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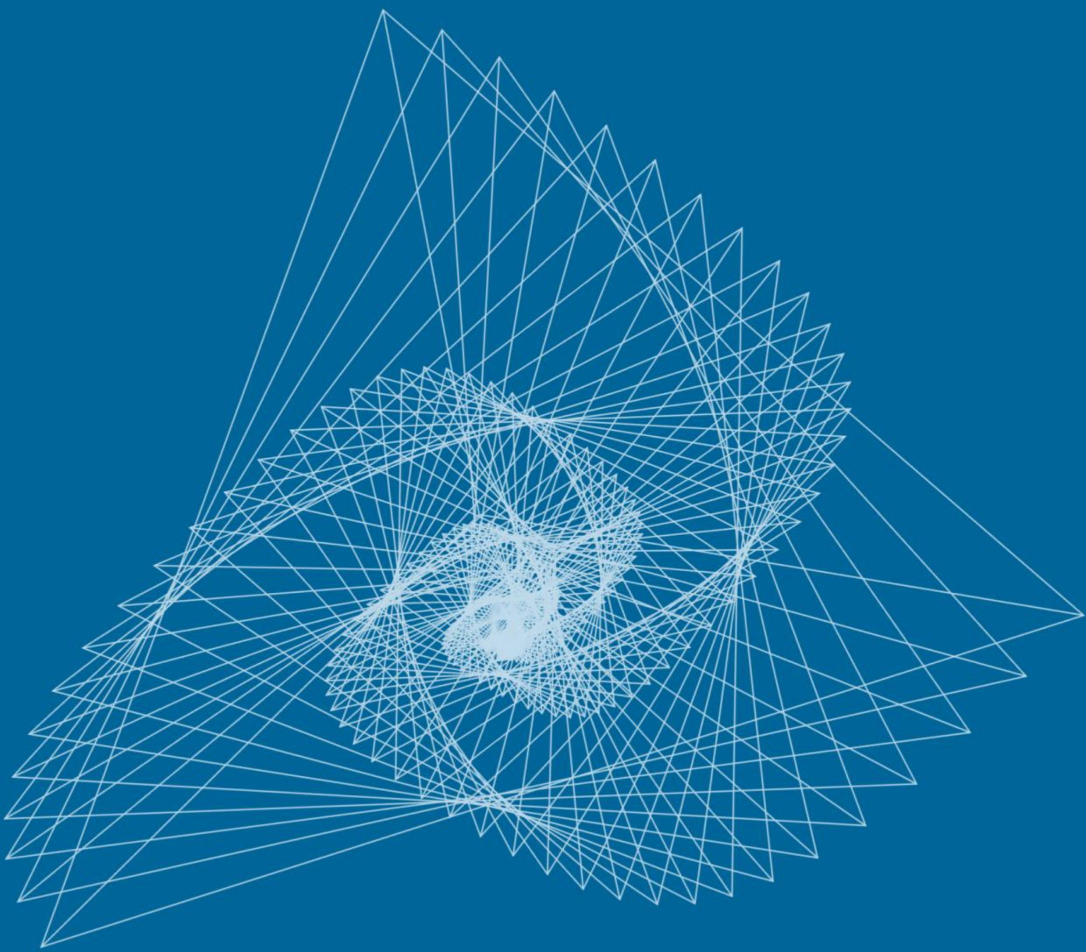

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N.N.Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences

Ural Federal University named after the first President of Russia B.N.Yeltsin

Contact Information

16 S. Kovalevskaya str., Ekaterinburg, Russia, 620990
Phone: +7 (343) 375-34-73 Fax: +7 (343) 374-25-81
Email: secretary@umjuran.ru
Web-site: https://umjuran.ru

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THE EKATERINBURG SEMINAR "ALGEBRAIC SYSTEMS": 50 YEARS OF ACTIVITIES

Lev N. Shevrin

Ural Federal University, Ekaterinburg, Russia,
lev.shevrin@urfu.ru

Abstract: The aim of the present article is to give a characterization of distinctive features of the scientific seminar founded and led by the author as well as to show the main sides of its activities during half a century.

Key words: algebraic systems, Ekaterinburg seminar, sides of activities.

1. Introducing remarks

The seminar indicated in the title of the article started its work in 1966. By that time several younger researchers had been grouped around the present writer at Ural State University. Naturally, I discussed with each of them different problems pertaining to the area of his/her research. However, besides these individual meetings, the natural need had arisen to gather regularly and to discuss the results obtained as well as diverse problems concerning our investigations in algebra.

It should be noted that Ekaterinburg (Sverdlovsk from 1924 till 1991) is a city with considerable scientific algebraic traditions. In a great degree the beginning of algebraic studies there was owing to the activities of Professor P. G. Kontorovich (1905–1968) who worked for several decades at Ural State University and was one of the leading Soviet algebraists. The scientific school created by P. G. Kontorovich gained notable recognition in mathematical community by the 1960s, and, in particular, it was not accidental that, after the first two All-Union Algebraic Conferences held in Moscow in 1958 and 1959, the third one was organized in Sverdlovsk in 1960, and Prof. Kontorovich was the Chairman of its Organizing Committee[1]. My scientific rise began under Kontorovich's supervision; I defended a dissertation for "Candidate of Sciences" degree in 1961 and a dissertation for "Doctor of Sciences" degree in 1966[2].

The seminar, which afterwards received the name "Algebraic Systems", had at first about 10 members. Since the 1970s the number of its regular participants remains at the level about 20, although in the 1980s it sometimes achieved up to 25–30 persons. Since the middle of the 1980s, side by side with students of the leader of the seminar, students of my students were becoming regular participants in the seminar. The number of such "scientific grandchildren" is steadily increasing; moreover, now there are already more than ten scientific great-grandchildren of mine.

[1]In the paper [1], I have written rather minutely about the formation of the Sverdlovsk algebraic school for the period from the end of the 1930s till the beginning of the 1960s, where some key figures were presented and some essential events and facts were mentioned. The articles [2] and [3] are devoted personally to P. G. Kontorovich. They are published in a special issue of the journal "Izvestiya Ural'skogo gosudarstvennogo universiteta" dedicated to the centenary of his birthday; the second of them is reproduced from an issue of "Matematicheskie zapiski" (Mathematical Transactions) of Ural State University (1970) dedicated to the memory of Kontorovich.

[2]For a foreign reader who may not be familiar with the system of Soviet (and now Russian) scientific degrees, I note that Candidate of Sciences approximately corresponds to Ph.D. in the Western World, while Doctor of Sciences is a considerably higher scientific degree.

Besides discussing problems and results, it became traditional for the seminar to discuss also abstracts of talks being prepared for various conferences. Furthermore, it became customary when the members of the seminar attended some conferences report at the seminar about these conferences. The leader of the seminar pays much attention to the development of disciples' skill in performing scientific talks and writing mathematical works.

Soon enough, since 1969, algebraists from other towns of the former Soviet Union began to appear as the speakers at our seminar; this takes place up to now more or less regularly. Since 1989 foreign speakers appear from time to time as well.

For half a century the community of regular participants in the seminar in different years got significant achievements, both in research and educational spheres as well as in various forms of scientific-organizing activities. So a natural desire has arisen to display a general picture of distinctive features of the seminar and to show these achievements. The aim of this article is to present such a picture. Sections 2–8 are devoted to the following its aspects: *topics and some peculiarities in research, dissertations, grants, publications, participation in conferences, organization of conferences, membership in editorial boards*; some information concerning meetings and speakers at the seminar is given in Section 9.

It should be noted that the first rather detailed information about this seminar in some publication was given in 1998 in [4]. As to the present article, it is an enlarged (and slightly revised) version of the article [5] placed in 2007 on the website of the European Academy of Sciences. I am grateful to Mikhail Volkov, Boris Vernikov, and Arseny Shur for their help in collecting the data concerning the last decade for this version.

2. Topics and some peculiarities in research

The objects of investigations carrying out at the seminar are a number of the main types of algebraic systems: semigroups, groups, rings and algebras (both associative and non-associative, in particular, Lie algebras), lattices, and some others. These types of systems are subjects of largely developed theories, which continue to develop intensively. The members of the seminar succeeded in fundamental contributing to several branches of these theories; it will be briefly characterized below. Since the 1990s the area of our interests had been broadened and began including some topics that it is customary to refer as belonging to discrete mathematics. I mention among them, first of all, certain problems of the theories of graphs and clones, in particular, problems of discrete optimization. Since the end of the last century considerable attention has been given to applied aspects of algebra, see item (x) below and item g) at the end of this section. One may distinguish the following main lines of our investigations (with a different degree of intensity in different periods of our work).

(i) Structure of systems and finiteness conditions[3].

(ii) Lattices properties, i.e. properties connected with considering lattices of subsystems for systems of a given class.

[3]Here it is, of course, impossible to give definitions for the main notions being mentioned. However, in some exclusive cases one may formulate definitions that can be understandable even for the reader who is not an algebraist. For instance, it concerns the notion "finiteness condition". Given a class of algebraic systems, by a *finiteness condition* is meant any property which is possessed by all finite systems of this class. Many infinite systems satisfy certain finiteness conditions, and investigation of systems with such conditions provides a possibility to obtain diverse results in much more general situations than for finite systems. Imposing finiteness conditions is a classical approach (within the 20th century) in investigations of algebraic systems of different kinds. For the reader who would like to consult some source with other algebraic notions mentioned in this article, I may recommend the handbook [6] (in Russian) or the handbook [7] (in English).

(iii) Varieties and similar classes: quasivarieties, pseudovarieties, and the like.
(iv) Algorithmic problems.
(v) Congruences.
(vi) Embeddings.
(vii) Transformations.
(viii) Independence of related structures: automorphism group, congruence lattice, subsystem lattice, etc.
(ix) Combinatorics, graph theory, and discrete optimization.
(x) Applications of an algebraic approach to some branches of computer science: computational complexity, synchronizability of finite automata, various problems for formal languages, etc.

In each of these lines, there are many essential results obtained by the members of the seminar. The most notable achievements, especially for the first four decades of our activity, concern lines (i)–(iv) which we accomplished quite a number of major series of works in. They have gained considerable recognition, which is reflected, in particular, in some summarizing publications at an international level, see Section 5 below.

I would like to give some comments for line (iii). The central concept here is a variety. A class of algebraic systems is called a *variety* if there is a set of identities such that this class consists of all systems satisfying all identities from this set. Development of the theory of varieties started in 1935 by a basic paper by G. Birkhoff. At the second half of the 20th century the theory of varieties became one of the main lines in general algebra. Plenty of investigations were devoted to this theory in many countries. In our seminar, such investigations had been begun since the end of the 1960s. They may be (conditionally) divided into five topics: identities, structural aspects, lattices of varieties, free systems, and algorithmic problems. In each of them, certain concrete parts may be distinguished. For instance, for the topic "identities", one of the central problems, called the *finite basis problem*, is to determine which varieties can be given by a finite set of identities. Some fundamental results obtained by several members of the seminar were devoted just to this problem for different classes of algebraic systems, to the problem of classification of varieties with certain restrictions on the lattice of subvarieties, to description of varieties whose elementary theory is decidable as well as to quite a number of other important problems.

The concept of a quasivariety is a certain generalization of the concept of a variety, the concept of a pseudovariety is an analogue of the concept of variety as applied to finite systems. Both these concepts also serve as the objects of fruitful investigations. A motivation for the study of pseudovarieties is caused in a great degree by deep connections between pseudovarieties and formal languages. A key impulse for the development in this direction was given by S. Eilenberg (1976) in the volume B of his well-known monograph "Automata, Languages and Machines".

The classification (i)–(x) is rather conditional: there are no clear-cut borders between these lines. Moreover, for the themes of many works accomplished at the seminar, it is just typical to combine idea motifs pertained to two or more lines. Examples of such interlacing are numerous, and there is no reason to try characterizing all corresponding situations within this article. I will touch only on one of subject lines — imposing finiteness conditions. This classical approach was applied in our research in diverse situations: in study of lattice properties of semigroups and groups; in considerations of congruences (for instance, in study of residually finite semigroups and rings, in particular, from the view of the theory of varieties); in a combination with problems of embeddings (a typical example is a search of conditions for embeddability in finitely generated systems with certain restrictions) as well as in investigations on algorithmic problems (for instance, in those cases when the main objects of attention are finitely presented systems).

One may note several specific features of research at the seminar.

a) *Predominance of semigroups*, especially during the first three decades of the seminar work. It is worth noticing in this connection that the theory of semigroups is one of the youngest fields of modern algebra. It had been formed by the 1960s, just then the first monographs appeared that were entirely devoted to semigroups: the book "*Semigroups*" by E. S. Ljapin in Soviet Union (1960) and two volumes of the classical monograph "*The Algebraic Theory of Semigroups*" in USA by A. H. Clifford and G. B. Preston (1961, 1967). As it can be seen from relevant information in item f) of this section and in Sections 3–8, diverse manifestations of our activities in semigroup theory are numerous.

b) At the same time, *dissemination of our interests and research to other types of algebraic systems.* See, in particular, comments given in item c) and especially in item f) containing a more concrete assertion that will be illustrated in the subsequent text of that paragraph. This illustration, together with information given in Sections 3–5, allows showing more minutely various subjects of research accomplished at the seminar.

c) *Permanent attention to lattices*, both as an independent type of systems and (mostly) as related structures: the lattices of subsystems, of congruences, of ideals, of clones, of subvarieties, of subquasivarieties, of subpseudovarieties, and the like.

d) In general, *attention to related structures of different kinds*: automorphism groups, endomorphism monoids, elementary theories, and the like.

e) *Permanent attention to algorithmic problems.*

f) During the first 3 decades of activities of the seminar, *investigations of the same problem for different types of algebraic systems*, both owing to research pursuing by different authors and in works carried out by one and the same author.

Here is a list of the themes of such "polysystem" series of works, each one due to the same author.

- Densely embedded ideals of semigroups, associative algebras (as well as some generalizations), and Lie algebras, by *L. N. Shevrin*, the 1960s – the beginning of the 1970s.
- Decidability of elementary theories of varieties of groups, semigroups, and associative rings, by *A. P. Zamyatin*, the 1970s.
- Independence of related structures for semigroups and lattices, by *V. A. Baransky*, the 1970s–1980s.
- Attainability and solvability for classes of algebras as applied to arbitrary universal algebras, semigroups, groups, modules, associative and Lie algebras, and unars, by *L. M. Martynov*, the 1970s–1980s.
- Bases of identities and lattices of varieties applied to associative rings and semigroups, by *M. V. Volkov*, the end of the 1970s – the beginning of the 1990s.
- Critical theories of certain classes of semigroups and rings, by *Yu. M. Vazhenin*, the 1980s – the beginning of the 1990s.
- The word problem for varieties of groups and Lie algebras, by *O. G. Kharlampovich*, the 1980s.
- Algorithmic problems for semigroups and associative algebras, by *M. V. Sapir*, the 1980s.
- Radicals and bands of semigroups and associative rings, by *A. V. Kelarev*, the 1980s.
- Lattices of varieties of associative rings and semigroups, by *B. M. Vernikov*, the 1980s.
- Representation of lattices by subsystem lattices as applied to semigroups, groups, rings, and lattices, by *V. B. Repnitskiĭ*, the end of the 1980s – the 1990s.
- Algorithmic problems for varieties of semigroups, monoids, groups, and rings, by

V. Yu. Popov, the 1990s – the beginning of the 2000s.

g) In the 21st century, *increased attention to using an algebraic approach in some related fields of mathematics, first of all in computer science.* As a result, the following topics have become rather typical for our investigations accomplished in the years of this century: algebraic theory of formal languages and automata, diverse problems concerning computational complexity (in particular, as applied to bioinformatics), extremal problems on graphs and working out effective algorithms for solution of them, intelligent systems and robotics. Some details of this trend can be observed in formulations of the subjects of relevant dissertations and grants (see Sections 3 and 4) as well as in the themes of three conferences held in Ekaterinburg since 2007 (see corresponding information in Section 7). I notice that a remote portent of these conferences was the Regional Conference of Young Scientists held in Sverdlovsk in 1982 (see its name in Section 7). Organization of the mentioned three conferences in Ekaterinburg may serve as one of the signs showing that our achievements in respective branches also, as properly algebraic ones, have received international recognition.

3. Dissertations

The results of our investigations have found a natural reflection in dissertations defended by the members of the seminar. By 2017 there have been 95 such dissertations in all, among them 81 for Candidate Degree and 14 for Doctor Degree.

I give below the list of all these dissertations. The components of this list concerning Candidate dissertations are ordered by the starting years for the corresponding supervisors and, within the group of dissertations with the same supervisor, chronologically by the dates of defences. As to Doctoral dissertations, they are ordered chronologically.

3.1. Candidate dissertations

Supervisor L. N. Shevrin:

1. *N. D. Filippov.* Partially ordered sets and certain algebraic systems connected with them, 1969.
2. *E. A. Golubov.* Finitely separable and residually finite semigroups, 1970.
3. *V. A. Baransky.* Lattice isomorphisms of semigroups, and certain semigroup-theoretic constructions, 1971.
4. *Yu. M. Vazhenin.* Semigroups of transformations of graphs, and the first order language, 1972.
5. *L. M. Martynov.* Verbal chains in universal algebras, 1972.
6. *A. N. Trakhtman.* On the system of proper subsemigroups of a semigroup, 1973.
7. *T. I. Ershova.* On lattice properties of inverse semigroups, 1974.
8. *A. S. Prosvirov.* Idealizers of subsemigroups, and the structure of a semigroup, 1977.
9. *V. B. Lender.* The operation of multipication on classes of lattices, and related topics, 1977.
10. *A. M. Gasanov.* Ternary semigroups of continuous and homeomorphic mappings, 1978.
11. *A. P. Zamyatin.* Decidability of elementary theories of varieties of groups, semigroups, and associative rings, 1979.
12. *V. N. Klimov.* Congruences of semigroups, 1979.
13. *A. J. Ovsyannikov.* Lattice isomorphisms of semigroups, and varieties of semigroups, 1980.
14. *M. V. Volkov.* Lattices of varieties of rings, 1980.

15. *B. P. Tanana.* On lattice properties of topological semigroups, 1980.
16. *E. V. Sukhanov.* Varieties and bands of semigroups, 1980.
17. *E. I. Kleiman.* Varieties of inverse semigroups, 1981.
18. *O. M. Mamedov.* Equational compactness in general algebras and algebras with an order relation, 1982.
19. *V. V. Rasin.* Varieties of Clifford semigroups, 1982.
20. *M. V. Sapir.* Quasivarieties of semigroups, 1983.
21. *S. I. Katsman.* Semigroups with certain types of subsemigroup lattices, 1983.
22. *T. A. Martynova.* The groupoid of varieties of semigroups with zero, 1983.
23. *O. G. Kharlampovich.* Algorithmic and other combinatorial problems for groups and Lie algebras, 1984.
24. *V. B. Repnitskiĭ.* Varieties of lattice-ordered semigroups, 1985.
25. *I. O. Koryakov.* Periodic linear semigroups, 1985.
26. *E. A. Perminov.* Rigid graphs and lattices, 1985.
27. *B. V. Rozenblat.* On elementary and positive theories of relatively free semigroups, 1985.
28. *A. N. Petrov.* Embeddings of semigroups, and varieties, 1987.
29. *A. V. Kelarev.* Radicals and bands of semigroups and associative rings, 1989.
30. *B. M. Vernikov.* Varieties of associative rings and semigroups with restrictions on the subvariety lattice, 1989.
31. *O. V. Knyazev.* On the theory of varieties of Clifford semigroups, 1991.

Supervisor V. A. Baransky:

1. *P. V. Shumyatsky.* Periodic groups whose automorphism groups are regular 2-groups, 1989.
2. *A. P. Zolotarev.* Helly, Radon, Caratheodory, and Goldi numbers in lattices, 1993.
3. *V. A. Shcherbakova.* The Steiner problem on a graduated directed graph, 1998.
4. *O. V. Rasin.* Polynomial algorithms of recognition of isomorphism in some classes of graphs, 2005.
5. *L. M. Volkov.* Models and algorithms of information handling in program complexes of electronic documents-circulation, 2006.
6. *T. A. Koroleva.* Lattices of integral partitions, and chromatically uniqueness of graphs, 2008.
7. *M. I. Naumik.* Congruences of the semigroup of linear relations, 2008.
8. *S. N. Pupyrev.* Models, algorithms, and a program complex for visualizing complex networks, 2010.
9. *T. A. Senchonok.* Classification and determinability of elements of small height in the lattices of complete multipartite graphs, 2012.
10. *O. E. Perminova.* Critical lattices, 2014.

Supervisor Yu. M. Vazhenin:

1. *S. V. Sizyi.* Quazivarieties of endomodels, and algorithmic problems,1990.
2. *B. Bayasgalan.* Decidable theories of related structures of semigroups, 1991.
3. *V. Yu. Popov.* Critical theories of varieties of rings, 1995.
4. *Yu. V. Nagrebetskaya.* Decidability of theories of the first order of matrix algebras and groups of transformations, 2000.

Supervisor E. V. Sukhanov:

1. *A. A. Bulatov.* Algebraic properties of the lattice of clones, 1995.
2. *A. M. Shur.* Algebraic and combinatorial properties of equational languages, 1998.
3. *A. A. Krokhin.* Intervals in lattices of clones, 1998.
4. *K. L. Safin.* Ideals of iterative algebras, 2000.
5. *A. P. Semigrodskikh.* Lattices of closed classes of functions on an infinite set, 2003.
6. *A. V. Klepinin.* On algebraic and applied aspects of the problem of search of information, 2005.

Supervisor M. V. Volkov:

1. *D. S. Ananichev.* Identities in the lattices of varieties of solvable Lie rings, 1997.
2. *O. B. Finogenova (Paison).* Indicator characterizations of certain properties of varieties of associative rings, 1998.
3. *I. A. Goldberg.* The finite basis problem for semigroups of transformations, 2006.
4. *G. V. Tanana.* Structural and equational properties of adjoint regular rings, 2007.
5. *V. S. Grishchenko.* Metrics of reputation: models and algorithms of construction of open information environments, 2007.
6. *S. V. Goldberg (Pleshcheva).* Complexity of the identity checking problem in finite semigroups, 2008.
7. *E. S. Skvortsov.* On effective algorithms for the problem CSP, and a program realization of them, 2008.
8. *E. V. Pribavkina.* Problems of optimality in the theory of synchronizing automata, 2009.
9. *G. A. Povarov.* Descriptive complexity of certain transformations of regular languages, 2010.
10. *I. A. Mikhailova.* Patterns being avoided by antichains of words, and algebraic applications of them, 2010.
11. *M. V. Berlinkov.* Approximating the lengths of synchronizing words for finite automata, 2011.
12. *Yu. I. Zaks.* Synchronizability of finite automata in extremal and mean cases, 2012.
13. *V. V. Gusev.* Extremal constructions in the theory of synchronizing automata, 2013.
14. *T. V. Pervukhina.* Varieties and pseudovarieties of semigroups of triangular matrices, 2014.
15. *M. I. Maslennikova.* Ideal languages and synchronizing automata, 2015.

Supervisor I. O. Koryakov:

1. *I. Yu. Zhil'tsov.* Pseudo-operations and pseudo-free semigroups, 1999.

Supervisor A. M. Shur:

1. *Yu. V. Gamzova.* Combinatorial properties of partial words (co-supervisor E. V. Sukhanov), 2006.
2. *A. N. Plyushchenko.* On combinatorial properties of Burnside semigroups, 2012.
3. *I. A. Gorbunova.* Constructing and enumerating extremal power-free words, and an estimate of the quantity of them, 2013.
4. *E. A. Petrova.* On combinatorial properties of power-free languages, 2016.
5. *M. V. Rubinchik.* Computational complexity of certain problems of string processing, 2016.

6. *D. A. Kosolobov.* Efficient algorithms for studying regularities in strings, 2016.

Supervisor D. S. Ananichev:

1. *P. V. Martyugin.* Bounds for the length and computational complexity of synchronization of finite automata, 2008.
2. *I. V. Petrov.* Universal synchronizing and universal collapsing words, 2009.

Supervisor V. Yu. Popov:

1. *Yu. S. Okulovsky.* A program complex for software of intelligent computations, 2009.
2. *A. A. Gorbenko.* Methods of combinatorial virtualization for mobile robots, 2014.
3. *A. S. Sheka.* Models, algorithms, and a program complex for software of intelligent experiment, 2014.

Supervisor S. V. Sizyi:

1. *E. A. Rogozinnikov.* Groups of transformations of curves, 2014.

Supervisor B. M. Vernikov:

1. *V. Yu. Shaprynskiĭ.* Special elements of lattices of semigroup varieties, 2015.
2. *D. V. Skokov.* Identities and special elements in the lattice of varieties of epigroups, 2016.

3.2. Doctoral dissertations

For such dissertations, as it is known, a person whose role is similar to that of a supervisor is called a scientific consultant. In cases 2–8 and 10–12, the scientific consultant was L. N. Shevrin, in cases 9 and 14 they were, respectively, Yu. M. Vazhenin and M. V. Volkov. Case 13 is unordinary: this is a dissertation for a scientific degree "Doctor of Engineering Sciences", and the scientific consultant was Prof. V. M. Saĭ, a specialist in certain topics pertaining to railway transport, who originally asked to help him in application of some mathematical methods in the problems that he considered. S. V. Sizyi responded to that request and, as a result, had involved in these problems, what ultimately led to his own achievements in the area considered. His dissertation is obviously far from algebra, but the author used a number of methods of related fields of mathematics: discrete mathematics, graph theory, probability theory, mathematical statistics, and the like.

1. *L. N. Shevrin.* Lattice properties of semigroups, 1966.
2. *V. A. Baransky.* Independence of related structures in classes of algebraic systems, 1987.
3. *O. G. Kharlampovich.* Word problem for groups and Lie algebras, 1990.
4. *Yu. M. Vazhenin.* Critical theories of the first order, 1992.
5. *L. M. Martynov.* Spectra of solvability for varieties of algebras, 1992.
6. *M. V. Volkov.* Identities in lattices of varieties of semigroups, 1994.
7. *D. A. Bredikhin.* Identities and quasi-identities of relation algebras, 1997.
8. *V. B. Repnitskiĭ.* Representations of lattices by subalgebra lattices, 1997.
9. *V. Yu. Popov.* Algorithmic problems for varieties of semigroups, monoids, groups, and rings, 2002.
10. *B. M. Vernikov.* Identities and quasi-identities in lattices of varieties of semigroups, and congruences related to them, 2004.

11. *A. A. Bulatov.* Algebraic methods in investigation of combinatorial problems, 2008.
12. *A. M. Shur.* Combinatorial characterization of formal languages, 2010.
13. *S. V. Sizyi.* Theory and methodology of formation of netting-organizational co-ordination on railway transport, 2011.
14. *O. B. Finogenova.* Properties of varieties of associative algebras given in a language of derived objects: indicator and equational characterizations, 2016.

In conclusion of this section, I would like to note that several former members of the seminar now work in other towns, mostly abroad. Some of them trained their own students who defended dissertations. They are the following persons; for each one, I give the number of dissertations for which he/she was the supervisor: L. M. Martynov (Omsk, Russia) – 3, M. V. Sapir (Nashville, USA) – 5, O. G. Kharlampovich (Montreal, Canada, later New York, USA) – 9, A. V. Kelarev (Hobart, Australia) – 2, P. V. Shumyatsky (Brasilia, Brazil) – 9, B. P. Tanana (Maputo, Mozambique) – 1, A. A. Bulatov (Vancouver, Canada) – 1, Krokhin (Durham, Great Britain) – 1.

From information given above it follows that by 2017 there are in all 113 disciples with scientific degrees in the "scientific tree" of the leader of the seminar. The number of persons in this tree will definitely increase for the near years: now there are fairly many post-graduated students who are doing research under supervision of some elder members of the seminar.

4. Grants

The subjects of our research are also reflected in the grants we had from sources both national and international. Note that in Russia a system of scientific grants was established only since the beginning of the 1990s. I give below a list of grants got by some, as a rule, small teams of researchers belonging to the seminar and point out, in the chronological order within a group of grants with the same distributor, i) the structures that distributed grants, ii) the research subjects of grants, iii) for each grant, the leader (the only researcher or one of the members of an international team of researches, if it is specially pointed out), and iv) the years of supporting.

State Committee of Higher Education (later Ministry of Education)

- Pseudovarieties of algebras: combinatorial-algebraic aspects, *Shevrin*, 1994–1995.
- Pseudovarieties: algorithmic and structural-topological aspects, *Shevrin*, 1996–1997.
- Combinatorial-algebraic properties of logical functions and formal languages, *Sukhanov*, 1996–1997.
- Combinatorial-algebraic aspects of the theory of logical functions and formal languages, *Sukhanov*, 1998–2000.
- New approaches in the theory of pseudovarieties of semigroups, *Shevrin*, 1998–2000.
- Pseudovarieties of semigroups, and their applications in computer science, *Shevrin*, 2001–2002.
- Profinite methods in the theory of pseudovarieties and symbolic dynamics, *Shevrin*, 2003–2004.

International Science Foundation

- Semigroup varieties: their lattices and free objects, *Shevrin*, 1994–1995.

The International Association for the promotion of cooperation with scientists from the New Independent States of the former Soviet Union (INTAS).

- Algebraic and logic models for computer science, *one of the researchers Volkov*, 1995–1996.
- Combinatorial and geometric theory of groups and semigroups, and its applications to computer science, *the coordinator of the Russian part of the project Volkov*, 2000–2003.
- Universal algebra and lattice theory, *one of the researchers Repnitskiĭ*, 2004–2006.

Ministry of Culture and Education of Hungary

- Semigroups and their classes, *one of the researchers Volkov*, 1997–2000.

Russian Foundation for Basic Research

- Idea of variety applied to finite and regular semigroups, *Shevrin*, 1997–1999.
- Lattices of varieties of classical algebras, *Volkov*, 2001–2003.
- Access control of information in computer systems, *Baransky*, 2003.
- Combinatorics on words and automata, and its applications in computer science and bioinformatics, *Volkov*, 2005–2007.
- Epigroups: structural and equational aspects, *Shevrin*, 2006–2008.
- Fundamental problems of the theory of algebraic systems, and applications in informatics, *Volkov*, 2009–2010.
- Lattice properties of semigroups and semigroup varieties, *Shevrin*, 2010–2012.
- Dynamics of finite automata and regular languages, *Volkov*, 2010–2012.
- Intelligent algorithms of planning, and correction of movement of a robot, *Okulovsky*, 2013.
- New aspects in dynamics of finite automata and symbolic sequences, *Volkov*, 2013–2015.
- Investigation of algorithms for intelligent robotics complexes, *Popov*, 2013–2015.
- Selected aspects of structural and equational theory of semigroups, *Shevrin*, 2014–2016.

The scientific program "Universities of Russia"

- Subsystems and congruences of algebraic systems, *Shevrin*, 1994–1995.
- Lattices as related structures, *Shevrin*, 1998–2000.
- Structural and combinatorial properties of algebraic systems, *Shevrin*, 2002–2003.
- Structural and combinatorial theory of algebraic systems, and its applications, *Shevrin*, 2004.

