functional analysis have been successfully extended to this structure. References [9, 14] provide the technical details of the modular spaces mentioned above. Functional equations of various kinds have been considered in the investigation of Hyers–Ulam–Rassias stability properties [8, 23, 29]. We study the stability of such equations in modular spaces without assuming the Δ_α-condition, using a fixed point technique. It may be noted that fixed point methods have already been applied to Hyers–Ulam–Rassias stability problems in [2, 24]. Here, we apply this approach to our problems in modular spaces.

2. Preliminaries

If X and Y are assumed to be a real vector space and a Banach space, respectively, then a mapping $f : X \to Y$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad \forall x, y \in X, \tag{2.1}$$

which is known as the quadratic functional equation.

Any solution of (2.1) is called a quadratic mapping. In particular, if $X = Y = R$, the quadratic form $f(x) = ax^2$ is a solution of (2.1).

We consider here a type of Euler–Lagrange functional equation known as the general k-quadratic Euler–Lagrange functional equation:

$$q(kx+y) + q(kx-y) = 2[q(x+y) + q(x-y)] + 2(k^2-2)q(x) - 2q(y), \quad \forall x, y \in X, \tag{2.2}$$

where $k \in \mathbb{N}$, and $q : X \to Y$ is a function from a real vector space X to a Banach space Y.

Here, we recall certain definitions, theorems, and results regarding modular spaces.

Definition 1 [16, 17]. *A generalized functional $m : X \to [0, \infty]$ is called a modular if, for any two elements $x, y \in X$, where X is considered as a vector space over a field $\mathbb{K}$ (in our case $\mathbb{R}$ or $\mathbb{C}$), the following conditions hold*:

(i) $m(x) = 0$ *if and only if* $x = 0$,
(ii) $m(cx) = m(x)$ *for every scalar* c *with* $|c| = 1$,
(iii) $m(x') \le m(x) + m(y)$ *whenever* x' *is a convex combination of* x *and* y,
(iii)' *if* $c_1, c_2 \ge 0$ *and* $c_1 + c_2 = 1$, *then* $m(c_1 x + c_2 y) \le c_1 m(x) + c_2 m(y)$, *and in this case, m is said to be a convex modular.*

Definition 2. *The modular space, denoted by X_m, is defined as*

$$X_m := \{x \in X : m(\alpha x) \to 0 \text{ as } \alpha \to 0\}.$$

Example 1. If $(X, \|\cdot\|)$ is a normed space, then $\|\cdot\|$ is a convex modular on X, but the converse is not necessarily true [15].

Definition 3. *If m is a convex modular, then the norm known as the Luxemburg norm is defined as*

$$\|x\|_m := \inf\left\{\alpha > 0 : m\left(\frac{x}{\alpha}\right) \le 1\right\}.$$

Definition 4. *Consider X_m as a modular space and let $\{x_n\}$ be a sequence in X_m. Then,*

(i) *the sequence $\{x_n\}$ is called m-convergent to a point $x \in X_m$, denoted $x_n \overset{m}{\longrightarrow} x$, if $m(x_n - x) \to 0$ as $n \to \infty$* [10];

(ii) *$\{x_n\}$ is called an m-Cauchy sequence if for any $\epsilon > 0$, $m(x_n - x_p) < \epsilon$ for sufficiently large $n, p \in \mathbb{N}$* [10];

(iii) *a subset $K(\subset X_m)$ is called m-complete if every m-Cauchy sequence in X_m is m-convergent to an element in K* [10].

Note that m-convergence does not imply m-Cauchy since m does not satisfy the triangle inequality. In fact, one can show that this implication holds if and only if m satisfies the Δ_2-condition.

(iv) *The modular m is said to have the Fatou property if $m(x) \le \lim_{n\to\infty} \inf m(x_n)$ whenever the sequence $\{x_n\}$ is m-convergent to x* [10];

(v) *a modular m is said to satisfy the Δ_α-condition if there exists $\kappa \ge 0$ such that $m(\alpha x) \le \kappa m(x)$ for all $x \in X_m$ and $\alpha \in \mathbb{N}$, $\alpha \ge 2$* [3].

Observations.

(i) $m(x) \le \delta\, m((1/\delta)x)$ for all $x \in X_m$, if m is a convex modular and $0 < \delta \le 1$;

(ii) in general, the modular m does not behave like a norm or a metric since it is not subadditive [16]; however, every norm on X is a modular on X.

Definition 5. *Consider a modular space X_m, a nonempty subset $C \subset X_m$, and a mapping $D : C \to C$. The orbit of D at a point $z \in X_m$ is the set*

$$\mathbb{O}(z) := \{z,\, Dz,\, D^2z,\, \dots\}.$$

The quantity

$$\delta_m(z) := \sup\{m(x - y) : x, y \in \mathbb{O}(z)\}$$

is called the orbit diameter of D at z. In particular, if $\delta_m(z) < \infty$, then D has a bounded orbit at z.

Definition 6. *Let the modular m be defined on the vector space X, and let $C \subset X_m$ be nonempty. A function $D : C \to C$ is called m-Lipschitzian if there exists a constant $L \ge 0$ such that*

$$m(D(x) - D(y)) \le L\, m(x - y), \quad \forall x, y \in C.$$

If $L < 1$, then D is called an m-contraction.

Definition 7 [11]. *Let C be a subset of a modular function space X_m. A function $D : C \to C$ is called an m-strict contraction if there exists a constant $\lambda < 1$ such that*

$$m(D(x) - D(y)) \le \lambda\, m(x - y), \quad \forall x, y \in C.$$

Theorem 1 [1] (**The Banach Contraction Mapping Principle in Modular Spaces**). *Assume that X_m is m-complete. Let C be a nonempty m-closed subset of X_m, and let $T : C \to C$ be an m-contraction mapping. Then T has a fixed point z if and only if T has an m-bounded orbit. Moreover, if*

$$m(x - z) < \infty,$$

then $\{T^n(x)\}$ m-converges to z for any $x \in C$.

If x_1 and x_2 are two fixed points of T such that $m(x_1 - x_2) < \infty$, then from the above theorem we conclude that $x_1 = x_2$. Furthermore, if C is m-bounded, then T has a unique fixed point in C.

3. The generalized Hyers–Ulam stability of (2.2) in modular spaces

Lemma 1. *Assume that X is a linear space, and let X_m be an m-complete convex modular space. Consider the set*

$$\mathbb{M} = \{h : X \to X_m : h(0) = 0\}$$

and define a mapping $\tilde{m}$ on $\mathbb{M}$ by

$$\tilde{m}(h) = \inf\{c > 0 : m(h(x)) \leq c\psi(x,x)\}, \quad h \in \mathbb{M},$$

where $\psi : X^2 \to [0,\infty)$. Then $M_{\tilde{m}}$ is a complete convex modular space.

P r o o f. It is easy to prove that $\tilde{m}$ is a convex modular on $\mathbb{M}$ [22].

For completeness, let $\{h_n\}$ be an $\tilde{m}$-Cauchy sequence in $\mathbb{M}_{\tilde{m}}$, and let $\epsilon > 0$ be given. Then there exists $k \in \mathbb{N}$ such that $\tilde{m}(h_n - h_p) \leq \epsilon$ for all $p, n \geq k$. Therefore,

$$m\left(h_n(x) - h_p(x)\right) \leq \epsilon\psi(x,x) \quad \text{for all} \quad x \in X \quad \text{and} \quad p, n \geq k. \tag{3.1}$$

This shows that $\{h_n(x)\}$ is an m-Cauchy sequence in X_m for each fixed $x \in X_m$. Since X_m is m-complete, it follows that $\{h_n(x)\}$ is m-convergent in X_m for each fixed $x \in X$. Thus, we can define $h : X \to X_m$ by

$$h(x) = \lim_{n\to\infty} h_n(x), \quad \text{for any} \quad x \in X.$$

Clearly, $h \in \mathbb{M}_{\tilde{m}}$. Since m has the Fatou property, taking the limit as $m \to \infty$ in (3.1), we obtain

$$m\left(h_n(x) - h(x)\right) \leq \epsilon\psi(x,x) \quad \text{for all} \quad x \in X \quad \text{and} \quad n \geq k.$$

Thus, $\tilde{m}(h_n - h) \leq \epsilon$ for all $n \geq k$, and therefore $\{h_n\}$ is an $\tilde{m}$-convergent sequence in $\mathbb{M}_{\tilde{m}}$. Hence, $\mathbb{M}_{\tilde{m}}$ is complete. □

Theorem 2. *Let X be a linear space, and X_m be an m-complete convex modular space. Suppose that $q : X \to X_m$ is a function with $q(0) = 0$ satisfying the inequality*

$$m\left(q(kx+y) + q(kx-y) - 2[q(x+y) + q(x-y)] - 2(k^2-2)q(x) + 2q(y)\,\right) \leq \psi(x,y) \tag{3.2}$$

for all $x, y \in X$ and some $k \in \mathbb{N}$, where $\psi : X^2 \to [0,\infty)$ is a function satisfying

$$\psi\left(kx, ky\right) \leq k^2 L\psi(x,y)$$

for all $x, y \in X$ and some L with $0 < L < 1$. Then there exists a unique mapping $P : X \to X_m$ satisfying (2.2) such that

$$m\left(2P(x) - q(x)\right) \leq \frac{1}{2k^2(1-L)}\psi(x,0). \tag{3.3}$$

P r o o f. Putting $y = 0$ in (3.2), we obtain

$$m\left(2q(kx) - 2k^2 q(x)\right) \leq \psi(x,0) \tag{3.4}$$

or equivalently,

$$m\left(q(kx) - k^2 q(x)\right) \leq \frac{1}{2}\psi(x,0). \tag{3.5}$$

Now,
$$m\Big(q(x)-\frac{q(kx)}{k^2}\Big)=m\Big(\frac{1}{2k^2}(2q(k\,x)-2k^2q(x))\Big)\ \le \frac{1}{2k^2}\psi(x,0).$$
Consider the set
$$\mathbb{M}=\{h:X\to X_m:h(0)=0\}$$
and define a function $\tilde{m}$ on $\mathbb{M}$ by
$$\tilde{m}(h)=\inf\{c>0:m(h(x))\le c\psi(x,x)\},\quad h\in\mathbb{M}.$$
By Lemma 1, $M_{\tilde{m}}$ is a complete convex modular space.

Also, consider the operator $S:\mathbb{M}_{\tilde{m}}\to\mathbb{M}_{\tilde{m}}$ defined by
$$Sh(x)=\frac{1}{k^2}h(kx)\quad\forall h\in\mathbb{M}_{\tilde{m}},\quad x\in X\quad\text{and}\quad k\in\mathbb{N}.$$
Thus,
$$S^n\,h(x)=\frac{1}{k^{2n}}h(k^n\,x)\quad\forall h\in\mathbb{M}_{\tilde{m}},\quad x\in X\quad\text{and}\quad k\in\mathbb{N}.$$

Let us show that S is an $\tilde{m}$-strictly contractive mapping. Let $h,z\in\mathbb{M}_{\tilde{m}}$, and suppose there exists a constant $c\in[0,\infty)$ such that
$$\tilde{m}(h-z)\le c.$$
Then,
$$m(h(x)-z(x))\le c\psi(x,x)\quad\forall x\in X.$$
Now,
$$\begin{aligned}m(Sh(x)-Sz(x))&=m\Big(\frac{1}{k^2}h(kx)-\frac{1}{k^2}z(kx)\Big)\le\frac{1}{k^2}m(h(kx)-z(kx))\\&\le\frac{1}{k^2}c\psi(kx,kx)\le cL\psi(x,x)\quad\forall x\in X.\end{aligned}$$
Therefore,
$$\tilde{m}(Sh-Sz)\le\ c\,L.$$
Hence,
$$\tilde{m}(Sh-Sz)\le L\,\tilde{m}(h-z)\quad\text{for all}\quad g,h\in\mathbb{M}_{\tilde{m}}.$$
That is, S is an $\tilde{m}$-strict contraction.

Now, we prove
$$\delta_{\tilde{m}}=\sup\{\tilde{m}(S^n\,(f)-S^m\,(f)):m,n\in\mathbb{N}\}\ <\infty.$$
From (3.5), we have
$$m\left(q(k^2x)-k^2q(kx)\right)\le\frac{1}{2}\psi(kx,0).\tag{3.6}$$
Thus,
$$\begin{aligned}m\left(\frac{q(k^2x)}{(k^2)^2}-q(x)\right)&=m\left(\frac{1}{(k^2)^2}(q(k^2x)-k^2q(kx))+\frac{1}{k^2}(q(kx)-k^2q(x))\right)\\&\le\frac{1}{(k^2)^2}\,m(q(k^2x)-k^2q(kx))+\frac{1}{k^2}\,m(q(kx)-k^2q(x))\\&\le\frac{1}{2(k^2)^2}\psi(kx,0)+\frac{1}{2k^2}\psi(x,0)\overset{(3.5),(3.6)}{=}\frac{1}{2}\sum_{i=0}^{1}\frac{1}{k^{2(i+1)}}\psi(k^ix,0)\quad\text{for all}\quad x\in X.\end{aligned}$$

Since

$$\frac{1}{2}\sum_{i=0}^{n-1}\frac{1}{k^{2(i+1)}} \leq 1,$$

for all $n \geq 0$, we have

$$m\Big(\frac{q(k^n x)}{k^{2n}} - q(x)\Big) = m\Big[\sum_{i=0}^{n-1}\Big(\frac{q(k^{i+1}x)}{k^{2(i+1)}} - \frac{q(k^i x)}{k^{2i}}\Big)\Big]$$

$$= \sum_{i=0}^{n-1}\frac{1}{2\,k^{2(i+1)}}\, m\left(2\,q(k^{i+1}x) - 2\,k^2\,q(k^i x)\right) = \sum_{i=0}^{n-1}\frac{1}{2\,k^{2(i+1)}}\psi(k^i x,\,0)$$

$$\overset{(3.4)}{\leq} \frac{\psi(x,0)}{2k^2}\sum_{i=0}^{n-1}L^i \leq \frac{\psi(x,0)}{2k^2(1-L)} \quad \text{since} \quad 0<L<1.$$

Hence,

$$m\Big(\frac{q(k^n x)}{k^{2n}} - q(x)\Big) \leq \frac{\psi(x,0)}{2k^2(1-L)} \quad \text{since} \quad 0<L<1 \tag{3.7}$$

$\forall x \in X$ and $n \in \mathbb{N}$. Thus, from (3.7) it follows that for any $n, p \in \mathbb{N}$,

$$m\Big(\frac{q(k^n x)}{2k^{2n}} - \frac{q(k^p x)}{2k^{2p}}\Big) \leq \frac{1}{2}m\Big(\frac{q(k^n x)}{k^{2n}} - q(x)\Big) + \frac{1}{2}m\Big(\frac{q(k^p x)}{k^{2p}} - q(x)\Big)$$

$$\leq \frac{1}{2}\cdot\frac{\psi(x,0)}{2k^2(1-L)} + \frac{1}{2}\cdot\frac{\psi(x,0)}{2k^2(1-L)} \leq \frac{\psi(x,0)}{2k^2(1-L)} \quad \text{for all} \quad x \in X \quad [\text{by } (3.7)].$$

This implies that

$$\tilde{m}\Big(S^n\Big(\frac{1}{2}q\Big) - S^p\Big(\frac{1}{2}q\Big)\Big) \leq \frac{1}{2K^2(1-L)} < \infty$$

for all $p, n \in \mathbb{N}$.

This shows that S has a bounded orbit at $1/2q$. Then,

$$m\Big(S^n(\frac{1}{2}q(x)) - \frac{1}{2}q(x)\Big) = m\Big(\frac{q(k^n x)}{2k^{2n}} - \frac{1}{2}q(x)\Big)$$

$$\leq \frac{1}{2}m\Big(\frac{q(k^n x)}{k^{2n}} - q(x)\Big) \leq \frac{1}{2}\cdot\frac{\psi(x,0)}{2k^2(1-L)} < \text{finite} \quad \forall x \in X \quad \text{and} \quad \forall k \in \mathbb{N} \quad [\text{by } (3.7)].$$

Thus, by applying Theorem 1,

(i) S has a fixed point $P \in \mathbb{M}$ at $1/2q$, that is, $SP = P$, or equivalently,

$$P(x) = \frac{1}{k^2}P(kx) \quad \text{for all} \quad x \in X;$$

(ii) the sequence $\{S^n\,(1/2q)\}$ $\tilde{m}$-converges to P.

Therefore,

$$\lim_{n\to\infty} m\Big(\Big(\frac{1}{2k^{2n}}q(k^n x)\Big) - P(x)\Big) = 0.$$

Thus, we can define

$$P(x) := \frac{1}{2}\lim_{n\to\infty}\frac{q(k^n x)}{k^{2n}}.$$

Again, replacing x and y by $k^n x$ and $k^n y$, respectively, in (3.2), we obtain

$$m\Big(\frac{1}{2k^{2n}}q(k^n(kx+y))+q(k^n(kx-y))-2[q(k^n(x+y))+q(k^n(x-y))]$$
$$-2(k^2-2)q(k^n x)+2q(k^n y)\Big)\leq\frac{1}{2k^{2n}}\psi(k^n x,k^n y)\leq\frac{1}{2}L^n\psi(x,y)\quad\forall x\in X,\quad n\in\mathbb{N}.$$

Now, taking the limit as $n\to\infty$ and applying the Fatou property, where $0<L<1$, we get

$$P(kx+y)+P(kx-y)=2[P(x+y)+P(x-y)]+2(k^2-2)P(x)-2P(y).$$

Thus, P is a k-quadratic Euler–Lagrange mapping.

Also, since m has the Fatou property, it follows from (3.7) that

$$m(2P(x)-q(x))\leq\frac{1}{2k^2(1-L)}\psi(x,0)\quad\forall x\in X.$$

To prove uniqueness, let $P':X\to X_m$ be another k-quadratic Euler–Lagrange functional mapping satisfying inequality (3.3). Then we have

$$m\left(P(x)-P'(x)\right)\leq\frac{1}{2}m\left(2P(x)-q(x)\right)+\frac{1}{2}m\left(2P'(x)-q(x)\right)\leq\frac{\psi(x,0)}{2k^2(1-L)}<\infty$$

for all $x\in X$ and $k\in\mathbb{N}$.

Again, let P and P' be two fixed points of S such that

$$m\left(P(x))-P'(x)\right)<\infty.$$

Then, by Theorem 1, we conclude that $P(x)=P'(x)$ for all $x\in X$.

This completes the proof of the theorem. □

Corollary 1. *Let X be a normed linear space, and let X_m be an m-complete convex modular space. Suppose $\theta\geq 0$. Let $q:X\to X_m$ be a function with $q(0)=0$ satisfying*

$$m\big(q(kx+y)+q(kx-y)-2[q(x+y)+q(x-y)]-2(k^2-2)q(x)+2q(y)\big)\leq\theta\big(\|x\|^p+\|y\|^p\big)$$

for all $x,y\in X$, $k\in\mathbb{N}$, and $0\leq p<1$. Then there exists a unique k-quadratic mapping $P:X\to X_m$ such that

$$m\big(2P(x)-q(x)\big)\leq\frac{\theta}{k^2(2-2^p)}\|x\|^p$$

for all $x\in X$.

P r o o f. Define

$$\psi(x,y)=\theta\big(\|x\|^p+\|y\|^p\big)$$

for all $x,y\in X$ and take $L=2^{p-1}$. Then the proof of the result follows similarly to Theorem 2. □

Corollary 2. *Let $\epsilon\geq 0$, X be a normed linear space, and X_m be an m-complete convex modular spaces. Suppose a function $q:X\to X_m$ with $q(0)=0$ satisfies*

$$m(q(kx+y)+q(kx-y)-2[q(x+y)+q(x-y)]-2(k^2-2)q(x)+2q(y)\,)\leq\epsilon$$

for all $x,y\in X$ and $k\in\mathbb{N}$. Then there exists a unique k-quadratic mapping $P:X\to X_m$ such that

$$m(2P(x)-q(x))\leq\frac{\epsilon}{k^2}$$

for all $x\in X$.

P r o o f. Define $\psi(x, y) = \epsilon$ for all $x, y \in X$ and take $L = 1/2$. Then the proof of the result follows similarly to Theorem 2. □

Corollary 3. *Let $\theta, \epsilon \geq 0$, X be a normed linear space, and let Y be a Banach space. Suppose that a mapping $q : X \to Y$ with $q(0) = 0$ satisfies the inequality*

$$\big\|q(kx+y) + q(kx-y) - 2[q(x+y) + q(x-y)] - 2(k^2-2)q(x) + 2q(y)\big\| \leq \epsilon + \theta(\|x\| + \|y\|)$$

for all $x, y \in X$ and $k \in \mathbb{N}$. Then there exists a unique k-quadratic mapping $P : X \to Y$ such that

$$\|(2P(x) - q(x))\| \leq \frac{\epsilon}{k^2(2-2^p)} + \frac{\theta}{k^2(2-2^p)}\|x\|^p$$

for all $x \in X$ and $0 \leq p < 1$.

P r o o f. Since every normed linear space is a modular space, we define $m(x) = \|x\|$ and

$$\psi(x, y) = \epsilon + \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ and take $L = 2^{p-1}$. Then the proof follows from Theorem 2. □

Example 2. Let $(X, \|\cdot\|)$ be a commutative Banach algebra, and let X_m be an m-complete convex modular space, where $m(x) = \|x\|$.

Define $q : X \to X_m$ by

$$q(x) = ax^2 + A\|x\|x_0$$

for all $x \in X$, where a, $A \in \mathbb{R}^+$ and x_0 is a unit vector in X. Then

$$\begin{aligned} m\big(q(kx+y) + q(kx-y) - 2[q(x+y) + q(x-y)] - 2(k^2-2)q(x) + 2q(y)\big) \\ \leq 2A\big[(k^2-k-2)\|x\| + 4\|y\|\big] \end{aligned}$$

for all $x, y \in X$.

Define

$$\psi(x, y) = 2A[(k^2-k-2)\|x\| + 4\|y\|]$$

for all $x, y \in X$ and take $L = 1/2$. Thus, all the conditions of Theorem 2 are satisfied. Then there exists a unique k-quadratic Euler–Lagrange function $P : X \to X_m$ such that

$$m(2P(x) - q(x) \leq \frac{2A(k^2-k-2)}{k^2}\|x\| \quad \forall\, x \in X.$$

Remark 1. Many of the Hyers–Ulam–Rassias stability results rely on the Δ_α-condition stated in part (v) of Definition 4 for various values of $\alpha \geq 2$. Our theorems are established without assuming this condition on the modular space. Omitting this condition makes the proof more involved. Furthermore, we have employed fixed point methods within the framework of modular spaces. Such an approach to stability problems in modular spaces has previously appeared in [22]. This methodology can also be adapted to other functional equations, potentially serving as a foundation for future research.

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A REMARK AND AN IMPROVED VERSION ON RECENT RESULTS CONCERNING RATIONAL FUNCTIONS[1]

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Abstract: This paper extends as a lemma an auxiliary result obtained by Singh and Chanam. Using it, we prove a refinement of the Turán-type inequality for rational functions obtained recently by Akhter et al. Next, using examples, we discuss the result of Mir et al.

Keywords: Rational function, Polynomial, Inequalities in complex domain.

1. Introduction

Let $\mathbb{C}$ denote the set of complex numbers z, and let $\Re(z)$ be the real part of z. Let $\mathcal{P}_n$ be the set of all complex polynomials

$$g(z) := \sum_{k=0}^{n} d_k z^k$$

of degree at most n, and let $g'(z)$ be the derivative of $g(z)$. Let $S_l := \{z : |z| = l\}$, and let R_l^- and R_l^+ be the interior and exterior of S_l, respectively. For $\gamma_k \in \mathbb{C}$, let

$$w(z) := \prod_{k=1}^{n} (z - \gamma_k); \quad V(z) := \prod_{k=1}^{n} \left(\frac{1 - \overline{\gamma_k} z}{z - \gamma_k} \right),$$

and let

$$\mathcal{R}_n = \mathcal{R}_n(\gamma_1, \gamma_2, \ldots, \gamma_n) := \left\{ \frac{g(z)}{w(z)} : g \in \mathcal{P}_n \right\}$$

be the set of rational functions having a finite limit as $z \to \infty$ and poles $\gamma_1, \gamma_2, \ldots, \gamma_n$, such that $\gamma_k \in R_1^+$. The well-known result of Bernstein [4] states the following.

Theorem 1 [4]. *For any $z \in \mathbb{C}$, if $g \in \mathcal{P}_n$, then*

$$\max_{z \in S_1} |g'(z)| \le n \max_{z \in S_1} |g(z)|.$$

Confining himself to the set of polynomials whose zeros all lie in $S_1 \cup R_1^+$, Erdös conjectured, which was later confirmed by Lax [5], that

$$\max_{z \in S_1} |g'(z)| \le \frac{n}{2} \max_{z \in S_1} |g(z)|.$$

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If all zeros of $g(z)$ are in $S_1 \cup R_1^-$, Turán [9] proved that

$$\max_{z\in S_1} |g'(z)| \geq \frac{n}{2} \max_{z\in S_1} |g(z)|.$$

Li et al. [6] derived inequalities similar to Bernstein inequalities for rational functions $q \in \mathcal{R}_n$, considering prescribed poles $\gamma_1, \gamma_2, \ldots, \gamma_n$ and replacing z^n by the Blashke product $V(z)$. They established the following result featuring these poles.

Theorem 2 [6]. *If $q \in \mathcal{R}_n$ has all its zeros in $S_1 \cup R_1^+$, then, for $z \in S_1$,*

$$|q'(z)| \leq \frac{1}{2}|V'(z)||q(z)|.$$

Equality holds for $q(z) = a_0 V(z) + b_0$ with $|a_0| = |b_0| = 1$.

Aziz and Shah [2] improved this inequality as follows.

Theorem 3 [2]. *Let $q \in \mathcal{R}_n$ and all its zeros lye in $S_1 \cup R_1^+$. If $e_1, e_2, \ldots, e_n$ are the zeros of $V(z) + \xi$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are the zeros of $V(z) - \xi$, $\xi \in S_1$, then, for $z \in S_1$,*

$$|q'(z)| \leq \frac{|V'(z)|}{2}\left\{\Big(\max_{1\leq k\leq n}|q(e_k)|\Big)^2 + \Big(\max_{1\leq k\leq n}|q(\epsilon_k)|\Big)^2\right\}^{1/2}. \tag{1.1}$$

Recently, Mir et al. [7] proved the following result, which gives a generalized and strengthened upper estimate than that in Theorem 3.

Theorem 4 [7]. *Let*

$$q(z) = \frac{g(z)}{w(z)} \in \mathcal{R}_n,$$

where

$$g(z) = z^s \sum_{k=0}^{m-s} d_k z^k$$

is an m-degree polynomial ($m \leq n$) having all its zeros in $S_l \cup R_l^+$, $l \geq 1$, except for a zero of multiplicity s at the origin. If $e_1, e_2, \ldots, e_n$ are the zeros of $V(z)+\xi$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are the zeros of $V(z) - \xi$, $\xi \in S_1$, then, for $z \in S_1$,

$$\begin{aligned} |q'(z)| \leq \frac{|V'(z)|}{2}\bigg\{&\Big(\max_{1\leq k\leq n}|q(e_k)|\Big)^2 + \Big(\max_{1\leq k\leq n}|q(\epsilon_k)|\Big)^2 \\ -4\bigg(\frac{l}{1+l}\bigg(\frac{|d_0| - l^{m-s}|d_{m-s}|}{|d_0| + l^{m-s}|d_{m-s}|}\bigg) - \frac{sl}{1+l} - \frac{2m - n(1+l)}{2(1+l)}\bigg)&\frac{|q(z)|^2}{|V'(z)|}\bigg\}^{1/2}. \end{aligned} \tag{1.2}$$

Furthermore, Li et al. [6] obtained the following inequality for rational functions, which generalizes the polynomial inequality of Turán [9].

Theorem 5 [6]. *If $q \in \mathcal{R}_n$ has all its zeros in $S_1 \cup R_1^-$, then, for $z \in S_1$,*

$$|q'(z)| \geq \frac{1}{2}|V'(z)||q(z)|.$$

Recently, Akhter et al. [1] obtained the following result by introducing a complex parameter α which provides an improvement and a generalization of Theorem 5.

Theorem 6 [1]. *Assume that*

$$q(z) = \frac{g(z)}{w(z)} \in \mathcal{R}_n,$$

where

$$g(z) = z^s \sum_{k=0}^{m-s} d_k z^k$$

is an m-degree polynomial ($m \leq n$) having all zeros in $S_l \cup R_l^-$, $l \leq 1$, and a zero of multiplicity s at the origin. Then, for every complex δ, $|\delta| \leq 1$, and $z \in S_1$,

$$\left|zq'(z) + \frac{(m-s)\delta}{1+l}q(z)\right| \geq \frac{1}{2}\left\{|V'(z)| + \frac{1}{1+l}\Big(l(2s-n) + 2m - n + 2(m-s)\Re(\delta)\Big)\right\}|q(z)|.$$

In this paper, we first establish a refined inequality of Theorem 6 by including certain coefficients of the polynomial, and then discuss Theorem 4 due to Mir et al. [7] using counterexamples that they claim improve the bound given by Theorem 3. The paper is organized as follows. Section 2 presents the main result, some remarks, and corollaries. In addition, we discuss the result due to Mir et al. [7]. Section 3 presents some auxiliary results necessary to establish the main result. Section 4 provides a proof of the main result. Section 5 concerns the conclusion.

2. Main result and discussion

Here, we present the following result concerning rational functions, which generalizes and sharpens the polynomial inequality of Turán [9].

Theorem 7. *Let*

$$q(z) = \frac{g(z)}{w(z)} \in \mathcal{R}_n,$$

where

$$g(z) = z^s \sum_{k=0}^{m-s} d_k z^k$$

is an m-degree polynomial ($m \leq n$) having all its zeros in $S_l \cup R_l^-$, $l \leq 1$, and a zero of multiplicity s at the origin. Then, for every complex δ, $|\delta| \leq 1$, and $z \in S_1$,

$$\begin{aligned}\left|zq'(z) + \frac{(m-s)\delta}{1+l}q(z)\right| \geq \frac{1}{2}\Bigg\{|V'(z)| + \frac{1}{1+l}\Bigg(& l(2s-n) + 2m - n \\ &+ 2l\left(\frac{\sqrt{l^{m-s}|d_{m-s}|} - \sqrt{|d_0|}}{\sqrt{l^{m-s}|d_{m-s}|}}\right) + 2(m-s)\Re(\delta)\Bigg)\Bigg\}|q(z)|.\end{aligned} \tag{2.1}$$

Remark 1. Since the zeros of the polynomial

$$h(z) = \frac{g(z)}{z^s} = \sum_{k=0}^{m-s} d_k z^k$$

are in $S_l \cup R_l^-$, $l \leq 1$, we have

$$\left|\frac{d_0}{d_{m-s}}\right| \leq l^{m-s},$$

which is equivalent to

$$\sqrt{l^{m-s}|d_{m-s}|} \geq \sqrt{|d_0|}. \tag{2.2}$$

On the right-hand side of inequality (2.1) of Theorem 7, there is an extra term contributed by the quantity

$$2l\left(\frac{\sqrt{l^{m-s}|d_{m-s}|} - \sqrt{|d_0|}}{\sqrt{l^{m-s}|d_{m-s}|}}\right),$$

which in view of (2.2) is nonnegative, and hence Theorem 7 refines Theorem 6.

Taking $\delta = 0$ and $m = n$ in Theorem 7, we obtain the following interesting result, which gives a generalization and an improvement of Theorem 5 due to Li et al. [6], and an improvement of the result established by Akhter et al. [1, Corollary 2.2].

Corollary 1. *Let*

$$q(z) = \frac{g(z)}{w(z)} \in \mathcal{R}_n,$$

where

$$g(z) = z^s \sum_{k=0}^{n-s} d_k z^k$$

is an n-degree polynomial having all its zeros in $S_l \cup R_l^-$, $l \leq 1$, and a zero of multiplicity s at the origin. Then, for $z \in S_1$,

$$|q'(z)| \geq \frac{1}{2}\left\{|V'(z)| + \frac{1}{1+l}\left(2ls + n(1-l) + 2l\left(\frac{\sqrt{l^{n-s}|d_{n-s}|} - \sqrt{|d_0|}}{\sqrt{l^{n-s}|d_{n-s}|}}\right)\right)\right\}|q(z)|.$$

Moreover, taking $l = 1$ in Theorem 7, we obtain a result that improves the known result [1, Corollary 2.4] obtained by Akhter et al.

Corollary 2. *Let*

$$q(z) = \frac{g(z)}{w(z)} \in \mathcal{R}_n,$$

where

$$g(z) = z^s \sum_{k=0}^{m-s} d_k z^k$$

is an m-degree polynomial ($m \leq n$) having all its zeros in $S_1 \cup R_1^-$ and a zero of multiplicity s at the origin. Then, for every complex δ, $|\delta| \leq 1$, and $z \in S_1$,

$$\left|zq'(z) + \frac{(m-s)\delta}{2}q(z)\right| \geq \frac{1}{2}\left\{|V'(z)| + (s+m-n) + \left(\frac{\sqrt{|d_{m-s}|} - \sqrt{|d_0|}}{\sqrt{|d_{m-s}|}}\right) + (m-s)\Re(\delta)\right\}|q(z)|.$$

Next, the claim that the bound in inequality (1.2) of Theorem 4 proved by Mir et al. [7] sharpens the bound in inequality (1.1) of Theorem 3 due to Aziz and Shah [2] follows in the case when the quantity

$$\left(\frac{l}{1+l}\left(\frac{|d_0| - l^{m-s}|d_{m-s}|}{|d_0| + l^{m-s}|d_{m-s}|}\right) - \frac{sl}{1+l} - \frac{2m - n(1+l)}{2(1+l)}\right) = A$$

on the right-hand side of inequality (1.2) of Theorem 4 is nonnegative. But this is not always the case, as the following counterexamples illustrate.

Example 1. Let $q \in \mathcal{R}_6$, where $g(z) = z^3(z^3 - z^2 + z - 1)$ has the zeros $\{1, i, -i\}$ on $|z| = 1$ and the remaining zeros at the origin. It can be easily seen that this polynomial gives $A = -1.5$ in Theorem 4.

Example 2. Let $q \in \mathcal{R}_5$, where $g(z) = z^3(z^2 - 4)$ has the zeros $\{-2, 2\}$ on $|z| = 2$ and the remaining zeros at the origin. For this polynomial, we have $A = -1.166\bar{6}$.

3. Lemmas

We must incorporate the following lemmas into our proof to demonstrate the theorem. Aziz and Zargar [3] established the first.

Lemma 1 [3]. *If*

$$V(z) = \prod_{k=1}^{n} \left(\frac{1 - \overline{\gamma_k} z}{z - \gamma_k} \right),$$

then, for $z \in S_1$,

$$\Re\left(\frac{z w'(z)}{w(z)} \right) = \frac{n - |V'(z)|}{2}.$$

Lemma 2. *If* $0 \le a \le 1$, $0 \le b \le 1$, *and* $0 \le l \le 1$, *then*

$$\frac{2}{1+a} \ge 1 + l\sqrt{b} - l\sqrt{ab}.$$

P r o o f. For $a = 1$, the inequality follows trivially. So, take $a < 1$, then

$$\frac{1+\sqrt{a}}{1+a} > 1 \ge l\sqrt{b};$$

that is,

$$\frac{1-a}{1+a} > l\sqrt{b}\frac{1-a}{1+\sqrt{a}} = l\sqrt{b} - l\sqrt{ab}.$$

Hence,

$$\frac{2}{1+a} > 1 + l\sqrt{b} - l\sqrt{ab}.$$

□

The following lemma we prove is a generalization of a finding by Singh and Chanam [8].

Lemma 3. *If* $g \in \mathcal{P}_n$ $(n \ge 1)$ *has all its zeros in* $S_l \cup R_l^-$, $l \le 1$, *then, for* $z \in S_1$ *such that* $g(z) \ne 0$,

$$\Re\left(z\frac{g'(z)}{g(z)} \right) \ge \frac{1}{1+l}\left\{ n + l\left(\frac{\sqrt{l^n|d_n|} - \sqrt{|d_0|}}{\sqrt{l^n|d_n|}} \right) \right\}. \tag{3.1}$$

Remark 2. As the abstract mentioned, for $l = 1$, this lemma reduces to Lemma 2 of Singh and Chanam [8].

P r o o f. For simplicity, suppose that $d_n = 1$. We apply mathematical induction on the degree of $g(z)$.

If $n = 1$, then $g(z) = z - z_0$, $z_0 \in S_l \cup R_l^-$, and, for $z \in S_1$ and $z \neq z_0$, we have

$$\Re\left(z\frac{g'(z)}{g(z)}\right) = \Re\left(\frac{z}{z - z_0}\right) \geq \frac{1}{1 + |z_0|}.$$

By basic computation, we can show that, for $z_0 \in S_l \cup R_l^-$,

$$\frac{1}{1 + |z_0|} \geq \frac{1}{1 + l}\left\{1 + l\left(\frac{\sqrt{l} - \sqrt{|z_0|}}{\sqrt{l}}\right)\right\}.$$

So,

$$\Re\left(z\frac{g'(z)}{g(z)}\right) \geq \frac{1}{1 + l}\left\{1 + l\left(\frac{\sqrt{l} - \sqrt{|z_0|}}{\sqrt{l}}\right)\right\},$$

which is inequality (3.1) for $n = 1$.

Suppose that (3.1) holds for all polynomials of degree $\leq M$.