Federal Agency of Russia on Science and Innovation (Rosnauka)

- Combinatorial characterizations of formal languages, *Shur*, 2006–2007.

A special Federal program "Scientific and scientific-educational specialists of innovative Russia"

- Intelligent algorithms of a calibration of robotics systems, *Okulovsky*, 2010.

The Ministry program of a support of post-graduate students

- The isomorphism problem for graphs, and dist-decompositions, *the researcher O. V. Rasin, the supervisor Baransky*, 2003–2004.
- The finite basis problem for some semigroups of transformations, *the researcher I. A. Goldberg, the supervisor Volkov*, 2004–2005.

Other projects of the Ministry of Education (or, later, of Education and Science)

- Combinatorial theory of varieties and pseudovarieties of semigroups, languages, and automata, and its applications in computer science and information security, *Shevrin*, 2003–2005.
- A new generation of a scientific school on algebra and discrete mathematics, *Shevrin*, 2005.
- Investigations on the theory of algebraic systems and its applications in computer science and bioinformatics, *Shevrin*, 2009–2011.
- Algebraic models of robotics systems, *Popov*, 2009–2011.
- Structural and combinatorial methods in the theory of algebraic systems, and its applications in computer science and bioinformatics, *Shevrin*, 2013.
- Combinatorial-logical methods in mathematical modeling and in computer science, *Volkov*, 2013.
- Intelligent systems of navigation and control for teams of autonomous mobile robots, *Popov*, 2012–2013.
- Structural and combinatorial methods of modern general algebra, *Vernikov*, 2014–2016.
- Applied aspects of combinatorial algebra: discrete modeling of informational and technological processes, *Volkov*, 2014–2016.

The President program of a support of young candidates of sciences

- Investigation of synchronizable finite automata and their generalizations, *the researcher Martyugin*, 2012–2013.
- Languages of synchronizing words of automata, and synchronizable colorings, *the researcher Pribavkina*, 2013–2015.

The President program of a support of young doctors of sciences

- Computational complexity of algorithmic problems, *the researcher Popov*, 2006–2007.
- Intelligent systems, high-performance computing, and computational complexity of algorithmic problems, *the researcher Popov*, 2008–2009.

The President program of a support of leading scientific schools of the Russian Federation

- Investigations of classical algebraic systems and algebraic methods in computer science, *Shevrin*, 2003–2005.
- Investigations on the theory of algebraic systems and its applications in computer science, *Shevrin*, 2014–2015.

These two grants, crowning the long sequence of our grants, have marked activities of the whole collective joined by the seminar “Algebraic Systems”.

5. Publications

Taking into account a long life of the seminar and a great quantity of persons who participated in its work in different years, it is not easy to give absolute exact numbers of scientific works of diverse kinds published by these persons (although during the first years of the seminar we kept a precise account concerning this matter). Anyway, I may assert that there are definitely over 900 papers in all published by the members of the seminar, not counting many hundreds of abstracts of talks at various conferences[4].

The main summands of this number are the following: over 800 research papers including more than 300 papers published in the central Russian mathematical journals, more than 200 papers published in other publications in Russian (among them more than 110 ones in "Matematicheskie zapiski" of Ural State University, which were published annually in the 1960s–1980s), and more than 300 papers published in international journals or proceedings of international conferences; further, over 60 papers of encyclopaedic character published in three encyclopaedias, one encyclopaedic dictionary, and one handbook; over 30 papers of information character about conferences or other mathematical events, about mathematicians, and the like (some typical examples are represented by the papers [1]–[5]); several scientific-popular papers.

As to the books written by regular members of the seminar, there are three considerable monographs and one large book chapter as well as about hundred books of teaching character (textbooks and brochures) pertaining to both university and school education.

Certain details concerning this side of our activity are displayed below.

It is reasonable to point out first of all our summarizing works. These are the survey articles [8]–[28] and the monographs [29, 30]. One should add to this list the monograph [31]: its content includes a number of results obtained by the present or the former (as, first of all, its author) members of the seminar. Moreover, mentioning the book [31] in such a list is justified additionally by the fact that Volkov fruitfully contributed to creation of this monograph[5]. All these works are devoted to topics which particular attention was given to at the seminar, and where we made an appreciable (or, in some points, even a crucial) contribution. Note that the surveys [19] and [22] pertain to some topical trends concerning applications of an algebraic approach to computer science, which entered into a sphere of our interests since the very end of the 1990s. The first two authors of [19] are former members of the seminar, however in the year of this publication the second of them was still a regular member of it. I also note that the work on the surveys [13]–[15] was begun where their authors were among the regular members of the seminar.

The works listed, except the papers [16, 17, 23], and [26], are comprehensive and present in a systematical form achievements in the corresponding areas belonging not only to the members of the seminar but to many other authors. The first two of the papers just particularly noted give a survey of results in the theory of clones obtained by their authors (Sukhanov and his students) in the 1990s; the other two papers are based also on results obtained by their author. The short survey [26], motivated by a jubilee date related to S. N. Chernikov, draws a fragment of a wide picture presented with due completeness in Chapter IV of the monographs [29] and [30]. As to the paper [23], this 6-pages survey is in fact a very long abstract (almost the full text) of a plenary talk at an international conference. It contains, with thorough comments, 8 open problems posed by the author in different years. The first two of them concern nilsemigroups; they are connected:

[4] I emphasize that this and subsequent numbers concern publications which appeared when their authors were regular participants in the seminar. It is by no means that each of the mathematicians enumerated in Section 3 was a regular member of the seminar in some time. Moreover, some of them gave the only talk at the seminar, namely, a summarizing talk of a post-graduate student presenting his/her dissertation, while all previous talks of such students were given at the corresponding "subseminars" mentioned in Section 10. Furthermore, some regular members of the seminar once ceased attending it by one or another reason; in particular, this concerns those ones who left Ekaterinburg in their time. So the further works of such persons are not taken into account in reviews of our publications that we do once in a while, and, respectively, these data are not reflected in the present article.

[5] As the author writes in Introduction, he is especially grateful to Mikhail Volkov "for writing about the road coloring problem, the Baer radical, the Kruse–L'vov theorem, and his important contributions to several other sections of the book". (A similar gratitude is addressed also to Victor Guba, and the author briefly noted these two contributions on the cover and on the title-page of the book.)

a counter-example for the former will be a counter-example for the latter. Both of them had been posed at the beginning of the 1960s (the first one was explicitly noticed in 1961 in some paper) and afterwards mentioned in several publications. There were attempts to solve them made by different authors who obtained (positive) results concerning certain partial cases. So in general case they have been open for 55 years.

In connection with the assertion of the preceding sentence, it is exciting that apparently the present “have been” in it may be changed by the past “had been”: the fact is that one of the young members of the seminar Shaprynskiĭ announced that he had constructed a counter-example solving these problems. By the way, his talk at the seminar, where he presented some principal ideas of his construction, took six (!) two-hours meetings in March–May, 2016. So, when the proof is carefully verified and turns out correct (we hope), one may suppose that the detailed text of a corresponding Shaprynskiĭ's paper will be many tens of pages long.

The next portion of publications being described in this section is a good deal of works of encyclopaedic character. This line was begun with the paper [32]. Later the present writer prepared a series of (41) papers on semigroups for the “Mathematical Encyclopaedia” [33]. Note that this encyclopaedia, published originally in Russian, later received rather wide spreading, since English, Spanish, and Chinese translations of it have appeared. A root paper *Semigroup* from the series mentioned was reproduced later in the “Mathematical Encyclopaedic Dictionary” [34]. The large book chapter [35] of a handbook on general algebra gives a comprehensive and detailed picture of semigroup theory (including applications to the theories of formal languages, automata, and codes) having formed by the beginning of the 1990s.

Later I made a contribution to the handbook [7] by preparing, partly with co-authors, 9 sections for Chapter “Semigroups” and 2 sections for Chapter “Universal Algebra”. These are the following sections: *Ideals and Green's Relations, Bands of Semigroups, Free semigroups, Simple Semigroups, Epigroups, Periodic Semigroups, Subsemigroup Lattices* (the last one jointly with Ovsyannikov), *Varieties of Semigroups* (with Volkov), *Applications of Semigroups* (with G. F. Pilz and P. G. Trotter), *Free Algebras* (with Sukhanov), *Varieties and Quasivarieties* (with Volkov).

For the encyclopaedia [36], a team consisting of Koryakov, Shevrin, Volkov, and Zamyatin prepared 11 papers: *Code, Finite Automaton, Pushdown automaton, Variety of Rational Languages, Rational Language, Syntactic monoid* (all by Koryakov), *Formal Grammar* (by Koryakov and Zamyatin), *Formal language* (by Koryakov and Shevrin), *Asynchronous Automaton and Trace Theory* (both by Volkov), *and Pseudovariety of Universal Algebras* (by Volkov and Shevrin).

A certain attention was given also to scientific-popular papers devoted to some interesting mathematical, especially algebraic, topics. The first experience in this direction was my paper *On periodic and locally finite groups and semigroups* (1979) published in the pamphlet “Methodical Recommendations and Instructions on Specialization” addressed to students-mathematicians of Ural State University. Later I published three papers in the “Soros Educational Journal”: *Identities in algebra* (1996), *What a semigroup is* (1997), *How groups appear when studying semigroups* (1997). The first of them has been reproduced in volume 3 of the encyclopaedia [37] prepared by the International Soros Science Education Program. Koryakov and Volkov have written some papers published in the journal for schoolchildren “MIF” (this abbreviation is derived from Russian “Matematika, Informatika i Fizika” — Mathematics, Informatics, and Physics) issued during several years by a special school (Lyceum) attached to Ural State University. It is worth especially noting the paper *The finite basis problem for identities* by Volkov (1997); the problem figuring in this title was mentioned above in Section 2. Lastly, I mention my paper *The aestheticism of mathematics* addressed to a broad circle of readers and published in the journal “Izvestiya Ural'skogo gosudarstvennogo universiteta”, No. 4 (1995).

All senior members of the seminar are university teachers, so some of them succeeded in writing teaching books for students. The most activity in this affair was manifested by Baransky, Ovsyannikov, Perminov, (V. V.) Rasin, Repnitskiĭ, Shur, Sizyi, Vernikov, and Zamyatin. There are over 50 such teaching books and methodical brochures written by different members of the seminar. They concern diverse subjects pertaining to both general and special courses: linear algebra, general algebra, geometry, number theory, mathematical logic, theory of algorithms, graph theory, theory of varieties, etc. Not giving any long list of such books and brochures, I mention here only several especially notable samples of textbooks.

The book [38] gives a good contemporary exposition of basic topics in graph theory as well as presents a detailed discussion of combinatorial algorithms solving optimization problems that arise frequently in applications. A revised edition of this book (in particular, with a number of simplified proofs) was printed by the Publishers Lan' (Fallow-deer) in 2010. The book [39], having joined especially many contributors from the seminar, covers all basic subjects of the corresponding courses and provides a rich choice of problems of different levels. The book [40] is the first textbook on this topic written in Russian. Its main content is devoted to consideration of combinatorial problems connected with the important notions "periodicity" and "unavoidability"; some combinatorial characteristics of formal languages are considered as well. The book [41] gives a systematic presentation of a number of important parts of the theory of formal languages and applications of this theory to the construction of compilers. The books [42] and [43], giving fundamentals of the corresponding mathematical disciplines, are characterized by a number of interesting peculiarities, in particular, by a lively style of presentation. The book [44], besides fundamentals of general algebra, presents different examples of its applications to such topics as binary codes (there is a corresponding separate section in the book), Boolean functions (a separate chapter), finite automata and regular languages (a separate chapter).

Another line of our activities concerns school mathematics. There are over 30 books and brochures for schoolchildren (or, partly, for school teachers) written by some members of the seminar. It would be hardly expediently to describe this line in detail within the present article; I will characterize apparently the most notable our work in this field. This is the book "*Mathematics 5–6. Textbook-Interlocutor*" created by a team headed by the present writer whose co-authors were Gein, Koryakov, and Volkov[6]. A manuscript of this textbook was awarded at the All-Union Competition in 1987 and was published in 1989. Subsequently the separate editions of the books *Mathematics 5* and *Mathematics 6* (that is, textbooks for the 5th and the 6th class, respectively) appeared in 1992–2004: four editions in Russian, one edition in Belorussian. Further, two editions of the Working Copy-Books attached to these textbooks as well as one edition of a book of methodical recommendations "*Mathematics 5. Book for Teacher*" were published. So there was, as it is customary to speak, a teaching-methodical complete set consisting of the books mentioned.

One may note in addition that the literary activities of the present writer in mathematics included also, since 1969, three teaching-belletristic books, jointly with V. G. Zhitomirsky, addressed to small children, in particular, under school age[7].

Yet another important line of our scientific publications concerns Russian translations of sev-

[6]A. G. Gein, originally a specialist on Lie algebras, formally does not belong to the scientific school led by the present writer, but he is closely connected with the seminar, repeatedly attended its meetings and gave several talks there in different years. Many years ago he as a student attended my lectures on general algebra and some special seminars, and since the 1986 he became the main co-author of mine in creation of our textbook together with all the works adjacent to it. Later he wrote quite a number of significant works devoted to school informatics and in 2000 defended a dissertation for a scientific degree "Doctor of Pedagogic Sciences".

[7]These are *Geometry for Kids*, *Mathematical ABC*, and *Travels in Geometry Land*; the last one is a considerably enlarged version of the first book. These books had in all 12 editions in Russian and 26 editions in 20 other languages.

eral books by foreign authors. These are the books [43]–[47]. This choice is explained by both our mathematical interests and a role of these books in the corresponding fields of science. The monograph [43], being, as already was mentioned in Section 2, one of the pioneering monographs on semigroup theory, is a classical work in this field of algebra, so it was very important to have this monograph in Russian. Importance of the book [44] (appeared in English in 1979) is caused by the fact that this is practically the first summarizing work treating applications of semigroups to the theories of automata, formal languages and codes in a consecutive and fundamental manner; by the way, it did not lose its significance up to now. The book [45] (appeared in English for the first time in 1984) is one of the first teaching books treating diverse applications of general algebra. Note that monographs [43] and [44] combine traits inherent in both a reference book and a teaching book, and, for instance, at Ural State University, we used them in a teaching process for lectures and special seminars. The book [46] (appeared in English in 1998) is the first monograph devoted to DNA computing, it opens quite a new trend on a junction of computer science and molecular biology. The book [47] devoted to very actual area is a professional and interactive tutorial.

6. Participation in conferences

As it follows from a remark at the beginning of Section 5 mentioning many hundreds of abstracts of talks at various conferences, the members of the seminar took part in numerous (definitely over 350) conferences, symposia, workshops, schools, etc. It is worth, first of all, mentioning among them All-Union Algebraic Conferences which were the most considerable meetings of Soviet algebraists for a long while. They were held regularly till 1991 in different algebraic centers of the Soviet Union and gathered up to several hundreds of participants. The first three such conferences were mentioned in Section 1. My students began to take part in All-Union Conferences from the 8th one, which was held in 1967; I participated in the preceding ones as well. Perhaps it is not without interest for the reader to learn a list of all these conferences.

Here is this list: I – Moscow, 1958; II – Moscow, 1959; III – Sverdlovsk, 1960; IV – Kiev, 1962; V – Novosibirsk, 1963; VI – Minsk, 1964; VII – Kishinev, 1965; VIII – Riga, 1967; IX – Gomel, 1968; X – Novosibirsk, 1969; XI – Kishinev, 1971; XII – Sverdlovsk, 1973; XIII – Gomel, 1975; XIV – Novosibirsk, 1977; XV – Krasnoyarsk, 1979; XVI – Leningrad,1981; XVII – Minsk, 1983; XVIII – Kishinev, 1985; XIX – Lvov, 1987; XX – Novosibirsk, 1989; XXI – Barnaul, 1991.

It should be noted that the last two conferences were in fact international; they were dedicated, respectively, to the 80th birthday of Academician A. I. Mal’cev (1909–1967) and to the 70th birthday of Corresponding Member of the Academy of Sciences of USSR A. I. Shirshov (1921–1981).

Almost every year certain conferences of All-Russian or regional status are attended by some representatives of the seminar. But there were very many international conferences, both in Russia and in foreign countries, which members of the seminar participated in. The leader of the seminar began to participate in conferences abroad in 1967. It was 1981 when I for the first time came to such a conference together with several of my students (Baransky, Martynov, Sukhanov, Trakhtman, and Vazhenin), it happened at the International Conference on Semigroup Theory in Szeged; I was a member of the Organizing Committee of that conference. Since the 1980s my disciples took part with increasing activity in various mathematical meetings abroad, and many members of the seminar visited on this occasion the following countries (quite a number of them repeatedly): Australia, Austria, Belgium, Brazil, Bulgaria, Canada, China, Czechoslovakia (since the beginning of the 1990s — separately the Czech Republic and Slovakia), Denmark, Estonia, Finland, France, Germany, Great Britain, Hong Kong, Hungary, Iceland, India, Ireland, Israel, Italy, Japan, Latvia, Mexico, Netherlands, Poland, Portugal, Romania, Serbia, Spain, Sweden, Ukraine, the USA. Such a list can be somewhat extended if one takes into account conferences attended by the former members of the seminar who work abroad now.

Obviously it would not be reasonable to give here a list of all conferences with our participation. I remark only that in the 21st century the seminar was represented, as a rule, at more than 10 international conferences every year, and I give as an illustration the corresponding data for the year 2003. The titles of a majority of the conferences listed there show in addition some components of the spectrum of our research at the beginning of the century.

The 7th International Conference "Developments in Language Theory", Szeged, Hungary, — a speaker Ananichev;

International Conference "Lattices, Universal Algebra, and Applications", Lisbon, Portugal, the 18th IEEE Symposium on Logic in Computer Science, Ottawa, Canada, the 18th International Joint Conference on Artificial Intelligence, Acapulco, Mexico, International Seminar "Graph Coloring", Castle Dagstuhl, Germany, and the 44th Annual IEEE Symposium on Foundations of Computer Science, Cambridge, USA, — a speaker Bulatov;

International Conference "Kolmogorov and Contemporary Mathematics", Moscow, Russia, — a speaker Volkov;

International meeting on Semigroups and Related Topics, Braga, Portugal, — a member of the Program Committee Volkov;

Euresco Conference "Symmetries and Ordered Structures under the Influence of Model Theory and Combinatorics", Hattingen, Germany, — an invited speaker Volkov;

The IV International Conference on Words, Turku, Finland, — an invited speaker Volkov, speakers Ananichev and Shur;

NATO Advanced Study Institute on Structural Theory of Automata, Semigroups, and Universal Algebra, Montreal, Canada, — invited lecturers Shevrin, Volkov, and a former member of the seminar Krokhin, a speaker Semigrodskih; among the listeners there were (I. A.) Goldberg, Pleshcheva, and Vernikov;

International Conference "Mal'cev Readings", Novosibirsk, Russia, — an invited speaker Shevrin, speakers Sukhanov and Vernikov.

Thus representatives of the seminar "Algebraic Systems" participated in 12 international conferences in 2003. In 2004–2016 they attended 197 international conferences; as a rule, in each year more than 10 and sometimes more than 20 conferences were attended by representatives of the seminar. The maximum for all the preceding years was reached in 2015, namely, 29 conferences, which took place in 15 countries, among them 6 conferences in Russia.

7. Organization of conferences

For the first four decades the seminar was involved (in full or in part) in organization of several algebraic conferences. First of all, a principal role was played by it in organization of all three All-Union Symposia on Semigroup Theory held in Sverdlovsk by Ural State University (1969, 1978, and 1988); in particular, the leader of the seminar was the Chairman of the Organizing Committees of these symposia. We held also the Regional Conference of Young Scientists "Algorithms, Automata, and Semigroups" in Sverdlovsk (1982). Further, the enlarged 500th meeting of our seminar (1985) may be regarded as a conference, see some details in Section 9.

Several members of the seminar took part in organization of the XII All-Union Algebraic Conference held in Sverdlovsk (1973) as well as two International Conferences on Semigroups held in St. Petersburg (1995, 1999): Ural State University was an official co-organizer of these conferences. The former of the conferences in St. Petersburg was dedicated to the 80th birthday of E. S. Ljapin (1914–2005), the latter was held also in honor of Ljapin.

The most considerable algebraic meeting held in Ekaterinburg was the International Algebraic Conference dedicated to the centenary of the birthday of P. G. Kontorovich and to the 70th birthday of the present writer. This conference was organized by Ural State University and the Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences and took place from August 29 to September 3, 2005. The Organizing Committee headed by the Rector of the University V. E. Tretjakov and the Director of the Institute V. I. Berdyshev included, among the others, several representatives of the seminar: Baransky (Vice-Chairman), Popov, Repnitskiĭ, Sukhanov, Volkov (Secretary). The seminar in all (with the exception of its leader who because of a natural reason was free of any organizing duties that time) was attached to the Organizing Committee for carrying out a lot of tasks usually arising in such arrangements. The Program Committee consisted of 20 major algebraists from Russia (9 members), the USA (3 members), Germany (2 members), Austria, Belorussia, Canada, Great Britain, Hungary, and Israel (by 1 member). The seminar was represented in the Program Committee by Baransky as one of the Co-Chairmen as well as by two former members of it, Kharlampovich and Sapir.

The conference gathered about 200 participants from 23 countries. The following five sections worked: Groups, Semigroups, Rings, Universal Algebras and Lattices, and Applications. 21 plenary lectures, 15 section lectures and 93 contributed talks at the sections were given. It is interesting to note that the seminar "Algebraic Systems" celebrated a certain jubilee at the conference: one of the plenary sessions was combined just with the 1000th meeting of our seminar; see the program of that meeting in Section 9.

The next international conferences in organization of which a key role was played by some members of the seminar were devoted to computer science and the theory of formal languages. The first of them belonged to the series of annual International Computer Science Symposia in Russia (CSR). This series started in 2006 (CSR 2006) by a symposium held in St. Petersburg. Just the second one, CSR 2007, was organized in Ekaterinburg by Ural State University and took place on September 3–7, 2007. The Organizing Committee chaired by Volkov included also his students (S. V.) Goldberg, Povarov, and Pribavkina. The Program Committee consisted of 35 leading computer scientists from 13 countries and was co-chaired by V. Diekert (Germany) and A. Voronkov (Great Britain). Besides plenary meetings, two parallel sections worked: Theoretical Computer Science, Applied Computer Science and Technology.

The Symposium CSR 2007 was accompanied by three satellite events: the Workshop on Computational Complexity and Decidability in Algebra, the Workshop on Infinite Words, Automata and Dynamics, and the Russian Summer School in Information Retrieval. The total number of participants of CSR 2007 and its satellite events was about 150, including 40 foreign participants from 18 countries.

The conference proceedings were published by Springer as volume 4649 of the series "Lecture Notes in Computer Science" (the editors Diekert, Volkov, and Voronkov).

The Symposia CSR 2008 – CSR 2012 took place, respectively, in Moscow, Novosibirsk, Kazan, St. Petersburg, and Nizhny Novgorod. The 8th Symposium CSR was held again in Ekaterinburg on June 25–29, 2013; the organizers were actually the same, by this time the name of the university-organizer had been changed and became Ural Federal University: this university appeared in 2011 as a result of merger of Ural State University and Ural Technical University. The Organizing Committee of the Symposium CSR 2013 chaired by Shur included, in particular, Pribavkina and Volkov. The Program Committee chaired by Bulatov consisted of 23 renowned specialists from 12 countries. It is interesting to note that among the main speakers of the symposium there was M. Szegedy (USA), two-time winner of the Godel Prize, who gave the opening lecture.

There were three satellite events: the 2nd Workshop on Current Trends in Cryptology, the 4th Workshop on Program Semantics, Specification and Verification: Theory and Applications, and the 6th School for students and young researchers Computer Science Ekaterinburg Days, the topic of the latter was "Algorithms and Complexity". Altogether, more than 100 scientists, including 30 foreign colleagues, participated in CSR 2013 and its satellite events.

The conference proceedings appeared as volume 7913 in Springer's series "Lecture Notes in

Computer Science" (the editors Bulatov and Shur).

Another series of conferences where we were to have a role of the organizers is the series of International Conferences Developments in Language Theory (DLT). This is the main conference series in formal language theory; it was founded by the famous computer scientists G. Rozenberg and A. Salomaa in 1993. Since 2001 DLT conferences became annual, and they take place in Europe in every odd year and outside Europe in every even year. The first such conference in Russia was held in Ekaterinburg by Ural Federal University on August 26–29, 2014; this was the 18th DLT conference. (Recall that Ekaterinburg is located in Asia, so that it was quite eligible to host DLT 2014 here.) In the Organizing Committee of the Conference DLT 2014 chaired by Volkov, a key role was played by the same team as at the Symposium CSR 2013: the chairman, Pribavkina, and Shur. The Program Committee co-chaired by Shur and Volkov included 18 prominent researchers from 12 countries.

The 7th School for students and young researchers Computer Science Ekaterinburg Days with the topic "Strings, Languages, Automata" was held as a satellite event of DLT 2014. About 70 participants in the conference and the school came from 20 countries including very remote ones, such as Brazil and New Zealand.

The conference proceedings were published by Springer as volume 8613 of "Lecture Notes in Computer Science" (the editors Shur and Volkov). Note that revised and expanded versions of selected papers presented at DLT 2014 formed a special issue of the "Journal of Foundations of Computer Science" (vol. 27, No. 2 (2016)), the guest editor of which was Shur.

It should be also noted that some representatives of the seminar were included in the organizing or program committees of many other conferences, both in Soviet Union (or Russia) and abroad. This concerns first of all the present writer who took part in such committees from the end of the 1960s. For the last two decades this kind of activities is rather typical for Volkov, and in the last years the same takes place for Shur as well. I mention also Baransky who had been participating for some years in the organization of several conferences, both All-Russian and regional, devoted to topics of information security.

8. Membership in editorial boards

Since 1972 the leader of the seminar enters the Editorial Board of the journal "Izvestiya VUZ. Matematika" which is one of the central All-Russian (before 1992 — All-Union) mathematical journals. By comparison with my membership in other editorial boards, the work in this Editorial Board was the most intensive. Indeed, during 45 years I had to consider over 650 papers submitted to this journal and devoted to algebraic (or, in very rare cases, number-theoretic) subjects. As usual, it was necessary to select an appropriate referee for each of them; for more than 60 papers, I myself acted as the referee.

It is notable that from the beginning of the 1980s till 2010 there was a tradition to publish systematically separate issues of this journal entirely devoted to one or another branch of mathematics and compiled by some member of the Editorial Board. Such issues were printed with a considerably greater circulation. I compiled and edited 5 such issues devoted to topical subjects. Here is the corresponding list, where a thematic peculiarity of each issue is given: i) 1982, No. 11 – varia; ii) 1985, No. 11 – the theory of varieties of algebraic systems; iii) 1989, No. 6 – again the theory of varieties; iv) 1995, No. 1 – the theory of pseudovarieties of algebraic systems; v) 2010, No. 1 – certain connections between the theories of finite automata and formal languages. I should note that the last issue was compiled and edited by me jointly with Volkov. Among the authors of papers in all these issues, there were quite a number of mathematicians specially invited for contributing to this enterprise. In the first three issues, they are such distinguished Soviet algebraists as Yu. A. Bakhturin, L. A. Bokut', Z. I. Borevich, A. R. Kemer, Yu. N. Mal'cev, A. V. Mikhalev, A. Yu. Ol'shanskii, L. A. Skornjakov, A. I. Starostin, and E. I. Zel'manov (future Fields medalist). In the other two issues, they are the following renowned foreign specialists in the corresponding areas of mathematics: J. Almeida, S. Margolis, J.-E. Pin, P. Weil, and, respectively, V. Diekert, T. Harju, D. Nowotka, M. Droste. The members of our seminar were represented among the authors in all these issues.

It was 1976 when I was invited to enter the Editorial Board of "Semigroup Forum", which is an international journal on the theory of semigroups printed in the USA. In 1979–1988 I was a member of the Editorial Board of "Simon Stevin", an international journal printed in Belgium. The current Editorial Board of "Semigroup Forum" contains two representatives of Russia, and both of them are members of our seminar: the present writer and Volkov who was invited in 1998 and became one of the Executive Editors since 2003. In 2010 Volkov entered the Editorial Board of "International Journal of Algebra and Computation" as well.

Now I mention the periodical editions on mathematics published by Ural State University. This is first of all "Matematicheskie zapiski" already mentioned above. After P. G. Kontorovich who was the first Editor-in-Chief since the beginning of the 1960s, the present writer carried out the same functions in 1969–1989. Since 1982 the Editorial Board of "Matematicheskie zapiski" was supplemented by Volkov, since 1987 it was supplemented by Koryakov. After a nine-year break publishing periodical mathematical editions at our University resumed with the journal "Izvestiya Ural'skogo gosudarstvennogo universiteta. Matematika i mehanika"; the Editorial Board consisted of 7 members including Shevrin as the Vice-Editor-in-Chief and Volkov as the Secretary. This journal was published till 2006. By the way, the first paper about our seminar [4] appeared just in the first issue of it. In that journal, there was a rubric "Scientific life", where we regularly published information about the meetings of our seminar.

In 2008–2010 a new series of "Izvestiya Ural'skogo gosudarstvennogo universiteta" existed, with the heading "Matematika, mehanika i informatika". Among 14 members of its Editorial Board, there were 4 representatives of our seminar: Baransky, Shevrin (one of two Vice-Editors-in-Chief), Shur, and Volkov (Secretaries).

There were some single editions that had the editorial boards entirely consisted of members of the seminar. These are mainly the materials of three All-Union Symposia on Semigroup Theory mentioned above. For each of these symposia, we prepared a collection of abstracts of talks at this symposium. Here are the Editorial Boards for these collections, everywhere with Shevrin as the Editor-in-Chief: for the 1st Symposium, Golubov, Shevrin, and Vazhenin; for the 2nd Symposium, Baransky, Golubov, Shevrin, Vazhenin, and Zamyatin; for the 3rd Symposium, Baransky, Golubov, Shevrin, Sukhanov, Vazhenin, Volkov, and Zamyatin. At the 2nd Symposium, a separate pamphlet with abstracts of plenary lectures was also printed (edited by Shevrin).

Another our useful publication was closely connected with the symposia mentioned. After each of them we prepared a collection of unsolved problems in semigroup theory. Some of them were posed directly at the symposium, mostly at a special session devoted to open problems; some problems were sent by their authors later, among such authors there were not only those ones who attended the symposium. For the first collection, a part of problems was taken also from a notebook started in 1965 by the present writer who proposed personally some semigroupists to write down open problems in this notebook for a subsequent publication somewhere. (It is so notable that Prof. A. H. Clifford (1908–1992), a patriarch in semigroup theory, was among the first ones who contributed to this collection. I first met him in 1966 at the International Congress of Mathematicians in Moscow.) The collection mentioned was entitled "Sverdlovsk Tetrad" (Sverdlovsk notebook) and published as a pamphlet, which afterwards was distributed among algebraists interested in semigroup theory. Thus there were three editions of "Sverdlovsk Tetrad" (1969, 1979, and 1989). Here are the Editorial Boards of them, with the same Editor-in-Chief as for the collections of abstracts at the symposia: for the 1st edition, Shevrin, Vazhenin; for the 2nd one, Baransky, Golubov, Shevrin, and Vazhenin; for the 3rd one, Baransky, Golubov, Shevrin, and Volkov. There is English translation of the 1st edition, which was revised by omitting some of the problems (as a rule, those ones which were solved by that time), see [48]. In each of the subsequent editions there was a special section with comments concerning problems from the previous edition(s) that had been (completely or partially) solved by that time.