Let $g(z) = (z - w)G(z)$, $w \in S_l \cup R_l^-$, where

$$G(z) = \sum_{k=0}^{M} d_k z^k$$

is a polynomial of degree M having all its zeros in $S_l \cup R_l^-$, then

$$\Re\left(z\frac{g'(z)}{g(z)}\right) = \Re\left(\frac{z}{z - w}\right) + \Re\left(z\frac{G'(z)}{G(z)}\right) \geq \frac{1}{1 + |w|} + \frac{1}{1 + l}\left\{M + l\left(\frac{\sqrt{l^M} - \sqrt{|d_0|}}{\sqrt{l^M}}\right)\right\}$$

for all $z \in S_1$ such that $g(z) \neq 0$.

It is required to show that, for $z \in S_1$,

$$\Re\left(z\frac{g'(z)}{g(z)}\right) \geq \frac{1}{1 + l}\left\{M + 1 + l\left(\frac{\sqrt{l^{M+1}} - \sqrt{|w||d_0|}}{\sqrt{l^{M+1}}}\right)\right\}. \tag{3.2}$$

Clearly, inequality (3.2) holds if

$$\frac{1}{1 + |w|} + \frac{1}{1 + l}\left\{M + l\left(\frac{\sqrt{l^M} - \sqrt{|d_0|}}{\sqrt{l^M}}\right)\right\} \geq \frac{1}{1 + l}\left\{M + 1 + l\left(\frac{\sqrt{l^{M+1}} - \sqrt{|w||d_0|}}{\sqrt{l^{M+1}}}\right)\right\},$$

which is equivalent to

$$\frac{1 + l}{1 + |w|} \geq 1 + l\sqrt{\frac{|d_0|}{l^M}} - l\sqrt{\frac{|w||d_0|}{l^{M+1}}}. \tag{3.3}$$

As the zeros of $g(z)$ are in $S_l \cup R_l^-$ and

$$0 \leq l \leq 1, \quad 0 \leq \frac{|d_0|}{l^M} \leq 1, \quad 0 \leq \frac{|w|}{l} \leq 1,$$

by Lemma 2,

$$\frac{2l}{l + |w|} \geq 1 + l\sqrt{\frac{|d_0|}{l^M}} - l\sqrt{\frac{|w||d_0|}{l^{M+1}}}. \tag{3.4}$$

Also,

$$\frac{1 + l}{1 + |w|} \geq \frac{2l}{l + |w|}. \tag{3.5}$$

From (3.4) and (3.5), inequality (3.3) follows, and this proves Lemma 3. □

4. Proof of the main result

Proof of Theorem 7. Since

$$q(z)=\frac{z^s h(z)}{w(z)}\in\mathcal{R}_n,$$

where

$$h(z)=\sum_{k=0}^{m-s} d_k z^k,$$

for every complex δ, $|\delta|\le 1$, we have

$$\frac{zq'(z)}{q(z)}+\frac{(m-s)\delta}{1+l}=s+\frac{zh'(z)}{h(z)}-\frac{zw'(z)}{w(z)}+\frac{(m-s)\delta}{1+l}.$$

Equivalently,

$$\Re\left(\frac{zq'(z)}{q(z)}+\frac{(m-s)\delta}{1+l}\right)=s+\Re\left(\frac{zh'(z)}{h(z)}\right)-\Re\left(\frac{zw'(z)}{w(z)}\right)+\frac{(m-s)\Re(\delta)}{1+l}.$$

Specially for $z\in S_1$, using Lemmas 3 and 1, we have

$$\Re\left(\frac{zq'(z)}{q(z)}+\frac{(m-s)\delta}{1+l}\right)\ge s+\frac{1}{1+l}\left\{m-s+l\left(\frac{\sqrt{l^{m-s}|d_{m-s}|}-\sqrt{|d_0|}}{\sqrt{l^{m-s}|d_{m-s}|}}\right)\right\}$$
$$-\left(\frac{n-|V'(z)|}{2}\right)+\frac{(m-s)\Re(\delta)}{1+l}$$
$$=\frac{1}{2}\left\{|V'(z)|+\frac{1}{1+l}\left(l(2s-n)+2m-n+2l\left(\frac{\sqrt{l^{m-s}|d_{m-s}|}-\sqrt{|d_0|}}{\sqrt{l^{m-s}|d_{m-s}|}}\right)+2(m-s)\Re(\delta)\right)\right\},$$

from which it is obvious that

$$\left|zq'(z)+\frac{(m-s)\delta}{1+l}q(z)\right|$$
$$\ge\frac{1}{2}\left\{|V'(z)|+\frac{1}{1+l}\left(l(2s-n)+2m-n+2l\left(\frac{\sqrt{l^{m-s}|d_{m-s}|}-\sqrt{|d_0|}}{\sqrt{l^{m-s}|d_{m-s}|}}\right)+2(m-s)\Re(\delta)\right)\right\}|q(z)|.$$

This proves Theorem 7. □

5. Conclusion

This paper investigates the bounds of the derivative of a class of rational functions on the unit disk while considering the contribution of certain coefficients of the underlying polynomial. We also discuss the result by Mir et al., recently published in the Ural Mathematical Journal, using some counterexamples.

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ON AN INITIAL BOUNDARY–VALUE PROBLEM FOR A DEGENERATE EQUATION OF HIGH EVEN ORDER

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Abstract: In this paper, we formulate and study an initial boundary-value problem of the type of the third boundary condition for a degenerate partial differential equation of high even order in a rectangle. Using the Fouriers method, based on separation of variables, a spectral problem for an ordinary differential equation is obtained. Using the Green's function method, the latter problem is equivalently reduced to the Fredholm integral equation of the second kind with a symmetric kernel, which implies the existence of eigenvalues and a system of eigenfunctions of the spectral problem. Using the found integral equation and Mercer's theorem, the uniform convergence of certain bilinear series depending on the eigenfunctions is proved. The order of the Fourier coefficients has been established. The solution to the considered problem has been written as a sum of the Fourier series over the system of eigenfunctions of the spectral problem. The uniqueness of the solution to the problem was proved using the method of energy integrals. An estimate for solution of the problem was obtained, which implies its continuous dependence on the given functions.

Keywords: Degenerate equation, Initial boundary-value problem, Method of separation of variables, Spectral problem, Green's function method, Integral equation, Fourier series.

1. Introduction

Recently, researchers have been paying more and more attention to degenerate partial differential equations. This trend is primarily driven by the intrinsic requirements of the theory of partial differential equations. Additionally, a multitude of problems in gas dynamics, hydrodynamics [4, 5], the theory of infinitesimal bending of surfaces, and the momentless theory of shells with alternating curvature [17], as well as in the theory of oscillations [8, 9], mathematical biology [12], filtration theory, boundary layer theory, and technical mechanics, necessitate the investigation of degenerate partial differential equations.

Currently, intensive research is underway on initial boundary value problems in quadrangular domains for degenerate partial differential equations of high even order in spatial variables. For instance, in [3], initial boundary value problems in a rectangle were formulated and investigated for the following degenerate equation:

$$\frac{\partial^l u}{\partial t^l} = (-1)^k \frac{\partial^k}{\partial x^k}\left(x^\alpha \frac{\partial^k u}{\partial x^k}\right) + f(x,t), \quad l=\overline{1,2}, \quad \alpha\in(0,2k). \tag{1.1}$$

Moreover, in [2] and [13], similar equations with generalizations were explored.

When considering initial boundary value problems for degenerate equations of type (1.1), the formulation of the problems is significantly influenced by the degree of degeneracy α [2, 3], and sometimes by the evenness and oddness of the number k. Additionally, as the order of the equation increases, the number of options for boundary conditions also increases. For instance, in [2, 3], when considering initial boundary value problems for equation (1.1) in the quadrilateral

$$\Omega = \{0 < x < 1,\ 0 < t < T\}$$

at $0 < \alpha < 1$, boundary conditions of the form

$$\left(\partial^j/\partial x^j\right) u\big|_{x=0} = 0, \quad j = \overline{0, k-1}; \quad \left(\partial^q/\partial x^q\right) u\big|_{x=1} = 0, \quad q = \overline{0, k-1} \tag{1.2}$$

were specified, at $\alpha \in (1, k)$, some boundary conditions at $x = 0$ are replaced by the boundedness condition, and at $\alpha \in (k, 2k)$ at $x = 0$ no boundary conditions were specified.

In [13], considering equation (1.1) for $\alpha \in (0, 1)$, boundary conditions of the form (1.2) were specified, but here $q = \overline{k, 2k-1}$.

In [6, 7], when considering a degenerate equation of a different type, boundary conditions (1.2) were adopted. In [15], for a specific degenerate equation, a problem with boundary conditions relating the values of the desired function and the derivatives with respect to x at $x = 0$ and $x = 1$ was formulated and studied. In [1] and [16], for equation (1.1) with $\alpha = 0$, $l = 2$, and for a degenerate fourth-order equation of type (1.1) respectively, conditions of the third type were specified for both $x = 0$ and $x = 1$. Moreover, in [14], a mixed problem was considered for a fourth-order degenerate equation with fractional case of l, namely for $1 < l < 2$, and the dependence of the degeneration degree of α to the formulation of the boundary conditions has been studied.

In this paper, an initial boundary value problem with conditions similar to the third boundary condition for a degenerate partial differential equation of high even order in a rectangle is formulated and investigated.

2. Formulation of the problem

In a rectangle

$$\Omega = \{(x, t) : 0 < x < 1;\ 0 < t < T\},$$

we consider the following degenerate equation of high even order

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^{2n}}{\partial x^{2n}}\left(x^\alpha \frac{\partial^{2n} u}{\partial x^{2n}}\right) = f(x, t), \tag{2.1}$$

where $u = u(x, t)$ is an unknown function, $f(x, t)$ is a given function, and α is a given real number, such that $0 < \alpha < 1$ and $n \in N$.

We study the following initial boundary-value problem:

Problem A. Find a function $u(x, t)$ such that:

1) $u_t, \left(\partial^j/\partial x^j\right) u, \left(\partial^j/\partial x^j\right)\left[x^\alpha \left(\partial^{2n}/\partial x^{2n}\right) u\right] \in C(\bar{\Omega}), \quad j = \overline{0, 2n-1};$
$\left(\partial^{2n}/\partial x^{2n}\right)\left[x^\alpha \left(\partial^{2n}/\partial x^{2n}\right) u\right], \quad u_{tt} \in C(\Omega);$
2) it satisfies the equation (2.1) in the domain Ω;
3) it satisfies the following initial conditions

$$u(x, 0) = \varphi_1(x), \quad x \in [0, 1], \quad u_t(x, 0) = \varphi_2(x), \quad x \in [0, 1] \tag{2.2}$$

and boundary conditions

$$\left.\begin{aligned} &\frac{\partial^{2j}}{\partial x^{2j}} u(0, t) = \frac{\partial^{2j+1}}{\partial x^{2j+1}} u(0, t), \quad \frac{\partial^{2j}}{\partial x^{2j}}\left(x^\alpha \frac{\partial^{2n}}{\partial x^{2n}} u(x, t)\right)\Big|_{x=0} = \frac{\partial^{2j+1}}{\partial x^{2j+1}}\left(x^\alpha \frac{\partial^{2n}}{\partial x^{2n}} u(x, t)\right)\Big|_{x=0};\\ &\frac{\partial^{2j}}{\partial x^{2j}} u(1, t) = \frac{\partial^{2j+1}}{\partial x^{2j+1}} u(1, t), \quad \frac{\partial^{2j}}{\partial x^{2j}}\left(x^\alpha \frac{\partial^{2n}}{\partial x^{2n}} u(x, t)\right)\Big|_{x=1} = \frac{\partial^{2j+1}}{\partial x^{2j+1}}\left(x^\alpha \frac{\partial^{2n}}{\partial x^{2n}} u(x, t)\right)\Big|_{x=1};\\ &\qquad j = \overline{0, n-1}, \quad t \in [0, T], \end{aligned}\right\} \tag{2.3}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are given continuous functions.

3. Investigation of the spectral problem

By formally applying the Fourier method to the problem A, we get the following spectral problem:

$$M[v(x)] \equiv \left(x^\alpha v^{(2n)}(x)\right)^{(2n)} = \lambda v(x), \quad 0 < x < 1; \tag{3.1}$$

$$\left.\begin{array}{c} v^{(j)}(x), \left(x^\alpha v^{(2n)}(x)\right)^{(j)} \in C[0,1], \quad j = \overline{0, 2n-1}; \\ v^{(2j)}(0) = v^{(2j+1)}(0), \quad \left[x^\alpha v^{(2n)}(x)\right]^{(2j)}\Big|_{x=0} = \left[x^\alpha v^{(2n)}(x)\right]^{(2j+1)}\Big|_{x=0}, \quad j = \overline{0, n-1}; \\ v^{(2j)}(1) = v^{(2j+1)}(1), \quad \left[x^\alpha v^{(2n)}(x)\right]^{(2j)}\Big|_{x=1} = \left[x^\alpha v^{(2n)}(x)\right]^{(2j+1)}\Big|_{x=1}, \quad j = \overline{0, n-1}. \end{array}\right\} \tag{3.2}$$

It is easy to verify that for any functions $v(x)$ and $w(x)$ satisfying the conditions (3.2), the equality

$$\int_0^1 w(x) M[v(x)] dx = \int_0^1 v(x) M[w(x)] dx$$

holds true. This implies that the problem with conditions $M[v(x)] = 0$ and (3.2) is self-adjoint.

Let $v(x)$ be a function satisfying conditions $\{(3.1), (3.2)\}$. Then, multiplying the equation (3.1) with the function $v(x)$ and integrating the resulting equality over the interval $[0,1]$, and subsequently applying the integration by parts rule and considering equalities (3.2), we arrive at

$$\lambda \int_0^1 v^2(x) dx = \int_0^1 x^\alpha \left[v^{(2n)}(x)\right]^2 dx. \tag{3.3}$$

If $\lambda = 0$, then from equality (3.3) it follows that

$$v^{(2n)}(x) = 0, \quad 0 < x < 1.$$

Hence, due to the conditions

$$v^{(2j)}(0) = v^{(2j+1)}(0), \quad v^{(2j)}(1) = v^{(2j+1)}(1), \quad j = \overline{0, n-1},$$

we have $v(x) \equiv 0, \quad 0 \le x \le 1$. If $\lambda < 0$, then from (3.3) it immediately follows that $v(x) \equiv 0$, $0 \le x \le 1$. Consequently, problem $\{(3.1), (3.2)\}$ can have nontrivial solutions only for $\lambda > 0$.

Assuming $\lambda > 0$, we prove the existence of eigenvalues of problem $\{(3.1), (3.2)\}$ using the Green's function method. The Green's function $G(x,s)$ of this problem has the following properties:

1) $\left(\partial^j/\partial x^j\right) G(x,s)$, $j = \overline{0, 2n-1}$ and $\left(\partial^j/\partial x^j\right)\left[x^\alpha \left(\partial^{2n}/\partial x^{2n}\right) G(x,s)\right]$, $j = \overline{0, 2n-2}$ are continuous for all $x, s \in [0,1]$;
2) in each of the intervals $[0,s)$ and $(s,1]$ there exists a continuous derivative $\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^\alpha \left(\partial^{2n}/\partial x^{2n}\right) G(x,s)\right]$, and at $x = s$ it has a jump:

$$\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^\alpha \left(\partial^{2n}/\partial x^{2n}\right) G(x,s)\right]_{x=s-0}^{x=s+0} = 1; \tag{3.4}$$

3) in the intervals $(0,s)$ and $(s,1)$ with respect to the argument x there exists a continuous derivative $MG(x,s)$ and the equality $MG(x,s) = 0$ holds;
4) for $s \in (0,1)$ with respect to x it satisfies the conditions

$$\left.\begin{array}{c} \dfrac{\partial^{2j} G(0,s)}{\partial x^{2j}} = \dfrac{\partial^{2j+1} G(0,s)}{\partial x^{2j+1}}, \\ \dfrac{\partial^{2j}}{\partial x^{2j}}\left(x^\alpha \dfrac{\partial^{2n}}{\partial x^{2n}} G(x,s)\right)\Big|_{x=0} = \dfrac{\partial^{2j+1}}{\partial x^{2j+1}}\left(x^\alpha \dfrac{\partial^{2n}}{\partial x^{2n}} G(x,s)\right)\Big|_{x=0}, \quad j = \overline{0, n-1}; \end{array}\right\}$$

$$\left.\begin{aligned}\frac{\partial^{2j}G(1,s)}{\partial x^{2j}}=\frac{\partial^{2j+1}G(1,s)}{\partial x^{2j+1}},\\ \frac{\partial^{2j}}{\partial x^{2j}}\Big(x^\alpha\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\Big)\Big|_{x=1}=\frac{\partial^{2j+1}}{\partial x^{2j+1}}\Big(x^\alpha\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\Big)\Big|_{x=1},\quad j=\overline{0,n-1}.\end{aligned}\right\}$$

As proven above, problem $\{(3.1),(3.2)\}$ for $\lambda=0$ has only a trivial solution. Then, according to [11, p. 39], there exists a unique Green's function $G(x,s)$ for this problem. Let us now prove that the Green's function $G(x,s)$, satisfying the above conditions 1–4, is symmetric with respect to its arguments.

Let

$$v(x),\ h(x)\in C^{2n-1}[0,1];\quad x^\alpha v^{(2n)}(x),\ x^\alpha h^{(2n)}(x)\in C^{2n-1}[0,1]\cap C^{2n}(0,1).$$

Let us introduce the following notation:

$$M[v(x)]\equiv\big(x^\alpha v^{(2n)}(x)\big)^{(2n)}=f(x),\quad M[h(x)]\equiv\big(x^\alpha h^{(2n)}(x)\big)^{(2n)}=g(x).$$

Then the following equality holds true

$$\begin{gathered}h(x)M[v(x)]-v(x)M[h(x)]=h(x)\big(x^\alpha v^{(2n)}(x)\big)^{(2n)}-v(x)\big(x^\alpha h^{(2n)}(x)\big)^{(2n)}\\ =\sum_{j=0}^{2n-1}\frac{d}{dx}\Big\{(-1)^j\Big[h^{(j)}(x)\big(x^\alpha v^{(2n)}(x)\big)^{(2n-1-j)}-v^{(j)}(x)\big(x^\alpha h^{(2n)}(x)\big)^{(2n-1-j)}\Big]\Big\}\\ =f(x)h(x)-g(x)v(x),\quad 0<x<1.\end{gathered}\tag{3.5}$$

If we assume $v(x)=G(x,s)$ and $h(x)=G(x,\xi)$, then at all the points of the interval $(0,1)$, except points $x\neq\xi,\ x\neq s$, the equalities $M[v(x)]=0$ and $M[h(x)]=0$ hold. Then equality (3.5) takes the form

$$\begin{gathered}\sum_{j=0}^{2n-1}\frac{d}{dx}\Big\{(-1)^j\Big[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\Big)\\ -\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big]\Big\}=0,\quad x\in(0,1)/\{s,\xi\}\end{gathered}\tag{3.6}$$

Without loss of generality, we assume that $s<\xi$. Then the segment $[0,1]$ is divided into three segments: $[0,s],\ [s,\xi],\ [\xi,1]$. Integrating the equality (3.6) over these segments, we obtain

$$\begin{aligned}&\sum_{j=0}^{2n-1}\Big\{(-1)^j\Big[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\Big)\\ &\quad-\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big]\Big\}\Big|_{x=0}^{x=s-0}\\ &+\sum_{j=0}^{2n-1}\Big\{(-1)^j\Big[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\Big)\\ &\quad-\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big]\Big\}\Big|_{x=s+0}^{x=\xi-0}\\ &+\sum_{j=0}^{2n-1}\Big\{(-1)^j\Big[\frac{d^j}{dx^j}G(x,\xi)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,s)\Big)\\ &-\frac{d^j}{dx^j}G(x,s)\frac{d^{2n-1-j}}{dx^{2n-1-j}}\Big(x^\alpha\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big]\Big\}\Big|_{x=\xi+0}^{x=1}=0.\end{aligned}$$

If we consider the properties 1 and 4 of the Green's function $G(x,s)$, then the last equality takes the form:

$$-\Big[G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,s)\Big)\Big]\Big|_{x=s-0}^{x=s+0}+\Big[G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big]_{x=s-0}^{x=s+0}$$
$$-\Big[G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,s)\Big)\Big]_{x=\xi-0}^{x=\xi+0}+\Big[G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big]\Big|_{x=\xi-0}^{x=\xi+0}=0.$$

According to the property 2 of the function $G(x,\eta)$, the derivative of $\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^{\alpha}\left(\partial^{2n}/\partial x^{2n}\right)G(x,\eta)\right]$ is continuous at $x\neq\eta$. Therefor we have the equality

$$\Big[G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,s)\Big)\Big|_{x=s-0}-G(x,\xi)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,s)\Big)\Big|_{x=s+0}\Big]$$
$$+\Big[G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big|_{x=\xi+0}-G(x,s)\frac{d^{2n-1}}{dx^{2n-1}}\Big(x^{\alpha}\frac{d^{2n}}{dx^{2n}}G(x,\xi)\Big)\Big|_{x=\xi-0}\Big]=0.$$

Hence, by virtue of equality (3.4), the equality

$$-G(s,\xi)+G(\xi,s)=0,$$

follows, which we need to prove.

In the special case when $n=1$, the Green's function $G\left(x,s\right)$ takes the following form:

$$G\left(x,s\right)=\begin{cases}\dfrac{sx^{3-\alpha}}{(2-\alpha)_2}+\dfrac{sx^{2-\alpha}}{(1-\alpha)_2}+\Big(\dfrac{s^{3-\alpha}}{(2-\alpha)_2}+\dfrac{s}{3-\alpha}+\dfrac{1}{3-\alpha}\Big)\left(x+1\right), & 0\le x\le s,\\ \dfrac{xs^{3-\alpha}}{(2-\alpha)_2}+\dfrac{xs^{2-\alpha}}{(1-\alpha)_2}+\Big(\dfrac{x^{3-\alpha}}{(2-\alpha)_2}+\dfrac{x}{3-\alpha}+\dfrac{1}{3-\alpha}\Big)\left(s+1\right), & s\le x\le 1.\end{cases}$$

Now, applying the method used in [11], it is easy to verify that problem $\{(3.1),(3.2)\}$ is equivalent to study of the following integral equation

$$v(x)=\lambda\int_0^1 G(x,s)v(s)ds. \tag{3.7}$$

Since the kernel is continuous, symmetric and positive, the integral equation (3.7), and therefore, the problem $\{(3.1),(3.2)\}$ both have a countable set of eigenvalues

$$0<\lambda_1\le\lambda_2\le\lambda_3\le\ldots\le\lambda_k<\ldots,\quad \lambda_k\to+\infty,$$

and the corresponding system of eigenfunctions $v_1(x)$, $v_2(x)$, $v_3(x)$, $\ldots$, $v_k(x)\ldots$ forms an orthonormal system in the space $L_2(0,1)$ [10].

In addition, it is not difficult to verify that the system of functions $x^{\alpha/2}v_k^{(2n)}(x)/\sqrt{\lambda_k}$, $k=1,2,\ldots$ also forms an orthonormal system in $L_2(0,1)$.

Lemma 1. *Let the function $g(x)$ satisfy the conditions (3.2) and $Mg(x)\in C(0,1)\cap L_2(0,1)$. Then, $g(x)$ can be expanded on the segment $[0,1]$ into the absolutely and uniformly convergent series in the system of eigenfunctions of the problem $\{(3.1),(3.2)\}$.*

P r o o f. Using the integration by parts rule, the properties of the Green's function $G(x,s)$, and the conditions imposed on the function $g(x)$, it is straightforward to verify the equality:

$$\int_0^1 G(x,s)Mg(s)ds=\int_0^1 G(x,s)\left[s^{\alpha}g^{(2n)}(s)\right]^{(2n)}ds=g(x).$$

Since $Mg(x) \in L_2(0,1)$, it follows from the last equality that $g(x)$ is a function representable through the kernel $G(x,s)$. Additionally, the function $G(x,s)$, i.e. the kernel of equation (3.7), is continuous in $\bar{\Omega}$. Then, based on Theorem 2 in [10, p. 153], the statement of Lemma 1 holds true. □

Lemma 2. *The following series converge uniformly on segment* $[0,1]$:

$$\sum_{k=1}^{+\infty} \left[v_k^{(j)}(x)\right]^2 / \lambda_k, \quad \sum_{k=1}^{+\infty} \left(\left[x^\alpha v_k^{(2n)}(x)\right]^{(j)}\right)^2 / \lambda_k^2, \quad j = \overline{0, 2n-1} \tag{3.8}$$

P r o o f. Considering the equality (3.1) and the properties of the function $G(x,s)$, from (3.7) at $v(x) \equiv v_k(x)$, we obtain

$$v_k^{(j)}(x) = \lambda_k \int_0^1 \frac{\partial^j}{\partial x^j} G(x,s) v_k(s) ds = \int_0^1 \left[s^\alpha v_k^{(2n)}(s)\right]^{(2n)} \frac{\partial^j}{\partial x^j} G(x,s) ds, \quad j = \overline{0, 2n-1}.$$

Hence, applying the rule of integration by parts $2n$ times, and then considering the conditions (3.2), we have

$$v_k^{(j)}(x) = \int_0^1 s^\alpha v_k^{(2n)}(s) \frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s) ds, \quad j = \overline{0, 2n-1},$$

which, due to $\lambda_k > 0$, implies the equality

$$\frac{v_k^{(j)}(x)}{\sqrt{\lambda_k}} = \int_0^1 \left(s^{\alpha/2} \frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s)\right) \left(\frac{s^{\alpha/2} v_k^{(2n)}(s)}{\sqrt{\lambda_k}}\right) ds, \quad j = \overline{0, 2n-1}. \tag{3.9}$$

From (3.9) it follows that $v_k^{(j)}(x)/\sqrt{\lambda_k}$ is the Fourier coefficient of the function by the orthonormal system

$$\left\{s^{\alpha/2} v_k^{(2n)}(s)/\sqrt{\lambda_k}\right\}_{k=1}^{+\infty}.$$

Therefore, according to Bessel's inequality [10], we obtain

$$\sum_{k=1}^{+\infty} \left[v_k^{(j)}(x)\right]^2 / \lambda_k \le \int_0^1 s^\alpha \left[\frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s)\right]^2 ds, \quad j = \overline{0, 2n-1}. \tag{3.10}$$

The integral on the right-hand side (3.10) can be rewritten as

$$\int_0^1 s^\alpha \left[\frac{\partial^{2n+j}}{\partial x^j \partial s^{2n}} G(x,s)\right]^2 ds = \int_0^1 s^{-\alpha} \left[\frac{\partial^j}{\partial x^j} \left(s^\alpha \frac{\partial^{2n}}{\partial s^{2n}} G(x,s)\right)\right]^2 ds, \quad j = \overline{0, 2n-1}.$$

Since

$$s^\alpha \frac{\partial^{2n} G(x,s)}{\partial s^{2n}}, \ \frac{\partial^j G(x,s)}{\partial x^j} \in C(\bar{\Omega}), \quad j = \overline{0, 2n-1},$$

the function in the square bracket is continuous on $\bar{\Omega}$. Then, due to $0 < \alpha < 1$, the integral on the right-hand side, and therefore the integral in (3.10), is uniformly bounded at $j = \overline{0, 2n-1}$, which implies that the first series in (3.8) converges uniformly.

The convergence of the remaining series can be proved similarly.

Lemma 2 has been proved. □

Lemma 3. *Let the conditions*

$$g^{(j)}(x) \in C[0,1], \quad j = \overline{0, 2n-1}, \quad x^{\alpha/2} g^{(2n)}(x) \in C(0,1) \cap L_2(0,1);$$
$$g^{(2j)}(0) = g^{(2j+1)}(0), \quad g^{(2j)}(1) = g^{(2j+1)}, \quad j = \overline{0, n-1}$$

be fulfilled, then the inequality

$$\sum_{k=1}^{+\infty} \lambda_k g_k^2 \leq \int_0^1 x^\alpha \left[g^{(2n)}(x)\right]^2 dx \tag{3.11}$$

holds true. Specifically, the series on the left-hand side converges, where

$$g_k = \int_0^1 g(x) v_k(x) dx, \quad k \in N.$$

P r o o f. By utilizing equation (3.1), we can write

$$\lambda_k^{1/2} g_k = \lambda_k^{1/2} \int_0^1 g(x) v_k(x) dx = \lambda_k^{-1/2} \int_0^1 g(x) \left[x^\alpha v_k^{(2n)}(x)\right]^{(2n)} dx.$$

Hence, by applying the integration by parts rule $2n$ times and considering the properties of the functions $g(x)$ and $v_k(x)$, we derive

$$\lambda_k^{1/2} g_k = \int_0^1 \left\{x^{\alpha/2} g^{(2n)}(x)\right\} \left\{\lambda_k^{-1/2} x^{\alpha/2} v_k^{(2n)}(x)\right\} dx.$$

This implies that $\lambda_k^{1/2} g_k$ is the Fourier coefficient of the function $x^{\alpha/2} g^{(2n)}(x)$ by the orthonormal system $\left\{x^{\alpha/2} v^{(2n)}(x)/\sqrt{\lambda_k}\right\}_{k=1}^{+\infty}$. Therefore, according to Bessel's inequality [10], inequality (3.11) holds true. Lemma 3 has been proved. □

Lemma 4. *Let the function $g(x)$ satisfy the conditions (3.2) and let*

$$Mg(x) \in C(0,1) \cap L_2(0,1),$$

then the following inequality holds true

$$\sum_{k=1}^{+\infty} \lambda_k^2 g_k^2 \leq \int_0^1 [Mg(x)]^2 dx. \tag{3.12}$$

Specifically, the series on the left side converges, where

$$g_k = \int_0^1 g(x) v_k(x) dx, \quad k \in N.$$

P r o o f. By virtue of the formula for g_k and equation (3.1), the equality

$$\lambda_k g_k = \lambda_k \int_0^1 g(x) v_k(x) dx = \int_0^1 g(x) \left[x^\alpha v_k^{(2n)}(x)\right]^{(2n)} dx$$

is valid.

Applying the rule of integration by parts $4n$ times to the integral on the right side and considering the properties of the functions $g(x)$ and $v_k(x)$, we get

$$\lambda_k g_k = \int_0^1 \left[x^\alpha g^{(2n)}(x)\right]^{(2n)} v_k(x)dx = \int_0^1 \left[Mg(x)\right] v_k(x)dx.$$

This implies that the value $\lambda_k g_k$ is the Fourier coefficient of the function $Mg(x)$ in the orthonormal system of functions $\{v_k(x)\}_{k=1}^{+\infty}$. Then, according to Bessel's inequality [10], inequality (3.12) holds true. Lemma 4 has been proved. □

Similarly to Lemma 3, one can prove the following

Lemma 5. *If the function $g(x)$ satisfies the conditions (3.2) and*

$$[Mg(x)]^{(j)} \in C[0,1], \quad j = \overline{0, 2n-1}; \quad x^{\alpha/2}[Mg(x)]^{(2n)} \in C(0,1) \cap L_2(0,1);$$

$$[Mg(x)]^{(2j)}\big|_{x=0} = [Mg(x)]^{(2j+1)}\big|_{x=0}, \quad [Mg(x)]^{(2j)}\big|_{x=1} = [Mg(x)]^{(2j+1)}\big|_{x=1}, \quad j = \overline{0, n-1},$$

then the inequality

$$\sum_{k=1}^{+\infty} \lambda_k^3 g_k^2 \le \int_0^1 x^\alpha \left\{[Mg(x)]^{(2n)}\right\}^2 dx$$

holds true, particularly, the series on the left side converges, where

$$g_k = \int_0^1 g(x) v_k(x) dx, \quad k \in N.$$

4. Existence, uniqueness and stability of a solution to Problem A

We will seek a solution to problem A in the form

$$u(x,t) = \sum_{k=1}^{+\infty} u_k(t) v_k(x), \tag{4.1}$$

where $v_k(x)$, $k \in N$ are the eigenfunctions of the problem $\{(3.1), (3.2)\}$, and $u_k(t)$, $k \in N$ are the unknown functions to be determined.

Substituting (4.1) into equation (2.1) and the initial conditions (2.2), with respect to $u_k(t)$, $k \in N$, we obtain the following problem

$$u_k''(t) + \lambda_k u_k(t) = f_k(t), \quad t \in (0,T), \quad k \in N,$$
$$u_k(0) = \varphi_{1k}, \quad u_k'(0) = \varphi_{2k},$$

where

$$\varphi_{jk} = \int_0^1 \varphi_j(x) v_k(x) dx, \quad j = \overline{1,2}; \quad f_k(t) = \int_0^1 f(x,t) v_k(x) dx, \quad k \in N.$$

It is known that the solution to the last problem exists, is unique and is determined by the following formula:

$$u_k(t) = \varphi_{1k}\cos\left(t\sqrt{\lambda_k}\right) + \varphi_{2k}\lambda_k^{-1/2}\sin\left(t\sqrt{\lambda_k}\right) + \lambda_k^{-1/2}\int_0^t f_k(\tau)\sin\left[(t-\tau)\sqrt{\lambda_k}\right]d\tau, \quad 0 \le t \le T. \tag{4.2}$$

From here, the following estimate

$$|u_k(t)| \le |\varphi_{1k}| + \frac{1}{\sqrt{\lambda_k}}|\varphi_{2k}| + \frac{1}{\sqrt{\lambda_k}}\sqrt{\int_0^T f_k^2(\tau)d\tau}, \quad 0 \le t \le T \tag{4.3}$$

easily follows.

Theorem 1. *Let the function $\varphi_1(x)$ satisfy the conditions of Lemma 5, the function $\varphi_2(x)$ satisfy the conditions of Lemma 4, and the function $f(x,t)$ satisfy the conditions of Lemma 4 with respect to the argument x uniformly in t. Then series (4.1), the coefficients of which are defined by the equalities (4.2), determines the solution to problem A.*

P r o o f. To do this, it is necessary to prove the uniform convergence in $\bar{\Omega}$ of series (4.1) and the following series, formally obtained from (4.1):

$$u_t(x,t) = \sum_{k=1}^{+\infty} u_k'(t)v_k(x),$$

$$\frac{\partial^j u(x,t)}{\partial x^j} = \sum_{k=1}^{+\infty} u_k(t)v_k^{(j)}(x), \quad j = \overline{1, 2n-1},$$

$$\frac{\partial^j}{\partial x^j}\left(x^\alpha \frac{\partial^{2n} u(x,t)}{\partial x^{2n}}\right) = \sum_{k=1}^{+\infty} u_k(t)\left(x^\alpha v_k^{(2n)}(x)\right)^{(j)}, \quad j = \overline{0, 2n-1}$$

and uniform convergence in any compact set of $\Omega_0 \subset \Omega$ the series

$$\frac{\partial^{2n}}{\partial x^{2n}}\left(x^\alpha \frac{\partial^{2n} u(x,t)}{\partial x^{2n}}\right) = \sum_{k=1}^{+\infty} u_k(t)\left(x^\alpha v_k^{(2n)}(x)\right)^{(2n)}, \tag{4.4}$$

$$u_{tt}(x,t) = \sum_{k=1}^{+\infty} u_k''(t)v_k(x). \tag{4.5}$$

Let us consider series (4.1). By virtue of (4.3) from (4.1), for any $(x,t) \in \bar{\Omega}$ we have

$$|u(x,t)| \le \sum_{k=1}^{+\infty} |u_k(t)|\,|v_k(x)| \le \sum_{k=1}^{+\infty} \frac{|v_k(x)|}{\sqrt{\lambda_k}}\left(\sqrt{\lambda_k}\,|\varphi_{1k}| + |\varphi_{2k}| + \sqrt{\int_0^T f_k^2(\tau)d\tau}\right).$$

From here, applying the Cauchy–Schwarz inequality, we obtain

$$|u(x,t)| \le \sqrt{\sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k}}\left(\sqrt{\sum_{k=1}^{+\infty} \lambda_k \varphi_{1k}^2} + \sqrt{\sum_{k=1}^{+\infty} \varphi_{2k}^2} + \sqrt{\int_0^T \sum_{k=1}^{+\infty} [f_k(\tau)]^2\, d\tau}\right). \tag{4.6}$$

The series on the right-hand sides of this inequality, due to the conditions of Theorem 1, according to Lemmas 2 and 3, converges uniformly. Therefore, the series on the left side, i.e. series (4.1), converges uniformly in $\bar{\Omega}$.