The Editorial Board of the collection of abstracts at the International Algebraic Conference held in Ekaterinburg in 2005 consisted of four persons, two of which are members of our seminar: Ovsyannikov and Vernikov (Editor-in-Chief). I remark also that two present members of the seminar and one former member were mentioned in Section 7 as the editors of the materials relating to the international conferences CSR 2007, CSR 2013, and DLT 2014 characterized in that section.

I add yet that the Editorial Board of the journal "MIF" mentioned in Section 5 included two representatives of the seminar: Rasin (Editor-in-Chief) and Volkov.

Lastly, I would like to note that some former members of the seminar working abroad were invited to enter the editorial boards of several journals: Kharlampovich — in "International Journal of Algebra and Computation", since 1997; Krokhin — in "Multiple-Valued Logic", 1999–2002; Sapir — in "Algebra Universalis", 1999–2004, "International Journal of Algebra and Computation", 2000–2015 (since 2010 Managing Editor), "Algebra and Discrete Mathematics", since 2003, as well as "Algebra and Combinatorics" (book series), since 2001.

9. Meetings and speakers

The first meeting of the seminar took place on November 2, 1966; so the 50th anniversary of our work was celebrated in November, 2016. Usually we hold about 25 meetings per year; by the anniversary mentioned 1246 meetings had been held. At each meeting, we, as a rule, listen and discuss one talk of duration either about 2 hours (with a break) or a "long" hour; sometimes there may be two shorter talks or, very rarely, a greater number of reports. On the other hand, separate talks may take 2–3 meetings; talks of such duration were rather frequent during the first years of the work of our seminar. The record of the duration of a talk (6 meetings) was mentioned in Section 8.

As was mentioned in the previous section, information about the meetings of our seminar was regularly published in the journal "Izvestiya Ural'skogo gosudarstvennogo universiteta. Matematika i mehanika" during the period of its existence (1998–2005). More concretely, this was done in issues 1–4 and 6–8 of this journal and embraced the meetings from the 800th one to the 1000th one. The corresponding reports were devoted to each meeting separately, they indicated its number and the date, often included the abstracts of talks, which usually were rather informative (not infrequently with the formulations of theorems and, if necessary, even with required definitions). By the by, abstracts of the foreign speakers were given in English.

There is a diary of the seminar which is kept by the secretary of the seminar. I should remember Vazhenin (1945–2003) who was the permanent secretary from the very beginning of our work up to his last days. After him Popov was the secretary for 4 years, and then Shur changed him in this job. From the diary we can derive statistics we would like to know. Every hundredth meeting has a special program: we sum up some statistics, discuss both certain results of the period passed and possible prospects of research for forthcoming years. Four meetings of the seminar were enlarged; information about them is given in the next paragraphs.

The 300th meeting took place on June 30, 1978, just after the 2nd All-Union Symposium on Semigroup Theory mentioned in section 7, and many participants in that symposium attended this meeting. The 415th meeting took place on June 30, 1982, and it was combined with a session of the Regional Conference “Algorithms, Automata, and Semigroups” also mentioned in section 7.

The 500th meeting was especially considerable: nominally one meeting, in reality it was divided into five long sessions held during three days, from January 31 to February 2, 1985. This meeting in essence turned into a peculiar All-Union conference, it gathered over 90 participants from 20 towns of the Soviet Union and had 38 speakers.

The 1000th meeting took place on September 1, 2005, and was continuing the whole day. As was mentioned in Section 7, it was combined with one of the plenary sessions of the International Algebraic Conference, so it gathered many algebraists from different countries. Here is the program of this jubilee meeting.

L. N. Shevrin (Ekaterinburg), *The seminar “Algebraic Systems” by the 1000th meeting.*

O. G. Kharlampovich (Montreal), *Decidability of the elementary theory of the free group.*

V. Yu. Popov (Ekaterinburg), *Status and diameter of semigroups.*

A. A. Bulatov (Vancouver), *Local methods in CSPs.*

L. Márki (Budapest), *Universal aspects of general radical theory.*

F. Pastijn (Milwaukee), *The lattice of varieties of idempotent semirings.*

A. M. Shur (Ekaterinburg), *On complexity of formal languages.*

R. Pöschel (Dresden), *Completeness and rigidity for operations and relations.*

As the reader can see, among the speakers at this meeting, besides the leader of the seminar, there were two of its current participants and two former members who came from Montreal and Vancouver. The other speakers belong to the group of our foreign colleagues who visited us before and gave their talks at the seminar in different years. For instance, L. Márki was just the first foreign speaker at our seminar and first gave his talk at the seminar on August 29, 1989.

When we were planning the 1200th meeting, which took place on December 4, 2014, a certain summarizing statistics had been prepared. Here are several principal details of this statistics. By that date there had been 301 speakers in all at the seminar who gave 1673 talks or reports. The set of speakers has the following partition into three groups: a) 135 speakers from Ekaterinburg (Sverdlovsk) including those ones who work in other towns now, in particular, abroad (some of them repeatedly visited the seminar by coming to Ekaterinburg for several days from the place of their current stay); b) 145 speakers from 48 other towns of the former Soviet Union; c) 21 speakers properly from 15 foreign countries, namely, from Australia, Austria, Canada, China, the Czech Republic, France, Germany, Great Britain, Hungary, India, Italy, Poland, Portugal, Spain, the USA.

The speakers from Ekaterinburg have given 1362 talks (or, sometimes, shorter reports), the speakers from the other two groups have given, respectively, 280 and 31 talks. Naturally, now all mentioned numbers have increased in due course.

For each town represented by at least one speaker at our seminar, we know the set of all such speakers and the number of talks given by every of them. As to Ekaterinburg, the total quantity of speakers and the total number of talks (reports) given by them are shown above. I show similar numbers for several towns from an upper part of the list of other towns (in brackets, the number of the corresponding talks is given): Novosibirsk – 19 (39), Moscow – 18 (28), Leningrad (later St. Petersburg) – 11 (14), Omsk – 10 (29), Saratov – 10 (20). It is interesting to notice that, as one easily can deduce, the speakers from these five towns form about a half of the whole set of speakers from 48 towns of the second mentioned group, and the total number of talks given by them is approximately in the same relation to the corresponding quantity 280 talks.

As to the regular members of the seminar, I give in the next paragraph an upper part (ten places) of a list of the most productive speakers and show the number of the talks given by each of the mentioned persons by the 1200th meeting, i. e., by December, 2014.

Volkov – 107; *Vazhenin* – 75, the second place for him may be considered as surprising, because he passed away 14 years ago, but during the first decades of our work he permanently was the most active speaker at the seminar; *Shevrin* – 72; *Sapir* – 66, which is surprising, since he left Ekaterinburg in 1990 and visited us only once afterwards (in 1990 he was the second one in the list under consideration); *Vernikov* – 60; *Kharlampovich* – 58, which may seem surprising, since she left Ekaterinburg in 1990, but in reality she visits the native city every year (sometimes twice) and always gives a talk at the seminar; *Baransky* – 50; *Repnitskiĭ* – 46; *Popov* – 46; *Golubov* – 44.

In a similar list of the foreign speakers at the seminar, the first place is occupied by *L. Márki* (Budapest), *F. Pastijn* (Milwaukee), *R. Pöschel* (Dresden), and *J. Tůma* (Prague) — by 3 talks.

Quite a number of my scientific disciples have been or had been the regular members of the seminar for several decades, giving talks, participating in discussions, and contributing to creation of a propitious atmosphere at our meetings. I would like to list here those of them who are connected with the seminar for many years and continue their regular participation in its work up to now; for each one, I indicate the year when he/she began attending the meetings of the seminar. They are *Vitaly Baransky* (since the first meeting, 1966), *Mikhail Volkov* (since 1973), *Vladimir Repnitskiĭ* (since 1974), *Alexander Ovsyannikov* (since 1975), *Eugene Perminov* (since 1977), *Boris Vernikov* (since 1980), *Dmitry Ananichev* (since 1991), *Olga Finogenova* (since 1992), *Arseny Shur* (since 1993). It may be said that these mathematicians, together with the present writer, form a current core of the seminar. One may regard Olga Kharlampovich as a person adjacent to this core. She began attending the seminar in 1978, and, as it was noted above, although she left Ekaterinburg more than quarter a century ago, she visits us every year and belongs to the most productive speakers at the seminar.

With warm feelings I also remember my talented and active students passed away for the last 14 years: *Yuri Vazhenin, Alexei Zamyatin, Igor Koryakov, Veniamin Rasin, Eugene Sukhanov.* The reader can observe that their names repeatedly appear in the text of this article including the list of references.

10. Concluding remarks

Viability of a scientific collective for a long time and its stability depend on many factors. Not discussing in detail this theme here, I want only to remark that one of such factors is more or less regular replenishment of the corresponding group with younger researches. Applied to the seminar under discussion, it was a matter of my permanent care. And there are reasons to be satisfied that several of my scientific children and grandchildren now continue this line and successfully train some representatives of a further generation for research work; a confirmation of this assertion can be seen in Section 3 and in some parts of Section 4. This is promoted, in particular, by means of organizing certain "subseminars". More than 20 years ago Baransky, Vazhenin, and Sukhanov organized, respectively, their seminars on combinatorics, algorithmic problems of algebra, and discrete mathematics. At present Baransky leads the seminar "Algorithms and Combinatorics", Volkov and Ananichev lead the seminar "Computer Science", Popov leads the seminar "Intelligent Systems", Shur leads the seminar "Discrete Mathematics".

Now representatives of four scientific generations may take part at a meeting of the seminar "Algebraic Systems". One may hope that various participants in the seminar will successfully continue their investigations both in topics that have become traditional for the seminar and in new topics being assimilated at present.

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AUTOMORPHISMS OF DISTANCE-REGULAR GRAPH WITH INTERSECTION ARRAY $\{25, 16, 1; 1, 8, 25\}$ [1]

Konstantin S. Efimov

Ural Federal University, Ekaterinburg, Russia,
Ural State University of Economics, Ekaterinburg, Russia
konstantin.s.efimov@gmail.com

Alexander A. Makhnev

N.N. Krasovskii Institute of Mathematics and Mechanics UB RAS, Ekaterinburg, Russia,
Ural Federal University, Ekaterinburg, Russia
makhnev@imm.uran.ru

Abstract: Makhnev and Samoilenko have found parameters of strongly regular graphs with no more than 1000 vertices, which may be neighborhoods of vertices in antipodal distance-regular graph of diameter 3 and with $\lambda = \mu$. They proposed the program of investigation vertex-symmetric antipodal distance-regular graphs of diameter 3 with $\lambda = \mu$, in which neighborhoods of vertices are strongly regular. In this paper we consider neighborhoods of vertices with parameters $(25, 8, 3, 2)$.

Key words: Strongly regular graph, Distance-regular graph.

Introduction

We consider undirected graphs without loops and multiple edges. Given a vertex a in a graph Γ, we denote by $\Gamma_i(a)$ the subgraph induced by Γ on the set of all vertices, that are at the distance i from a. The subgraph $[a] = \Gamma_1(a)$ is called the *neighborhood of the vertex* a. Let $\Gamma(a) = \Gamma_1(a)$, $a^\perp = \{a\} \cup \Gamma(a)$. If graph Γ is fixed, then instead of $\Gamma(a)$ we write $[a]$. For the set of vertices X of graph Γ through $X^\perp$ denote $\cap_{x\in X} x^\perp$.

Let Γ be an antipodal distance-regular graph of diameter 3 and $\lambda = \mu$, in which neighborhoods of vertices are strongly-regular graphs. Then Γ has intersection array $\{k, \mu(r-1), 1; 1, \mu, k\}$, and spectrum $k^1, \sqrt{k}^f, -1^k, -\sqrt{k}^f$, where $f = (k+1)(r-1)/2$. In the case $r = 2$ we obtain Taylor's graph, in which $k' = 2\mu'$. Conversely, for any strongly regular graph with parameters $(v', 2\mu', \lambda', \mu')$ there exists a Taylor's graph, in which neighborhoods of vertices are strongly regular with relevant parameters.

In [1]there were chosen strongly-regular graphs with no more than 1000 vertices, which may be neighborhoods of vertices of antipodal distance-regular graph of diameter 3 and $\lambda = \mu$. There is provided a research program of the study of vertex-symmetric antipodal distance-regular graphs of diameter 3 with $\lambda = \mu$, in which neighborhoods of vertices are strongly regular with parameters from Proposition 1.

Proposition 1. *Let Δ be a strongly-regular graph with parameters (v, k, λ, μ). If $(r-1)k = v-k-1$, $v \le 1000$ and number $(v+1)(r-1)$ is even, then either $r = 2$, or parameters (v, k, λ, μ, r) belong to the following list:*

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(1) $(16,5,0,2,3)$, $(25,8,3,2,3)$, $(49,12,5,2,4)$, $(64,21,8,6,3)$, $(81,16,7,2,5)$, $(81,20,1,6,4)$, $(85,14,3,2,6)$, $(99,14,1,2,7)$, $(100,33,8,12,3)$, $(121,20,9,2,6)$, $(121,30,11,6,4)$, $(121,40,15,12,3)$, $(126,25,8,4,5)$, $(133,44,15,14,3)$, $(169,24,11,2,7)$, $(169,42,5,12,4)$, $(169,56,15,20,3)$, $(176,25,0,4,7)$, $(196,39,14,6,5)$, $(196,65,24,20,3)$;

(2) $(225,28,13,2,8)$, $(225,56,19,12,4)$, $(243,22,1,2,11)$, $(256,51,2,12,5)$, $(256,85,24,30,3)$, $(261,52,11,10,5)$, $(288,41,4,6,7)$, $(289,32,15,2,9)$, $(289,48,17,6,6)$, $(289,72,11,20,4)$, $(289,96,35,30,3)$, $(305,76,27,16,4)$, $(325,54,3,10,6)$, $(351,50,13,6,7)$, $(351,70,13,14,5)$, $(352,39,6,4,9)$, $(361,36,17,2,10)$, $(361,72,23,12,5)$, $(361,90,29,20,4)$, $(361,120,35,42,3)$;

(3) $(400,57,20,6,7)$, $(400,133,48,42,3)$, $(441,40,19,2,11)$, $(441,88,7,20,5)$, $(441,110,19,30,4)$, $(484,161,48,56,3)$, $(495,38,1,3,13)$, $(505,84,3,16,6)$, $(507,46,5,4,11)$, $(512,73,12,10,7)$, $(529,44,21,2,12)$, $(529,66,23,6,8)$, $(529,88,27,12,6)$, $(529,132,41,30,4)$, $(529,176,63,56,3)$, $(540,49,8,4,11)$, $(576,115,18,24,5)$;

(4) $(625,48,23,2,13)$, $(625,156,29,42,4)$, $(625,208,63,72,3)$, $(640,71,6,8,9)$, $(649,72,15,7,9)$, $(649,216,63,76,3)$, $(676,75,26,6,9)$, $(676,135,14,30,5)$, $(704,37,0,2,19)$, $(729,52,25,2,14)$, $(729,104,31,12,7)$, $(729,182,55,42,4)$, $(736,105,20,14,7)$, $(768,59,10,4,13)$, $(784,261,80,90,3)$;

(5) $(837,76,15,6,11)$, $(841,56,27,2,15)$, $(841,84,29,6,10)$, $(841,140,39,20,6)$, $(841,168,47,30,5)$, $(841,210,41,56,4)$, $(841,280,99,90,3)$, $(847,94,21,9,9)$, $(848,121,24,16,7)$, $(901,60,3,4,15)$, $(961,60,29,2,16)$, $(961,120,35,12,8)$, $(961,160,9,30,6)$, $(961,192,23,42,5)$, $(961,240,71,56,4)$, $(961,320,99,100,3)$, $(1000,111,14,12,9)$.

Graphs with local subgraphs having parameters $(64,21,8,6)$, $(81,16,7,2)$, $(85,14,3,2)$ and $(99,14,1,2)$ were investigated in [2], [3], [4] and [5]. In this article we investigate parameters $(25,8,3,2,3)$, i.e. this graph is locally 5×5-grid. In [6] it is proved that distance-regular locally 5×5-grid of diameter more then 2 is either isomorphic to the Johnson's graph $J(10,5)$ or has an intersection array $\{25,16,1;1,8,25\}$.

Theorem 1. *Let Γ be a distance-regular graph with intersection array $\{25,16,1;1,8,25\}$, $G=\mathrm{Aut}(\Gamma)$, g is an element of prime order p in G and $\Omega=\mathrm{Fix}(g)$ contains exactly s vertices in t antipodal classes. Then $\pi(G)\subseteq\{2,3,5,13\}$ and one of the following assertions holds:*

(1) *Ω is empty graph and $p\in\{2,3,13\}$;*

(2) *$p=5$, $t=1$, $\alpha_3(g)=0$, $\alpha_1(g)=50l+25$ and $\alpha_2(g)=50-50l$;*

(3) *$p=3$, $s=3$, $t=2,5,8$, $\alpha_3(g)=0$, $\alpha_1(g)=30l+16-11t$ and $\alpha_2(g)=62-30l+8t$;*

(4) *$p=2$, and either $s=1$, Ω is t-clique, $t=2,4,6$, $\alpha_3(g)=2t$, $\alpha_1(g)=20l-t+6$ and $\alpha_2(g)=72-20l-2t$, or $s=3$, $t\le 8$, t is even, $\alpha_3(g)=0$, $\alpha_1(g)=20l-11t+6$ and $\alpha_2(g)=72-20l+8t$.*

Corollary 1. *Let Γ be a distance-regular graph with intersection array $\{25,16,1;1,8,25\}$ and a group $G=\mathrm{Aut}(\Gamma)$ acts transitively on the set of vertices of Γ. Then one of the following assertions holds:*

(1) *Γ is a Cayley graph, G is the a Frobenius group with the kernel of order* 13 *and with the complement of order* 6*;*

(2) *Γ is a arc-transitive Maton's graph and the socle of G is isomorphic to $L_2(25)$;*

(3) *G is an extension of a group Q of order 2^{12} by the group $T=L_3(3)$, $|Q:Q_{\{F\}}|=2$, $T_{\{F\}}$ is an extension of group E_9 by $SL_2(3)$, T acts irreducibly on Q and for an element f of order* 13 *in G we have $C_Q(f)=1$.*

1. Proof of the Theorem

Note that there is Delsarte boundary (proposition 4.4.6 from [7]) of maximum order of clique in distance-regular graph with intersection array $\{25, 16, 1; 1, 8, 25\}$ and spectrum $25^1, 5^{26}, -1^{25}, -5^{26}$ no more than $1 - k/\theta_d = 1 + 25/5 = 6$. If C is 6-clique in Γ, then each vertex not in C is adjacent to 0 or to $b_1/(\theta_d + 1) + 1 - k/\theta_d = 2$ vertices in C.

Lemma 1. *Let Γ be a distance-regular graph with intersection array $\{25, 16, 1; 1, 8, 25\}$, $G = \mathrm{Aut}(\Gamma)$ and $g \in G$. If ψ is the monomial representation of a group G in $GL(78, \mathbf{C})$, χ_1 is the character of the representation ψ on subspace of eigenvectors of dimension 26, corresponding to the eigenvalue 5, χ_2 is the character of the representation ψ on subspace of dimension 25, then $\chi_1(g) = (10\alpha_0(g)+2\alpha_1(g)-\alpha_2(g)-5\alpha_3(g))/30$, $\chi_2(g) = (\alpha_0(g)+\alpha_3(g))/3-1$. If $|g| = p$ is prime, then $\chi_1(g) - 26$ and $\chi_2(g) - 25$ are divided by p.*

P r o o f. We have

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 26 & 26/5 & -13/5 & -13 \\ 25 & -1 & -1 & 25 \\ 26 & -26/5 & 13/5 & -13 \end{pmatrix}.$$

Therefore $\chi_1(g) = (10\alpha_0(g) + 2\alpha_1(g) - \alpha_2(g) - 5\alpha_3(g))/30$. Substituting $\alpha_2(g) = 78 - \alpha_0(g) - \alpha_1(g) - \alpha_3(g)$, we obtain $\chi_1(g) = (11\alpha_0(g) + 3\alpha_1(g) - 4\alpha_3(g))/30 - 13/5$.

Similarly, $\chi_2(g) = (25\alpha_0(g) - \alpha_1(g) - \alpha_2(g) + 25\alpha_3(g))/78$. Substituting $\alpha_1(g) + \alpha_2(g) = 78 - \alpha_0(g) - \alpha_3(g)$, we obtain $\chi_2(g) = (\alpha_0(g) + \alpha_3(g))/3 - 1$.

The remaining assertions follow from Lemma 1 in [8]. The proof is complete. □

Let further in the paper Γ be a distance-regular graph with intersection array $\{25, 16, 1; 1, 8, 25\}$, $G = \mathrm{Aut}(\Gamma)$, g is an element of prime order p in G and $\Omega = \mathrm{Fix}(g)$.

Lemma 2. *If Ω is an empty graph, then either $p = 13$, $\alpha_1(g) = 26$ and $\alpha_2(g) = 52$, or $p = 3$, $\alpha_3(g) = 9s+6$, $s < 8$, $\alpha_1(g) = 54+12s-30l$ and $\alpha_2(g) = 18-21s+30l$, $l \le 5$, or $p = 2$, $\alpha_3(g) = 0$, $\alpha_1(g) = 20l + 6$ and $\alpha_2(g) = 72 - 20l$, $l \le 3$.*

P r o o f. Let Ω be an empty graph and $\alpha_i(g) = pw_i$ for $i > 0$. Since $v = 78$, we have $p \in \{2, 3, 13\}$.

Let $p = 13$. Then $\alpha_3(g) = 0$, $\alpha_1(g) + \alpha_2(g) = 78$ and $\chi_1(g) = (2\alpha_1(g) - \alpha_2(g))/30 = 13(w_1 - 2)/10$. This implies $\alpha_1(g) = 26$ and $\alpha_2(g) = 52$.

Let $p = 3$. Then $\chi_2(g)-25 = \alpha_3(g)/3-26$ is divided by 3, $\alpha_3(g) = 9s+6$, $s \le 8$ and $\alpha_2(g) = 72-9s-\alpha_1(g)$. Furthermore, the number $\chi_1(g) = (2\alpha_1(g)-\alpha_2(g)-45s-30)/30 = (3w_1-12s-34)/10$ is congruent to 2 modulo 3. This implies $\alpha_1(g) = 54 + 12s - 30l$ and $\alpha_2(g) = 18 - 21s + 30l$, $l \le 5$. In case $s = 8$ we have $\alpha_3(g) = 78$ and $\langle g\rangle$ acts regularly on each antipodal class. By lemma 4 in [9] 3 must divide $k + 1 = 26$, we have a contradiction.

Let $p = 2$. Then $\alpha_3(g) = 0$, $\alpha_1(g) + \alpha_2(g) = 78$, the number $\chi_1(g) = (\alpha_1(g) - 26)/10$ is even, $\alpha_1(g) = 20l + 6$ and $\alpha_2(g) = 72 - 20l$, $l \le 3$. □

In Lemmas 3–6 it is assumed that there are t antipodal classes intersecting the Ω on s vertices. Then p divides $26 - t$ and $3 - s$. Let F be an antipodal class, containing the vertex $a \in \Omega$, $F \cap \Omega = \{a, a_2, ..., a_s\}$, $b \in \Omega(a)$. By $F(x)$ we denote an antipodal class containing vertex x.

Lemma 3. *The following assertions hold:*
(1) *if* $t = 1$, *then* $p = 5$, $\alpha_3(g) = 0$, $\alpha_1(g) = 50l + 25$ *and* $\alpha_2(g) = 50 - 50l$;
(2) *if* p *more than* 3, *then* $p = 5$ *and* $t = 1$;
(3) *if* $s = 1$, *then* $p = 2$, $t = 2, 4, 6$, $\alpha_3(g) = 2t$, $\alpha_1(g) = 20l - t + 6$ *and* $\alpha_2(g) = 72 - 20l - 2t$.

P r o o f. If $s = 3$, then each vertex from $\Gamma - \Omega$ is adjacent to t vertices in Ω, so $t \leq 8$.

Let $t = 1$. As p divides $26 - t$, then $p = 5$, $s = 3$, $\alpha_2(g) = 75 - \alpha_1(g)$, the number $\chi_1(g) = (\alpha_1(g) - 15)/10$ is congruent to 1 modulo 5. This implies $\alpha_1(g) = 50l + 25$.

Let $p > 3$, $\alpha_1(g) = pw_1$. Then $s = 3$, $|\Omega| = 3t$, Ω is a regular graph by degree $t - 1$ and p divides $26 - t$.

If $p > 7$, then Ω is a distance-regular graph with intersection array $\{t - 1, 16, 1; 1, 8, t - 1\}$, we come to a contradiction.

Let $p = 7$. As p divides $26 - t$, then $t = 5$, the subgraph $\Omega(b)$ contains 2 vertices in $a^\perp$ and a vertex from $[a_2]$ and from $[a_3]$, so Ω is a distance-regular graph with intersection array $\{4, 1, 1; 1, 1, 4\}$, it is a contradiction with the fact that $r = 3$.

Let $p = 5$. As p divides $26 - t$, then $t = 1, 6$. If $t = 6$, then the subgraph $\Omega(b)$ contains a vertex in $a^\perp$, 3 vertices from $[a_2]$ and 3 vertices from $[a_3]$, we come to a contradiction.

Let $s = 1$. Then $p = 2$, $t \leq 6$, $\alpha_3(g) = 2t$, $\alpha_2(g) = 78 - \alpha_1(g) - 3t$, and $\chi_1(g) = (\alpha_1(g) + t - 26)/10$ is even. This implies that $\alpha_1(g) = 20l - t + 6$. □

Lemma 4. *If* $p = 3$, *then* $s = 3$, $t = 2, 5, 8$, $\alpha_3(g) = 0$, $\alpha_1(g) = 30l + 16 - 11t$ *and* $\alpha_2(g) = 62 - 30l + 8t$.

P r o o f. Let $p = 3$. Then $s = 3$, $t = 2, 5, 8$, $\alpha_2(g) = 78 - \alpha_1(g) - 3t$, and the number $\chi_1(g) = (11t + \alpha_1(g) - 26)/10$ is congruent to 2 modulo 3. This implies that $\alpha_1(g) = 30l + 16 - 11t$. In the case $t = 2$ graph Ω is a union of 3 isolated edges. □

Lemma 5. *If* $p = 2$, $s = 3$, *then* t *is even,* $t \leq 8$, $\alpha_3(g) = 0$, $\alpha_1(g) = 20l - 11t + 6$ *and* $\alpha_2(g) = 72 - 20l + 8t$.

P r o o f. Let $p = 2$, $s = 3$. Then t is even, $t \leq 8$, $\alpha_3(g) = 0$, $\alpha_2(g) = 78 - 3t - \alpha_1(g)$.
The number $\chi_1(g) = (11t + \alpha_1(g) - 26)/10$ is even, so $\alpha_1(g) = 20l - 11t + 6$. □

Lemmas 2–5 imply the proof of the Theorem.

2. Proof of Corollary

Let the group G acts transitively on the set of vertices of the graph Γ. Then for a vertex $a \in \Gamma$ subgroup $H = G_a$ has index 78 in G. By Theorem we have $\{2, 3, 13\} \subseteq \pi(G) \subseteq \{2, 3, 5, 13\}$.

Lemma 6. *Let* f *be an element of order* 13 *in* G. *Then* $\mathrm{Fix}(f)$ *is an empty graph,* $\alpha_1(f) = 26$ *and the following assertions hold:*
(1) *if* g *is an element of prime order* $p \neq 13$ *in* $C_G(f)$, *then* $p = 2$, Ω *is an empty graph,* $\alpha_1(g) = 26$ *and* $|C_G(f)|$ *is not divided by* 4;
(2) *either* $|G| = 78$ *or* $F(G) = O_2(G)$;
(3) *if* G *is nonsolvable group, then the socle* $\bar{T}$ *of the group* $\bar{G} = G/F(G)$ *is isomorphic to* $L_2(25)$, $L_3(3)$, $U_3(4)$, $L_4(3)$ *or* ${}^2F_4(2)'$.

P r o o f. By Lemma 2 Fix(f) is an empty graph and $\alpha_1(f) = 26$.

Suppose that g is an element of prime order $p \neq 13$ in $C_G(f)$. As f acts without fixed points on Ω then by Theorem Ω is an empty graph, $p = 2$ and $\alpha_1(g) = 20l + 6$ divided by 13. This implies that $\alpha_1(g) = 26$ and $|C_G(f)|$ is not divided by 4.

Let $Q = O_p(G) \neq 1$. If $p = 13$, then $|G|$ divides $26 \cdot 12$. In this case $C_G(f) = \langle f \rangle$, otherwise for an involution g of $C_G(f)$ we obtain a contradiction with the action of element of order 3 of G on $\{u \mid d(u, u^g) = 1\}$. Let the involution g inverts f, h is an element of order 3 in $C_G(g)$. From action h on $\{u \mid d(u, u^g) = 1\}$ it follows that $\alpha_1(g) = 20l + 6$ is divided by 3. In each case $\alpha_1(g)$ is not divided by 4 and $|G| = 78$.

If $p = 3$, then Q fixes some antipodal class. This implies that Q fixes each antipodal class. By Lemma 3 in [9] G does not contain subgroups of order 3, which are regular on each antipodal class, we come to a contradiction. So, if $|G| \neq 78$ we have $F(G) = O_2(G)$.

Let $\bar{T}$ be the socle of the group $\bar{G} = G/F(G)$. Note that 13 divides $|\bar{T}|$ and by Theorem 1 in [10] group $\bar{T}$ is isomorphic to $L_2(25)$, $L_3(3)$, $U_3(4)$, $L_4(3)$, ${}^2F_4(2)'$. □

Let us to prove the Corollary. As $\bar{T}$ contains a subgroup of index dividing 26, then the group $\bar{T}$ is isomorphic to $L_2(25)$ (and $\bar{T}_{\{F\}}$ is the extension of a group of order 25 by group of order 12) or $L_3(3)$ (and $\bar{T}_{\{F\}}$ is the extension of a group of order 9 by $SL(2, 3)$).

In the first case $F(G)$ fixes each antipodal class and $F(G) = 1$. This implies that Γ is the arc-transitiv Maton's graph.

In the second case for $Q = F(G)$ we have $|Q : Q_{\{F\}}| = 2$ and $\bar{T}$ acts irreducibly on Q. Further, for the element f of order 13 of G by Lemma 6 the number $|C_Q(f)|$ divides 2. As Q is either 12-dimensional module over F_2, or 16-dimensional module over F_{16}, or 26-dimensional module over F_2, then $|Q| = 2^{12}$ and $C_Q(f) = 1$. The Corollary is proved.

3. Conclusion

We found possible automorphisms of a distance regular graph with intersection array $\{25, 16, 1; 1, 8, 25\}$. This completes the research program of vertex-symmetric antipodal distance-regular graphs of diameter 3 with $\lambda = \mu$, in which neighborhoods of vertices are strongly regular with parameters from Proposition 1.

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DISPERSIVE RAREFACTION WAVE WITH A LARGE INITIAL GRADIENT[1]

Alexander E. Elbert

Krasovskii Institute of Mathematics and Mechanics
of Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russia
E-mail: aee@imm.uran.ru

Sergey V. Zakharov

Krasovskii Institute of Mathematics and Mechanics
of Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russia
E-mail: svz@imm.uran.ru

Abstract: Consider the Cauchy problem for the Korteweg–de Vries equation with a small parameter at the highest derivative and a large gradient of the initial function. Numerical and analytical methods show that the obtained using renormalization formal asymptotics, corresponding to rarefaction waves, is an asymptotic solution of the KdV equation. The graphs of the asymptotic solutions are represented, including the case of non-monotonic initial data.

Key words: The Korteweg–de Vries, Cauchy problem, Asymptotic behavior, Rarefaction wave.

1. Introduction

Consider the Cauchy problem for the Korteweg–de Vries equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \varepsilon\frac{\partial^3 u}{\partial x^3} = 0, \quad t \geqslant 0, \quad \varepsilon > 0, \tag{1.1}$$

$$u(x, 0, \varepsilon, \rho) = \Lambda\left(\frac{x}{\rho}\right), \quad x \in \mathbb{R}, \quad \rho > 0, \tag{1.2}$$

with a bounded initial function Λ, which has finite limits $\Lambda_0^{\pm} = \lim\limits_{s\to\pm\infty} \Lambda(s)$, $\Lambda_0^- < \Lambda_0^+$, and its derivative quickly enough tends to zero at infinity. This is a classic model of nonlinear waves in the medium with small dispersion. For the case of discontinuous initial functions the asymptotics was studied by A.B. Gurevich and L.P. Pitaevskii [1]. The asymptotic formulas for the evolution of rarefaction waves were also found in the works [2, 3] by the Whitham method and in the work [4] by the inverse scattering method. For the initial step-like function the asymptotic formulas are obtained by the method of the inverse scattering [5–7]. In the case of a smoothed step-like initial function the asymptotic expansion was constructed by the method of matching in the work [8]. We emphasize that the study of analytic properties of solutions of the KdV equation and their asymptotic behavior is one of the interesting problems of the modern mathematical physics [9],[17]-[26]. It should be noted that the KdV equation remains a most important model in hydrodynamics [31] and physics of plasma up to subnuclear scales [29]. In particular, the latest research was aimed to the study of rarefaction waves [34, 35].

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We assume that a smooth initial function $\Lambda : \mathbb{R} \to \mathbb{R}$ has finite limits $\Lambda_0^\pm = \lim_{\sigma\to\pm\infty} \Lambda(\sigma)$, and the asymptotic expansions satisfy the asymptotic formula

$$\Lambda(\sigma) = \sum_{n=0}^{\infty} \frac{\Lambda_n^\pm}{\sigma^n}, \quad \sigma \to \pm\infty. \tag{1.3}$$

The existence of a classical smooth solution of problem (1.1)–(1.2) is guaranteed by the Kappeler theorem [10] if

$$\int_{-\infty}^{0} |\Lambda(x) - \Lambda_0^-|(1 - x^3)dx < \infty, \quad \int_{0}^{\infty} |\Lambda(x) - \Lambda_0^+|(1 + x^3)dx < \infty.$$

Here, however, it is possible not to assume these restrictions, but we may consider only the formal asymptotic solution; moreover, in a special class of functions the existence of solutions is proved for unbounded initial data [11].

In this article the approximation constructed in the work [12] is refined, and it is shown in Section 3 that it is the asymptotic solution for the problem (1.1)–(1.2) as $\varepsilon \to 0$, $\rho \to 0$ and the ratio of the parameters $\rho^2/\varepsilon \to 0$, and also numerical analysis has been performed for this solution in Section 4. These results show the efficiency of the renormalization approach used in paper [12]. One of the purposes of the paper is to understand and determine the precise mathematical meaning of "formal asymptotic solutions" of the KdV equation. Since for a similar problem for a parabolic equation the closeness of the asymptotics obtained by renormalization, to the exact solution, was proved [16], so there is some reason to suggest that for the KdV equation the formal asymptotic solution, which is found by the same method, also approximate the exact solution.

It is clear that the structure of the asymptotics must essentially depend on the ratio of the parameters ε and ρ. Here we assume the following conditions:

$$\mu = \frac{\rho}{\sqrt{\varepsilon}} \to 0.$$

A similar problem for compression waves was studied in the work [33]. In the present paper, we have to use another definition of the asymptotic solution (different from [33]) because of the specificity of its behavior.

2. Renormalization

It is known that in some cases the behavior of the solutions of singularly perturbed differential equations with a small parameter at the highest derivative becomes in some sense self-similar. Then the analysis of solutions using the method of renorm-group [13] becomes effective. This approach has the advantage that we immediately get the uniform approximation of the problem, which eliminates the need to build asymptotic ansatzes in different areas. For example, the composite asymptotic solution of the Cauchy problem with the condition (1.2) was obtained by matching [14] for a quasilinear parabolic equation [15], and in [16] it is shown that the renormalization approximation of the solution is asymptotically close to the composite asymptotic solution.

The relationship between self-similarity of the solutions and the parameter $\mu = \dfrac{\rho}{\sqrt{\varepsilon}}$ was shown already in the work of Yu. A. Berezin and V. Karpman [28] in connection with the study of the evolution of perturbations in plasma.

We will construct asymptotic solution of the problem (1.1)–(1.2), using a technique similar to the method of renormalization group, in the most simple version. Let us introduce the inner variables

$$x = \sqrt{\varepsilon}\,\eta, \quad t = \sqrt{\varepsilon}\,\theta, \tag{2.1}$$

since this takes into account all terms of equation (1.1). We take the solution of the equation

$$\frac{\partial Z}{\partial \theta} + Z\frac{\partial Z}{\partial \eta} + \frac{\partial^3 Z}{\partial \eta^3} = 0, \tag{2.2}$$

with the initial condition

$$Z(\eta, 0) = \begin{cases} \Lambda_0^-, & \eta < 0, \\ \Lambda_0^+, & \eta > 0 \end{cases} \tag{2.3}$$

as a "starting" function. We seek the expansion of the solution in the following form

$$u(x, t, \varepsilon, \rho) = Z(\eta, \theta) + \mu W(\eta, \theta, \mu) + \dots, \tag{2.4}$$

where the additive $\mu W(\eta, \theta, \mu)$ is supposed to eliminate the "starting" function singularity at the initial time. From equations (1.1) and (2.2) it follows that the function W satisfies the linear equation

$$\frac{\partial W}{\partial \theta} + \frac{\partial (ZW)}{\partial \eta} + \frac{\partial^3 u}{\partial \eta^3} = 0. \tag{2.5}$$

Differentiating the equation (2.2) with respect to the variable η, we see that the expression

$$G(\eta, \theta) = \frac{1}{\Lambda_0^+ - \Lambda_0^-} \frac{\partial Z(\eta, \theta)}{\partial \eta}$$

satisfies the equation (2.5). Moreover, G is the Green function, since

$$\lim_{\theta \to +0} \int_{-\infty}^{\infty} G(\eta, \theta) f(\eta)\, d\eta = -\frac{1}{\Lambda_0^+ - \Lambda_0^-} \int_{-\infty}^{\infty} Z(\eta, 0) f'(\eta)\, d\eta = f(0)$$

for any finite functions f.

We will find a solution W in the form of a convolution with the Green function G such that the asymptotic approximation will satisfy the initial condition (1.2). Then

$$W = \frac{1}{\Lambda_0^+ - \Lambda_0^-} \int_{-\infty}^{\infty} \frac{\partial Z(\eta - \mu s, \theta)}{\partial \eta} \left[\Lambda(s) - Z(s, 0)\right] ds.$$

After integration by parts and substitution of the expansion (2.4) we get the desired expression

$$u \approx \frac{1}{\Lambda_0^+ - \Lambda_0^-} \int_{-\infty}^{\infty} Z\left(\frac{x - \rho s}{\sqrt{\varepsilon}}, \frac{t}{\sqrt{\varepsilon}}\right) \Lambda'(s)\, ds, \tag{2.6}$$

which clarifies the structure of the asymptotic solution in the parameters ε and ρ in the leading approximation. Despite the fact that the functions Z and W do not depend explicitly on $\sqrt{\varepsilon}$, the result depends on μ and $\sqrt{\varepsilon}$ because of the change (2.1) since the asymptotic solution (2.6) is also considered at finite values of time t.

These calculations are given here for the convenience of the reader and a logical passage to the next section, where we use the result of [12] based on formula (2.6) to obtain a more suitable expression for numerical computations.

3. Formal asymptotic solution

We seek the asymptotic solution $\tilde{u}$ in the form

$$\tilde{u}(x,t) = R\left(\sigma, \mu^2\omega\right) + S(\sigma, \omega), \tag{3.1}$$

$$\sigma = \frac{x}{\rho}, \quad \omega = \frac{\varepsilon t}{\rho^3}, \quad \mu = \frac{\rho}{\sqrt{\varepsilon}}, \tag{3.2}$$

where the function

$$R\left(\sigma, \frac{t}{\rho}\right) = \int\limits_0^1 \Lambda\left(\sigma + \frac{cty}{\rho}\right) dy, \tag{3.3}$$

according to [12], is obtained by substitution of the asymptotics of the Gurevich-Pitaevskii solution [1] for the rarefaction wave for $\Lambda_0^+ = 0$ and $\Lambda_0^- = -\dfrac{c}{6}$ $(c > 0)$ in the formula (2.6).

The function S satisfies the equation

$$\frac{\partial S}{\partial \omega} + \frac{\partial^3 S}{\partial \sigma^3} = -\frac{\partial^3 R}{\partial \sigma^3}, \quad S(\sigma, 0) = 0 \tag{3.4}$$

to compensate the leading term with $\dfrac{\partial^3 R}{\partial x^3}$ in the KdV equation (1.1). The solution S can be written in the following form:

$$S = -\int\limits_0^\omega \frac{1}{\sqrt[3]{3(\omega-\omega')}} \int\limits_{-\infty}^\infty \mathrm{Ai}\left(\frac{\sigma-\sigma'}{\sqrt[3]{3(\omega-\omega')}}\right) \frac{\partial^3 R(\sigma', \mu^2\omega')}{\partial \sigma'^3} d\sigma' d\omega'. \tag{3.5}$$

Proposition 1. *Let $\mu < 1$, $\Lambda \in C^6(\mathbb{R})$,*

$$\lim_{s\to-\infty} \Lambda(s) = -\frac{c}{6}, \quad \lim_{s\to\infty} \Lambda(s) = 0,$$

and

$$\lim_{x\to\pm\infty} \left|\Lambda^{(k)}(x)\cdot x\right| < \infty, \quad k = 1\ldots 5. \tag{3.6}$$

Then there exists a constant M such that S is approximated by the formula

$$\begin{aligned}
S = \int\limits_0^\infty \Big[A_1\Big(\frac{-s}{\sqrt[3]{3\omega}}\Big)\Big(\Lambda'(s+\sigma) + \frac{c\mu^2\omega}{2}\Lambda''(s+\sigma) + \frac{c\mu^2 s^3}{12}\Lambda''(s+\sigma)\Big) - \\
-\mathrm{Ai}\Big(\frac{-s}{\sqrt[3]{3\omega}}\Big)\frac{(3\omega)^{2/3}}{2s^2}\cdot\frac{c\mu^2 s^3}{6}\Lambda''(s+\sigma) + \mathrm{Ai}'\Big(\frac{-s}{\sqrt[3]{3\omega}}\Big)\frac{(3\omega)^{1/3}}{2s}\cdot\frac{c\mu^2 s^3}{6}\Lambda''(s+\sigma)\Big] ds - \\
-\int\limits_{-\infty}^0 \Big[A_2\Big(\frac{-s}{\sqrt[3]{3\omega}}\Big)\Big(\Lambda'(s+\sigma) + \frac{c\mu^2\omega}{2}\Lambda''(s+\sigma) + \frac{c\mu^2 s^3}{12}\Lambda''(s+\sigma)\Big) + \\
+\mathrm{Ai}\Big(\frac{-s}{\sqrt[3]{3\omega}}\Big)\frac{(3\omega)^{2/3}}{2s^2}\cdot\frac{c\mu^2 s^3}{6}\Lambda''(s+\sigma) - \mathrm{Ai}'\Big(\frac{-s}{\sqrt[3]{3\omega}}\Big)\frac{(3\omega)^{1/3}}{2s}\cdot\frac{c\mu^2 s^3}{6}\Lambda''(s+\sigma)\Big] ds + \varphi(\sigma,\omega), \\
|\varphi(\sigma,\omega)| \le M\mu^4\omega^2 = M(\frac{t}{\rho})^2,
\end{aligned}$$

where

$$A_1(x) = \int\limits_{-\infty}^x \mathrm{Ai}(z)dz, \quad A_2(x) = \int\limits_x^\infty \mathrm{Ai}(z)dz.$$

P r o o f. Conditions (3.6) provide the same decaying of R and its derivatives as $\sigma \to \pm\infty$. After integration of (3.5) by parts twice, using the properties of Airy functions, we have

$$S = -\int_0^{\omega}\int_{-\infty}^{\infty} \frac{\sigma-\sigma'}{(3(\omega-\omega'))^{4/3}} \mathrm{Ai}\left(\frac{\sigma-\sigma'}{\sqrt[3]{3(\omega-\omega')}}\right) \frac{\partial R(\sigma',\mu^2\omega')}{\partial\sigma'} d\sigma' d\omega'.$$

We split the integral with respect to σ' from $-\infty$ to ∞ into two integrals (from $-\infty$ to σ and from σ to ∞), let us also make change of variables

$$s = \sigma' - \sigma, \quad z = \frac{-s}{\sqrt[3]{3(\omega-\omega')}},$$

and change the order of integration. All integrals above exist due to (3.6), (3.3). We have:

$$S = -\int_{-\infty}^{0}\int_{\frac{-s}{\sqrt[3]{3\omega}}}^{\infty} \mathrm{Ai}\,(z)\, \frac{\partial R(s+\sigma,\mu^2\omega')}{\partial s} dz ds + \int_0^{\infty}\int_{-\infty}^{\frac{-s}{\sqrt[3]{3\omega}}} \mathrm{Ai}\,(z)\, \frac{\partial R(s+\sigma,\mu^2\omega')}{\partial s} dz ds,$$
$$\omega' = \omega + \frac{1}{3}\left(\frac{s}{z}\right)^3. \tag{3.7}$$

For $\dfrac{\partial R}{\partial \sigma}$, using (3.3), we have:

$$\frac{\partial R}{\partial\sigma} = \frac{1}{c\mu^2\omega'}\int_0^1 \frac{d\Lambda\left(\sigma + c\mu^2\omega' y\right)}{dy} dy = \frac{\Lambda\left(\sigma + c\mu^2\omega'\right) - \Lambda(\sigma)}{c\mu^2\omega'}. \tag{3.8}$$

For small μ we expand this expression into the Taylor series:

$$\frac{\partial R}{\partial\sigma} = \Lambda'(\sigma) + \frac{c\mu^2\omega'}{2}\Lambda''(\sigma) + O(\mu^4\omega'^2) \tag{3.9}$$

and place it in S. We have:

$$S = \int_0^{\infty}\int_{-\infty}^{\frac{-s}{\sqrt[3]{3\omega}}} \mathrm{Ai}\,(z)\left(\Lambda'(s+\sigma) + \frac{c\mu^2\omega'}{2}\Lambda''(s+\sigma) + O(\mu^4\omega'^2)\right) dz ds -$$
$$-\int_{-\infty}^{0}\int_{\frac{-s}{\sqrt[3]{3\omega}}}^{\infty} \mathrm{Ai}\,(z)\left(\Lambda'(s+\sigma) + \frac{c\mu^2\omega'}{2}\Lambda''(s+\sigma) + O(\mu^4\omega'^2)\right) dz ds.$$

For A_1, A_2 it holds

$$\int_{-\infty}^{x} \frac{\mathrm{Ai}(z)}{z^3} dz = -\frac{x^{-2}}{2}\mathrm{Ai}(x) - \frac{x^{-1}}{2}\mathrm{Ai}'(x) + \frac{1}{2}A_1(x),$$

$$\int_{x}^{\infty} \frac{\mathrm{Ai}(z)}{z^3} dz = \frac{x^{-2}}{2}\mathrm{Ai}(x) + \frac{x^{-1}}{2}\mathrm{Ai}'(x) + \frac{1}{2}A_2(x).$$

So

$$
\begin{aligned}
S = \int\limits_0^{\infty} \Big[A_1\Big(\frac{-s}{\sqrt[3]{3\omega}}\Big)\Big(\Lambda'(s+\sigma) + \frac{c\mu^2\omega}{2}\Lambda''(s+\sigma) + \frac{c\mu^2 s^3}{12}\Lambda''(s+\sigma)\Big) - \\
-\mathrm{Ai}\Big(\frac{-s}{\sqrt[3]{3\omega}}\Big)\frac{(3\omega)^{2/3}}{2s^2}\cdot\frac{c\mu^2 s^3}{6}\Lambda''(s+\sigma) + \mathrm{Ai}'\Big(\frac{-s}{\sqrt[3]{3\omega}}\Big)\frac{(3\omega)^{1/3}}{2s}\cdot\frac{c\mu^2 s^3}{6}\Lambda''(s+\sigma)\Big]ds + O(\mu^4\omega^2) - \\
-\int\limits_{-\infty}^{0} \Big[A_2\Big(\frac{-s}{\sqrt[3]{3\omega}}\Big)\Big(\Lambda'(s+\sigma) + \frac{c\mu^2\omega}{2}\Lambda''(s+\sigma) + \frac{c\mu^2 s^3}{12}\Lambda''(s+\sigma)\Big) + \\
+\mathrm{Ai}\Big(\frac{-s}{\sqrt[3]{3\omega}}\Big)\frac{(3\omega)^{2/3}}{2s^2}\cdot\frac{c\mu^2 s^3}{6}\Lambda''(s+\sigma) - \mathrm{Ai}'\Big(\frac{-s}{\sqrt[3]{3\omega}}\Big)\frac{(3\omega)^{1/3}}{2s}\cdot\frac{c\mu^2 s^3}{6}\Lambda''(s+\sigma)\Big]ds + O(\mu^4\omega^2).
\end{aligned}
\tag{3.10}
$$

This formula admits formal differentiation with respect to σ since (3.8), (3.9) give the same formulas

$$
\frac{\partial^2 R}{\partial\sigma^2} = \frac{\Lambda'\left(\sigma + c\mu^2\omega'\right) - \Lambda'(\sigma)}{c\mu^2\omega'} = \Lambda''(\sigma) + \frac{c\mu^2\omega'}{2}\Lambda'''(\sigma) + O(\mu^4\omega'^2).
$$

Also

$$
\frac{\partial^3 R}{\partial\sigma^3} = \Lambda'''(\sigma) + \frac{c\mu^2\omega'}{2}\Lambda^{(4)}(\sigma) + O(\mu^4\omega'^2), \quad \frac{\partial^4 R}{\partial\sigma^4} = \Lambda^{(4)}(\sigma) + \frac{c\mu^2\omega'}{2}\Lambda^{(5)}(\sigma) + O(\mu^4\omega'^2).
$$

$$
\frac{d}{d\omega'}\frac{\partial R(\sigma,\mu^2\omega')}{\partial\sigma} = \frac{c\mu^2}{2}\Lambda''(\sigma) + O(\mu^4\omega').
$$

Therefore

$$
\frac{\partial S}{\partial\omega} = \int\limits_{-\infty}^{\infty} \mathrm{Ai}\left(\frac{-s}{\sqrt[3]{3\omega}}\right)\Lambda'(s+\sigma)ds - \int\limits_{-\infty}^{0}\int\limits_{\frac{-s}{\sqrt[3]{3\omega}}}^{\infty} \mathrm{Ai}\,(z)\,H(s,z)dzds + \int\limits_{0}^{\infty}\int\limits_{-\infty}^{\frac{-s}{\sqrt[3]{3\omega}}} \mathrm{Ai}\,(z)\,H(s,z)dzds.
$$

$$
H(s,z) = \frac{d}{d\omega'}\frac{\partial R(s+\sigma,\mu^2\omega')}{\partial s}\cdot\frac{d\omega'}{d\omega} = \frac{d}{d\omega'}\frac{\Lambda\left(s+\sigma+c\mu^2\omega'\right) - \Lambda(s+\sigma)}{c\mu^2\omega'}
$$

that proves Proposition 1.

For the approximate solution $\tilde{u} = R + S$ we denote

$$
\begin{aligned}
\mathcal{K}\tilde{u} = \frac{\partial\tilde{u}}{\partial\omega} + \mu^2\tilde{u}\frac{\partial\tilde{u}}{\partial\sigma} + \frac{\partial^3\tilde{u}}{\partial\sigma^3}, \quad m_1(\sigma,\omega) = \left|\mathcal{K}\tilde{u}\right|, \\
m_2(\sigma,\omega) = \left|\frac{\partial\tilde{u}}{\partial\omega}\right| + \left|\mu^2\tilde{u}\frac{\partial\tilde{u}}{\partial\sigma}\right| + \left|\frac{\partial^3\tilde{u}}{\partial\sigma^3}\right|, \quad A_\mu(\omega) = \frac{\|m_1(\cdot,\omega)\|_{L_1}}{\|m_2(\cdot,\omega)\|_{L_1}}.
\end{aligned}
\tag{3.11}
$$

Due to the equation (3.4) we have $\mathcal{K}\tilde{u} = \dfrac{\partial R}{\partial\omega} + \mu^2\tilde{u}\dfrac{\partial\tilde{u}}{\partial\sigma}$. We say that $\tilde{u}$ is an L_1-asymptotic solution, if the function $A_\mu(\omega)$ is uniformly small for $\omega \in (0,\omega_1)$.

Exactly this ratio is important, but not the value of $\mathcal{K}\tilde{u}$ by itself, because generally speaking, it can be great. Note that the used notion of L_1-asymptotic solution differs from the standard definition of the formal asymptotic solution in form of infinite number of terms, giving arbitrarily small error by substituting it into the equation.

In the case where the asymptotic solution is constructed in the form of a function (not a series), even the smallness of the remainders (after the substitution of this function into the equation) by itself cannot serve as a criterion of its suitability, since derivatives can be of the same order of smallness; hence, the value of the remainders should be compared with something. It is naturally to compare with absolute values of individual terms, included in the equation.

4. Numerical computations

In this section there are graphs of the L_1-asymptotic solutions of the problem. The system Matlab R2012a on the "Uran" supercomputer of IMM UB RAS was used for computations.

The double integration of (3.7) and approximate single integration (3.10) were performed. We used the method of trapezes with the integration step 0.1.

1. Initial function was chosen in the form $\Lambda = \dfrac{c}{6\pi}\left(\operatorname{arctg} x - \dfrac{\pi}{2}\right)$. In this case

$$R = -\frac{c}{12} + \frac{\rho}{6\pi t}\left[\left(\sigma + \frac{ct}{\rho}\right)\operatorname{arctg}\left(\sigma + \frac{ct}{\rho}\right) - \sigma\operatorname{arctg}\sigma - \frac{1}{2}\ln\left(1 + \frac{2ct\rho\sigma + c^2t^2}{\rho^2(1+\sigma^2)}\right)\right]$$

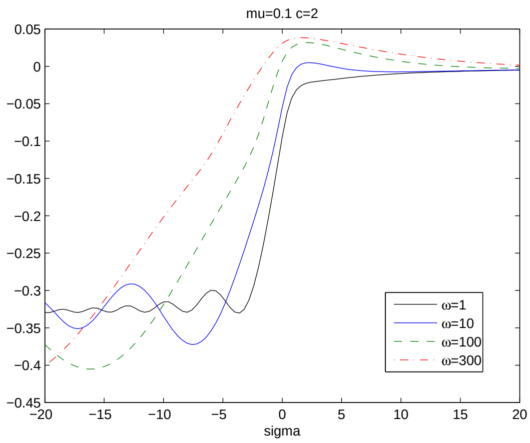

Figure 1. Function $\tilde{u}$ for different values of ω

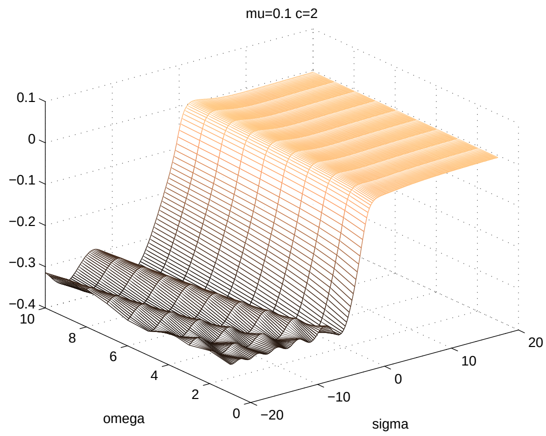

Figure 2. Function $\tilde{u}$ for $\omega \in (1, 10)$, $\sigma \in (-20, 20)$.

2. Numerical estimation of the KdV operator. The functions $m1, m2$ and $A_\mu(\omega)$ are introduced in (3.11).

As we can see in Fig. 3, 4, the value of $A_\mu(\omega)$ is of the order of 0.01 for $\mu = 0.1$, and of the order of 0.0001 in the case $\mu = 0.01$. Numerical computations for smaller μ show that A_μ has the order $\mu^2\omega$. We emphasize that it is important for the value of $A_\mu(\omega)$ to be small, and the value m_1 does not necessary have to be small. Thus, according to (3.2), the obtained asymptotic solution describes only the initial stage of the propagation of the rarefaction wave for $t \ll \rho$ and further investigation is needed.

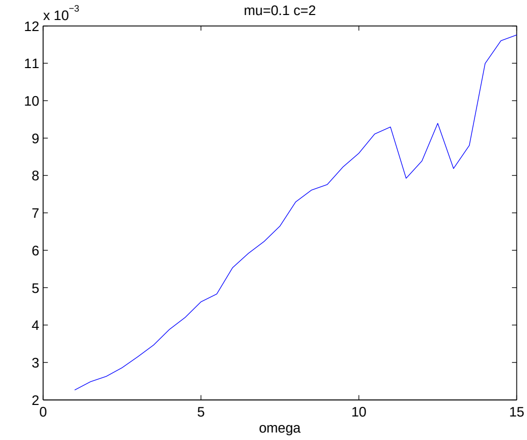

Figure 3. $A_\mu(\omega)$ for $\mu = 0.1$.

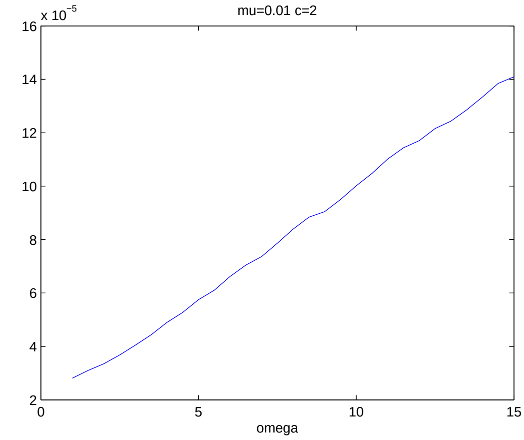

Figure 4. $A_\mu(\omega)$ for $\mu = 0.01$.

3. In this section we compare two ways of integration. While obtaining of S we may use one of two choices: the formulas (3.7), (3.8), and the formula (3.10) after approximation (3.9). This approximation helps us to simplify the computing due to removing the integration with respect to z, and the integral (3.10) with respect to s only remains. It reduces the duration of the computations up to 30 times.

We denote (3.7) as $f_{1-} + f_{1+}$ and (3.10) as $f_{2-} + f_{2+}$ and compare them. We have

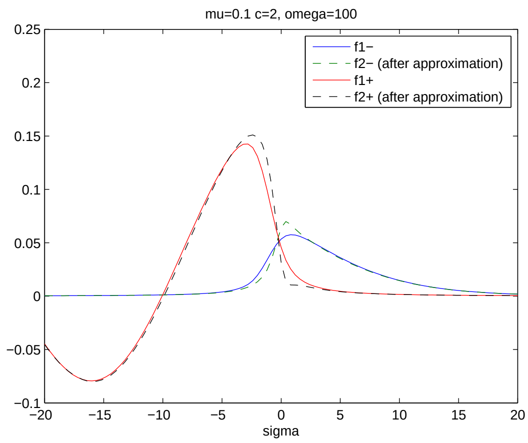

Figure 5. Functions $f_{1\pm}$, $f_{2\pm}$ for $\omega = 100$.

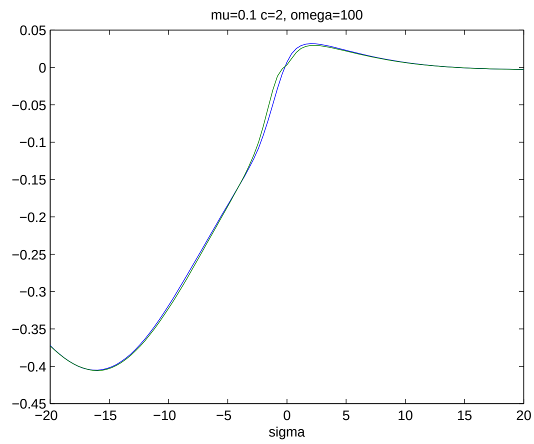

Figure 6. Functions $\tilde{u} = R + S$ for $\omega = 100$ before and after approximation (3.10).

We see that although $\mu^2\omega = 1$ for $\mu = 0.1$, $\omega = 100$, we get satisfactory result for applying the approximation (3.10).

The following graph shows that for $\omega = 10$ the functions $f_{1\pm}$ almost match $f_{2\pm}$.

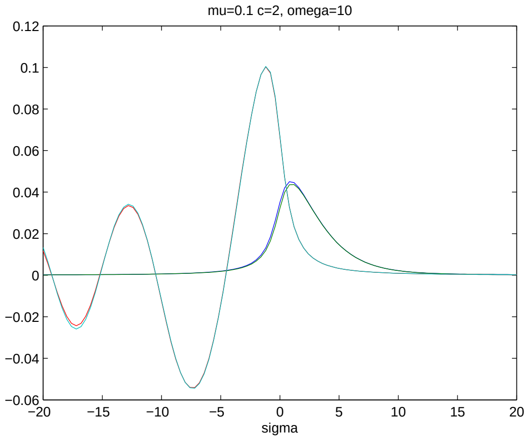

Figure 7. Functions $f_{1\pm}$, $f_{2\pm}$ for $\omega = 10$.