Now, we consider the series (4.4). By virtue of equation (3.1), in any compact set Ω_0 the series in (4.4) may be written in the form

$$\sum_{k=1}^{+\infty} \lambda_k u_k(t)v_k(x). \tag{4.7}$$

To prove the uniform convergence of series (4.7), according to (4.3), it is enough to prove the absolute and uniform convergence of the series

$$\sum_{k=1}^{+\infty} \lambda_k \varphi_{1k} v_k(x), \quad \sum_{k=1}^{+\infty} \sqrt{\lambda_k}\varphi_{2k} v_k(x), \quad \sum_{k=1}^{+\infty} \sqrt{\lambda_k}\sqrt{\int_0^T [f_k(\tau)]^2\, d\tau}\, v_k(x). \tag{4.8}$$

In Ω_0, we apply the Cauchy-Schwarz inequality to each of these series:

$$\Big|\sum_{k=1}^{+\infty}\lambda_k\varphi_{1k}v_k(x)\Big| \le \sum_{k=1}^{+\infty}\Big|\sqrt{\lambda_k^3}\varphi_{1k}\frac{v_k(x)}{\sqrt{\lambda_k}}\Big| \le \Big[\sum_{k=1}^{+\infty}\lambda_k^3\varphi_{1k}^2\sum_{k=1}^{\infty}\frac{v_k^2(x)}{\lambda_k}\Big]^{1/2},$$

$$\Big|\sum_{k=1}^{+\infty}\sqrt{\lambda_k}\varphi_{2k}v_k(x)\Big| \le \sum_{k=1}^{+\infty}\Big|\lambda_k\varphi_{2k}\frac{v_k(x)}{\sqrt{\lambda_k}}\Big| \le \Big[\sum_{k=1}^{+\infty}\lambda_k^2\varphi_{2k}^2\cdot\sum_{k=1}^{\infty}\frac{v_k^2(x)}{\lambda_k}\Big]^{1/2},$$

$$\Big|\sum_{k=1}^{+\infty}\sqrt{\lambda_k}\sqrt{\int_0^T[f_k(\tau)]^2d\tau\cdot v_k(x)}\Big| \le \sum_{k=1}^{+\infty}\Big|\sqrt{\lambda_k^2\int_0^T[f_k(\tau)]^2d\tau\cdot\frac{v_k(x)}{\sqrt{\lambda_k}}}\Big|$$

$$\le\Big[\int_0^T\sum_{k=1}^{+\infty}\lambda_k^2[f_k(\tau)]^2d\tau\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\Big]^{1/2}.$$

The series on the right-hand sides of these inequalities, due to the conditions of Theorem 1, according to Lemmas 2, 4 and 5, converges uniformly. Then the series located on the left sides, i.e. series (4.8) converges absolutely and uniformly in Ω_0. Therefore, the series (4.7), and therefore the series in (4.4), converges uniformly in the compact set Ω_0. The uniform convergence in Ω_0 of series (4.5) follows from the convergence of series (4.4) and the validity of equation (2.1).

The uniform convergence of the remaining series is similarly proved. Theorem 1 has been proved. □

Theorem 2. *A problem a cannot have more than one solution.*

P r o o f. Let us assume that there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ of problem A. We denote their difference by $u(x,t)$. Then the function $u(x,t)$ satisfies the equation (2.1) for $f(x,t)\equiv 0$, and conditions (2.2) and (2.3) for $\varphi_1(x)\equiv\varphi_2(x)\equiv 0$.

Let $\forall\, T_0\in(0,T]$,

$$\Omega_0=\{(x,t):0<x<1,\ 0<t<T_0\}.$$

It is obvious that $\bar\Omega_0\subset\bar\Omega$. Let us introduce the following function:

$$\omega(x,t)=-\int_t^{T_0}u(x,\xi)d\xi,\quad (x,t)\in\bar\Omega_0.$$

This function has the following properties:

1) $\omega_t,\,\omega_{tt},\,\dfrac{\partial^j\omega}{\partial x^j},\,\dfrac{\partial^j}{\partial x^j}\left(x^\alpha\dfrac{\partial^{2n}\omega}{\partial x^{2n}}\right)\in C\left(\bar\Omega_0\right),\quad j=\overline{0,2n-1};$

2) it satisfies the conditions (2.3) at $t\in[0,T_0]$.

Let us consider the equation (2.1) for $f(x,t)\equiv 0$ and multiply it by the function $\omega(x,t)$, and then integrate the resulting equality over the domain Ω_0 :

$$\int_{\Omega_0}\omega(x,t)\Big\{u_{tt}(x,t)+\frac{\partial^{2n}}{\partial x^{2n}}\Big[x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\Big]\Big\}dtdx=0.$$

We rewrite this equality as

$$\int_0^{T_0}dt\int_0^1\omega(x,t)\frac{\partial^{2n}}{\partial x^{2n}}\Big[x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\Big]dt+\int_0^1dx\int_0^{T_0}\omega(x,t)u_{tt}(x,t)dt=0.$$

Now, applying the rule of integration by parts, we obtain

$$\int_0^{T_0}\left[\omega(x,t)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)-\frac{\partial\omega(x,t)}{\partial x}\frac{\partial^{2n-2}}{\partial x^{2n-2}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)+\ldots\right.$$
$$\left.+\ldots-\frac{\partial^{2n-1}\omega(x,t)}{\partial x^{2n-1}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)\right]_{x=0}^{x=1}dt+\int_0^{T_0}dt\int_0^1 x^\alpha\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}dx+$$
$$+\int_0^1\left[\omega(x,t)\frac{\partial u(x,t)}{\partial t}\Big|_{t=0}^{t=T_0}-\int_0^{T_0}\frac{\partial\omega(x,t)}{\partial t}\frac{\partial u(x,t)}{\partial t}\right]dx=0,$$

from which, due to the properties of functions $\omega(x,t)$ and $u(x,t)$, the equality

$$\int_0^{T_0}dt\int_0^1 x^\alpha\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}dx-\int_0^1 dx\int_0^{T_0}\frac{\partial\omega(x,t)}{\partial t}\frac{\partial u(x,t)}{\partial t}dt=0$$

follows.

Hence, taking into account equalities

$$u=\frac{\partial\omega}{\partial t},\quad \frac{\partial^{2n}u}{\partial x^{2n}}=\frac{\partial^{2n+1}\omega}{\partial x^{2n}\partial t},$$

we have

$$\int_0^1 x^\alpha dx\int_0^{T_0}\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n+1}\omega(x,t)}{\partial x^{2n}\partial t}dt-\int_0^1 dx\int_0^{T_0}u(x,t)\frac{\partial u(x,t)}{\partial t}dt=0.$$

Further, taking into account the equalities

$$u(x,t)\frac{\partial u(x,t)}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}[u(x,t)]^2,\quad \frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n+1}\omega(x,t)}{\partial x^{2n}\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\right]^2,$$

and applying the rule of integration by parts to integrals over t, taking into account $\omega(x,T_0)=0$, $u(x,0)=0$, we obtain

$$\int_0^1 u^2(x,T_0)\,dx+\int_0^1 x^\alpha\left[\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\right]_{t=0}^2 dx=0.$$

It follows that $u(x,T_0)\equiv 0$, $x\in[0,1]$. Since we considered $\forall\,T_0\in[0,T]$, then $u(x,t)\equiv 0$, $(x,t)\in\bar\Omega$. Then $u_1(x,t)\equiv u_2(x,t)$, $(x,t)\in\bar\Omega$. Theorem 2 is proven. □

Theorem 3. *Let functions $\varphi_1(x)$, $\varphi_2(x)$ and $f(x,t)$ satisfy the conditions of Theorem 1. Then for the solution of Problem A the following estimates*

$$\|u(x,t)\|^2_{L_2(0,1)}\le K_0\left[\|\varphi_1(x)\|^2_{L_2(0,1)}+\|\varphi_2(x)\|^2_{L_2(0,1)}+\|f(x,t)\|^2_{L_2(\Omega)}\right],\tag{4.9}$$
$$B\|u(x,t)\|_{C(\Omega)}\le K_1\left[\|\varphi_1^{(2n)}(x)\|_{L_{2,r}(0,1)}+\|\varphi_2(x)\|_{L_2(0,1)}+\|f(x,t)\|_{L_2(\Omega)}\right],\tag{4.10}$$

are valid, where

$$\|\varphi_1(x)\|_{L_{2,r}(0,1)}=\left[\int_0^1 x^\alpha\left[\varphi_1(x)\right]^2dx\right]^{1/2}$$

and $r=r(x)=x^\alpha$, and K_0 and K_1 are some real positive numbers.

P r o o f. Here, taking into account the orthonormality of the system $\{v_k(x)\}_{k=1}^{+\infty}$ and inequality (4.3) followed from (4.1), we obtain

$$\|u(x,t)\|^2_{L_2(0,1)} = \sum_{k=1}^{+\infty} u_k^2(t) \le \sum_{k=1}^{+\infty} \left[|\varphi_{1k}| + \frac{1}{\sqrt{\lambda_k}} |\varphi_{2k}| + \frac{1}{\sqrt{\lambda_k}} \|f_k(t)\|_{L_2(0,T)} \right]^2$$

$$\le 3\sum_{k=1}^{+\infty} \left[\varphi_{1k}^2 + \frac{1}{\lambda_k}\varphi_{2k}^2 + \frac{1}{\lambda_k} \|f_k(t)\|^2_{L_2(0,T)} \right] \le 3\sum_{k=1}^{+\infty} \left[\varphi_{1k}^2 + \frac{1}{\lambda_1}\varphi_{2k}^2 + \frac{1}{\lambda_1} \|f_k(t)\|^2_{L_2(0,T)} \right].$$

Hence, considering Bessel's inequality, we get

$$\|u(x,t)\|^2_{L_2(0,1)} \le K_0 \left(\|\varphi_1(x)\|^2_{L_2(0,1)} + \|\varphi_2(x)\|^2_{L_2(0,1)} + \sum_{k=1}^{+\infty} \|f_k(t)\|^2_{L_2(0,T)} \right), \tag{4.11}$$

where $K_0 = 3C,\ C = \max\left(1, 1/\lambda_1\right).$

Taking into account the following easily verifiable equality

$$\|f(x,t)\|^2_{L_2(\Omega)} = \sum_{n=1}^{+\infty} \|f_k(t)\|^2_{L_2(0,T)},$$

from (4.11), we obtain inequality (4.9).

Further, according to the statements of Lemmas 2 and 3, from (4.6) it follows

$$\|u(x,t)\|_{C(\bar{)}} = \sup_{\Omega} |u(x,t)| \le K_1 \left\{ \sqrt{\int_0^1 x^\alpha [\varphi_1^{(2n)}(x)]^2 dx} + \sqrt{\sum_{k=1}^{+\infty} \varphi_{2k}^2} + \sqrt{\int_0^{T} \sum_{k=1}^{+\infty} \left[f_k(\tau)\right]^2 d\tau} \right\},$$

where

$$K_1 = \sup_{[0,1]} \sqrt{\sum_{k=1}^{+\infty} v_k^2\left(x\right)/\lambda_k}\ .$$

From here, due to the introduced notation, inequality (4.10) follows. Theorem 3 has been proved. □

5. Conclusion

In a quadrilateral, an initial boundary-value problem has been considered for a high-order partial differential equation that degenerates at the boundary of the domain. The uniqueness of the solution to the problem was proved by the method of energy integrals. The solution to the problem was found in the form of a Fourier series. The sufficient conditions for the given functions have been identified that ensure the existence of a solution to the problem. The estimates for the solution of the problem in spaces $L_2\left[0,1\right]$ and $C\left[0,1\right]$ have been obtained.

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THE IMPACT OF TOXICANTS IN THE MARINE THREE ECOLOGICAL FOOD-CHAIN ENVIRONMENT: A MATHEMATICAL APPROACH

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Abstract: To explore the impact of toxicants on a marine ecological food chain system consisting of three species, this work develops and analyzes a non-linear mathematical model. The model consists of five state variables: phytoplankton, zooplankton, fish, environmental toxicant, and organismal toxicant. We have incorporated the Monod-Haldane functional response as a predation function for each species. Using the Jacobian matrix, the stability analysis was conducted, and necessary constraints were obtained for the system's local and global stability. Hopf bifurcation analysis was performed for carrying capacity (K) and the rate of decrease in the growth rate of phytoplankton due to the presence of toxicants (r_1). Also, phase portraits are presented for different parameters of the model. In addition, numerical simulations are executed using MATLAB to prove theoretical findings and explore the impact of parameter variation on ecological species behavior.

Keywords: Environmental toxicant, Marine food chain, Stability, Hopf-bifurcation, Lyapunov function.

1. Introduction

It is well known that environmental contamination poses a significant threat to marine ecosystems. The main causes of it are industrial discharge and chemical spills. The rapid expansion of modern industry and agriculture significantly contributes to environmental pollution and habitat degradation. These pollutants contain harmful elements such as cadmium, zinc, copper, iron and mercury. As a result of the destruction of their natural ecosystems and increased exposure to dangerous pollutants, many species face serious risks to their survival, and many are on the verge of becoming extinct. Therefore, it is essential to study toxic substances in marine ecosystems from an environmental and conservational point of view.

In recent decades, mathematical models have become tremendously helpful in understanding and assessing the feeding relationships between species within ecosystems. In [2], Babu et al. explored the dynamic difficulties of a three-species food chain model. From the stability analysis, sufficient constraints for the survival and extinction of the population under toxicant stress have been revealed. Zhang et al. [22] considered an experimental marine food chain with three levels (microalgae $\rightarrow$ zooplankton $\rightarrow$ fish) to evaluate how feeding selectivity affects the transmission of methylmercury ($MeHg^+$) across the food chain system. In [11], Misra and Babu proposed and examined a three-species mathematical model in the presence of environmental and organismal toxicants. They found that Hopf bifurcation occurs at the predation rate of the intermediate predator. They also note that the system containing toxicants appears to be more stable than the toxicant-free system. Kalyan Das et al. [5] determine how the nanoparticle influences the interaction between phytoplankton and zooplankton. They observed that when zooplankton consumes phytoplankton, the growth of the zooplankton is slowed down by nanoparticles. Majeed and Kadhim [13] discussed the occurrence of local bifurcation and persistence under suitable food chain conditions, including a model of prey-first predator-second predator under the influence of toxins on all species. Talb et al. [20] considered a three-species aquatic food chain model in a polluted environment. It is noted that there are rich dynamics in the proposed food chain model, including periodic and chaotic. Kavita Yadav et al. [21] examined a marine tri-trophic food chain system that has distributed delay and environmental toxicants. They observed that distributed delay and environmental toxicants are crucial variables in the occurrence of Hopf bifurcation. Mandal et al. [14] created a mathematical model to study the control of the harmful effects of toxicants on the phytoplankton-zooplankton system by raising public awareness among people. They reveal that a moderate level of anthropogenic pollution might cause the phytoplankton-zooplankton system to become unstable. However, the contaminated system becomes stable due to public awareness. Smith and Weis [18] have observed that fish from polluted environments have much higher mortality rates than fish from unpolluted areas when they were exposed to a predator (blue crab Callinectes sapidus Rathbun).

Although several mathematical models may be used to explain the dynamics of interacting species, predator-prey theory is still based on the predator's functional response. Pal et al. [17] developed a simplified Monad Haldane (MH) functional response for toxin-producing phytoplankton and zooplankton populations and investigated how the toxication process of phytoplankton affects bloom creation and termination. Lui and Tan [9] where MH functional response is used for group defense theory. Several studies, based on theoretical and experimental data, have examined tri-trophic food chain systems, focusing on the impact of toxicants on the system's survival or extinction [1, 3, 4, 6–8, 10, 15, 16, 19]. So, these investigations encourage us to investigate the dynamics of the fish, phytoplankton, and zooplankton systems when toxicants are present.

In this paper, we formulated a mathematical model to study the impact of toxicants in a three-species marine food chain system considering Monad–Haldane functional responses. The existence of several equilibrium points has been examined. Then we established the local stability of the system using the Jacobian matrix. We also use the Lyapunov function and the Routh–Hurwitz criteria to assess the global stability and durability of the system.

2. Model formulation

Here, we consider an ecological model with three marine species. There are two ways through which toxicants can enter an organism. It can be absorbed by the population through resources (food chain) or directly from the environment. The model assumes that organismal toxicants have a negative impact on the growth rate of prey populations. In the absence of organismal toxicants, the prey's population growth follows logistic growth. In the model there are five state variables: $x(t)$ density of phytoplankton, $y(t)$ density of zooplankton, $z(t)$ density of fish, $c_e(t)$ concentration of environmental toxicants and $c_0(t)$ concentration of organism toxicant in the prey population. By considering these as state variables, we formulate a mathematical model to investigate the effects of toxicants on a three-species marine food chain system using the following system of non-linear ordinary differential equations

$$\frac{dx}{dt} = xr(c_0)\left(1 - \frac{x}{K}\right) - \frac{axy}{\alpha x^2 + m}, \tag{2.1}$$

$$\frac{dy}{dt} = \frac{bxy}{\alpha x^2 + m} - d_1 y - \frac{cyz}{\beta y^2 + h} - g_1 y^2, \tag{2.2}$$

$$\frac{dz}{dt} = \frac{dyz}{\beta y^2 + h} - d_2 z - g_2 z^2, \tag{2.3}$$

$$\frac{dc_e}{dt} = q_0 - a_1 c_e - a_2 x c_e + v x c_0, \tag{2.4}$$

$$\frac{dc_0}{dt} = a_2 x c_e - b_1 c_0 - v x c_0, \tag{2.5}$$

with $x(0) \geq 0,\ \ y(0) \geq 0,\ \ z(0) \geq 0,\ \ c_0 \geq 0,\ \ c_e(0) > 0$. Here, we assumed that the growth of phytoplankton is negatively affected by organismal toxicants, we consider

$$r(c_0) = r_0 - r_1 c_0,$$

where r_0 denotes the intrinsic growth rate of phytoplankton, r_1 is the constant that determines the rate of decrease in the growth rate of phytoplankton due to the presence of toxicants, and K is the environmental capacity.

The expression $axy/(\alpha x^2 + m)$ describes the predation of phytoplankton by zooplankton following Monad Haldane functional response, a is the predation rate, m is the saturation constant which is scaling the impact of the predator interference, food chain and food weighting factor, α denotes the inhibitory effect.

As the zooplankton population consumes the phytoplankton population, the growth is directly related to the rate at which phytoplankton is consumed, *i.e.*, response function for zooplankton is $bxy/(\alpha x^2 + m)$, where b is conversion coefficient, d_1 is the natural death rate of zooplankton and g_1 is the intraspecies competition coefficient among zooplankton population.

The term $cyz/(\beta y^2 + h)$ describes the predation of zooplankton by fish, c denotes the predation rate, h is the saturation constant which is scaling the impact of the predator interference, food chain and food weighting factor, and β denotes the inhibitory effect.

As zooplankton is consumed by the fish population, so the growth of fish is $dyz/(\beta y^2 + h)$, where d is the conversion coefficient of zooplankton to fish, d_2 is the natural death rate of fish population and g_2 is the intraspecies competition coefficient among fish population.

Let q_0 represents the external input of toxicant into the environment. The parameter v denotes the removal rate of a toxicant from the prey population (phytoplankton) due to its death. The parameter a_2 denotes the removal rate of a toxicant from the environment due to uptake by the phytoplankton (prey) populations. Furthermore, b_1 and a_1 denote the washout rates of organismal and environmental toxicant, respectively.

3. Boundedness of the Model

Determining the boundedness of solutions is essential to ensuring the system's biological feasibility. It guarantees that all population densities remain finite and non-negative for all time. Now we will determine the region of attraction, where our system is bounded.