4. Consider non-monotonic case with the initial function in the form

$$\Lambda = \frac{c}{6\pi}\left(\operatorname{arctg} x - \frac{\pi}{2} + \frac{5}{1+x^2}\right).$$

We get

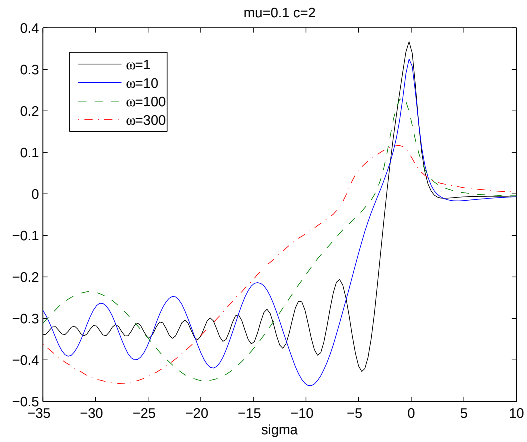

Figure 8. Function $\tilde{u}$ for different values of ω in non-monotonic case.

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AN ALGORITHM FOR COMPUTING BOUNDARY POINTS OF REACHABLE SETS OF CONTROL SYSTEMS UNDER INTEGRAL CONSTRAINTS[1]

Mikhail I. Gusev

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S.Kovalevskaya str., 620990, Ekaterinburg, Russia
gmi@imm.uran.ru

Abstract: In this paper we consider a reachability problem for a nonlinear affine-control system with integral constraints, which assumed to be quadratic in control variables. Under controllability assumptions it was proved in [8] that any admissible control that steers the control system to the boundary of its reachable set is a local solution to an optimal control problem with an integral cost functional and terminal constraints. This leads to the Pontriagyn maximum principle for boundary trajectories. We propose here a numerical algorithm for computing the reachable set boundary based on the maximum principle and provide some numerical examples.

Key words: Optimal control, Reachable set, Integral constraints, Boundary points, Pontriagyn maximum principle.

Introduction

We consider here the reachable sets of a nonlinear affine-control system with joint integral constraints on the state and the control. The numerical algorithms for constructing approximations of reachable sets of control systems were investigated in many works (see, for example [2, 4, 7, 9–12, 14, 15, 17]). The properties of reachable sets under integral constraints and algorithms for their construction were studied in [1, 5, 6, 16]. For systems with pointwise constraints on the control it is known (see, for example, [13]) that the control, which steers the trajectory to the boundary of the reachable set, satisfies the Pontryagin maximum principle. In the paper [8] we have considered the reachability problem for a nonlinear affine-control system with constraints on the control variables given by the quadratic integral inequality. Assuming the controllability property of the linearized system, we proved that any admissible control that steers the control system to the boundary of its reachable set is a local solution to an optimal control problem with an integral cost functional and a terminal constraint. This leads to the maximum principle for boundary trajectories. The last result admits a generalization to the case of joint integral constraints on the state and the control given by the inequality

$$J(u(\cdot)) = \int_{t_0}^{t_1} f_0(t, x(t), u(t))dt \leq \mu^2.$$

The reachable set in this case may be considered as the solution to the inverse optimal control problem: to find the terminal states reachable from the given initial state by the trajectories satisfying the constraints on the value of the cost functional. The aim of the present paper is to propose a numerical algorithm for computing boundary points of the reachable set. This algorithm is based on the solution of equations following from the maximum principle for boundary trajectories.

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1. Notation and definitions

Further by $A^\top$ we denote the transpose of a real matrix A, I_n is an identity $n \times n$-matrix, 0_n is a zero $n \times n$-matrix, 0 stands for a zero vector of appropriate dimension. For $x, y \in \mathbb{R}^n$ let $(x, y) = x^\top y$ denotes the inner product, $x^\top = (x_1, \ldots, x_n)$, $\|x\| = (x, x)^{\frac{1}{2}}$ be the Euclidean norm, and $B_r(\bar{x})$: $B_r(\bar{x}) = \{x \in \mathbb{R}^n : \|x - \bar{x}\| \leq r\}$ be a ball of radius $r > 0$ centered at $\bar{x}$. For a set $S \subset \mathbb{R}^n$ let ∂S be the boundary of S; $\dfrac{\partial f}{\partial x}(x)$ is the Jacobi matrix of a vector-valued function $f(x)$. For a real $k \times m$ matrix A a matrix norm is denoted as $\| A \|$. The symbol $\mathbb{R}^{n \times r}$ denotes a space of $n \times r$ real matrices, the symbols $\mathbb{L}_1$, $\mathbb{L}_2$ and $\mathbb{C}$ stand for the spaces of summable, square summable and continuous vector-functions respectively. The norms in these spaces are denoted as $\| \cdot \|_{\mathbb{L}_1}$, $\| \cdot \|_{\mathbb{L}_2}$, $\| \cdot \|_{\mathbb{C}}$.

We consider the control system

$$\dot{x}(t) = f_1(t, x(t)) + f_2(t, x(t))u(t), \quad x(t_0) = x^0, \tag{1.1}$$

on the fixed interval $[t_0, t_1]$, where $t_0 \leq t \leq t_1$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $f_1 : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $f_2 : \mathbb{R}^{n+1} \to \mathbb{R}^{n \times r}$ are continuous mappings.

The functions f_1 and f_2 are assumed to be continuously differentiable in x and satisfying the following conditions:

$$\| f_1(t, x) \| \leq l_1(t)(1 + \| x \|), \quad \| f_2(t, x) \| \leq l_2(t), \tag{1.2}$$

where $l_1(\cdot) \in \mathbb{L}_1$, $l_2(\cdot) \in \mathbb{L}_2$. Under these assumptions for any $u(\cdot) \in \mathbb{L}_2$ there exists a unique absolutely continuous solution $x(t)$ of system (1.1) which satisfies the initial condition $x(t_0) = x_0$ and is defined on the interval $[t_0, t_1]$.[2]

Denote as $J(u(\cdot))$ the following integral functional

$$J(u(\cdot)) = \int_{t_0}^{t_1} (Q(t, x(t)) + u^\top(t) R(t, x(t)) u(t)) dt.$$

Here $x(t)$ is a solution of system (1.1) corresponding to the control $u(t)$ and the initial vector x^0. The function $Q(t, x)$ and the positive definite symmetric matrix $R(t, x)$ are assumed to be continuous on $[t_0, t_1] \times \mathbb{R}^n$ and satisfying the inequalities $Q(t, x) \geq 0$, $u^\top R(t, x) u \geq \alpha \|u\|^2$ for some $\alpha > 0$ and any $(t, x, u) \in [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^r$.

Define the set

$$U = \{u(\cdot) \in \mathbb{L}_2 : J(u(\cdot)) \leq \mu^2\},$$

where $\mu > 0$ is a given number, and let P be a $m \times n$ full rank real matrix, $m \leq n$. Denote by $G(t_1)$ the (output) reachable set of the system (1.1) at the time t_1 for the fixed x^0 and the integral constraints:

$$G(t_1) = \big\{ y \in \mathbb{R}^m : \exists u(\cdot) \in U, \ y = Px(t_1, u(\cdot)) \big\},$$

where $x(t, u(\cdot))$ is a trajectory of system (1.1), corresponding to $u(\cdot)$.

The reachable set is a compact set in $\mathbb{R}^m$, but it may be empty.

Recall the following definitions: the linear control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in [t_0, t_1], \quad x(t_0) = x^0,$$

a) is said to be *controllable* on $[t_0, t_1]$ with respect to the output $y = Px$ if for any $y^1 \in \mathbb{R}^m$ there exists a control $u(\cdot) \in \mathbb{L}_2$ that transfers the system from the zero initial state $x(t_0) = 0$ to the final

[2] We use the same notation for the space $\mathbb{L}_2$ in the case of a scalar function $l_2(\cdot)$ and a vector-function $u(\cdot)$.

state $x(t_1)$ such that $Px(t_1) = y^1$;
b) is said to be the *linearization* of the system $\dot{x} = F(t,x,u)$ *along the trajectory* $x(t), u(t)$ if

$$A(t) = \frac{\partial F}{\partial x}(t, x(t), u(t)), \quad B(t) = \frac{\partial F}{\partial u}(t, x(t), u(t)).$$

2. The Maximum Principle for Boundary Trajectories

2.1. Extremal Properties of Boundary Points

Let us show that any admissible control that steers the control system to the boundary of its reachable set is a local solution to an optimal control problem with an integral cost functional and terminal constraints.

Theorem 1. *Assume that:*

1) $y^1 \in \partial G(t_1)$*;*

2) $u(\cdot) \in U$ *is a control that steers the system from the state* $x(t_0) = x^0$ *to the point* $x(t_1)$, $Px(t_1) = y^1$, $x(t)$ *is the corresponding trajectory;*

3) the linearization along $(x(t), u(t))$ *of system (1.1) is controllable on* $[t_0, t_1]$ *w.r.t. output* $y = Px$*;*

Then there exists $\sigma > 0$ *such that* $J(v(\cdot)) \geq \mu^2$ *for any* $v(\cdot) \in B(u(\cdot), \sigma) \subset \mathbb{L}_2$ *satisfying the condition* $Px(t_1) = y^1$*. Since* $J(u(\cdot)) \leq \mu^2$*, this implies that* $J(u(\cdot)) = \mu^2$ *and the control* $u(\cdot)$ *provides a local minimum in the optimal control problem*

$$J(u(\cdot)) \to \min, \quad u(\cdot) \in \mathbb{L}_2, \quad x(t_0) = x^0, \quad Px(t_1) = y^1 \tag{2.3}$$

with terminal constraint $Px(t_1) = y^1$*.*

P r o o f. The proof follows the scheme of the proof of the Theorem 1 [8] and uses the Graves theorem [3]. $\square$

Since the local minimum in $\mathbb{L}_2$ admits the needle variations of the control, the local $\mathbb{L}_2$-minimizer satisfies Pontryagin's maximum principle. Introduce the Pontryagin function (Hamiltonian) associated with (2.3)

$$H(p, t, x, u) = -p_0 f_0(t, x, u) + p^\top (f_1(t, x) + f_2(t, x)u),$$

$p_0 \geq 0$, $f_0(t, x, u) = Q(t, x) + u^\top R(t, x)u$. Assume additionally that $Q(t, x)$, $R(t, x)$ are continuously differentiable in x. A locally optimal control for (2.3) satisfies the maximum principle: there exist $p_0 \geq 0$, $l \in \mathbb{R}^m$, $(p_0, l) \neq 0$, and a function $p(t)$ such that

$$H(p(t), t, x(t), u(t)) = \max_{v \in \mathbb{R}^r} H(p(t), t, x(t), v),$$

$$\dot{p(t)} = -\frac{\partial H}{\partial x}(p(t), x(t), u(t)) = -A^\top(t)p(t) + p_0 \frac{\partial f_0}{\partial x}(t, x(t), u(t)), \quad p(t_1) = P^\top l.$$

Since the terminal constraints are regular ($rankP = m$), we have $p_0 + \|p(t)\| \neq 0$, $t \in [t_0, t_1]$. As previously, we denote here by $(A(t), B(t))$ the matrices of the linearization along $(x(t), u(t))$ of system (1.1). Applying the maximum principle to the solution of problem (2.3) we come the following

Corrolary 1. *Suppose that $u(t)$ satisfies the assumptions of Theorem 1. Then there exist $l \in \mathbb{R}^m$, $l \neq 0$ and a function $p(t)$ such that*

$$\dot{p(t)} = -\frac{\partial H}{\partial x}(p(t), x(t), u(t)) = -A^\top(t)p(t) + \frac{1}{2}\frac{\partial f_0}{\partial x}(t, x(t), u(t)), \quad p(t_1) = P^\top l,$$

$$u(t) = R^{-1}(t, x(t)) f_2^\top(t, x(t)) P(t), \quad t \in [t_0, t_1].$$

P r o o f. If a pair $(A(t), B(t))$ is controllable w.r.t. $y = Px$, then $p_0 > 0$. Indeed, if it turned out that $p_0 = 0$, then $p(\cdot)$ is a non zero solution of the equation

$$\dot{p}(t) = -A^\top(t)p(t), \quad p(t_1) = P^\top l,$$

and from the maximum principle we would obtain

$$p^\top(t)B(t)u(t) = \max_{v \in \mathbb{R}^r} p^\top(t)B(t)v$$

almost everywhere in t. The last is valid only if $p^\top(t)B(t) \equiv 0$. Represent $p(t)$ in the form $p(t) = X^\top(t_1, t)P^\top l$, then $\|l^\top PX(t_1, t)B(t)\|^2 = 0$, $t \in [t_0, t_1]$. Integrating both sides of the last equality over $[t_0, t_1]$, we get $l^\top Vl = 0$. This contradicts to the controllability of $(A(t), B(t))$ w.r.t. $y = Px$, since $l \neq 0$. Thus we can take $p_0 = \frac{1}{2}$, from the maximum principle it follows that $H_u(p(t), t, x(t), u(t)) = 0$, hence $u(t) = u(t, x(t), p(t))$, where $u(t, x, p) = R^{-1}(t, x) f_2^\top(t, x)p$. □

2.2. Algorithm

Let us describe the following algorithm for calculating boundary points of reachable sets based on the results of previous subsection. Further we assume that $P = [I_m, 0]$ if $m < n$ or $P = I_n$ if $m = n$. In this case the transversality conditions $p(t_1) = P^\top l$ take the form $p_i(t_1) = 0$, $i = m+1, .., n$. Letting

$$\dot{x}_0(t) = f_0(t, x(t), u(t)), \quad x_0(t_0) = 0,$$

we get $J(u(\cdot)) = x_0(t_1)$. Substituting $u(t, x, p)$ into differential equations, we obtain the following system

$$\begin{aligned} \dot{x}(t) &= f_1(t, x(t)) + f_2(t, x(t))u(t, x(t), p(t)), \quad x(t_0) = x^0, \\ \dot{p}(t) &= -\frac{\partial f}{\partial x} H(p(t), x(t), u(t, x(t), p(t))), \quad p(t_0) = q, \\ \dot{x}_0(t) &= f_0(t, x(t), u(t, x(t), p(t))), \quad x_0(t_0) = 0. \end{aligned} \tag{2.1}$$

Denote by X the following $(2n+1)$-column vector $X = [x; p; x_0]$ and write equations (2.1) as the system

$$\dot{X}(t) = F(t, X(t)), \quad X(t_0) = [x_0; q; 0], \tag{2.2}$$

By $F(t, X)$ we denote the right-hand side of (2.1). Since x^0 is fixed, the solution of (2.2) depends only on the vector $q \in \mathbb{R}^n$, denote this solution as $X(t, q) = [x(t, q); p(t, q); x_0(t, q)]$. These functions have continuous derivatives $X_q(t, q)$ with respect to q, which can be found by integrating the linearization of (2.2) along the trajectory $X(t, q)$

$$\dot{X}_q(t, q) = \frac{\partial F}{\partial X}(t, X(t, q))X_q(t, q), \quad X_q(t_0, q) = [0_n; I_n; 0]. \tag{2.3}$$

The integration of equations (2.1)and (2.2) over the interval $[t_0, t_1]$ may be performed simultaneously. To this end, we unite both systems into one system of dimension $(2n+1)(n+1)$

$$\begin{aligned} \dot{X}(t) &= F(t, X(t)), \quad X(t_0) = [x_0; q; 0], \\ \dot{X}_q(t, q) &= \frac{\partial F}{\partial X}(t, X(t, q))X_q(t, q), \quad X_q(t_0, q) = [0_n; I_n; 0]. \end{aligned} \tag{2.4}$$

Consider the following continuously differentiable functions

$$\psi_0(q) = x_0(t_1, q) - \mu^2, \quad \psi_i(q) = p_{m+i}(t_1, q), \quad i = 1, ..., n-m,$$

their derivatives in q may be found by numerical integration of differential equations (2.4). The calculations of boundary points require the solution of the system of equations

$$\psi_i(q) = 0, \quad i = 0, ..., n-m, \tag{2.5}$$

and also the integration of system (2.1) with zeros of system (2.5) as the initial points for (2.1). In case $m = n$ the system (2.5) consists of a single equation $\psi_0(q) = 0$.

Let us describe a simple version of the algorithm for calculating zeros of $\psi_i(q)$ in the case $m = n = 2$. Represent $q \in \mathbb{R}^2$ in polar coordinates: $q_1(\theta) = r(\theta)\cos(\theta + \theta_0) + q_1^0$, $q_2(\theta) = r(\theta)\sin(\theta + \theta_0) + q_2^0$. Here $r(\theta)$ is a distance from a reference point q^0 and θ is an angle between $q - q^0$ and the reference direction $\bar{q} = (\cos\theta_0, \sin\theta_0)$. Differentiating the identity $\psi_0(q(\theta)) = 0$, we get a differential equation for $r(\theta)$

$$\dot{r}(\theta) = r(\theta)\frac{\psi_{0q_1}(q(\theta))\sin(\theta + \theta_0) - \psi_{0q_2}(q(\theta))\cos(\theta + \theta_0)}{\psi_{0q_1}(q(\theta))\cos(\theta + \theta_0) + \psi_{0q_2}(q(\theta))\sin(\theta + \theta_0)}, \quad 0 \le \theta \le 2\pi. \tag{2.6}$$

To start the solution we use a one-dimensional search procedure for finding the root of equation $\psi(q^0 + r\bar{q}) = 0$ and after this take this root as the initial state for differential equation (2.6).

3. Examples

Here we illustrate the above procedure for two examples of 2-dimensional control systems.

E x a m p l e 1. Consider the Duffing equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \varphi(x_1) + u, \quad t \in [0, t_1], \quad x_1(0) = 0, \quad x_2(0) = 0, \tag{3.1}$$

$\varphi(x_1) = -\alpha x_1 - \beta x_1^3, \quad \alpha, \beta > 0$, which describes the motion of nonlinear stiff spring on impact of an external force u. Consider the integral constraint on the state and the control

$$\int_0^{t_1} (a x_1^2(t) + b x_2^2(t) + u^2(t))dt \le 2,$$

where a, b are nonegative parameters and take $P = I_2$.

It is easy to verify that the controllability assumptions of Theorem 1 are satisfied here. Really, consider any trajectory $(x(t), u(t))$ of (3.3). The linearization of (3.3) along $(x(t), u(t))$ has the matrices

$$A(t) = \begin{pmatrix} 0 & 1 \\ \varphi'(x_1(t)) & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

An adjoint system $\dot{s} = -A^\top(t)s$ is as follows

$$\begin{aligned} \dot{s}_1(t) &= -\varphi'(x_1(t))s_2(t), \\ \dot{s}_2(t) &= -s_1(t). \end{aligned}$$

Thus, the identity $s^\top(t)B(t) = s_2(t) \equiv 0$ for $t \in [t_0, t_1]$ implies $s_1(t) \equiv 0$. This means the controllability of the pair $(A(t), B(t))$.

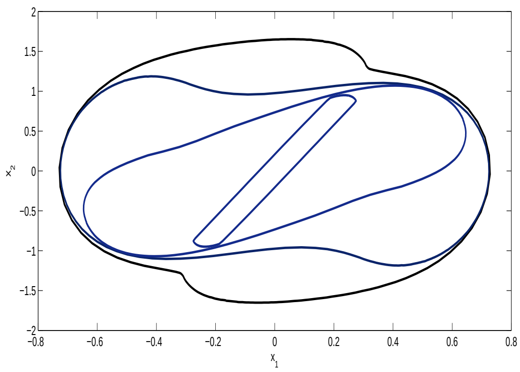

Figure 1. Reachable sets for different values of t_1.

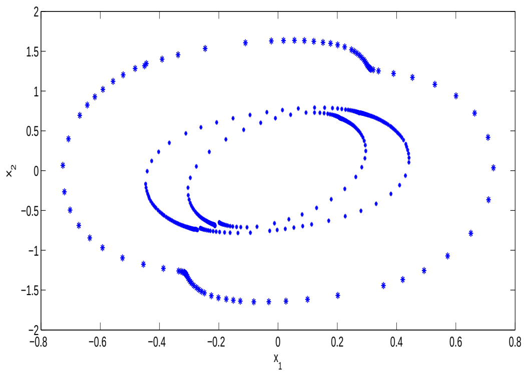

Figure 2. Reachable sets for different values of a, b.

The system (2.4) takes the following form

$$\begin{aligned}
\dot{X}_1 &= X_2,\\
\dot{X}_2 &= \varphi(X_1) + X_4,\\
\dot{X}_3 &= aX_1 - \varphi'(X_1)X_4,\\
\dot{X}_4 &= bX_2 - X_3,\\
\dot{X}_5 &= aX_1^2 + bX_2^2 + X_4^2,\\
\dot{X}_{5+i} &= X_{6+i},\\
\dot{X}_{6+i} &= \varphi'(X_1)X_{5+i} + X_{8+i},\\
\dot{X}_{7+i} &= aX_{5+i} - \varphi''(X_1)X_4X_{5+i} - \varphi'(X_1)X_{8+i},\\
\dot{X}_{8+i} &= bX_{6+i} - X_{7+i},\\
\dot{X}_{9+i} &= 2aX_1X_{5+i} + 2bX_2X_{6+i} + 2X_4X_{8+i}.
\end{aligned} \tag{3.2}$$

In equations (3.2) $i = 1, 6$, so (3.2) is a system of 15-th order. Integrating this system over $[0, t_1]$ for initial state $X^\top(0) = (0, 0, q_1, q_2, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0)$ we get

$$\psi_0(q) = X_5(t_1, q) - \mu^2, \quad \frac{\partial \psi_0}{\partial q_1}(q) = X_{10}(t_1, q), \quad \frac{\partial \psi_0}{\partial q_2}(q) = X_{15}(t_1, q), \quad q^\top = (q_1, q_2).$$

Since $x(0) = 0$ and $\varphi(x_1)$ is an odd function having even derivative it is not difficult to prove that the set $\{q : \psi_0(q) = 0\}$ is symmetric with respect to the origin. In this case it is natural to take the reference point $q^0 = 0$. As the reference direction we choose $\bar{q} = (1, 0)$. The results of numerical simulations for the case $\alpha = 1, \beta = 10$ are shown in Fig. 1–2.

The Fig. 1 shows the plot of the reachable sets boundaries for $t_1 = 0.5$, 1, 1.5, and 2 respectively, and for $a = 0$, $b = 0$. The reachable sets boundaries for the values of $a = 0$, $b = 0$; $a = 5$, $b = 10$; $a = 30$, $b = 15$ and $t_1 = 2$ are presented in Fig. 2.

E x a m p l e 2. Consider the following system [16]

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \varphi(x_1) + u, \quad t \in [0, 2\pi], \quad x_1(0) = 0, \quad x_2(0) = 0, \tag{3.3}$$

where $\varphi(x_1) = -\sin x_1$. The integral constraint on the state and the control are given by the inequality

$$\int_0^{2\pi} (ax_1^2(t) + bx_2^2(t) + u^2(t))dt \le 2$$

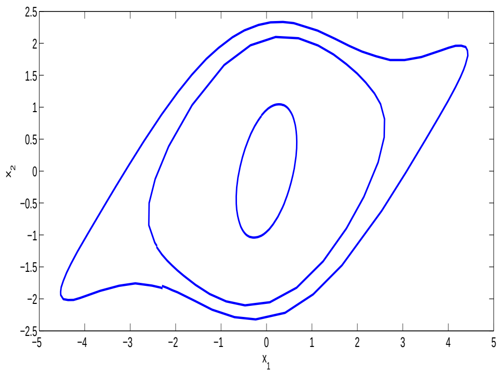

Figure 3. Reachable sets for different values of a, b

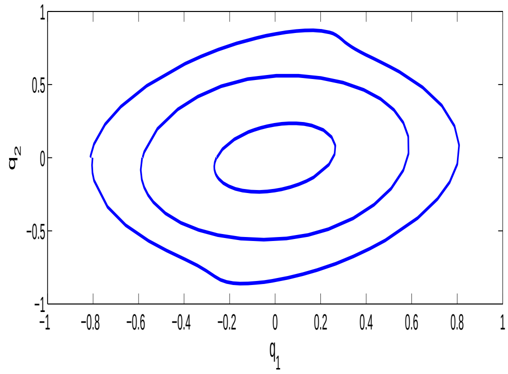

Figure 4. Zero-level lines of $\psi_0(q)$ for different values of a, b

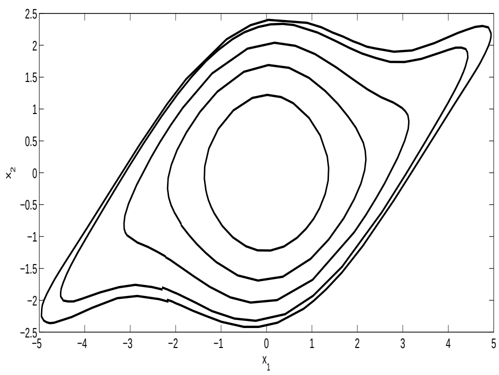

Figure 5. Reachable sets for different values of μ^2.

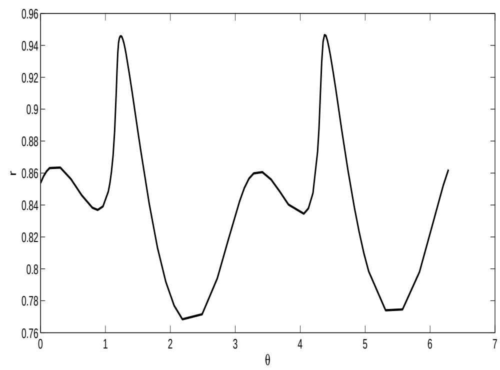

Figure 6. Graph of the function $r(\theta)$.

as in Example 1. As above the controllability assumptions of Theorem 1 are satisfied for the considered system.

The results of numerical simulation are shown in the Fig. 3–6. The Fig. 3 shows the plot of the reachable sets boundaries for $t_1 = 2$, and for $a = 0$, $b = 0$; $a = 0.1$, $b = 0$; $a = 0.5$, $b = 0.1$ respectively. This plot demonstrates that reachable sets are nonconvex for $a = 0$, $b = 0$ and became convex under increase of parameters a, b.

The next plot (Fig. 4) exhibits the zero-level lines of $\psi_0(q)$ corresponding to the curves of Fig. 3.

The Fig. 5 demonstrates the dependence of reachable sets on the value $\mu^2 = 0.5, 1, 1.5, 2, 2.2$. It shows that reachable sets that are convex for small μ^2 loose their convexity as μ^2 increases (see [16]). In this example the method fails for $\mu^2 > 2.2$ because a numerical integration of (2.6) unable to meet integration tolerances. Note that the considered procedure may by applied if the zero-level line $\psi_0(q) = 0$ is a differentiable curve. Differentiability can be violated in the points where $\psi_{0q_1}(q) = \psi_{0q_2}(q) = 0$ or the right-hand side of (2.6) is singular. The graph of the solution of (2.6) corresponding to the value $\mu^2 = 2.2$ is shown in Fig. 6.

4. Conclusion

This paper describes an algorithm for computing the boundaries of the reachable sets under joint integral constrains on state and control variables. The reachable set may be considered here as the solution to the inverse optimal control problem: to find the terminal states reached from the given initial state by the trajectories satisfying the constraints on the value of the cost functional. The Pontriagyn maximum principle for boundary trajectories is applied to construct a numerical algorithm for computing the boundary points. The results of numerical simulation for two examples of second order nonlinear control systems are presented.

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FINITE NILSEMIGROUPS WITH MODULAR CONGRUENCE LATTICES

Alexander L. Popovich

Ural Federal University, Ekaterinburg, Russia,
alexanderpopovich@urfu.ru

Abstract: This paper continues the joint work [2] of the author with P. Jones. We describe all finitely generated nilsemigroups with modular congruence lattices: there are 91 countable series of such semigroups. For finitely generated nilsemigroups a simple algorithmic test to the congruence modularity is obtained.

Key words: Semigroup, Nilsemigroup, Congruence lattice.

Introduction

In [2] the characterization of nilsemigroups with distributive and modular congruence lattices had been obtained. The basic notion in that result was *the width of a semigroup*, considered as a poset under division. Recall that *the width* of a poset is the maximal integer n such that the poset contains an antichain of n elements. It was proved in [2] that the congruence lattice of a nilsemigroup is distributive [modular and not distributive] if and only if it has the width 1 [the width 2].

A poset of the width 1 is a chain. Semigroups, whose congruence lattice form a chain, were investigated in the works [1, 3, 4]. There is no complete classification for such semigroups, in the same time some important cases (finite semigroups, commutative semigroups, permutative semigroups) were considered. It is known that finitely generated nilsemigroups whose congruence lattices form a chain are cyclic nilsemigroups. Thus we have a description of finitely generated nilsemigroups with distributive congruence lattices.

In this paper we describe all finitely generated nilsemigroups with the modular congruence lattice up to isomorphism or dual isomorphism. The set of all such semigroups has been splited into series (almost all of them are infinite), each of them has 4 or less natural parameters. The list of all series is given in the table below.

We prove the following theorem:

Theorem 1. *Let S be a finitely generated nilsemigroup. Then the following are equivalent:*

a) $\operatorname{Con} S$ *is modular and not distributive;*

b) S is generated by two elements a and b and the poset $\{a^2, ab, ba, b^2\}$ under division has the width 2;

c) S is isomorphic or dually isomorphic to a suitable semigroup in the following table:

N	Name	Presentation	Restrictions
1	$A(n)$	$a^2 = ab = ba = b^2$, $a^n = 0$	$n \geqslant 2$
2	$B_1(n)$	$a^2 = ab = b^2$, $a^n = 0$	$n \geqslant 3$
3	$B_{2.1}(m,n)$	$a^2 = b^2$, $ab = ba$, $a^m = ba^n = 0$	$m \geqslant 3$, $n \geqslant 2$, $\|m-n\| = 1$
4	$B_{2.2}(m,n)$	$a^2 = b^2$, $ab = ba$, $a^m = ba^{m-1}$, $a^n = 0$	$n \geqslant m \geqslant 3$
5	$B_{3.1}(m,n)$	$a^2 = ab = ba$, $a^m = b^n$	$n > m \geqslant 3$
6	$B_{3.2}(m,n)$	$a^2 = ab = ba$, $a^m = b^n = 0$	$n, m \geqslant 3$, $n \geqslant m-1$, $n \neq m$
7	$B_{3.3}(m,k)$	$a^2 = ab = ba$, $a^m = b^m$, $a^k = 0$	$k \geqslant m \geqslant 3$
8	$B_{4.1}(m,n)$	$a^2 = ab$, $b^2 = ba$, $a^m = b^n = 0$	$\|m-n\| = 1$; $m, n \geqslant 3$
9	$B_{4.2}(m,n)$	$a^2 = ab$, $b^2 = ba$, $a^m = b^m$, $a^k = 0$	$k \geqslant m \geqslant 3$
10	C_1	$a^2 = ab$, $b^2 = ba = 0$	
11	C_2	$a^2 = b^2 = ab$, $ba = 0$	
12	C_3	$a^2 = b^2 = ab = 0$	
13	C_4	$a^2 = b^2$, $ab = ba = 0$	
14	C_5	$a^2 = ab = ba$, $b^2 = 0$	
15	C_6	$ab = ba$, $a^2 = b^2 = 0$	
16	$C_{7.1}(n)$	$a^2 = ab = ba = b^n$	$n \geqslant 3$
17	$C_{7.2}(n)$	$a^2 = ab = ba = b^n = 0$	$n \geqslant 3$
18	$D_{1.1}(m,n)$	$b^2 = ba = a^m = a^n b$	$m \geqslant 3$, $m-1 \geqslant n \geqslant 2$
19	$D_{1.2}(m,n)$	$b^2 = ba = a^m$, $a^n b = 0$	$m \geqslant n \geqslant 2$, $m \geqslant 3$
20	$D_{2.1}(m,n,k)$	$ab = ba = a^m$, $a^n = b^k = 0$	$n > m \geqslant 3$, $k \geqslant 3$, $n \leqslant k(m-1)+1$
21	$D_{2.2}(m,n,k)$	$ab = ba = a^m$, $a^n = b^k = 0$, $a^{n-1} = b^{k-1}$	$m, k \geqslant 3$, $m \leqslant n-2 \leqslant (k-1)(m-1)$, $n \neq (m-1)(k-1)+1$
22	$D_{2.3}(m,n,q)$	$ab = ba = a^m$, $a^{(m-1)q} = b^q$, $a^n = 0$	$m \geqslant 3$,$q \geqslant 2$, $n \geqslant (m-1)q+1$
23	$D_{3.1}(m,n,k)$	$ab = ba$, $b^2 = a^m$, $a^n = a^k b = 0$	$m \geqslant 3$, $k \geqslant 2$, $k+m \geqslant n \geqslant k$, $n \geqslant m+1$
24	$D_{3.2}(n,k,q)$	$ab = ba$, $b^2 = a^{2(n-k)}$, $a^n = a^k b = 0$, $a^{q+n-k} = a^q b$	$n \geqslant 3$, $k \geqslant 2$, $n \geqslant k$, $n \geqslant 2n-2k+1$, $k > q \geqslant 2$
25	$D_{3.3}(m,n,k,q)$	$ab = ba$, $b^2 = a^m$, $a^{n-k+q} = a^q b$, $a^n = a^k b = 0$	$m \geqslant 3$, $k \geqslant 2$, $k+m \geqslant n \geqslant k$, $k \leqslant \min(n-k+q, q+m)$, $k > q \geqslant 2$
26	$D_4(n)$	$ba = b^n$, $a^2 = ab$	$n \geqslant 3$
27	$E_{1.1}(m,n,k)$	$b^2 = ba = a^m b$, $a^n = a^k b = 0$	$m \geqslant 2$, $2m \geqslant k \geqslant m$, $n \geqslant k$, $n \geqslant 3$
28	$E_{1.2}(m,n,k)$	$b^2 = ba = a^m b$, $a^n = a^k b$	$m \geqslant 2$, $2m \geqslant k \geqslant m$, $n > k$, $n \geqslant 3$
29	$E_{2.1}(m)$	$a^2 = b^2 = (ab)^{\frac{m}{2}} = (ba)^{\frac{m}{2}} = 0$	$m \geqslant 3$
30	$E_{2.2}(m)$	$a^2 = b^2 = (ab)^{\frac{m}{2}} = 0$	$m \geqslant 3$
31	$E_{2.3}(m)$	$a^2 = b^2 = (ab)^{\frac{m}{2}}$, $(ba)^{\frac{m}{2}} = 0$	$m \geqslant 3$
32	$E_{2.4}(m)$	$a^2 = b^2 = (ab)^{\frac{m}{2}} = (ba)^{\frac{m}{2}}$, $(ba)^{\frac{m+1}{2}} = 0$	$m \geqslant 3$
33	$E_{2.5}(m)$	$a^2 = b^2 = (ab)^{\frac{m}{2}}$, $(ba)^{\frac{m}{2}} = 0$	$m \geqslant 3$, m is odd
34	$E_{2.6}(m)$	$a^2 = b^2 = (ab)^{\frac{m}{2}}$, $(ab)^{\frac{m+1}{2}} = (ba)^{\frac{m+1}{2}} = 0$	$m \geqslant 3$

N	Name	Presentation	Restrictions
35	$E_{3.1}(n,m)$	$ab = ba = a^n = b^m$	$n, m \geqslant 3$
36	$E_{3.2}(n,m)$	$ab = ba = a^n = b^m = 0$	$n, m \geqslant 3$
37	E_4	$a^2 = b^2$, $ba = 0$	
38	$E_{5.1}(m,n,k)$	$ab = ba$, $b^2 = a^m b$, $a^n = a^k b = 0$	$n \geqslant k \geqslant m \geqslant 2$
39	$E_{5.2}(m,n,k,q)$	$ab = ba$, $b^2 = a^m b$, $a^{n-k+q} = a^q b$, $a^k = a^n = 0$	$n \geqslant k \geqslant m \geqslant 2$, $n \geqslant 3$, $n-k \neq m$, $k \leqslant \min(n-k+q, q+m)$, $q \geqslant 2$
40	$E_{5.3}(m,n,q)$	$ab = ba$, $b^2 = a^m b$, $a^n = 0$, $a^{m+q} = a^q b$	$n \geqslant m \geqslant 2$, $n \geqslant 3$, $q \geqslant 2$
41	$E_{6.1}$	$a^2 = ab$, $b^2 = ba^2$	
42	$E_{6.2}$	$a^2 = ab$, $b^2 = 0$	
43	$E_7(n)$	$a^2 = ab$, $ba = 0$, $b^n = 0$	$n \geqslant 3$
44	$G(m,n)$	$ab = a^m$, $ba = b^n$	$m, n \geqslant 3$
45	$H_{1.1}(m,n,k)$	$ba = a^m b$, $b^2 = a^n$, $a^{n+1} = a^k b = 0$	$n \geqslant k > m \geqslant 2$
46	$H_{1.2}(m,n,k)$	$ba = a^m b$, $b^2 = a^n = a^{k-1} b$, $a^{n+1} = a^k b = 0$	$n \geqslant k > m \geqslant 2$
47	$H_{2.1}(m,n,k,l)$	$b^2 = ba^m$, $ab = a^n$, $a^k = ba^l = 0$	$m \geqslant 2$, $k > n > m+1$, $m+n \geqslant k \geqslant l > m$
48	$H_{2.2}(m,n,k,l)$	$b^2 = ba^m$, $ab = a^n$, $a^{k-1} = ba^{l-1}$, $a^k = ba^l = 0$	$m \geqslant 2$, $k > n > m+1$, $m+n \geqslant k \geqslant l > m$, $k \neq l+n-1$
49	$H_{2.3}(m,n,k,q)$	$b^2 = ba^m$, $ab = a^n$, $a^{q+n-1} = ba^q$, $a^k = ba^l = 0$	$m \geqslant 2$, $n > m+1$, $m+n \geqslant k \geqslant q+n-1$
50	$H_{2.4}(m,k,l)$	$b^2 = ba^m$, $ab = a^{m+1}$, $a^k = ba^l = 0$	$m \geqslant 2$, $k \geqslant l \geqslant m+1$
51	$H_{2.5}(m,k,l)$	$b^2 = ba^m$, $ab = a^{m+1}$, $a^{k-1} = ba^{l-1}$, $a^k = ba^l = 0$	$m \geqslant 2$, $k > m+1$, $k \geqslant l > m$, $k \neq l+m$
52	$H_{2.6}(m,k,q)$	$b^2 = ba^m$, $ab = a^{m+1}$, $a^{q+m} = ba^q$, $a^k = 0$	$m \geqslant 2$, $n > m$, $k \geqslant q+n-1$, $q \geqslant 2$
53	$I_{1.1}(n)$	$a^2 = (ab)^{\frac{2n}{2}}$, $b^2 = (ba)^{\frac{2n}{2}}$	$n \geqslant 2$
54	$I_{1.2}(n)$	$a^2 = (ba)^{\frac{2n+1}{2}}$, $b^2 = (ab)^{\frac{2n+1}{2}}$	$n \geqslant 1$
55	$I_{1.3}(n)$	$a^2 = (ba)^{\frac{2n+1}{2}}$, $b^2 = (ab)^{\frac{2n+1}{2}}$, $(ab)^{\frac{2n+2}{2}} = (ba)^{\frac{2n+2}{2}} = 0$	$n \geqslant 1$
56	$I_{1.4}(n,m,k)$	$a^2 = (ab)^{\frac{2n+1}{2}}$, $b^2 = (ba)^{\frac{2n+1}{2}}$, $(ab)^{\frac{m}{2}} = (ba)^{\frac{k}{2}} = 0$	$n \geqslant 1$, $k, m > 2n+1$, $\lvert k-m \rvert \leqslant 1$
57	$J_{1.1}(n,m)$	$ab = ba = a^m$, $b^2 = a^n$	$m \geqslant 3$, $2m-2 > n > m$
58	$J_{1.2}(n,m)$	$ab = ba = a^m$, $b^2 = a^n = 0$	$m \geqslant 3$, $2m-2 > n > m$
59	$J_{1.3}(m,k)$	$ab = ba = a^m$, $b^2 = a^{2m-2}$, $a^k = 0$	$m \geqslant 3$, $k > 2m-2$
60	$J_2(n)$	$b^2 = a^n$, $ab = ba = a^{n+1} = 0$	$n \geqslant 3$
61	$J_3(n)$	$b^2 = ab = a^n$, $ba = a^{n+1} = 0$	$n \geqslant 3$
62	$J_4(n)$	$ab = a^n$, $b^2 = ba = a^{n+1} = 0$	$n \geqslant 3$
63	$L_1(n,m)$	$b^2 = a^m$, $ba^n = 0$, $ab = 0$	$m \geqslant 3$, $m+1 \geqslant n \geqslant 2$
64	$L_{2.1}(n,m)$	$ba = a^m$, $b^n = ab = 0$	$m \geqslant 3$, $n \geqslant 3$
65	$L_{2.2}(n,m)$	$ba = a^m = b^{n-1}$, $a^{m+1} = b^n = 0$	$m \geqslant 3$, $n \geqslant 4$
66	$L_{3.1}(m,k,n)$	$ab = a^m$, $b^2 = a^k$, $ba^{n+1} = 0$	$k > m \geqslant 3$, $k \neq 2m-2$, $k \geqslant n \geqslant k-m$
67	$L_{3.2}(m,k,q)$	$ab = a^m$, $b^2 = a^k = ba^q$	$k > m \geqslant 3$, $k \neq 2m-2$, $k > q \geqslant k-m+1$

N	Name	Presentation	Restrictions
68	$L_{3.3}(m,k,q)$	$ab=a^m$, $b^2=a^k$, $a^{q+m-1}=ba^q$	$k>m\geqslant 3$, $k\neq 2m-2$, $k-m+1>q\geqslant k-2m+2$
69	$L_{3.4}(m,n,l)$	$ab=a^m$, $b^2=a^{2m-2}$, $ba^l=a^n=0$	$m\geqslant 3$, $l\geqslant m-1$, $l+m\geqslant n\geqslant 2m-1$
70	$L_{3.5}(m,n,l)$	$ab=a^m$, $b^2=a^{2m-2}$, $ba^l=a^n=0$, $a^{n-1}=ba^{l-1}$	$m\geqslant 3$, $l\geqslant m-1$, $l+m\geqslant n\geqslant 2m-1$
71	$L_{3.6}(m,n,q)$	$ab=a^m$, $b^2=ba^{2m-2}$, $a^{q+m-1}=ba^q$, $a^n=0$	$m\geqslant 2$, $q\geqslant 2$, $n\geqslant q+m$
72	$L_{3.7}(m,l,k,n)$	$ab=a^m$, $b^2=ba^l$, $a^n=ba^k=0$	$2m-1\geqslant n\geqslant k>l\geqslant m\geqslant 3$
73	$L_{3.8}(m,l,k,n)$	$ab=a^m$, $b^2=ba^l$, $a^n=ba^k=0$, $a^{n-1}=ba^{l-1}$	$2m-1\geqslant n\geqslant k>l\geqslant m\geqslant 3$
74	$L_{3.9}(m,l,q,n)$	$ab=a^m$, $b^2=ba^l$, $a^n=0$, $ba^{q+m-1}=ba^q$	$2m-1\geqslant n>l\geqslant m\geqslant 3$, $q\geqslant 2$
75	$L_{3.10}(m,n,k)$	$ab=a^m$, $b^2=0$, $a^n=ba^k=0$	$n>m\geqslant 3$, $m+k\geqslant n\geqslant k\geqslant 2$
76	$L_{3.11}(m,n,k)$	$ab=a^m$, $b^2=0$, $a^n=ba^k=0$, $a^{n-1}=ba^{k-1}$	$n>m\geqslant 3$, $m+k\geqslant n\geqslant k\geqslant 2$
77	$L_{3.12}(m,n,q)$	$ab=a^m$, $b^2=0$, $a^n=0$, $a^{q+m-1}=ba^q$	$n>m\geqslant 3$, $q\geqslant 2$
78	$N_{1.1}(m,l,n,k)$	$b^2=a^mb$, $ba=a^lb$, $a^n=a^kb=0$	$n\geqslant k\geqslant l>m\geqslant 2$, $m+l\geqslant k$, $2m>l$
79	$N_{1.2}(m,l,n,k)$	$b^2=a^mb$, $ba=a^lb$, $a^n=a^kb=0$, $a^{n-1}=a^{k-1}b$	$n\geqslant k\geqslant l>m\geqslant 2$, $m+l\geqslant k$, $2m>l$
80	$N_{2.1}(m,l,n,k)$	$ba=a^mb$, $b^2=a^lb$, $a^n=a^kb=0$	$n\geqslant k\geqslant l>m\geqslant 2$, $m+l\geqslant k$
81	$N_{2.2}(m,l,n,k)$	$ba=a^mb$, $b^2=a^kb$, $a^n=a^lb=0$, $a^{n-1}=a^{k-1}b$	$n\geqslant k\geqslant l>m\geqslant 2$, $m+l\geqslant k$
82	$N_{3.1}(m,k)$	$a^2=(ab)^{\frac{2m+1}{2}}$, $b^2=(ab)^{\frac{k}{2}}$	$k>2m+1$, $m>1$
83	$N_{3.2}(m,k)$	$a^2=(ab)^{\frac{2m+1}{2}}$, $b^2=(ab)^{\frac{k}{2}}$, $b^2a=ab^2=b^3=0$	$k>2m$, $m>1$
84	$N_{3.3}(m,k)$	$a^2=(ab)^{\frac{2m+1}{2}}$, $b^2=(ab)^{\frac{k}{2}}=(ba)^{\frac{k}{2}}$	$k>2m$, $m>1$
85	$N_{3.4}(m,k)$	$a^2=(ab)^{\frac{2m+1}{2}}$, $b^2=(ab)^{\frac{k}{2}}$, $(ba)^{\frac{k}{2}}=0$	$k>2m$, $m>1$
86	$N_{3.5}(m,k,n,l)$	$a^2=(ab)^{\frac{2m+1}{2}}$, $b^2=(ba)^{\frac{2k+1}{2}}$ $(ab)^{\frac{n}{2}}=(ba)^{\frac{l}{2}}=0$	$k>m$, $m>1$, $n,l\geqslant k$, $\lvert n-l\rvert\leqslant 1$
87	$N_{3.6}(m,n,l)$	$a^2=(ab)^{\frac{2m+1}{2}}$, $b^2=(ab)^{\frac{n}{2}}=(ba)^{\frac{l}{2}}=0$	$m>1$, $n,l\geqslant m$, $\lvert n-l\rvert\leqslant 1$
88	$N_{3.7}(m)$	$a^2=(ba)^{\frac{m}{2}}$, $b^2=(ba)^{\frac{m+1}{2}}$	$m\geqslant 3$
89	$N_{3.8}(m)$	$a^2=(ab)^{\frac{m}{2}}$, $b^2=0$	$m\geqslant 3$
90	$N_{3.9}(m)$	$a^2=(ab)^{\frac{m}{2}}=(ba)^{\frac{m}{2}}$, $b^2=0$	$n\geqslant 3$
91	$N_4(m,n)$	$ab=a^n=b^m$, $ba=0$	$m,n\geqslant 3$

We show later that every row in this table, with some constants fixed, gives us exactly one semigroup up to isomorphism or dually isomorphism. Some rows have no parameters, which means that such rows defines only one finite semigroup.

Let us note that any two semigroups in this table are not isomorphic and are not dually isomorphic. Indeed, every nilsemigroup has exactly one basis, i.e. a minimal set of generators. Every generator is a maximal element under division order $\leqslant$. Conversely, every maximal element of $(S,\leqslant)$ is an element of any basis. So, the set of maximal elements is the unique basis of S. Then every automorphism of S maps the basis onto itself, which means that it preserves the presentation of S. All semigroups in the table have distinct presentations, that can be revised by a careful check.

It is easy to check that all semigroups in this table have a width 2. It gives us the implication from c) to a). The implication from a) to b) is proved in [2]. The rest of the paper is directed to prove that b) leads c).

Theorem 1 provides a simple test to determine whether the congruence lattice of a finite nilsemigroup is modular by checking the condition (b) or by searching the corresponding semigroup in the Table.

Theorem 1 has an important corollary for the class of nilpotent semigroups. Every nilpotent semigroup S satisfy the ascending chain condition under $\leqslant$. Then S has a basis, which consists of maximal elements of S under $\leqslant$. This basis form an antichain, so by result of [2], it has 1 or 2 elements. From Theorem 1 we have the following corollary:

Corollary 1. *Every nilpotent semigroup with modular congruence lattice is finite. It is isomorphic or dually isomorphic to a suitable semigroup in the Table.*

1. Preliminaries

We consider the division relation $\leqslant$ on a semigroup S defined as $a \leqslant b$ iff there exist $s, t \in S^1$ such that $b = sat$. Since every nilsemigroup is $\mathscr{J}$-trivial, the relation $\leqslant$ is an order relation on a nilsemigroup.

Our starting point is the following statements that was proved in [1] as Corollary 2.

Proposition 1. *Let S be a nilsemigroup such that* $\operatorname{Con} S$ *is modular. If S is finitely generated, then it is finite. If S is not cyclic, then it is generated by two elements a, b and the poset $\{a^2, ab, ba, b^2\}$ has width at most two.*

We assume further in the paper that S is a finite nilsemigroup generated by two distinct elements a and b.

We say that an element $x \in S$ is *an atom*, if x covers 0, i.e. $x > 0$ and, for every $z \in S$, the condition $0 < z \leqslant x$ implies $z = x$. Put

$$x \gg y \text{ iff there exist } s, t \in S^1 \text{ such that } y = sxt \text{ and } st \neq 1.$$

The relation $\gg$ on S is antisymmetric and transitive. It is easy to see that, for $x, y \in S$, $x > y$ implies $x \gg y$, and $x \gg y$ implies $x \geqslant y$ (the converse is false, since $0 \gg 0$, but $0 \not> 0$).

Lemma 1. *1) An element $x \in S$ is equal to zero if and only if $x \gg x$.*
2) For every $s \in S$ either $s = s'a$ or $s = s'b$ for some $s' \in S^1$.
3) For every $t \in S$ either $t = at'$ or $t = bt'$ for some $t' \in S^1$.
4) If $x \in S$ satisfies $xa = ax = xb = bx = 0$, then x is an atom or a zero.

The proof is obvious.

Let u be a word of n letters. Define $u^{\frac{p}{n}}$ for $0 \leqslant p \leqslant n-1$ as a p-element prefix of u. For an arbitrary positive integer p, put $u^{\frac{p}{n}} = u^{[p/n]} u^{\frac{p \bmod n}{n}}$.

Lemma 2. *Let c, d be letters and let p be a positive integer. Then:*
1) $(cd)^{\frac{p}{2}} = c(dc)^{\frac{p-1}{2}}$.
2) $(cd)^{\frac{p}{2}} = (cd)^{\frac{p-1}{2}} c$, if p is odd.
3) $(cd)^{\frac{p}{2}} = (cd)^{\frac{p-1}{2}} d$, if p is even.
4) $(cd)^{\frac{p}{2}} (dc)^{\frac{q}{2}} = (cd)^{\frac{p+q}{2}}$, if p is odd.
5) $(cd)^{\frac{p}{2}} (cd)^{\frac{q}{2}} = (cd)^{\frac{p+q}{2}}$, if p is even.

The proof is obvious.

Lemma 3. *1) If $a^2 \lessdot ab$, then either $a^2 \lessdot b^2$ or $a^2 \lessdot ba$.*
2) If $ba \lessdot ab$, then either $ba \lessdot a^2$ or $ba \lessdot b^2$.
3) If $a^2 > b^2 > ab$, $ba \not> b^2$ and $ab \neq 0$, then $ab < ba$.
4) If $a^2 > ab > b^2$, $ba \not> ab$ and $b^2 \neq 0$, then $b^2 < ba$.
5) If $a^2 > ab > ba$ and $ba \neq 0$, then $b^2 > ba$.

P r o o f. 1) Let $a^2 \lessdot ab$. Then $a^2 = sabt$ for some $s, t \in S^1$. If $s = s'a$ for some $s' \in S^1$, then $a^2 \lessdot a^2$, which implies $a^2 = 0 \lessdot b^2$. If $s = s'b$ for some $s' \in S^1$, then $a^2 = s'babt$ and $a^2 \lessdot ba$. Let $s = 1$ and $a^2 = abt$. If $t = at'$ for some $t' \in S^1$, then $a^2 \lessdot ba$. If $t = bt'$ for some $t' \in S^1$, then $a^2 \lessdot b^2$.

2) The proof is similar to 1).

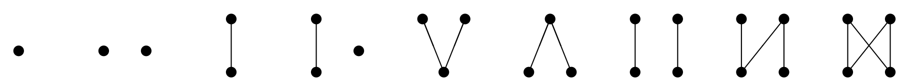
3) If $a^2 > b^2$, then $b^2 = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ for some $s' \in S^1$, then $b^2 < ba$, a contradiction. If $t = bt'$ for some $t' \in S^1$, then $b^2 < ab$, a contradiction. So $b^2 = a^k$ for some $k \geqslant 3$. Then $ab < b^2 = a^k$, so $ab = ua^kv$ for some $u, v \in S^1$. If $u = u'a$ or $v = av'$ for some $u', v' \in S^1$, then $ab < a^{k+1} = ab^2$, i.e. $ab = 0$, a contradiction. If $v = bv'$ for some $v' \in S^1$, then $ab < ab$ and $ab = 0$. If $u = u'b$ for some $u' \in S^1$, then $ba > ab$.

4) If $a^2 > ab$, then $ab = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ for some $s' \in S^1$, then $ab < ba$, a contradiction. If $t = bt'$ for some $t' \in S^1$, then $ab = 0$, contrary to $ab > b^2$. So $ab = a^k$ for some $k \geqslant 3$. Then $b^2 < ab = a^k$, so $b^2 = ua^kv$ for some $u, v \in S^1$. If $u = u'a$ or $v = av'$ for some $u', v' \in S^1$, then $ab < a^{k+1} = aba$, i.e. $ab < b^2$, a contradiction. If $v = bv'$ for some $v' \in S^1$, then $b^2 < a^kb = ab^2$ and $b^2 = 0$, a contradiction. If $u = u'b$ for some $u' \in S^1$, then $ba > b^2$.

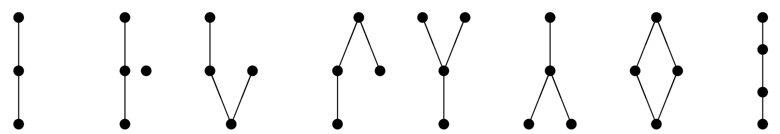
5) If $a^2 > ab$, then $ab = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ for some $s' \in S^1$, then $ab < ba$, a contradiction. If $t = bt'$ for some $t' \in S^1$, then $ab = 0$, contrary to $ab > ba$. So $ab = a^k$ for some $k \geqslant 3$. Then $ba < ab = a^k$, so $ba = ua^kv$ for some $u, v \in S^1$. If $u = u'a$ or $v = av'$ for some $u', v' \in S^1$, then $ba \leqslant a^{k+1} = aba \lessdot ba$, i.e. $ba = 0$, a contradiction. If $v = bv'$ for some $v' \in S^1$, then $ba \leqslant a^kb = ab^2 < b^2$. If $u = u'b$ for some $u' \in S^1$, then $ba \lessdot ba$, which means $ba = 0$, a contradiction. □

2. Finite nilsemigroups of width 2

The elements a^2, ab, ba, b^2 form a subposet of S. This subposet has no more than 4 elements and has no antichains with 3 elements. We enumerate all such posets in the following list.



A **B** **C** **D** **E** **F** **G** **H** **I**



J **K** **L** **M** **N** **O** **P** **Q**

For each poset **A-Q** we examine all possibilities of mapping the set $\{a^2, b^2, ab, ba\}$ onto the poset. We consider two cases be equal if one of them can be obtained from another either by replacing a to b and vice versa or by replacing ab to ba and vice versa. Indeed, these cases give us isomorphic or dually isomorphic semigroups. Some cases are forbidden by Lemma 3, we don't mention them.

Let us note that in cases **B**, **D**, **F**, **G**, **H**, **I**, **K**, **M**, **O** the elements a^2, b^2, ab, ba are not equal to zero, since every element of a nilsemigroup divides zero.

Series A. $a^2 = b^2 = ab = ba$. Then every element of S, except b, can be written as a^p for some positive p. Let n be the least positive integer such that $a^n = 0$. Then $S \cong A(n)$.

Series B. The following cases are possible:

$a^2 = b^2 = ab$	ba	$a^2 = b^2$	$ab = ba$
B1		**B2**	
$a^2 = ab = ba$	b^2	$a^2 = ab$	$b^2 = ba$
B3		**B4**	

Case B1. Let x be an element of S. If a or b^2 is a left divisor for x, then $x = a^p$ for some p. If ba^2 is a left divisor for x, then $x = a^p$ for some p, since $ba^2 = b^3$. So, every element of S, except b and ba, can be written as a^p for some p. Let n be the least positive integer such that $a^n = 0$. We obtain the semigroup $B_1(n)$.

Case B2. It is easy to show that every element can be written as a^p or ba^p for some $p \geqslant 0$. Let n and l be the least positive integers such that $a^n = ba^l = 0$. Then $|n - l| \leqslant 1$ and $n \geqslant 3$, $l \geqslant 2$. If $|n - l| = 1$, then $S \cong B_{2.1}(n, l)$.

Let $a^m = ba^{m-1}$ for some $m \geqslant 3$. Then $a^p = ba^{p-1}$ for all $p \geqslant m$. Let n be the least positive integer such that $a^n = 0$. We obtain the semigroup $B_{2.2}(m, n)$.

Case B3. In this case every element can be written as ab^{p-1} or b^p for some $p \geqslant 1$. Let m be the least positive integer such that $ab^{m-1} = b^n$ for some $n \geqslant 3$, $m \geqslant 3$ and $n \geqslant m - 1$. The following cases are possible:

Case B3.1 $m \neq n$. Then $ab^m = a(ab^{m-1}) = ab^n$, so $ab^m = 0$. The element $ab^{m-1} = b^n$ is a single atom or a zero. Then $S \cong B_{3.1}(m, n)$ or $S \cong B_{3.2}(m, n)$ respectively.

Case B3.2. $m = n$. Then $ab^{p-1} = b^p$ for all $p \geqslant m$. Let k be the least positive integer such that $b^k = 0$. We have that $S \cong B_{3.3}(m, k)$.

Case B4. In this case every element can be written as $ab^{p-1} = a^p$ or b^p for some $p \geqslant 0$. Let m and n be the least positive integers such that $a^m = b^n$. If $m < n$ then $a^m \lessdot ba^m = b^{m+1} \leqslant b^n$, so $a^m = b^n = 0$ and $n = m+1$. If $m > n$, then $b^n \lessdot ab^n = a^{n+1} \leqslant a^m$, so $a^m = b^n = 0$ and $m = n+1$. We got $|m - n| = 1$ and $S \cong B_{4.1}$. If $m = n$, then $a^p = b^p$ for all $p \geqslant n$. Let k be the least positive integer such that $a^k = 0$. We deduce that $S \cong B_{4.2}(m, k)$.

Series C. The following cases are possible:

$a^2 = ab$	$a^2 = b^2 = ab$	ba
$b^2 = ba$	ba	$a^2 = b^2 = ab$
C1	**C2**	**C3**

$a^2 = b^2$	$a^2 = ab = ba$	$ab = ba$	b^2
$ab = ba$	b^2	$a^2 = b^2$	$a^2 = ab = ba$
C4	**C5**	**C6**	**C7**

Case C1. Since $b^2 < a^2$, we have $b^2 = sa^2t$ for some $s, t \in S^1$. If $s = s'a$ for some $s' \in S^1$, then $ba = b^2 = s'a^3t = s'abat$, which means that $ba \gtrdot ba$, so $b^2 = 0$. Cases $s = s'b$ and $t = at'$ for some $s', t' \in S^1$ are similar. If $t = bt'$ for some $t' \in S^1$, then $ba = sa^2bt' = sa^3t' = sabat'$, which implies $ba \gtrdot ba$. So, $b^2 = ba = 0$.

An element a^2 is a single atom. Indeed, $a^2b = a^3 = aba = 0$ and $ba^2 = 0$. Then $S \cong C_1$.

Case C2. Since $ba < a^2$, then $ba = sa^2t$ for some $s, t \in S^1$. If $s = s'a$ for some $s' \in S^1$, then $ba = s'a^3t = s'abat$, which impies $ba = 0$. Cases $s = s'b$, $t = bt'$ and $t = at'$ for some $s', t' \in S^1$ are similar. So, $ba = 0$.

An element a^2 is a single atom. Indeed, $a^2b = a^3 = aba = 0$ and $ba^2 = 0$. Then $S \cong C_2$.

Case C3. By the same arguments as before, we have $a^2 = b^2 = ab = 0$. The element ba is a single atom. Then $S \cong C_3$.

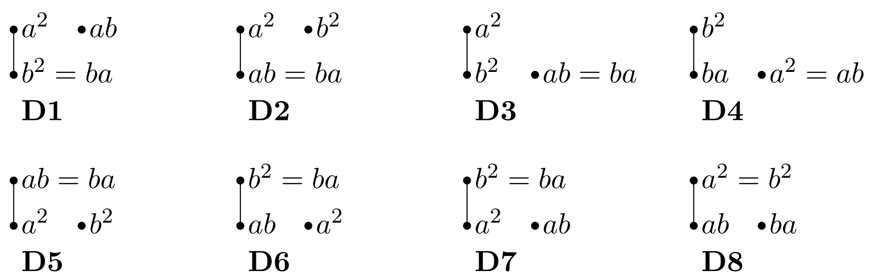
Case C4. Using arguments of case C1, we have $ab = ba = 0$. The element $a^2 = b^2$ is a single atom. Then $S \cong C_4$.