Theorem 1. *Let the set*

$$\Omega = \Big\{(x, y, z, c_e, c_o) \in \mathbb{R}^5 :\ x(t) \leq K,\ x(t) + \frac{a}{b} y(t) + \frac{ac}{bd} z(t) \leq K_1,$$
$$c_e(t) + c_0(t) \leq K_2,\ c_e(t) \geq K_3,\ x(t) + c_e(t) \geq K_4\Big\},$$

then all solutions of the system are bounded in the region Ω, *where*

$$K_1 = \frac{(r_0 + 1)K}{\phi_1}, \quad K_2 = \frac{q_0}{\phi_2}, \quad K_3 = \frac{q_0}{a_1 + a_2 K}, \quad K_4 = \frac{(q_0 - aK_1)}{\phi_3},$$
$$\phi_1 = \min\{d, d_2, 1\}, \quad \phi_2 = \min\{a_1, b_1\}, \quad \phi_3 = \max\{r_1 K_2 - r_0, a_1 + a_2 K\}.$$

P r o o f. From (2.1), we get

$$\frac{dx}{dt} \le xr_0\left(1 - \frac{x}{K}\right).$$

By the usual comparison theorem, we get as $t \to \infty$,

$$x(t) \le K.$$

Now, let us consider the following function:

$$F(t) = x(t) + \frac{a}{b}y(t) + \frac{ac}{bd}z(t)$$

by using (2.1), (2.2) and (2.3), we get

$$\frac{dF}{dt} + \phi_1 F \le K(r_0 + 1),$$

where $\phi_1 = \min\{1,\ d,\ d_2\}$ then, by the usual comparison theorem, we get as $t \to \infty$

$$F(t) \le \frac{K(r_0+1)}{\phi_1}, \quad F(t) = x(t) + \frac{a}{b}y(t) + \frac{ac}{bd}z(t) \le K_1, \quad K_1 = \frac{K(r_0+1)}{\phi_1}.$$

Again, consider the following function:

$$G(t) = c_e(t) + c_0(t),$$

then by using (2.4), (2.5), we get

$$\frac{dG}{dt} + (a_1 c_e + b_1 c_0) \le q_0,$$

then again using usual comparison theorem, we get as $t \to \infty$,

$$G(t) \le \frac{q_0}{\phi_2},$$

where $\phi_2 = \min\{a_1,\ b_1\}$, and hence

$$c_e(t) + c_0(t) \le K_2, \quad K_2 = \frac{q_0}{\phi_2}.$$

From (2.4) we get,

$$\frac{dc_e}{dt} + (a_1 + a_2 K)c_e \ge q_0,$$

then, we get as $t \to \infty$,

$$c_e(t) \ge K_3, \quad K_3 = \frac{q_0}{a_1 + a_2 K}.$$

Now let us consider the following function:

$$H(t) = x(t) + c_e(t),$$

by using (2.1) and (2.4) we get,

$$\frac{dH}{dt} + \phi_3 H \ge (q_0 - aK_1),$$

where

$$\phi_3 = \max\{r_1 K_2 - r_0,\ a_1 + a_2 K\},$$

then we get as $t \to \infty$,

$$H(t) \ge (q_0 - aK_1),$$

and hence,

$$x(t) + c_e(t) \ge K_4, \quad K_4 = \frac{(q_0 - aK_1)}{\phi_3}.$$

Hence, all the solutions of the system are bounded in the region Ω. □

4. Analysis of Model

4.1. Existence of equilibrium points

In steady-state solutions, where population densities do not change over time, the system's equilibrium points are found. These can be determined by solving the system of algebraic equations obtained by setting the right-hand sides of differential equations to zero. The set of four equilibrium points considered in this study includes all biologically feasible configurations of species survival and extinction under the influence of toxicants. Specifically, we examine: (*i*) the trivial equilibrium where no species survive, (*ii*) boundary equilibria representing partial survival of one or two species, and (*iii*) the interior equilibrium where all species coexist. Thus, the mathematical model has the following four positive equilibrium points, namely, $E_0(0,0,0,c_e,0)$, $\bar{E}_1(\bar{x},0,0,\bar{c_e},\bar{c_0})$, $\hat{E}_2(\hat{x},\hat{y},0,\hat{c_e},\hat{c_0})$, $E_3^\star(x^\star,y^\star,z^\star,c_e^\star,c_0^\star)$.

- *For the equilibrium point* $E_0(0,0,0,c_e,0)$:
  - from (2.4) we get $c_e = q_0/a_1$. When only an environmental toxicant is present, then the equilibrium point is $E_0(0,0,0,q_0/a_1,0)$.
- *In the absence of Zooplankton and Fish* $\bar{E}_1(\bar{x},0,0,\bar{c_e},\bar{c_0})$:
  - from (2.1) $\bar{x} = K$;
  - from (2.5) $\bar{c_0} = a_2K\bar{c_e}/(b_1+vK)$;
  - from (2.4)
  $$\bar{c_e} = \frac{q_0}{a_1 + a_2K - a_2vK^2/(b+vK)},$$
  $\bar{c_e} > 0$ if $(a_1+a_2K)(b+vK) > a_2vK^2$.
- *In the absence of Fish* $\hat{E}_2(\hat{x},\hat{y},0,\hat{c_e},\hat{c_0})$:
  - from (2.2) we get
  $$\hat{y} = \frac{1}{g_1}\Big[\frac{b\hat{x}}{\alpha\hat{x}^2+m} - d_1\Big] \tag{4.1}$$
  $\hat{y} > 0$ if $b\hat{x} > (\alpha\hat{x}^2+m)d_1$;
  - from (2.4)
  $$\hat{c_e} = \frac{q_0(b_1+v\hat{x})}{(a_1+a_2\hat{x})(b_1+v\hat{x}) - va_2\hat{x}^2}$$
  $\hat{c_e} > 0$ provided $(a_1+a_2\hat{x})(b_1+v\hat{x}) > va_2\hat{x}^2$;
  - from (2.5)
  $$\hat{c_0} = \frac{a_2\hat{x}\hat{c_e}}{b_1+v\hat{x}}; \tag{4.2}$$
  - from (2.1) we get an algebraic equation in $\hat{x}$ variable,
  $$(r_0 - r_1\hat{c_0})(\alpha\hat{x}^2+m)\Big(1-\frac{\hat{x}}{K}\Big) - a\hat{y} = 0.$$

  A positive solution is obtained by solving the above equation for $\hat{x}$ and then the values of $\hat{c_0}$, $\hat{c_e}$, $\hat{y}$ can be computed from equations (4.1) to (4.2).
- *When all the species are present* (non-trivial equilibrium point) $E_3^\star(x^\star,y^\star,z^\star,c_e^\star,c_0^\star)$: the existence of the equilibrium point $E_3^\star$ has been established through the isocline method [12],

– from (2.1)

$$c_0^\star = \frac{K}{r_1(K-x)}\Big[r_0\Big(1-\frac{x}{K}\Big) - \frac{ay}{\alpha x^2+m}\Big] = m_1(x,y); \tag{4.3}$$

– from (2.4) and (2.5),

$$c_e^\star = \frac{1}{a_1}[q_0 - b_1 m_1(x,y)] = m_2(x,y);$$

– from (2.2),

$$z^\star = \frac{\beta y^2+h}{c}\Big[\frac{bx}{\alpha x^2+m} - d_1 - g_1 y\Big] = m_3(x,y). \tag{4.4}$$

Now, considering two functions (from (2.2) to (2.4)),

$$S_{11}(x,y) = q_0 - (a_1+a_2x)m_2(x,y) + vxm_1(x,y),$$

$$S_{12}(x,y)\frac{bdxy}{\alpha x^2+m} + vxm_1(x,y) + q_0 - d_1y(d+g_1y) - cz(d_2+g_2z) - (a_1+a_2x)m_2(x,y).$$

For the existence of $x^\star$ and $y^\star$, the two isoclines,

$$S_{11}(x,y) = 0, \tag{4.5}$$

$$S_{12}(x,y) = 0, \tag{4.6}$$

must intersect. We note that

$$S_{11}(0,0) = \frac{br_0}{r_1} > 0, \quad S_{12}(0,0) = \frac{br_0}{r_1} + hd_1d_2 - \frac{g_2h^2d_1^2}{c},$$

$$S_{12}(0,0) > 0 \quad \text{if} \quad \frac{br_0}{r_1} + hd_1d_2 > \frac{g_2h^2d_1^2}{c}.$$

Also considering, $S_{11}(x,0) = 0$ then x will be a positive root (say) ϕ_1, from the following value of x,

$$x = \frac{ba_1r_0}{a_2(br_0 - r_1q_0) - va_1r_0} > 0,$$

if $a_2(br_0 - r_1q_0) - va_1r_0 > 0$.
Now, consider $S_{11}(0,y) = 0$ then,

$$y = \frac{mr_0}{a} = \phi_2.$$

Now, let us consider $S_{12}(x,0) = 0$, then x will have one positive root (say) ϕ_3, from the following cubic equation of x,

$$\alpha Bx^3 + \alpha Ax^2 + (\alpha mB - bh)x + mA = 0,$$

if $\alpha mB < bh$ and $mA > 0$, where,

$$A = \frac{r_0b_1}{r_1} + d_1h > 0, \quad B = \Big[\frac{r_0v}{r_1} - \frac{a_2}{a_1}\Big(q_0 - \frac{b_1r_0}{r_1}\Big)\Big].$$

Now $S_{12}(0,y) = 0$, then y will have one positive root (say) ϕ_4, from the following equation of y,

$$A_1y^6 + A_2y^5 + A_3y^4 - A_4y^3 + A_5y^2 + A_6y - A_7 = 0,$$

$$A_1 = \frac{g_2\beta^2}{c}, \quad A_2 = \frac{2d_1g_1\beta^2g_2}{c}, \quad A_3 = \frac{2g_2\beta g_1^2h}{c} + \frac{g_2\beta^2d_1^2}{c},$$

$$A_4 = g_1d_2\beta - \frac{4g_2g_1d_1h\beta}{c}, \quad A_5 = \frac{2\beta hd_1^2g_2}{c} - \frac{g_1^2g_2h^2}{c} - d_1d_2\beta + g_1d_1,$$

$$A_6 = \frac{2g_1g_2d_1h^2}{c} - d_2hg_1 + dd_1 + \frac{ab_1}{r_1m}, \quad A_7 = \frac{b_1r_0}{r_1} + d_1d_2h - \frac{g_2d_1^2h^2}{c},$$

if $A_4 > 0$, $A_5 < 0$, A_6 and $A_7 > 0$. Thus, both the isoclines intersect each other in the region ω

$$\omega = \{(x, y) : 0 < x < \phi_3,\ 0 < y < \phi_2\},$$

in the following two cases (see Fig. 1):

$$(i): \quad \phi_3 > \phi_2, \quad \phi_1 > \phi_4,$$
$$(ii): \quad \phi_3 < \phi_2, \quad \phi_1 < \phi_4.$$

This point of intersection will give $x^\star$, $y^\star$. For the uniqueness of the $(x^\star, y^\star)$, we must have $dy/dx < 0$ for the curves in the region ω. For the curve (4.5),

$$\frac{dy}{dx} = \frac{(\alpha x^2+m)}{aKF_2}\Big(F_1 r_1(K-x)(\alpha x^2+m) - F_2 K\Big(-\frac{r_0(K-x)}{K} + \frac{2a\alpha xy}{\alpha x^2+m} + A_8\Big)\Big) < 0, \quad (4.7)$$

where

$$F_1 = \frac{a_2}{a_1}(q_0 - b_1 m_1) - v m_1, \quad F_2 = \frac{a_1 + a_2 x}{a_1} b_1 + vx, \quad A_8 = r_0\Big(1 - \frac{x}{K}\Big) - \frac{ay}{\alpha x^2 + m}$$

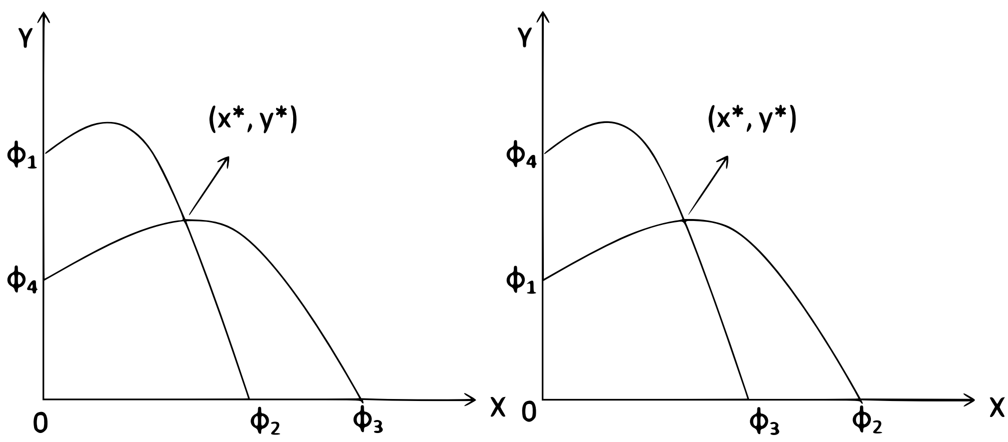
and for curve (4.6)

$$\frac{dy}{dx} = \frac{G_1 - G_2 - cm_3'(x,y)(d_2 + 2g_2 m_3) - bdy/(\alpha x^2+m)}{d_1(d+2gy) - bd/(\alpha x^2+m)} < 0, \quad (4.8)$$

where

$$G_1 = m_1'(x,y)\Big[vx + \frac{b_1(a_1 + a_2 x)}{a_1}\Big], \quad G_2 = m_1(x,y)\Big[v + \frac{a_2 b_1}{a_1} - \frac{a_2 q_0}{a_1}\Big].$$

In case (i), the absolute value of dy/dx given by (4.7) is less than the absolute value of dy/dx given by (4.8). For the case (ii), the condition is the opposite. Knowing the value of $x^\star$, $y^\star$; $z^\star$, $c_e^\star$ and $c_0^\star$ can be computed from the (4.3) to (4.4).



Case (i): $\phi_3 > \phi_2,\ \phi_1 > \phi_4$. Case (ii): $\phi_3 < \phi_2,\ \phi_1 < \phi_4$.

Figure 1. Existence of equilibrium point $E_3^\star$ of the Model.

4.2. Local stability of the Model

Local stability analysis investigates the behavior of solutions in proximity to equilibrium points through the examination of the Jacobian matrix. To validate the local stability of the equilibrium, the eigenvalues of the Jacobian matrix are computed at each equilibrium point. If all eigenvalues have a negative real part, the equilibrium point is locally asymptotically stable.

The Jacobian matrix associated with the Model is

$$J = \begin{bmatrix} d_{11} & -d_{12} & 0 & -d_{13} & 0 \\ d_{21} & -d_{22} & -d_{23} & 0 & 0 \\ 0 & d_{32} & d_{33} & 0 & 0 \\ d_{41} & 0 & 0 & d_{44} & d_{45} \\ d_{51} & 0 & 0 & d_{54} & d_{55} \end{bmatrix},$$

$$d_{11} = r(c_0)\Big(1 - \frac{2x}{K}\Big) - \frac{ay(m-\alpha x^2)}{(\alpha x^2+m)^2}, \quad d_{12} = \frac{ax}{\alpha x^2+m}, \quad d_{13} = r_1 x\Big(1-\frac{x}{K}\Big),$$

$$d_{21} = \frac{by(m-\alpha x^2)}{(\alpha x^2+m)^2}, \quad d_{22} = d_1 + 2g_1 y + \frac{cz(h-\beta y^2)}{(\beta y+h)^2}, \quad d_{23} = \frac{cy}{\beta y^2+h},$$

$$d_{32} = \frac{dz(h-\beta y^2)}{(\beta y+h)^2}, \quad d_{33} = \frac{dy}{\beta y^2+h} - d_2 - 2g_2 z,$$

$$d_{44} = xv, \quad d_{41} = -a_2 c_e + vc_0, \quad d_{45} = -a_1 - a_2 x,$$

$$d_{51} = a_2 c_e - vc_0, \quad d_{54} = -b_1 - vx, \quad d_{55} = a_2.$$

- **At $\mathbf{E_0}$**, the eigenvalues of the characteristic equation are r_0,$-d_1$, $-d_2$ and $\pm\sqrt{a_1 b_1}$, showing the instability of E_0 since one eigenvalue is positive.
- **At $\mathbf{\bar{E}_1}$**, two eigenvalues of the characteristic equation are,$-d_1$,$-d_2$, and the remaining three eigenvalues are given by the roots of the following cubic equation

  $$\lambda^3 + S_1\lambda^2 + S_2\lambda + S_3 = 0,$$

  where

  $$S_1 = \frac{\bar{x} r(\bar{c_0})}{K} - (a_1 + a_2\bar{x}) - r(\bar{c_0})\Big(1-\frac{\bar{x}}{K}\Big),$$
  $$S_2 = c_1\bar{x}(a_2+v) + a_{13}(v\bar{c_0} - a_2\bar{c_e}) - a_2 b_1\bar{x} - a_1 b_1 - a_1 v\bar{x},$$
  $$S_3 = a_{13}a_1(v\bar{c_0} - a_2\bar{c_e}) + c_1(a_2 b_1\bar{x} + a_1 b_1 + a_1 v\bar{x}),$$
  $$c_1 = \frac{\bar{x} r(\bar{c_0})}{K} - (a_1 + a_2\bar{x}) - r(\bar{c_0})\Big(1-\frac{\bar{x}}{K}\Big).$$

  According to Routh Hurwitz criteria $\bar{E_1}$ is locally asymptotically stable if $S_1 > 0$ and $S_1 S_2 - S_3 > 0$.
- **At $\mathbf{\hat{E}_2}$**, one of the eigenvalues of the characteristic equation is $d\hat{y}/(\beta\hat{y}^2+h) - d_2$ and the remaining four eigenvalues are given by the roots of the following equation

  $$\lambda^4 + Q_1\lambda^3 + Q_2\lambda^2 + Q_3\lambda + Q_4 = 0,$$

  where

  $$Q_1 = d_1 + 2g_1\hat{y} - (a_2+v)\hat{x} - \frac{ab\hat{x}\hat{y}(m-\alpha\hat{x}^2)}{(\alpha x^2+m)^3} - w_1,$$
  $$Q_2 = -w_1\Big[d_1 + 2g_1\hat{y} - \frac{ab\hat{x}\hat{y}(m-\alpha\hat{x}^2)}{(\alpha\hat{x}^2+m)^3}\Big] - a_1 b_1 - (a_1 v + a_2 b_1)\hat{x}$$
  $$-(a_2+v)\hat{x}\Big[d_1 + 2g_1\hat{y} - \frac{ab\hat{x}\hat{y}(m-\alpha\hat{x}^2)}{(\alpha\hat{x}^2+m)^3} - w_1\Big],$$

$$Q_3 = \hat{x}(a_2+v)w_1\Big[d_1+2g_1\hat{y}-\frac{ab\hat{x}\hat{y}(m-\alpha\hat{x}^2)}{(\alpha\hat{x}^2+m)^3}\Big]-(a_1v+a_2b_1)\hat{x}$$
$$\Big[d_1+2g_1\hat{y}-\frac{ab\hat{x}\hat{y}(m-\alpha\hat{x}^2)}{(\alpha\hat{x}^2+m)^3}-w_1\Big],$$
$$Q_4 = a_1b_1+(a_1v+a_2b_1)\hat{x}-w_1\Big[d_1+2g_1\hat{y}-(a_2+v)\hat{x}-\frac{ab\hat{x}\hat{y}(m-\alpha\hat{x}^2)}{(\alpha\hat{x}^2+m)^3}\Big],$$
$$w_1 = r(\hat{c_0})\Big(1-\frac{\hat{x}}{K}\Big)+\frac{\hat{x}r(c_0)}{K}+\frac{a\hat{y}(m-\alpha\hat{x}^2)}{(\alpha\hat{x}^2+m)^2}.$$