Case C5. Using arguments of case C2, we have $b^2 = 0$. The element $a^2 = ab = ba$ is a single atom. Then $S \cong C_5$.

Case C6. Using arguments of case C2, we have $a^2 = b^2 = 0$. The element $ab = ba$ is a single atom. Then $S \cong C_6$.

Case C7. We have $a^2 = ab = ba = b^n$ for some $n \geqslant 3$. The element a^2 is an atom or a zero, since $ab^2 = ba^2 = a^2b = a^3 = ab^k \lessdot ab^2$. If a^2 is an atom, then $S \cong C_{7.1}(n)$. If a^2 is a zero, then $S \cong C_{7.2}(n)$.

Series D. The following cases are possible:



Case D1. We have $b^2 < a^2$, so $b^2 = ba = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $ba < ab$ or $ba < ba$, a contradiction. So, $b^2 = ba = a^m$ for some $m \geqslant 3$. The element $a^{m+1} = ba^2 = b^2a = ba^m = a^{2m-1}$ with $m \neq 2$ divides itself, which implies that it is a zero. The elements $bab = b^3 = b^2a$ and $aba = a^{m+1}$ are also equal to zero, which means that ba is an atom. The element $a^{m-1}b$ is an atom or a zero.

Every element of S can be written as a^p or $a^{p-1}b$ for some $p \leqslant m$. Let $q \geqslant 1$ and $1 < n < m$ be the least positive integers such that $a^q = a^nb$. If $q < m$, then $a^m = a^qa^{m-q} = a^nba^{m-q} = a^na^ma^{m-q-1} < a^m$, so $a^m = ba = 0$, a contradiction. If $q = m$, then $a^{n+1}b = a^{m+1} = 0$, so a^m is a single atom and $S \cong D_{1.1}(m, n)$. If $q > m$, then $a^nb = 0$ and $S \cong D_{1.2}(m, n)$.

Case D2. We have $ab < a^2$, so $ab = ba = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $ba < ab$, a contradiction. So, $ab = ba = a^m$ for some $m \geqslant 3$. Then every element of S can be written as a^p or b^p for some p. Let n and k be the least positive integers such that $a^n = 0$ and $b^k = 0$. Then $n \leqslant k(m-1) + 1$ and $k \geqslant 3$.

If $a^p = b^q$ implies $a^p = 0$, then $S \cong D_{2.1}(m, n, k)$. Let p, q be the least positive integers such that $a^p = b^q \neq 0$. Then $a^{p+1} = b^qa = a^{(m-1)q+1}$. If $p \neq (m-1)q$, then $a^{p+1} = 0$ and $p + 1 = n$, $q + 1 = k$. In this case $S \cong D_{2.2}(m, n, k)$. If $p = (m-1)q$, then $a^{r(m-1)} = b^r$ for all $r \geqslant p$, so $k = [n/(m-1)]$ and $S \cong D_{2.3}(m, n, q)$.

Case D3. We have $b^2 = a^m$ for some $m \geqslant 3$. Every element of S can be written in the form a^p or a^pb for some p. Let n be the least positive integer such that $a^n = 0$ and let k be the least positive integer such that $a^kb = 0$. Since $a^kb^2 = a^{k+m}$, we have $k + m \geqslant n \geqslant k$.

If $a^p = a^q b$ for some p, q implies $a^p = 0$, then $S \cong D_{3.1}(m, n, k)$. Let p, q be the least positive integers such that $a^p = a^q b \neq 0$. Then $a^{p+r} = a^{q+r}b$ for all $r \geqslant 0$, which implies $n - p = k - q$, so $p = n - k + q$. We have $a^p b = a^q b^2 = a^{q+m}$, so either $q + m - p = p - q$ or $a^p b = 0$. In the former case $p = q + m/2$ and $m = 2(n-k)$, whence $S \cong D_{3.2}(n, k, q)$. In the latter case $k \leqslant \min(p, q+m)$ and $S \cong D_{3.3}(m, n, k, q)$.

Case D4. We have $ba = b^n$ for some $n \geqslant 3$. Then $bba = b^{n+1} = bab = baa = b^n a = b^{2n-1} \lessdot b^{n+1}$, so $bba = bab = baa = 0$. Also $a^3 = aba = ab^n = a^{n+1} \lessdot a^3$, so $aba = 0$. We got that ba is an atom. The element a^2 is also an atom, because $ba^2 = 0$, $a^3 = a^2 b = 0$. Elements of S are equal to a, or to a^2, or to b^i for $i = 1 \dots n$. We got a semigroup $D_4(n)$.

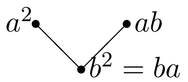
Case D5. We have $a^2 < ab = ba$, so $a^2 = sabt$ for some $s, t \in S^1$. If $s = s'a$ or $t = at'$ for some $s', t' \in S^1$, then $a^2 \lessdot a^2$ and $a^2 = 0 < b^2$, a contradiction. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $a^2 < b^2$, a contradiction.

Case D6. We have $ab < a^2 = b^2$, so $ab = sa^2t$ for some $s, t \in S^1$. If $s = s'a$ or $t = bt'$ for some $s', t' \in S^1$, then $ab \lessdot ab$ and $ab = 0 < ba$, a contradiction. If $s = s'b$ or $t = at'$ for some $s', t' \in S^1$, then $ab < ba$, a contradiction.

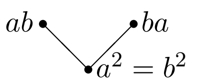
Case D7. We have $a^2 < b^2$, so $a^2 = sb^2t$ for some $s, t \in S^1$. If $s = s'a$ for some $s' \in S^1$, then $a^2 < ab$, a contradiction. If $s = s'b$ for some $s' \in S^1$, then $a^2 = s'b^3t = s'babt < ab$, a contradiction. Let $s = 1$. If $t = at'$ or $t = bt'$ for some $t' \in S^1$, then $a^2 = b^2at' = b^3t' = babt' < ab$, a contradiction.

Case D8. We have $ab < a^2$, so $ab = sa^2t$ for some $s, t \in S^1$. If $s = s'a$ or $t = at'$ for some $s', t' \in S^1$, then $ab \leqslant a^3 = ab^2 < ab$, a contradiction. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $ab \leqslant b^3 = a^2b < ab$, a contradiction.

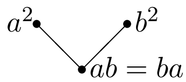
Series E. The following cases are possible:



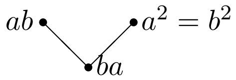
E1



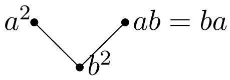
E2



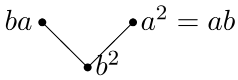
E3



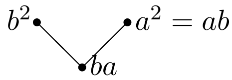
E4



E5



E6



E7

Case E1. We have $b^2 < ab$, so $b^2 = sabt$ for some $s, t \in S^1$. If $s = s'b$ or $t = at'$ or $t = bt'$ for some $s', t' \in S^1$, then $b^2 \lessdot ba = b^2$, which implies $b^2 = 0$. Anyway, there exists $m \geqslant 2$ such that $b^2 = ba = a^m b$. Then every element of the semigroup S can be written as a^p or $a^p b$ for some p.

Let k and n be the minimal numbers such that $a^k b = a^n = 0$. Obviously, $n \geqslant k$ and $k \geqslant m$. We have $a^{2m}b = a^m b^2 = b^3 = b^2 a = ba^m b = b^2 a^{m-1} b$. Since $m \neq 1$, the element $b^2 a$ divides itself, which means $a^{2m}b = 0$ and $k \leqslant 2m$.

If $a^q = a^r b$ implies $a^q = 0$, then $S \cong E_{1.1}(m, n, k)$. Let $a^q = a^r b \neq 0$ for some $q > r > 0$. Then $a^{r+1}b = a^{q+1} = a^r ba = a^{r+m}b \lessdot a^{r+1}b$, so $a^{q+1} = a^{r+1}b = 0$, which means $q = n-1$ and $r = k-1$. Then $S \cong E_{1.2}(m, n, k)$.

Case E2. We have $a^2 < ab$ and $a^2 < ba$, so $a^2 = b^2 = (ab)^{m/2}$ or $a^2 = b^2 = (ba)^{m/2}$ for some $m \geqslant 3$. Without loss of generality we suppose that $a^2 = b^2 = (ab)^{m/2}$. Then every element of S can be written in the form $(ab)^{p/2}$ or in the form $(ba)^{p/2}$ for some p. The following cases are possible:

Case E2.1. $m = 2n+1$ for some $n \geqslant 1$, so $a^2 = b^2 = (ab)^{\frac{2n+1}{2}}$. Then $a^3 = (ab)^{\frac{2n+1}{2}}a = (ab)^{\frac{2n}{2}}a^2 = (ab)^{\frac{2n}{2}}b^2 = (ab)^{\frac{2n-1}{2}}b^3 = (ab)^{\frac{2n-2}{2}}a^3 b$, so $a^3 = 0$. Therefore $a^2 b = b^3 = ba^2$, which means $(ab)^{\frac{2n+2}{2}} = (ba)^{\frac{2n+2}{2}}$. Then $(ab)^{\frac{2n+3}{2}} = (ab)^{\frac{2n+2}{2}}a = (ba)^{\frac{2n+2}{2}}a = b(ab)^{\frac{2n}{2}}a^2 = 0$ and, analogously,

$(ba)^{\frac{2n+3}{2}} = 0$. So, a^2b is an atom or a zero. If $a^2 = a^2b = (ba)^{\frac{m}{2}} = 0$, then $S \cong E_{2.1}(2n+1)$. If $a^2 = 0$ and $(ba)^{\frac{m}{2}} \neq 0$, then $S \cong E_{2.2}(2n+1)$. If $a^2 \neq 0$ and $(ba)^{\frac{m}{2}} = 0$, then $S \cong E_{2.3}(2n+1)$. If $a^2 = (ba)^{\frac{m}{2}} \neq 0$ and $a^2b = 0$, then $S \cong E_{2.4}(2n+1)$. If $a^2 \neq (ba)^{\frac{m}{2}}$ and $a^2b \neq 0$, then $S \cong E_{2.5}(2n+1)$. If $a^2 \neq 0$, $(ba)^{\frac{m}{2}} \neq 0$, $a^2 \neq (ba)^{\frac{m}{2}}$ and $a^2b = 0$, then $S \cong E_{2.6}(2n+1)$.

Case E2.2. $m = 2n$ for some $n \geqslant 2$, so $a^2 = b^2 = (ab)^{\frac{2n}{2}}$. Then $a^3 = a(ab)^{\frac{2n}{2}} = a^2(ba)^{\frac{2n-1}{2}} = b^3(ab)^{\frac{2n-2}{2}} = ba^3(ba)^{\frac{2n-3}{2}}$, so $a^3 = 0$. From here we obtain $ba^2 = b^3 = a^2b = (ab)^{\frac{2n}{2}}b = (ab)^{\frac{2n-2}{2}}ab^2 = (ab)^{\frac{2n-2}{2}}a^3 = 0$, which means that a^2 is an atom or a zero. If $a^2 = (ba)^{\frac{m}{2}} = 0$, then $S \cong E_{2.1}(2n)$. If $a^2 = 0$ and $(ba)^{\frac{m}{2}} \neq 0$, then $S \cong E_{2.2}(2n)$. If $a^2 \neq 0$ and $(ba)^{\frac{m}{2}} = 0$, then $S \cong E_{2.3}(2n)$. If $a^2 = (ba)^{\frac{m}{2}} \neq 0$, then $S \cong E_{2.4}(2n)$. If $a^2 \neq 0$, $(ba)^{\frac{m}{2}} \neq 0$ and $a^2 \neq (ba)^{\frac{m}{2}}$, then $S \cong E_{2.6}(2n)$.

Case E3. We have $ab < a^2$, whence $ab = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $ab \lessdot ab$ or $ba \lessdot ba$, which means $ab = ba = 0$. Anyway, $ab = a^m$ for some $m \geqslant 3$. Analogously, $ab = b^n$ for some $n \geqslant 3$. Every element of the semigroup S can be written in the form a^p or b^p for some p. Let n be the least positive integer such that $ab = ba = a^n$ and m be the least positive integer such that $ab = ba = b^m$. Then $aba = a^{n+1} = ab^m = a^nb^{m-1} = a^{2n-1}b^{m-2} \lessdot a^{n+1}$, so $aba = 0$. By the same arguments, $abb = 0$, which means that ab is either an atom or a zero. If ab is an atom, $S \cong E_{3.1}(n, m)$. If ab is a zero, $S \cong E_{3.2}(n, m)$.

Case E4. We have $ba < a^2 = b^2$, so $ba = sa^2t$ for some $s, t \in S^1$. If $s = s'a$ or $t = at'$ for some $s', t' \in S^1$, then $ba \leqslant a^3 = b^2a \lessdot ba$. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $ba \leqslant b^3 = ba^2 \lessdot ba$, which implies $ba = 0$. Now we have $ab^2 = a^3 = b^2a = 0$, $a^2b = b^3 = ba^2 = 0$, $aba = bab = 0$, so the semigroup S consists only of five elements and $S \cong E_4$.

Case E5. We have $b^2 < ab = ba$, so $b^2 = sabt$ for some $s, t \in S^1$. If $s = s'b$ or $t = bt'$ for some $s', t' \in S^1$, then $b^2 \lessdot b^2$, which means $b^2 = 0$. Anyway, there exists m such that $b^2 = a^mb$. Let m be the minimal integer with such a property.

Every element of S can be written as a^p or a^pb for a suitable p. Let n and k be minimal positive integers such that $a^n = 0$ and $a^kb = 0$.

If $a^p = a^qb$ implies $a^p = 0$ for some p, q, then $S \cong E_{5.1}(m, n, k)$.

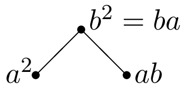
Let p, q be the least positive integers such that $a^p = a^qb \neq 0$. Then $p < n$, $q < k$ and $a^{p+r} = a^{q+r}b$ for all $r \geqslant 0$, so $n - p = k - q$, which means $p = n - k + q$. We have $a^{n-k+q} = a^qb$, so $a^{n-k+q}b = a^{q+m}b$. If $n - k \neq m$, then $a^{n-k+q}b = a^{q+m}b = 0$, so $k \leqslant \min(n - k + q, q + m)$. In this case $S \cong E_{5.2}(m, n, k, q)$. If $n - k = m$, then $S \cong E_{5.3}(m, n, q)$.

Case E6. Since $a^2 = ab$, every element of S can be written as a^p or b^qa^p for some p. Then $b^2 = 0$ or $b^2 = ba^n$ for some n. If $n \geqslant 3$, then $b^2 \lessdot a^3 = ab^2 \lessdot b^2$, which implies $b^2 = 0$. So, $b^2 = ba^2 \neq 0$ or $b^2 = 0$.

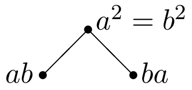
Let $b^2 = ba^2 \neq 0$. Then $a^3 = ab^2 = aba^2 = a^4$, so $a^3 = 0$. Thus, $b^2a = ba^3 = 0$, $b^3 = ba^2b = ba^3 = 0$ and $ab^2 = a^3 = 0$, so b^2 is an atom. Then $S \cong E_{6.1}$. If $b^2 = 0$, then $a^3 = 0$. Hence a^2 and ba are atoms and $S \cong E_{6.2}$.

Case E7. Using the same arguments as in case E6, for some $m \geqslant 2$, we have $ba = a^mb \lessdot a^3 = aba \lessdot ba$, so $ba = 0$. Let n be the index of b. Then S consist of elements $a, a^2, b, b^2, \ldots, b^{n-1}, 0$ and is isomorphic to $E_7(n)$.

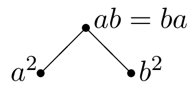
Series F. The following cases are possible:



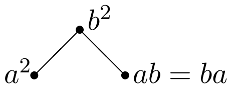
F1



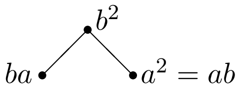
F2



F3



F4



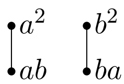
F5

Case F1. We have $a^2 < b^2 = ba$, but $a^2 \not< ab$. So, $a^2 \neq 0$ and $a^2 = b^k a$ for some $k \geqslant 2$. These arguments are true for ab, so $ab = b^l a$ for some $l \geqslant 2$. Then $a^2 \geqslant ab$ or $a^2 \leqslant ab$, a contradiction.

Case F2. $ab < a^2$, so $ab = sa^2t$ for some $s, t \in S^1$. If $s = s'b$ for $s' \in S^1$, then $ab < ba$. If $t = bt'$ for $t' \in S^1$, then $ab < ab$. If $s = s'a$ or $t = at'$ for some $s', t' \in S^1$, then $ab \leqslant a^3 = ab^2 < ab^2$. All the possibilities lead to a contradiction.

Cases **F3-F5** are analogous to **F1** or **F2**.

Series G. Only one case is possible:

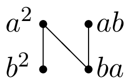


G1

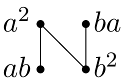
Case G1. We have $ab < a^2$, so $ab = a^m$ for some $m \geqslant 3$. Similarly, $ba = b^n$ for some $n \geqslant 3$. Then $a^{m+1} = a^2b = aba = ab^n = a^m b^{n-1} = a^{2m-1}b^{n-2} \lessdot a^{m+1}$, so $a^{m+1} = a^m b = ba^m = 0$. Similarly, $b^{n+1} = bab = b^2a = 0$. So, ab and ba are atoms.

Every element of the semigroup S can be written as a^p or b^p for a suitable p. Let $a^q = b^r$ for some $q < m$ and $r \leqslant n$. Then $a^{q+1} = b^r a = 0$ and $a^m = 0$, a contradiction. If $q = m$ and $r = n$, we have $ab = ba$, a contradiction. We obtain that $S \cong G_1(m, n)$.

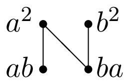
Series H. The following cases are possible:



H1



H2



H3

Case H1. We have $b^2 < a^2$, so $b^2 = a^n$ for $n \geqslant 3$. Since $ba < ab$ and $ba \not< b^2$, the equality $ba = a^m b$ holds for some $n > m \geqslant 2$. Every element of the semigroup S can be written as a^p or $a^p b$ for some p. Now $a^{n+1} = b^2 a = ba^m b = a^{m^2} b^2 = a^{m^2+n} \lessdot a^{n+1}$, so $b^2 = a^{n+1} = 0$, a contradiction. Therefore $b^2 = ba^n = a^{mn} = 0$ and b^2 is an atom. Let k be the least positive integer such that $a^k b = 0$. We have $a^n b = b^3 = 0$, so $m < k \leqslant n$.

If the equality $a^p = a^q b$ for some $p \leqslant n$ and $q \leqslant k$ implies $a^p = 0$, then $S \cong H_{1.1}(m, n, k)$. Let $a^p = a^q b$ for some $p \leqslant n$ and $q \leqslant k$. Then $a^{q+1}b = a^{p+1} = a^q ba = a^{q+m} b$, so $a^{p+1} = 0$. If $p < n$, then $b^2 = 0$, a contradiction. Let $p = n$ and $q > m$. We have $a^{q+1}b = 0$, so $q = k - 1$. We deduce $S \cong H_{1.2}(m, n, k)$.

Case H2. We have $ab = a^n$ for $n \geqslant 3$ and $b^2 = ba^m$ for $m \geqslant 2$. Since $b^2 \not> ab$, then $n \geqslant m + 1$. Every element of the semigroup S can be written as a^p or ba^p for some p.

Let $n > m+1$. Then $a^{n+m} = aba^m = ab^2 = a^n b = a^{2n-1} \lessdot a^{n+m}$, which implies $a^{n+m} = 0$. Let k and l be the minimal integers such that $a^k = 0$ and $ba^l = 0$. Therefore $n < k$ and $m < l \leqslant k \leqslant m+n$.

If $a^p = ba^q$ implies $a^p = 0$, then $S \cong H_{2.1}(m, n, k, l)$. Let $a^p = ba^q \neq 0$. Since $ab \not< ba$, then $p > n$. Therefore $ba^{q+1} = a^{p+1} = aba^q = a^{q+n}$. If $p + 1 \neq q + n$, then $a^{p+1} = 0$. This implies

$p+1=k$ and $q+1=l$, so $S \cong H_{2.2}(m,n,k,l)$. Let $p+1=q+n$. Then $a^{q+n-1+r}=ba^{q+r}$ for every $r \geqslant 0$, so $l=k-n+1$ and $k>q+n-1$. We obtain $S \cong H_{2.3}(m,n,k,q)$.

Let $n=m+1$. Let k and l be the minimal integers such that $a^k=0$ and $ba^l=0$. It is obvious that $m<k-1$, $m<l$ and $k \geqslant l$.

If $a^p=ba^q$ implies $a^p=0$, then $S \cong H_{2.4}(m,k,l)$. Let $a^p=ba^q \neq 0$. Since $ab \not< ba$, we have $p>m+1$. Therefore $ba^{q+1}=a^{p+1}=aba^q=a^{q+m+1}$. If $p+1 \neq q+m+1$, then $a^{p+1}=0$. This means $p+1=k$ and $q+1=l$, so $S \cong H_{2.5}(m,k,l)$. Let $p+1=q+n$. Then $a^{q+m+r}=ba^{q+r}$ for every $r \geqslant 0$, so $l=k-m$ and $k>q+m$, whence we get $S \cong H_{2.6}(m,k,q)$.

Case H3. We have $ab=a^m$ for $m \geqslant 3$ and $ba=a^n$ for $n \geqslant 3$. Then $ab>ba$ or $ba>ab$, a contradiction.

Series I. The following cases are possible:

I1 **I2** **I3** **I4**

Case I1. We have $a^2<ab$, $a^2<ab$, but $a^2 \not< b^2$, which means that $a^2=(ab)^{l/2}$ or $a^2=(ba)^{l/2}$ for some $l \geqslant 3$. We suppose without loss of generality that $a^2=(ab)^{l/2}$. Then $b^2=(ba)^{l/2}$. Every element can be written in the form $(ab)^p$ or $(ba)^p$ for some p. Two cases are possible:

Case I1.1: $a^2=(ab)^{\frac{2n}{2}}$ and $b^2=(ba)^{\frac{2n}{2}}$ for some $n \geqslant 2$. Then $(ab)^{\frac{2n+1}{2}}=a^2a=a^3=aa^2=a(ab)^{\frac{2n}{2}}=a^2(ba)^{\frac{2n-1}{2}}=(ab)^{\frac{2n}{2}}(ba)^{\frac{2n-1}{2}}=(ab)^{\frac{2n-1}{2}}b^2(ab)^{\frac{2n-2}{2}}=(ab)^{\frac{2n-1}{2}}(ba)^{\frac{2n}{2}}(ab)^{\frac{2n-2}{2}}=(ab)^{\frac{4n-1}{2}}(ab)^{\frac{2n-2}{2}} \lessdot (ab)^{\frac{2n+1}{2}}$, so $a^3=0$. Analogously, $b^3=0$. Then $a^2b=(ab)^{\frac{2n}{2}}b=(ab)^{\frac{2n-1}{2}}b^2=(ab)^{\frac{2n-1}{2}}(ba)^{\frac{2n}{2}}=(ab)^{\frac{4n-1}{2}} \lessdot a^3$, so $a^2b=0$. Similarly, $ba^2=ab^2=b^2a=0$, which means that a^2 and b^2 are atoms. Hence $S \cong I_{1.1}(n)$.

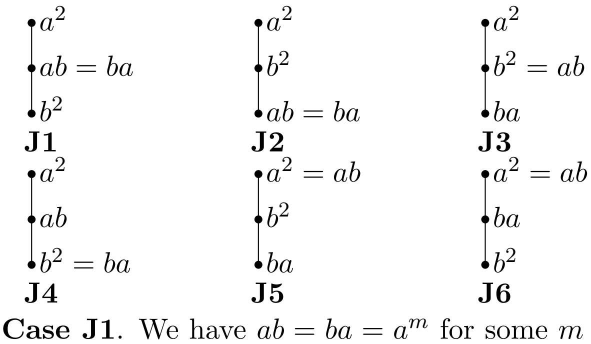
Case I1.2: $a^2=(ba)^{\frac{2n+1}{2}}$ and $b^2=(ab)^{\frac{2n+1}{2}}$ for some $n \geqslant 1$. Then $(ba)^{\frac{2n+2}{2}}=a^2a=aa^2=(ab)^{\frac{2n+2}{2}}$. It is easy to see that $(ba)^{\frac{2n+2}{2}}=(ab)^{\frac{2n+2}{2}}$ is either an atom or a zero. If it is an atom, then $S \cong I_{1.2}(n)$. If it is a zero, then $S \cong I_{1.3}$.

Case I1.3: $a^2=(ab)^{\frac{2n+1}{2}}$ and $b^2=(ba)^{\frac{2n+1}{2}}$ for some $n \geqslant 1$. Let m be the least positive integer such that $(ab)^{\frac{m}{2}}=0$ and k be the least positive integer such that $(ba)^{\frac{k}{2}}=0$. Then $|m-k| \leqslant 1$ and $m>2n+1$, $k>2n+1$. Hence $S \cong I_{1.4}(n,m,k)$.

Case I2. We have $b^2<ab$ and $b^2<a^2$, so $b^2=a^kb$ for some $k \geqslant 3$. By the same arguments $ba=a^lb$ for some $l \geqslant 3$. Then b^2 and ba are comparable, a contradiction.

Cases **I3** and **I4** lead to a contradiction in a similar way.

Series J. The following cases are possible:



Case J1. We have $ab=ba=a^m$ for some $m \geqslant 3$. Then $b^2=a^n$ for $m<n \leqslant 2m-2$. Hence $a^{n+1}=ab^2=a^mb=a^{2m-1}$. If $n<2m-2$, then $ab^2=0$ and $b^3=0$. So, b^2 is an atom or zero, which means that $S \cong J_{1.1}(n,m)$ or $S \cong J_{1.2}(n,m)$ respectively.

Let $n = 2m-2$ and let k be the index of a. Then $S \cong J_3(m,k)$.

Case J2. We have $b^2 = a^n$ for some $n \geqslant 3$. Also, $ab < b^2$, so $ab = sb^2t$ for some $s,t \in S^1$. If $s = s'a$ or $t = at'$ for $s',t' \in S^1$, then $ab \leqslant a^{n+1} = ab^2 \lessdot ab$. Cases $s = s'b$ or $t = at'$ for $s',t' \in S^1$ are similar. So, $ab = ba = a^{n+1} = 0$ and $S \cong J_2(n)$.

Case J3. In this case $0 < b^2 = ab < a^2$ implies that $b^2 = ab = a^n$ for some $n \geqslant 3$. Then $a^{n+1} = b^2a < ba$ and $a^nb = b^3 = ba^n \lessdot ba$. So, $ba = a^{n+1} = 0$ and $S \cong J_3(n)$.

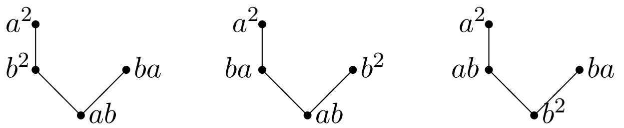
Case J4. We have $ab = a^n$ for some $n \geqslant 3$. Then $a^{n+1} = aba \lessdot ba$ and $a^nb = ab^2 \lessdot b^2 = ba$. But $ba < ab$, so $ba = 0$ and $S \cong J_4(n)$.

Case J5. The inequality $b^2 < a^2$ implies that $b^2 = sa^2t$ for some $s,t \in S^1$. If $s = s'b$ for $s' \in S^1$, then $b^2 < ba$, a contradiction. In other cases we have $b^2 < a^3 = ab^2$, which implies $b^2 = 0$ and $b^2 < ba$, a contradiction.

Case J6. As in the previous case, $ba \leqslant a^3 = aba$, whence $ba = 0$ and $ba < b^2$, a contradiction.

Series K. All cases from this series are impossible by Lemma 3.

Series L.



L1 **L2** **L3**

Case L1. We have $b^2 = a^m$ for some $m \geqslant 3$. If $ab = a^p$ for $p > m$, then $ab = a^p = a^{p-m}b^2 \lessdot ab$. If $ab = ba^q$ for $q \geqslant m$, then $ab = ba^q = b^3a^{q-m} = a^mba^{q-m} \lessdot ab$. Anyway, $ab = 0$. Every element of the semigroup S can be written in the form a^p or ba^p for some p.

Since $a^{m+1} = ab^2 = 0$ and $ba^m = b^3 = a^mb = 0$, the element b^2 is an atom. Let n be the minimal integer such that $ba^n = 0$. Then ba^{n-1} is an atom and $n \leqslant m+1$. Every element of the semigroup S can be written in the form a^p or ba^p for some p.

Let $a^p = ba^q$ for $p,q \leqslant n$. Then $b^2 = a^n = a^{n-p}a^p = a^{n-p}ba^q = 0$, a contradiction. So, $S \cong L_1(m,n)$.

Case L2. We have $ba = a^m$ for some $m \geqslant 3$. Then $a^{m+1} = aba \lessdot ab$, $a^mb \lessdot ab$, $ba^m = a^{2m-1} \lessdot ab$, but $ab < ba$, so $ab = 0$ and $ba = a^m$ is an atom. Every element of the semigroup S can be written in the form a^p or b^p for some p. Let n be the least positive integer such that $b^n = 0$.

If $a^p = b^q$ for some p,q leads to $a^p = 0$, then $S \cong L_{2.1}(m,n)$. Let $a^p = b^q$ for $p \leqslant m$ and $q < k$. Then $a^{p+1} = ab^q = 0 = a^{m+1}$, so $p = m$ and $q = n-1$. Then $S \cong L_{2.2}(m,n)$.

Case L3. $ab = a^m$ for some $m \geqslant 3$. Since $b^2 < ba$ and $b^2 < ab = a^m$, either $b^2 = a^n \neq 0$ for some $n \geqslant m+1$ or $b^2 = ba^l$ for some $l \geqslant m$ or $b^2 = 0$.

Case L3.1. $b^2 = a^k \neq 0$ and $k \neq 2m-2$. Then $b^2a = a^{k+1} = ab^2 = a^mb = a^{2m-1}$. Since $k+1 \neq 2m-1$, a^{k+1}=0. Then $b^2 \neq 0$, so b^2 is an atom. Every element of the semigroup can be written in the form a^p or ba^p for some p. Note that $ba^k = b^3 = a^kb = a^{m+k-1} = 0$. Let n be the least positive integer such that $ba^{n+1} = 0$. Then $k-m \leqslant n \leqslant k$, since $aba^l = a^{l+m}$ for all l.

If $a^p = ba^q$ implies $a^p = 0$ for all p,q, then $S \cong L_{3.1}(m,k,n)$. Suppose that $a^p = ba^q \neq 0$ for some p,q and let p,q be the least positive integers with such a property. Then $k \geqslant p > q$. We have $a^{p+1} = aba^q = a^{q+m}$, so either $p = k$ or $p = q+m-1$. If $p = k$, then $a^k = ba^q$ and $a^{k+1} = a^{q+m}$, so $q \geqslant k-m+1$ and $S \cong L_{3.2}(m,k,q)$. If $p = q+m-1$, then $a^{r+m-1} = ba^r$ for all $r \geqslant q$. But $ba^{q+m-1} = b^2a^q = 0$, so $a^{q+2m-1} = 0$, which implies $q \geqslant k-2m+2$. We got $S \cong L_{3.3}(m,k,q)$.