Applying Routh–Hurwitz criteria, it is found that $\hat{E_2}$ is locally asymptotically stable if the following conditions hold:

$$\frac{d\hat{y}}{\beta\hat{y}^2+h}<d_2,$$
$$Q_1>0,\quad Q_1Q_2>Q_3,\quad Q_1Q_2Q_3>Q_3^2+Q_1^2Q_4.$$

- *The characteristic equation of* $E_3^\star$ is given as:

$$\lambda^5+R_1\lambda^4+R_2\lambda^3+R_3\lambda^2+R_4\lambda+R_5=0,$$

where

$$R_1 = -(a_{44}+a_{55}+a_{11}+a_{22}+a_{33}),$$
$$R_2 = a_{44}a_{55}-a_{51}a_{45}+(a_{44}+a_{55})(a_{22}+a_{33}+a_{11})+a_{22}a_{33}$$
$$-a_{23}a_{32}+a_{11}(a_{22}+a_{33})+a_{12}a_{21},$$
$$R_3 = -[(a_{44}a_{55}-a_{51}a_{45})(a_{22}+a_{33}+a_{11})+(a_{44}+a_{55})(a_{22}a_{33}-a_{23}a_{32}$$
$$+a_{11}(a_{22}+a_{33})+a_{12}a_{21})]+a_{13}(a_{44}a_{55}-a_{51}a_{45})+a_{41}a_{13}(a_{22}+a_{33}),$$
$$R_4 = (a_{44}a_{55}-a_{51}a_{45})(a_{22}a_{33}-a_{23}a_{32}+a_{11}(a_{22}+a_{33})+a_{12}a_{21})+$$
$$(a_{44}+a_{55})(a_{12}a_{21}a_{33}+a_{11}(a_{22}a_{33}-a_{32}a_{23})),$$
$$R_5 = -(a_{44}a_{55}-a_{51}a_{45})(a_{12}a_{21}a_{33}+a_{11}(a_{22}a_{33}-a_{32}a_{23}))-(a_{41}a_{55}-a_{51}a_{45})$$
$$(a_{13}^2a_{23}a_{32}-a_{13}a_{22}a_{33}).$$

and

$$a_{11} = r(c_0^\star)\left(1-\frac{x^\star}{K}\right)-\frac{x^\star r(c_0^\star)}{K}-\frac{ay^\star(m-\alpha x^{\star 2})}{(\alpha x^{\star 2}+m)^2},\quad a_{12}=\frac{ax^\star}{\alpha x^{\star 2}+m},$$
$$a_{13} = r_1x^\star\left(1-\frac{x^\star}{K}\right),\quad a_{21}=\frac{by^\star(m-\alpha x^{\star 2})}{(\alpha x^{\star 2}+m)^2},\quad a_{22}=d_1+2g_1y^\star+\frac{cz^\star(h-\beta y^{\star 2})}{(\beta y^\star+h)^2},$$
$$a_{23} = \frac{cy^\star}{\beta y^{\star 2}+h},\quad a_{32}=\frac{dz^\star(h-\beta y^{\star 2})}{(\beta y^\star+h)^2},\quad a_{33}=\frac{dy^\star}{\beta y^{\star 2}+h}-d_2-2g_2z^\star,$$
$$a_{41}=-a_2c_e^\star+vc_0^\star,\quad a_{44}=vx^\star,\quad a_{45}=-a_1-a_2x^\star,$$
$$a_{51}=a_2c_e^\star-vc_0^\star,\quad a_{54}=-b_1-vx^\star,\quad a_{55}=a_2x^\star.$$

According to Routh–Hurwitz criterion, the equilibrium point $E_3^\star$ is locally asymptotically stable if

$$R_1>0,\quad R_1R_2-R_3>0,\quad R_1R_2R_3>R_3^2+R_1^2R_4,\quad R_1R_2R_3+R_1R_5>R_3^2+R_1^2R_4.$$

5. Global stability

Global stability is analyzed using Lyapunov functions, ensuring that the system will settle into a steady-state solution over time.

Theorem 2. *If the following constraints are satisfied in the region* Ω :

$$r(c_0^\star)\eta_1 > K\alpha\alpha y^\star(x_l + x^\star), \tag{5.1}$$

$$(d_1 + g_1(y_u + y^\star)) > M_4, \tag{5.2}$$

$$\eta_2(d_2 + g_2(z_u + z^\star)) > dy^\star(h - \beta y_u y^\star), \tag{5.3}$$

$$\Big(\frac{r(c_0^\star)}{K} - \frac{a\alpha y^\star(x_u + x^\star)}{\eta_1}\Big)M_1 > M_3, \tag{5.4}$$

$$M_1 M_2 \eta_2 + d(hz_u + \beta y_u y^\star z^\star) > cy^\star(h + \beta y_l y^{\star 2}), \tag{5.5}$$

$$(b + x^\star)(a_1 + a_2 x^\star) > (a_2 + v)x^\star, \tag{5.6}$$

$$(b + x^\star)\Big(\frac{r(c_0^\star)}{K} - \frac{a\alpha y^\star(x_u + x^\star)}{\eta_1}\Big) > (a_2(c_{e_l} - vc_{0_u}), \tag{5.7}$$

where

$$M_1 = (d_1 + g_1(y_u + y^\star)) - \Big(\frac{x^\star(1 + x_u\alpha b)}{\eta_1} - \frac{c(z_u h - \beta y_u y^\star z^\star)}{\eta_2}\Big),$$

$$M_2 = d_2 + g_2(z_u + z^\star) - \frac{dy^\star(h - \beta y_u y^\star)}{\eta_2},$$

$$M_3 = \Big[\frac{a(m + \alpha x^{\star 2})}{\eta_1} - \frac{b(my_u + \alpha x_u x^\star y^\star)}{\eta_2}\Big]^2,$$

$$M_4 = \Big(\frac{x^\star(1 + x_l\alpha b)}{\eta_1} - \frac{c(z_l h - \beta y_l y^\star z^\star)}{\eta_2}\Big),$$

$$\eta_1 = (\alpha x_u^2 + m)(\alpha x^{\star 2} + m), \quad \eta_2 = (\beta y_u^2 + h)(\beta y^{\star 2} + h),$$

where x_l *and* x_u, y_l *and* y_u, c_{e_l} *and* c_{0_u}, z_u *denote the lower* (l) *and upper* (u) *bounds of the respective state variables,*

$$x_l = K_4 - K_2, \quad x_u = K, \quad c_{e_l} = K_3, \quad c_{0_u} = K_2, \quad y_l = \frac{b(K_4 - K_2)}{a}, \quad y_u = K_1, \quad z_u = \frac{K_1 bd}{ac},$$

(*where values of* K_i, $i = 1, 2, 3, 4$ *can be seen at Theorem 1*) *then the positive equilibrium point* $E_3^\star$ *is globally asymptotically stable in the region* Ω.

P r o o f. We assumed the following positive definite function about $E_3^\star$:

$$L_1 = \Big(x - x^\star - x^\star \ln\Big(\frac{x}{x^\star}\Big)\Big) + \frac{n_1}{2}(y - y^\star)^2 + \frac{n_2}{2}(z - z^\star)^2 + \frac{n_3}{2}(c_e - c_e^\star)^2 + \frac{n_4}{2}(c_0 - c_0^\star)^2.$$

Differentiating L_1 with respect to time t, we get

$$\frac{dL_1}{dt} = \Big(\frac{x - x^\star}{x}\Big)\frac{dx}{dt} + n_1(y - y^\star)\frac{dy}{dt} + n_2(z - z^\star)\frac{dz}{dt} + n_3(c_e - c_e^\star)\frac{dc_e}{dt} + n_4(c_0 - c_0^\star)\frac{dc_0}{dt}.$$

After performing some algebraic manipulations using system of equations (2.1), (2.5), we obtain

$$\frac{dL_1}{dt} = -(x - x^\star)^2\Big(\frac{r(c_0^\star)}{K} - \frac{a\alpha y^\star(x + x^\star)}{\eta_1}\Big)$$
$$-(y - y^\star)^2\Big[n_1 d_1 + n_1 g_1(y + y^\star) - \Big(\frac{x^\star(1 + x\alpha b)}{\eta_1} - \frac{c(zh - \beta yy^\star z^\star)}{\eta_2}\Big)\Big]$$
$$-(z - z^\star)^2\Big[n_2(d_2 + g_2(z + z^\star)) - \frac{n_2 dy^\star(h - \beta yy^\star)}{\eta_2}\Big]$$

$$
\begin{aligned}
&-(c_e - c_e^\star)^2 n_4(a_1 + a_2 x^\star) - (c_0 - c_0^\star)^2 n_3(b + x^\star) \\
&-(x - x^\star)(y - y^\star)\Big[\frac{a(m + \alpha x^{\star 2})}{\eta_1} - \frac{n_1 b(my + \alpha x x^\star y^\star)}{\eta_2}\Big] \\
&-(y - y^\star)(z - z^\star)\frac{1}{\eta_2}\big(n_1 c(hy^\star + \beta y y^{\star 2}) - n_2 d(hz + \beta y y^\star z^\star)\big) \\
&-(x - x^\star)(c_0 - c_0^\star)\Big(r_1 - \frac{r_1 x}{K} - n_3 a_2 c_e + n_3 v c_0\Big) \\
&-(x - x^\star)(c_e - c_e^\star)n_4(a_2 c_e - v c_0) + (c_0 - c_0^\star)(c_e - c_e^\star)x^\star(a_2 + n_4 v),
\end{aligned}
$$

where

$$
\eta_1 = (\alpha x^2 + m)(\alpha x^{\star 2} + m), \quad \eta_2 = (\beta y^2 + h)(\beta y^{\star 2} + h).
$$

Now dL_1/dt can further be written as sum of the quadratic forms as

$$
\begin{aligned}
\frac{dL_1}{dt} \le -\big[&(b_{11}/2)(x - x^\star)^2 - b_{12}(x - x^\star)(y - y^\star) + (b_{22}/2)(y - y^\star)^2 \\
&+(b_{11}/2)(x - x^\star)^2 + b_{14}(x - x^\star)(c_e - c_e^\star) + (b_{44}/2)(c_e - c_e^\star)^2 \\
&+(b_{11}/2)(x - x^\star)^2 - b_{15}(x - x^\star)(c_0 - c_0^\star) + (b_{55}/2)(c_0 - c_0^\star)^2 \\
&+(b_{22}/2)(y - y^\star)^2 + b_{23}(y - y^\star)(z - z^\star) + (b_{33}/2)(z - z^\star) \\
&+(b_{44}/2)(c_e - c_e^\star)^2 - b_{45}(c_e - c_e^\star)(c_0 - c_0^\star) + (b_{55}/2)(c_0 - c_0^\star)^2\big],
\end{aligned}
$$

where

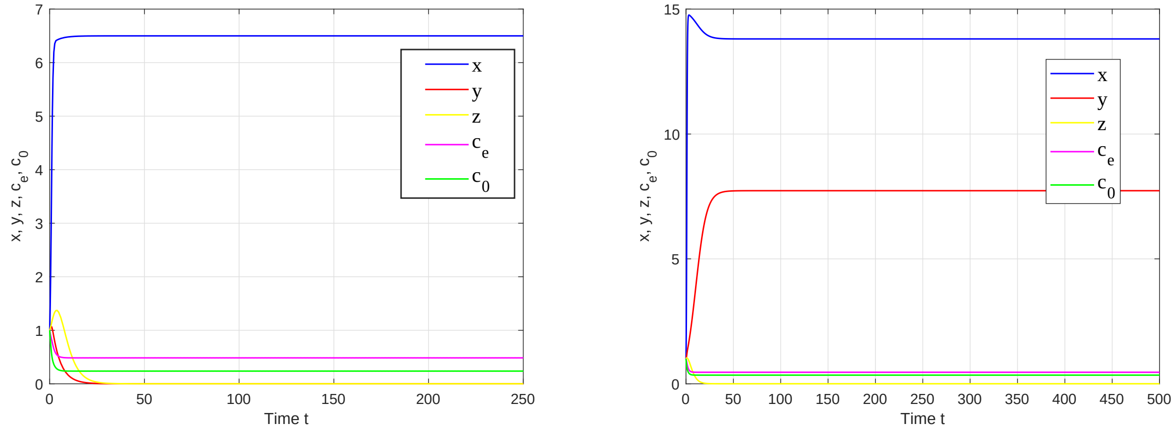
$$
\begin{aligned}
&b_{11} = \frac{r(c_0^\star)}{K} - \frac{a\alpha y^\star(x + x^\star)}{\eta_1}, \quad b_{22} = n_1 d_1 + n_1 g_1(y + y^\star) - \left(\frac{x^\star(1 + x\alpha b)}{\eta_1} - \frac{c(zh - \beta y y^\star z^\star)}{\eta_2}\right), \\
&b_{33} = n_2(d_2 + g_2(z + z^\star)) - \frac{n_2 d y^\star(h - \beta y y^\star)}{\eta_2}, \quad b_{44} = n_4(a_1 + a_2 x^\star), \quad b_{55} = n_3(b + x^\star), \\
&b_{12} = \frac{a(m + \alpha x^{\star 2})}{\eta_1} - \frac{n_1 b(my + \alpha x x^\star y^\star)}{\eta_2}, \quad b_{23} = \frac{1}{\eta_2}(n_1 c(hy^\star + \beta y y^{\star 2}) - n_2 d(hz + \beta y y^\star z^\star)), \\
&b_{45} = x^\star(a_2 + n_4 v), \quad b_{15} = (r_1 - \frac{r_1 x}{K} - n_3 a_2 c_e + n_3 v c_0).
\end{aligned}
$$

Now, by using Sylvesters criteria and by choosing

$$
n_1 = \frac{a(m + \alpha x^{\star 2})\eta_2}{\eta_1 b(my + \alpha x x^\star y^\star)} > 0
$$

and $n_2 = n_3 = n_4 = 1$ we get dL_1/dt is negative definite under the following conditions:

$$b_{11} > 0, \tag{5.8}$$
$$b_{22} > 0, \tag{5.9}$$
$$b_{33} > 0, \tag{5.10}$$
$$b_{11}b_{22} > b_{12}^2, \tag{5.11}$$
$$b_{11}b_{44} > b_{14}^2, \tag{5.12}$$
$$b_{22}b_{33} > b_{23}^2, \tag{5.13}$$
$$b_{11}b_{55} > b_{15}^2, \tag{5.14}$$
$$b_{44}b_{55} > b_{45}^2. \tag{5.15}$$



(a) Stable graph around the equilibrium point $\bar{E}_1$.

(b) Stable graph around the equilibrium point $\hat{E}_2$.

Figure 2. Stable graph around the equilibrium points $\bar{E}_1$ and $\hat{E}_2$

It is observed that the fourth inequality, *i.e.*, $b_{11}b_{22} > b_{12}^2$ is satisfied due to the proper choice of n_1, and for other inequalities, (5.1) $\Rightarrow$ (5.8), (5.2) $\Rightarrow$ (5.9), (5.3) $\Rightarrow$ (5.10), (5.4) $\Rightarrow$ (5.12), (5.5) $\Rightarrow$ (5.13), (5.6) $\Rightarrow$ (5.14), (5.7) $\Rightarrow$ (5.15). Hence L_1 is a Lyapunov function with respect to $E_3^\star$, whose domain contains the region of attraction Ω, which proves the theorem. $\square$

6. Simulations and discussion

In this section, we numerically explore the effects of key parameters on population interaction using MATLAB and MATHEMATICA software.

We have taken the following parameter values for $\bar{E}_1$:

$$r_0 = 3.05, \quad r_1 = 0.75, \quad K = 6.5, \quad a = 1.12, \quad \alpha = 0.49, \quad m = 1.48, \quad c = 0.01,$$
$$b = 1.21, \quad d_1 = 0.571, \quad g_1 = 0.02, \quad d = 3.1, \quad \beta = 1.42, \quad h = 7, \quad d_2 = 0.223,$$
$$g_2 = 0.025, \quad q_0 = 0.515, \quad v = 0.21, \quad a_1 = 0.81, \quad a_2 = 0.142, \quad b_1 = 0.52.$$

It has been found that under the above set of parameters, the equilibrium point $\bar{E}_1$ is locally asymptotically stable (see Fig. 2a).

$$\bar{x} = 6.5, \quad \bar{y} = 0, \quad z = 0, \quad \bar{c_e} = 0.4837, \quad \bar{c_0} = 0.2368.$$

We select the following parameter values for the equilibrium $\hat{E}_2$:

$$r_0 = 3.65, \quad r_1 = 0.52, \quad K = 15, \quad a = 1.99, \quad \alpha = 0.25, \quad m = 8.0458, \quad c = 0.01,$$
$$b = 1.01, \quad d_1 = 0.0571, \quad g_1 = 0.025, \quad d = 1.0571, \quad \beta = 2.192, \quad h = 0.1568, \quad d_2 = 0.35,$$
$$g_2 = 0.0351, \quad q_0 = 0.515, \quad v = 0.821, \quad a_1 = 0.92881, \quad a_2 = 0.63, \quad b_1 = 0.252.$$

It has been observed that under the above set of parameters, the equilibrium point $\hat{E}_2$ is locally asymptotically stable (see Fig. 2b).

$$\hat{x} = 13.85, \quad \hat{y} = 7.4350, \quad z = 0, \quad \hat{c_e} = 0.4611, \quad \hat{c_0} = 0.3453.$$

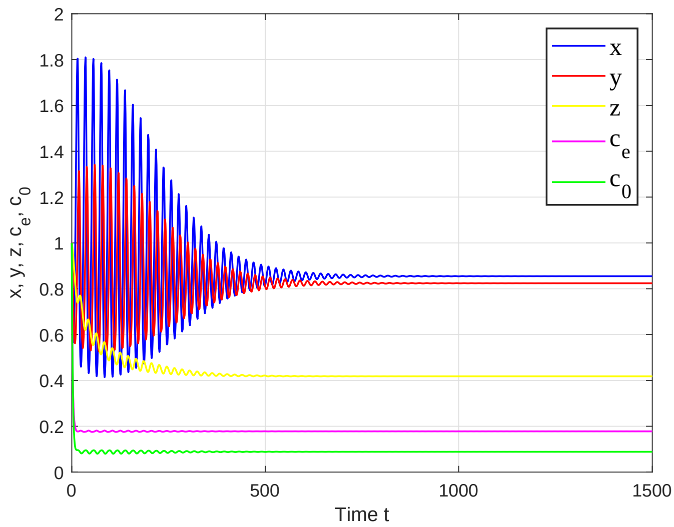


Figure 3. Stable graph around the equilibrium point $E_3^{\star}$.

We choose the following parameter values for $E_3^{\star}$:

$$r_0 = 0.58, \quad r_1 = 0.26, \quad K = 10, \quad a = 2.891, \quad \alpha = 0.653, \quad m = 4.2, \quad c = 0.671,$$
$$b = 1.46, \quad d_1 = 0.171, \quad g_1 = 0.085, \quad d = 0.59, \quad \beta = 0.52, \quad h = 10.53, \quad d_2 = 0.03,$$
$$g_2 = 0.0351, \quad q_0 = 0.155, \quad v = 0.8421, \quad a_1 = 0.81, \quad a_2 = 0.492, \quad b_1 = 0.1252.$$

It has been found that under the above set of parameters, the equilibrium point $E_3^{\star}$ is locally asymptotically stable (see Fig. 3 and Fig. 4).

$$x^{\star} = 0.7446, \quad y^{\star} = 0.9126, \quad z = 0.5445, \quad c_e^{\star} = 0.1780, \quad c_0^{\star} = 0.08689.$$

The bifurcation diagrams of phytoplankton, zooplankton, and fish with respect to K are presented in Fig. 5 and Fig. 6, where

$$r_0 = 0.58, \quad r_1 = 0.26, \quad a = 2.891, \quad \alpha = 0.653, \quad m = 4.2, \quad c = 0.671,$$
$$b = 1.46, \quad d_1 = 0.171, \quad g_1 = 0.085, \quad d = 0.59, \quad \beta = 0.52, \quad h = 10.53, \quad d_2 = 0.03,$$
$$g_2 = 0.0351, \quad q_0 = 0.155, \quad v = 0.8421, \quad a_1 = 0.81, \quad a_2 = 0.492, \quad b_1 = 0.1252.$$

For the above set of parameter values, we observed that if we change K from $6 \le K \le 7.5$ the system remains stable but shows oscillatory behavior in $7.55 \le K \le 10$.

Again, let us choose the following parameters

$$r_0 = 3.28, \quad K = 10, \quad a = 12.891, \quad \alpha = 0.0653, \quad m = 4.2, \quad c = 9.8671,$$
$$b = 11.46, \quad d_1 = 0.9971, \quad g_1 = 0.07685, \quad d = 5.59, \quad \beta = 2.952, \quad h = 10.53, \quad d_2 = 0.39,$$
$$g_2 = 0.015351, \quad q_0 = 0.151, \quad v = 0.8421, \quad a_1 = 0.81, \quad a_2 = 0.493, \quad b_1 = 0.1252.$$

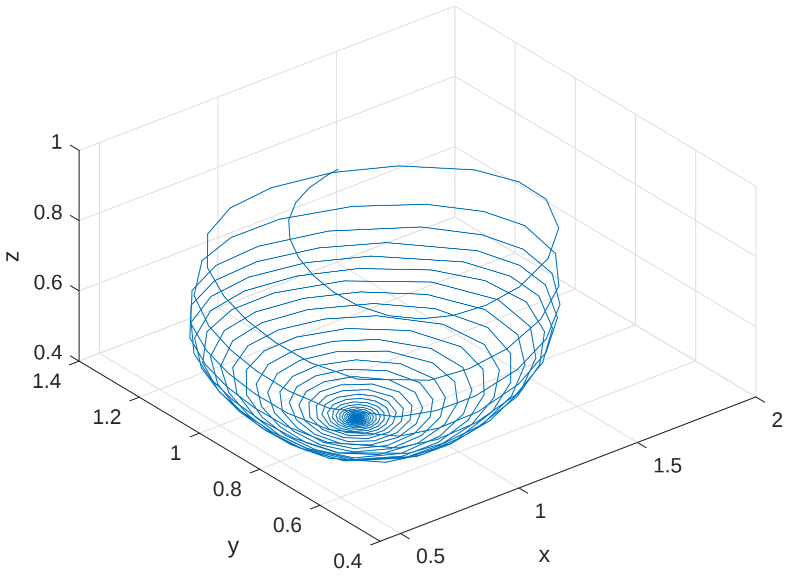


Figure 4. Phase graph around the equilibrium point $E_3^\star$.

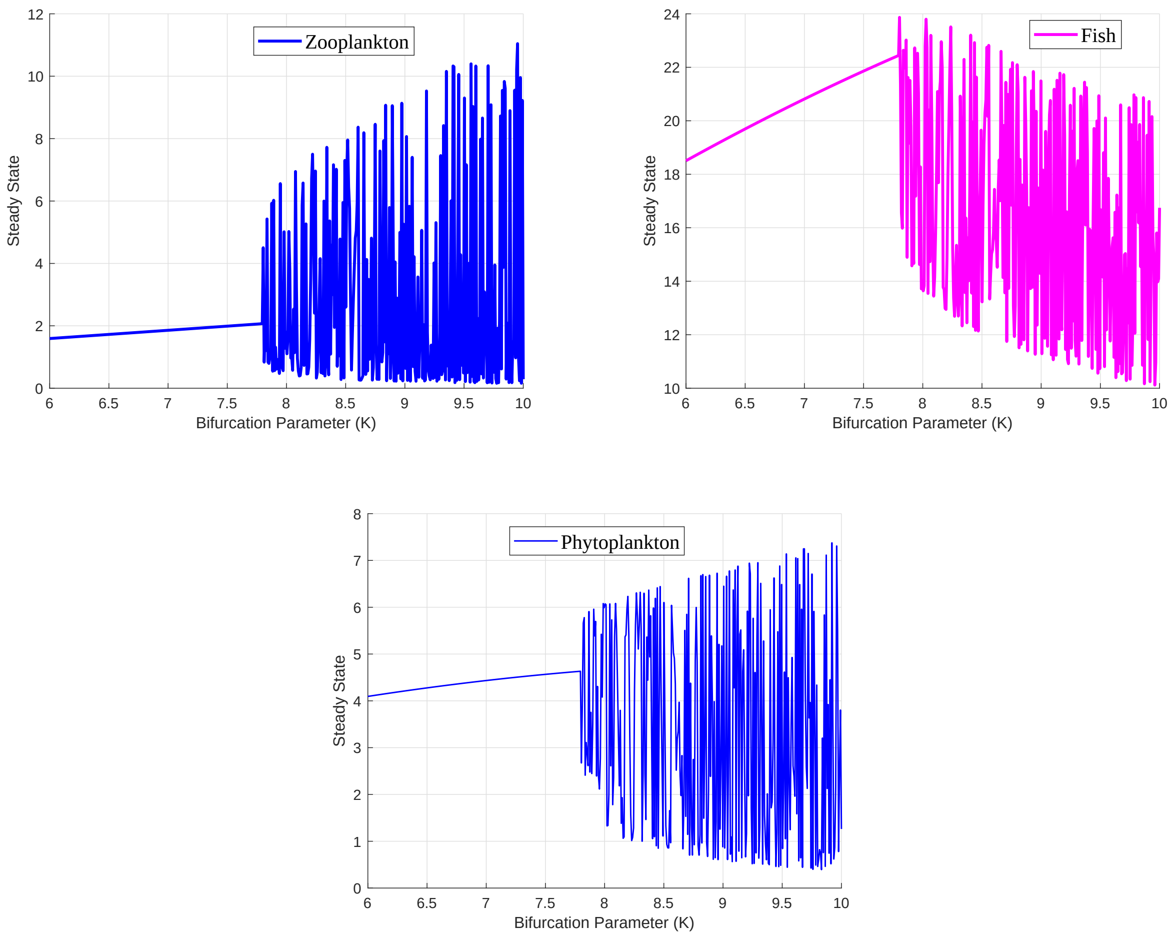


Figure 5. Bifurcation diagram of the model with respect to K.

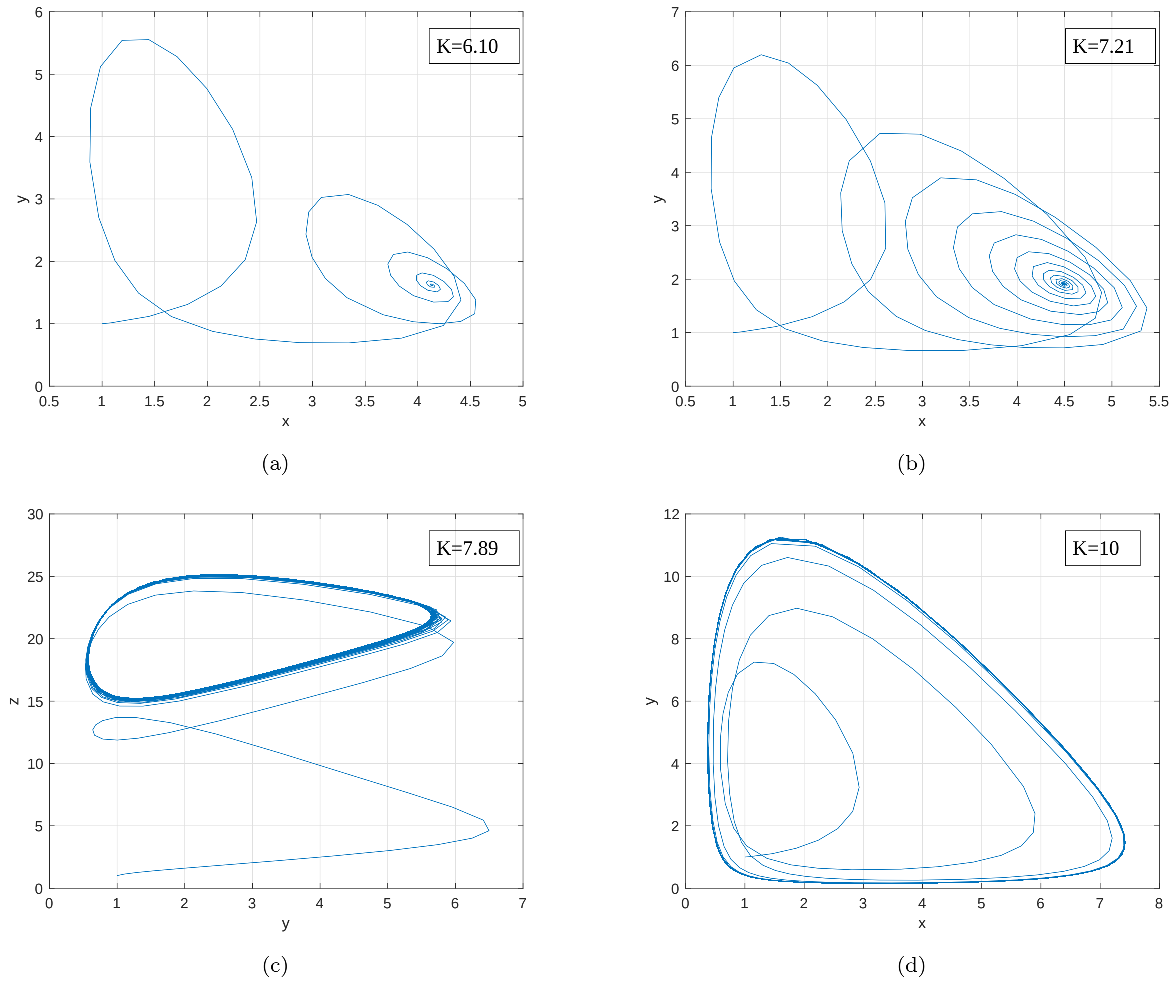


Figure 6. Phase graph of the system for different values of K.

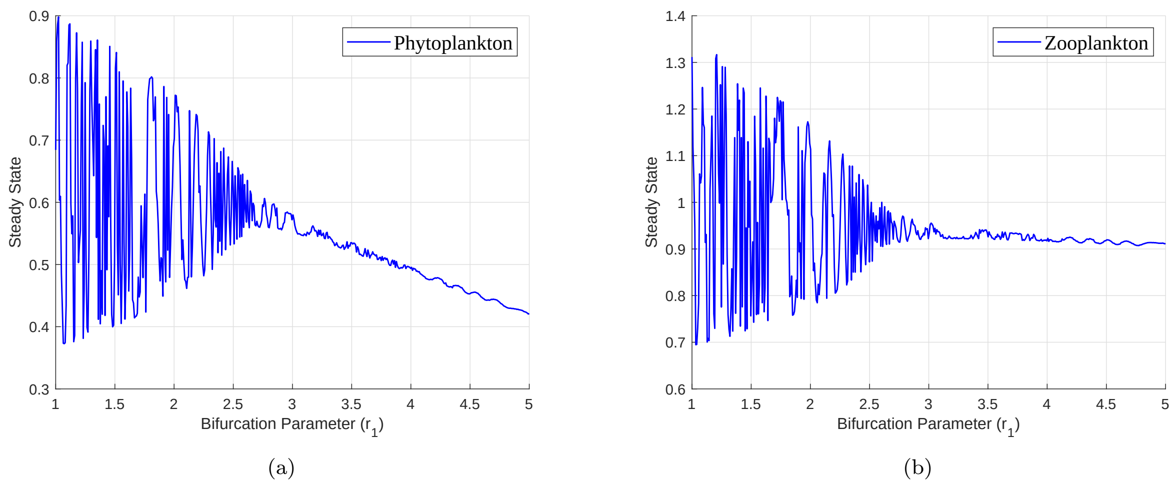


Figure 7. Bifurcation diagram of the system with respect to different values of r_1.

Bifurcation diagrams of phytoplankton and zooplankton with respect to r_1 are presented in Fig. 7a and 7b. Phase graphs for different values of r_1 showing limit cycle behavior are given at Fig. 8.

For the above set of parameter values, we observed that if we change r_1 from $1 \leq r_1 \leq 2.55$ the system shows oscillatory behavior, but is stable in $2.55 \leq r_1 \leq 10$.

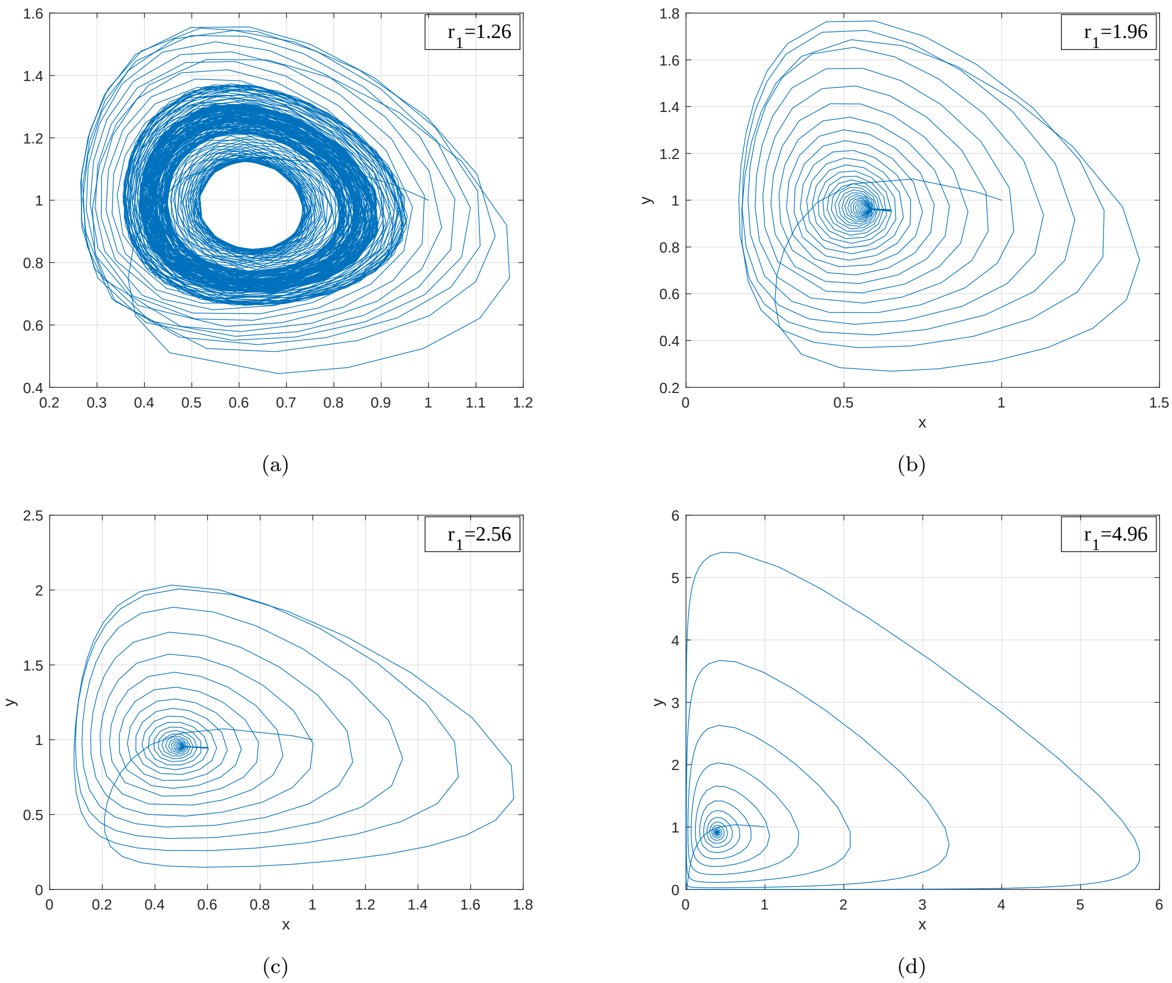


Figure 8. Phase graph of the system with respect to different values of r_1.

7. Conclusion

In this study, we proposed a mathematical model to explore the impact of toxicants in a tri-trophic marine food chain system. We established the boundedness of the system, which ensures that the population of the species remains within the feasible region. The local stability of the equilibrium point in the model has been analyzed using the Jacobian matrix. From the stability of $\bar{E}_1$, it can be concluded that the only population of phytoplankton will survive, and the population of zooplankton and fish would tend to go extinct (see Fig. 2a). The stability of $\hat{E}_2$ indicates that the phytoplankton and zooplankton population will survive and the fish will extinct (see Fig. 2b). The interior equilibrium point $E_3^\star$ is locally and globally stable, showing coexistence of all three populations (see Fig. 3). From this analysis, it is seen that some parameter associated with our proposed model can make the system unstable. Our investigation shows that a few parameters related to our suggested model have the potential to cause system instability. The numerical simulation indicates that increasing the system's carrying capacity K keeps it stable up to a critical value, after which it becomes unstable (Fig. 5). Also, it is concluded that r_1 has a significant role in the stability of the ecosystem (Fig. 7). Phase portraits are also presented, which show the limit cycle behavior of the system for different values of the parameters.

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