Case L3.2. $b^2 = a^{2m-2} \neq 0$. Every element of the semigroup can be written in the form a^p or ba^p for some p. Then $ba^{2m-2} = b^3 = a^{2m-2}b = a^{3m-3}$ and $ba^{2m-2+r} = a^{3m-3+r}$ for all $r \geqslant 1$. Let

n be the least positive integer such that $a^n = 0$. Then $n \geqslant 2m-1$, since $b^2 \neq 0$. Let l be the least positive integer such that $ba^l = 0$. Then $l \geqslant m-1$ and $n \leqslant l+m$.

If $a^p = ba^q$ implies $a^p = 0$ for all p, q, then $S \cong L_{3.4}(m,n,l)$. Let $a^p = ba^q \neq 0$ for some p, q and let p, q be the least positive integers with such a property. Then $a^{p+1} = aba^q = a^{q+m}$, so either $a^{p+1} = a^{q+m} = 0$ or $p = q+m-1$. Let $a^{p+1} = a^{q+m} = 0$. Then $p+1 = n$, $q+1 = l$ and $q+m \geqslant n$. Therefore $S \cong L_{3.5}(m,n,l)$.

Let $p = q+m-1$. Then $q \geqslant 2$, $n \geqslant q+m$ and we have $S \cong L_{3.6}(m,n,q)$.

Case L3.3. $b^2 = ba^l \neq 0$ for some l. Since $ab > b^2$, $l \geqslant m$. Every element of the semigroup S can be written either in the form a^p or in the form ba^p for some p. Then $a^{2m-1} = a^m b = ab^2 = aba^l = a^{l+m}$. Since $l \geqslant m$, $a^{2m-1} = a^{m+l} = 0$. Let n be the least positive integer such that $a^n = 0$ and k be the least positive integer such that $ba^k = 0$. Then $l < k \leqslant n \leqslant 2m-1$.

If $a^p = ba^q$ for some p, q implies $a^p = ba^q = 0$, then $S \cong L_{3.7}(m,l,k,n)$. Let p, q be the least positive integers such that $a^p = ba^q \neq 0$. Obviously, $q \geqslant 2$. Then $a^{p+1} = aba^q = a^{q+m}$, so either $a^{p+1} = a^{q+m} = 0$ or $p = q+m-1$. In the former case we have $p = n-1$ and $q = l-1$, so $S \cong L_{3.8}(m,l,k,n)$. In the latter case we have $a^{r+m-1} = ba^r$ for all $r \geqslant q$, then $k = n-m+1$ and $S \cong L_{3.9}(m,l,q,n)$.

Case L3.4. $b^2 = 0$. Every element of the semigroup can be written in the form a^p or ba^p for some p. Let n be the least positive integer such that $a^n = 0$ and k be the least positive integer such that $ba^k = 0$. Then $k \leqslant n \leqslant k+m$.

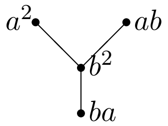
Let p, q be the least positive integers such that $a^p = ba^q$. Then one of the following possibilities holds:

1) $a^p = b^q = 0$;
2) $a^{p+1} = a^{q+m} = 0$;
3) $p = q+m-1$ and $n > p+1$.

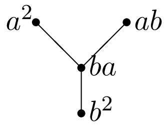
In the first case we have $p = n$ and $q = k$, so $S \cong L_{3.10}(m,n,k)$. In the second case we have $p+1 = n$ and $q+1 = k$, whence $S \cong L_{3.11}(m,n,k)$. In the third case we have $a^{r+m-1} = ba^r$ for all $r \geqslant q$, so $k = n-m+1$ and $S \cong L_{3.12}(m,n,q)$.

Series M. By Lemma 3, all cases lead to a contradiction.

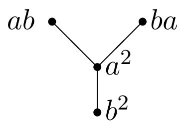
Series N.



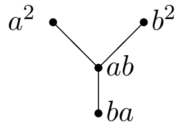
N1



N2



N3



N4

Case N1. We have $b^2 < ab$, so $b^2 = sabt$ for some $s, t \in S^1$. If $s = s'b$ or $t = at'$ for $s', t' \in S^1$, then $b^2 < ba$, a contradiction. If $t = bt'$ for some $t' \in S^1$, then $b^2 < b^2$, so $b^2 = 0 < ba$, a contradiction. Therefore $b^2 = a^m b$ for some $m \geqslant 2$. By the same arguments either $ba = 0$ or $ba = a^l b \neq 0$ for some $l > m$.

Case N1.1. $ba = a^l b \neq 0$ for some $l > m$. Every element of the semigroup can be written in the form a^p or $a^p b$ for some p. Note that $a^{2m} b = a^m b^2 = b^3 = ba^m b \lessdot ba$, so $l < 2m$. Let n be the least positive integer such that $a^n = 0$ and let k be the least positive integer such that $ba^k = 0$. Obviously, $l < k \leqslant n$. Since $a^{m+l} b = a^m ba = b^2 a = ba^l b = a^{l^2} b^2 = a^{l^2+m} b$, we obtain $k \leqslant m+l$.

Let p, q be the least positive integers such that $a^p = a^q b$. Then $a^{q+1} b = a^{p+1} = a^q ba = a^{q+l} b = 0$, so either $p = n$, $q = k$ and $S \cong N_{1.1}(m,l,n,k)$ or $p = n-1$ and $q = k-1$. This means that $S \cong N_{1.2}(m,l,n,k)$.

Case N1.2. $ba = 0$. Every element of the semigroup S can be written either in the form a^p or in the form $a^p b$ for some p. Let n be the least positive integer such that $a^n = 0$ and let k be the least positive integer such that $ba^k = 0$. Trivially, $k \leqslant n$.

Let p, q be the least positive integers such that $a^p = a^q b$. Then $a^{p+1} = a^q ba = 0$, so either $p = n$, $q = k$ and $S \cong N_{1.1}(m, l, n, l)$ or $p = n - 1$ and $q = k - 1$, i.e. $S \cong N_{1.2}(m, l, n, l)$.

Case N2. We have $ba < ab$, so $ba = sabt$ for some $s, t \in S^1$. If $s = s'b$ or $t = at'$ for some $s', t' \in S^1$, then $ba = 0 < b^2$, a contradiction. If $t = bt'$ for some $t' \in S^1$, then $ba < b^2$, a contradiction. Therefore $ba = a^m b$ for some $m \geqslant 2$. By the same arguments either $b^2 = 0$ or $b^2 = a^l b \neq 0$ for some $l > m$.

Case N2.1. $b^2 = a^l b \neq 0$ for some $l > m$. Every element of the semigroup can written in the form a^p or $a^p b$ for some p. Let n be the least positive integer such that $a^n = 0$ and let k be the least positive integer such that $a^k b = 0$, then $l < k \leqslant n$. Since $a^{m+l} b = a^l ba = b^2 a = ba^m b = a^{m^2+l} b \lessdot a^{m+l} b$, we have $k \leqslant m + l$.

Let p, q be the least positive integers such that $a^p = a^q b$. Then $a^{q+1} b = a^{p+1} = a^q ba = a^{q+m} b$, so either $p = n$, $q = k$ and $S \cong N_{2.1}(m, l, n, k)$ or $p = n - 1$ and $q = k - 1$, i.e. $S \cong N_{2.2}(m, l, n, k)$.

Case N2.2. $b^2 = 0$. Every element of the semigroup can be written in the form a^p or $a^p b$ for some p. Let n be the least positive integer such that $a^n = 0$ and let k be the least positive integer such that $a^k b = 0$. Then $k \leqslant n$.

Let p, q be the least positive integers such that $a^p = a^q b$. Then $a^{q+1} b = a^{p+1} = a^q ba = a^{q+m} b = 0$, so either $p = n$, $q = k$ and $S \cong N_{2.1}(m, l, n, l)$ or $p = n - 1$ and $q = k - 1$, i.e. $S \cong N_{2.2}(m, l, n, l)$.

Case N3. We have $a^2 < ab$ and $a^2 < ba$, but $a^2 \nless b^2$. So $a^2 = (ab)^{\frac{p}{2}}$ or $a^2 = (ba)^{\frac{p}{2}}$ for some p. Obviously, every element of the semigroup can be written in the same form.

Case N3.1. $a^2 = (ab)^{\frac{2m+1}{2}}$, $b^2 = (ab)^{\frac{2k+1}{2}} \neq 0$ for some $k > m > 1$. Then $(ab)^{\frac{2k+2}{2}} = (ab)^{\frac{2k+1}{2}} b = b^3 = b(ab)^{\frac{2k+1}{2}} = (ba)^{\frac{2k+2}{2}}$. So, $(ab)^{\frac{2k+3}{2}} = (ab)^{\frac{2k+2}{2}} a = (ba)^{\frac{2k+2}{2}} a = (ba)^{\frac{2k+1}{2}} a^2 = (ba)^{\frac{2k+1}{2}} (ab)^{\frac{2k+1}{2}} \lessdot (ab)^{\frac{2k+3}{2}}$, whence $(ab)^{\frac{2k+3}{2}} = 0$. Analogously, $(ba)^{\frac{2k+3}{2}} = 0$.

If $(ab)^{\frac{2k+1}{2}} \neq (ba)^{\frac{2k+1}{2}}$ and $(ab)^{\frac{2k+1}{2}} \neq 0$, then $S \cong N_{3.1}(m, 2k+1)$. If $(ab)^{\frac{2k+1}{2}} \neq (ba)^{\frac{2k+1}{2}} \neq 0$ and $(ab)^{\frac{2k+1}{2}} = 0$, then $S \cong N_{3.2}(m, 2k+1)$. If $(ab)^{\frac{2k+1}{2}} = (ba)^{\frac{2k+1}{2}}$, then $S \cong N_{3.3}(m, 2k+1)$. If $(ab)^{\frac{2k+1}{2}} \neq 0$ and $(ba)^{\frac{2k+1}{2}} = 0$, then $S \cong N_{3.4}(m, 2k+1)$.

Case N3.2. $a^2 = (ab)^{\frac{2m+1}{2}}$, $b^2 = (ab)^{\frac{2k+2}{2}} \neq 0$ for some $k \geqslant m > 1$. Then $(ba)^{\frac{2k+3}{2}} = b(ab)^{\frac{2k+2}{2}} = b^3 = (ab)^{\frac{2k+2}{2}} b = (ab)^{\frac{2k+1}{2}} b^2 = (ab)^{\frac{2k+1}{2}} (ab)^{\frac{2k+2}{2}} = (ab)^{\frac{2k}{2}} a^2 (ba)^{\frac{2k+1}{2}} = (ab)^{\frac{2k}{2}} (ab)^{\frac{2m+1}{2}} (ba)^{\frac{2k+1}{2}} = (ab)^{\frac{4k+2m+2}{2}} \lessdot (ba)^{\frac{2k+2}{2}}$, so $(ba)^{\frac{2k+3}{2}} = b^3 = 0$. Therefore $b^2 ab \lessdot b^3$ and $b^2 ab = 0$.

If $b^2 a \neq 0$, then $S \cong N_{3.1}(m, 2k+2)$. If $b^2 a = 0$ and $(ba)^{\frac{2k+2}{2}} \neq 0$, then $S \cong N_{3.2}(m, 2k+2)$. If $(ab)^{\frac{2k+2}{2}} = (ba)^{\frac{2k+2}{2}} \neq 0$, then $S \cong N_{3.3}(m, 2k+2)$. If $(ab)^{\frac{2k+2}{2}} \neq 0$ and $(ba)^{\frac{2k+2}{2}} = 0$, then $S \cong N_{3.4}(m, 2k+2)$.

Case N3.3. $a^2 = (ab)^{\frac{2m+1}{2}}$, $b^2 = (ba)^{\frac{2k+1}{2}}$ for some $m, k \geqslant 2$ and $k > m$. Let n and l be the least positive integers such that $(ab)^{\frac{n}{2}} = (ba)^{\frac{l}{2}} = 0$. Clearly, $n, l \geqslant 2k+1$ and $|n - l| \leqslant 1$. Then $S \cong N_{3.5}(m, k, n, l)$.

Case N3.4. $a^2 = (ab)^{\frac{2m+1}{2}}$, $b^2 = 0$. Let n and l be the least positive integers such that $(ab)^{\frac{n}{2}} = 0$ and $(ab)^{\frac{l}{2}} = 0$. Obviously, $|n - l| \leqslant 1$. Then $S \cong N_{3.6}(m, n, l)$.

Case N3.5. $a^2 = (ab)^{\frac{2m}{2}}$ for some $m > 1$. Then $(ab)^{\frac{2m+1}{2}} = a^3 = a(ab)^{\frac{2m}{2}} = a^2 (ba)^{\frac{2m-1}{2}} = (ab)^{\frac{2m}{2}} (ba)^{\frac{2m-1}{2}} = (ab)^{\frac{2m-1}{2}} b^2 (ab)^{\frac{2m-2}{2}}$. It is easy to see that $(ab)^{\frac{2m-1}{2}} b^2 (ab)^{\frac{2m-2}{2}} \lessdot (ab)^{\frac{2m+1}{2}}$, so $(ab)^{\frac{2m+1}{2}} = 0$. Therefore $(ba)^{\frac{2m+2}{2}} = 0$. Then $b^2 = (ba)^{\frac{2m+1}{2}}$ or $b^2 = 0$.

If $b^2 = (ba)^{\frac{2m+1}{2}} \neq 0$, then $S \cong N_{3.7}(2m)$. If $(ba)^{\frac{2m+1}{2}} \neq 0$ and $b^2 = 0$, then $S \cong N_{3.8}(2m)$. If $b^2 = (ba)^{\frac{2m+1}{2}} = 0$, then $S \cong N_{3.9}(2m)$.

Case N3.6. $a^2 = (ba)^{\frac{2m+1}{2}}$. Then $(ab)^{\frac{2m+2}{2}} = a^3 = (ba)^{\frac{2m+2}{2}}$, so $(ab)^{\frac{2m+3}{2}} = (ba)^{\frac{2m+3}{2}} = 0$. Therefore $b^2 = (ab)^{\frac{2m+2}{2}}$ or $b^2 = 0$.

If $b^2 = (ab)^{\frac{2m+2}{2}} \neq 0$, then $S \cong N_{3.7}(2m+1)$. If $(ab)^{\frac{2m+2}{2}} \neq 0$ and $b^2 = 0$, then $S \cong N_{3.8}(2m+1)$. If $b^2 = (ab)^{\frac{2m+2}{2}} = 0$, then $S \cong N_{3.9}(2m+1)$.

Case N4. We have $ab = a^n = b^m$ for some $n, m \geqslant 3$. Then $bab \lessdot ba$, $a^2b = a^{n+1} = aba \lessdot ba$, $ab^2 = b^{m+1} = bab \lessdot ba$, but $ba < ab$. So, $ba = 0$ and ab is an atom. If $a^p = b^q$ for $1 < p < n$ and $1 < q < m$, then $a^{p+1} = 0$, which means $ab = a^n = 0 = ba$, a contradiction. Then $S \cong N_4(n, m)$.

Series O, P, Q leads to a contradiction by Lemma 3 or by arguments from series **F**.

Theorem 1 is now proved. □

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ASYMPTOTIC EXPANSION OF A SOLUTION FOR ONE SINGULARLY PERTURBED OPTIMAL CONTROL PROBLEM IN $\mathbb{R}^n$ WITH A CONVEX INTEGRAL QUALITY INDEX

Alexander A. Shaburov

Institute of Natural Sciences and Mathematics, Ural Federal University,
Ekaterinburg, Russia,
alexandershaburov@mail.ru

Abstract: The paper deals with the problem of optimal control with a convex integral quality index for a linear steady-state control system in the class of piecewise continuous controls with a smooth control constraints. In a general case, for solving such a problem, the Pontryagin maximum principle is applied as the necessary and sufficient optimum condition. In this work, we deduce an equation to which an initial vector of the conjugate system satisfies. Then, this equation is extended to the optimal control problem with the convex integral quality index for a linear system with a fast and slow variables. It is shown that the solution of the corresponding equation as $\varepsilon \to 0$ tends to the solution of an equation corresponding to the limit problem. The results received are applied to study of the problem which describes the motion of a material point in $\mathbb{R}^n$ for a fixed period of time. The asymptotics of the initial vector of the conjugate system that defines the type of optimal control is built. It is shown that the asymptotics is a power series of expansion.

Keywords: Optimal control, Singularly perturbed problems, Asymptotic expansion, Small parameter.

Introduction

The paper is devoted to studying the asymptotics of the initial vector of a conjugated state and an optimal value of the quality index in the optimal control problem [1]–[3] for a linear system with a fast and slow variables (see review [4]), convex integral quality index [3, Chapter 3], and smooth geometrical constraints for control.

Singularly perturbed problems of optimal control have been considered in different settings in [5]–[7].

The method of boundary function that was developed in [4, 10] allows effectively constructing an asymptotics of solutions for problems with an open control area and smooth controlling actions.

The solving of problems with a closed and bounded control area meets certain difficulties. That is why the problems with fast and slow variables and closed constraints for control have been studied to a less extent. A significant contribution to solving these problems was made by Dontchev and Kokotovic.

Problems of fast operation and terminal control with constraints for control in the form of a polygon are dealt with in [5, 7]. The structure of such optimal control is a relay function with values in the apexes of the polygon. No optimal control with constraints in the form of a sphere, which is a continuous function with a finite and countable number of discontinuity points, has been considered so far.

The asymptotics of solutions of the perturbed control problem was formulated differently in papers [7, 9].

In the present work, the basic equation for searching for the asymptotics of the initial vector of the conjugated state of the problem under consideration and optimal control is obtained. General relationships are applied to the case of the optimal control with a point of a small mass in an n-dimensional space under the action of a bounded force.

1. General statement of problem and condition for optimality

Let us consider a problem that belongs to the class of piecewise continuous controls – optimal control problem for a linear stationary system with a convex integral quality index:

$$\begin{cases} \dot{z} = \mathcal{A}z + \mathcal{B}u, \quad z(0) = z^0, \quad \|u(t)\| \leqslant 1, \quad t \in [0;T], \\ J(u) = \varphi(z(T)) + \int\limits_0^T \|u(t)\|^2 \, dt \to \min, \end{cases} \tag{1.1}$$

where $z \in \mathbb{R}^{\widetilde{n}}$, $u \in \mathbb{R}^r$, $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^r$, $\mathcal{A}$, $\mathcal{B}$ are constant matrices of the corresponding dimensional, and $\varphi(\cdot)$ is the convex function that is continuously differentiable in $\mathbb{R}^{\widetilde{n}}$.

Note that in the considered convex integral quality index J, where the first term can be interpreted as a fine for the control error at a finite time instant T, whereas the second, as an account of an energy spent for the realization of control.

Condition 1. Let us assume that a pair $(\mathcal{A}, \mathcal{B})$ is quite controllable,

$$rank\big(\mathcal{B}, \mathcal{A}\mathcal{B}, \dots, \mathcal{A}^{\widetilde{n}-1}\mathcal{B}\big) = \widetilde{n}.$$

Under the conditions stated, the Pontryagin maximum principle in the problem (1.1) is the necessary and sufficient criterion of optimality. In this case, the problem has the unique solution [3, p. 3.5, Theorem 14]: if z, η is the unique solution to (1.1) and

$$\dot{\eta} = -\mathcal{A}^*\eta, \quad \eta(T) = -\nabla\varphi(z(T)), \tag{1.2}$$

then the optimal control u^o is determined from the maximum principle

$$-\|u^o(t)\|^2 + \langle \mathcal{B}^*\eta(t), u^o(t)\rangle = \max_{\|u\|\leqslant 1} \big(-\|u\|^2 + \langle \mathcal{B}^*\eta(t), u\rangle\big). \tag{1.3}$$

Here $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^r$.

Calculating maximum in (1.3), we find

$$u^o(t) = \frac{\mathcal{B}^*\eta(t)}{S(\|\mathcal{B}^*\eta(t)\|)}, \qquad \text{where} \quad S(\xi) := \begin{cases} 2, & 0 \leqslant \xi \leqslant 2, \\ \xi, & \xi > 2. \end{cases} \tag{1.4}$$

Note that the determination of function $S(\cdot)$ leads to the validity of inequality

$$\forall\, w_1, w_2 \in \mathbb{R}^r \qquad \left\| \frac{w_1}{S(\|w_1\|)} - \frac{w_2}{S(\|w_2\|)} \right\| \leqslant \|w_1 - w_2\|. \tag{1.5}$$

Let $\lambda := \eta(T)$. Then

$$\eta(t) = e^{-\mathcal{A}^*(t-T)}\lambda, \quad z(t) = e^{\mathcal{A}t}z^0 + \int\limits_0^t e^{\mathcal{A}(t-s)}\mathcal{B}u^o(s)ds.$$

At a finite time instant $t = T$ we have

$$z(T) = e^{\mathcal{A}T}z^0 + \int\limits_0^T \frac{e^{\mathcal{A}(T-s)}\mathcal{B}\mathcal{B}^*e^{\mathcal{A}^*(T-s)}\lambda}{S(\|\mathcal{B}^*e^{\mathcal{A}^*(T-s)}\lambda\|)}ds.$$

Replacing the variable $\tau := T - s$, we obtain

$$z(T) = e^{\mathcal{A}T} z^0 + \int\limits_0^T \frac{e^{\mathcal{A}\tau}\mathcal{B}\mathcal{B}^* e^{\mathcal{A}^*\tau}\lambda}{S(\|\mathcal{B}^* e^{\mathcal{A}^*\tau}\lambda\|)} d\tau.$$

Thus, the following is valid:

Statement 1. Let condition 1 be valid, $z(t)$, $u(t)$ be a solution of the system from Problem (1.1), and $\eta(t)$ be a solution of the system (1.2). Then $z(t)$, $\eta(t)$, $u(t)$ is the solution of the maximum principle problem (1.1), (1.2), (1.3) if and only if when $\eta(T) = \lambda$, $u(t)$ is determined by the formula (1.4), and a vector λ is the unique solution of equation

$$-\lambda = \nabla\varphi\left(e^{\mathcal{A}T} z^0 + \int\limits_0^T e^{\mathcal{A}\tau}\mathcal{B} \frac{\mathcal{B}^* e^{\mathcal{A}^*\tau}\lambda}{S(\|\mathcal{B}^* e^{\mathcal{A}^*\tau}\lambda\|)} d\tau \right). \tag{1.6}$$

Besides $u(t)$ is the unique optimal control in the problem (1.1).

The vector λ that satisfies the equation (1.6) will be called as *a vector determining the optimal control* in the problem (1.1).

Statement 2. Let $u^o(t)$ be the optimal control in (1.1). Then $u^o(t)$ is continuous on $[0;T]$ and infinitely differentiable at points $\widetilde{t}$ such that $\|\mathcal{B}^* e^{\mathcal{A}^*(T-\widetilde{t})}\lambda\| \neq 2$. Here λ is a vector determining the optimal control in problem (1.1).

P r o o f. The validity of statement follows from (1.4) and analytical form of the matrix exponent $e^{\mathcal{A}^* t}$. □

2. Optimal control problem with fast and slow variables

Consider a particular case of problem (1.1), when the system under control contains fast and slow variables and the terminal part of the quality index depends only on slow variables:

$$\begin{cases} \dot{x}_\varepsilon = A_{11}x_\varepsilon + A_{12}y_\varepsilon + B_1 u, & t \in [0,T], \quad \|u\| \leqslant 1, \\ \varepsilon\dot{y}_\varepsilon = A_{21}x_\varepsilon + A_{22}y_\varepsilon + B_2 u, & x_\varepsilon(0) = x^0, \quad y_\varepsilon(0) = y^0, \\ J(u) := \sigma(x_\varepsilon(T)) + \int\limits_0^T \|u(t)\|^2\, dt \to \min, & \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^r$; A_{ij}, B_i $(i,j = 1,2)$ are the constant matrices of the corresponding dimensions, and $\sigma(\cdot)$ is the convex function that is continuously differentiable in $\mathbb{R}^n$.

Condition 2. All eigenvalues of matrix A_{22} have negative real parts.

For each fixed $\varepsilon > 0$ the problem (2.1) coincides with the problem (1.1):

$$z_\varepsilon(t) = \begin{pmatrix} x_\varepsilon(t) \\ y_\varepsilon(t) \end{pmatrix}, \quad z_\varepsilon^0 = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}, \quad \mathcal{A}_\varepsilon = \begin{pmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{pmatrix}, \quad \mathcal{B}_\varepsilon = \begin{pmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{pmatrix},$$

$$\widetilde{n} = n + m, \quad \varphi(z_\varepsilon) = \sigma(x_\varepsilon).$$

As a limit problem for (2.1), the following problem is introduced

$$\begin{cases} \dot{x}_0 = A_0 x_0 + B_0 u, & t \in [0,T], \quad \|u\| \leqslant 1, \\ A_0 := A_{11} - A_{12}A_{22}^{-1}A_{21}, & B_0 := B_1 - A_{12}A_{22}^{-1}B_2, \quad x_0(0) = x^0, \\ J(u) := \sigma(x_0(T)) + \int\limits_0^T \|u(t)\|^2\, dt \to \min\,. \end{cases} \tag{2.2}$$

Condition 3. Pairs (A_0, B_0) and (A_{22}, B_2) are quite controllable.

If the Conditions 2-3 are satisfied, then there exists $\varepsilon_0 > 0$ such that the pair $(\mathcal{A}_\varepsilon, B_\varepsilon)$ is quite controllable at any $\varepsilon : 0 < \varepsilon \leqslant \varepsilon_0$ [5, Theorem 1].

Note that since $\nabla\varphi(z_\varepsilon) = \begin{pmatrix} \nabla\sigma(x_\varepsilon) \\ 0 \end{pmatrix}$, then the vector λ_ε, which determines the optimal control in the problem (2.1), has the form $\lambda_\varepsilon = \begin{pmatrix} l_\varepsilon \\ 0 \end{pmatrix}$, $l_\varepsilon \in \mathbb{R}^n$.

The vector l_ε also will be called *as determining the optimal control in problem* (2.1).

Let

$$e^{\mathcal{A}_\varepsilon t} := \begin{pmatrix} \mathcal{W}_\varepsilon^{11}(t) & \mathcal{W}_\varepsilon^{12}(t) \\ \mathcal{W}_\varepsilon^{21}(t) & \mathcal{W}_\varepsilon^{22}(t) \end{pmatrix}, \tag{2.3}$$

then, by virtue of (2.3) the equation (1.6) transforms into

$$-l_\varepsilon = \nabla\sigma\Bigg(\mathcal{W}_\varepsilon^{11}(T)x^0 + \mathcal{W}_\varepsilon^{12}(T)y^0 + \\ \int\limits_0^T \big(\mathcal{W}_\varepsilon^{11}(t)B_1 + \varepsilon^{-1}\mathcal{W}_\varepsilon^{12}(t)B_2\big) \frac{\big(B_1^*(\mathcal{W}_\varepsilon^{11}(t))^* + \varepsilon^{-1}B_2^*(\mathcal{W}_\varepsilon^{12}(t))^*\big)l_\varepsilon}{S\big(\big\|\big(B_1^*(\mathcal{W}_\varepsilon^{11}(t))^* + \varepsilon^{-1}B_2^*(\mathcal{W}_\varepsilon^{12}(t))^*\big)l_\varepsilon\big\|\big)} dt\Bigg). \tag{2.4}$$

Note that the optimal control $u_\varepsilon^o(t)$ in the problem (2.1) is expressed through the vector l_ε as follows:

$$u_\varepsilon^o(T-t) = \frac{\big(B_1^*(\mathcal{W}_\varepsilon^{11}(t))^* + \varepsilon^{-1}B_2^*(\mathcal{W}_\varepsilon^{12}(t))^*\big)l_\varepsilon}{S\big(\big\|\big(B_1^*(\mathcal{W}_\varepsilon^{11}(t))^* + \varepsilon^{-1}B_2^*(\mathcal{W}_\varepsilon^{12}(t))^*\big)l_\varepsilon\big\|\big)}. \tag{2.5}$$

Theorem 1. Let the Conditions 2 and 3 be valid. Then $l_\varepsilon \to l_0$ as $\varepsilon \to +0$, where l_ε is the unique solution of the equation (2.4), and l_0 is the unique solution of the equation

$$-l_0 = \nabla\sigma\left(e^{A_0 T}x^0 + \int\limits_0^T e^{A_0 t}B_0 \frac{B_0^* e^{A_0^* t} l_0}{S\big(\|B_0^* e^{A_0^* t} l_0\|\big)} dt \right). \tag{2.6}$$

P r o o f. It is known that the attainability set for the controllable system under control from (2.1) is uniformly bounded by the time instant T at $\varepsilon \in (0; \varepsilon_0]$ (see, for example, [6, theorem 3.1]). Hence, by virtue of (2.4) vectors $\{l_\varepsilon\}$ are also bounded at $\varepsilon \in (0; \varepsilon_0]$. Therefore, to prove the theorem, it is sufficient to show that all partial limits $\{l_\varepsilon\}$ as $\varepsilon \to +0$ are equal to l_0.

As follows from the A. B. Vasil'eva's results (see, for example [10, Chapter 3]) there is $\gamma > 0$ such that

$$\mathcal{W}_\varepsilon^{11}(t) = e^{A_0 t} + O(\varepsilon), \qquad \mathcal{W}_\varepsilon^{12}(t) = -\varepsilon e^{A_0 t}A_{12}A_{22}^{-1} + O(\varepsilon e^{-\gamma t/\varepsilon}) + O(\varepsilon^2), \\ \mathcal{W}_\varepsilon^{21}(t) = -A_{22}^{-1}A_{21}e^{A_0 t} + O(e^{-\gamma t/\varepsilon}) + O(\varepsilon), \qquad \mathcal{W}_\varepsilon^{22}(t) = O(e^{-\gamma t/\varepsilon}). \tag{2.7}$$

Moreover, asymptotic estimates are uniform in $t \in [0; T]$.

Hence, by virtue of (2.2) which determines the matrices A_0 and B_0 and by formulas (2.7) the expression standing $\nabla\sigma$ for the formula (2.4) has the form

$$e^{A_0T}x^0 + O(\varepsilon) + \int\limits_0^T \big(e^{A_0t}B_0 + O(e^{-\gamma t/\varepsilon}) + O(\varepsilon)\big)\frac{\big(B_0^*e^{A_0^*t} + O(e^{-\gamma t/\varepsilon}) + O(\varepsilon)\big)l_\varepsilon}{S\big(\big\|\big(B_0^*e^{A_0^*t} + O(e^{-\gamma t/\varepsilon}) + O(\varepsilon)\big)l_\varepsilon\big\|\big)}dt. \quad (2.8)$$

Let us divide the integral from (2.8) into two terms $\int_0^T = \int_0^{\sqrt{\varepsilon}} + \int_{\sqrt{\varepsilon}}^T$. Then, taking into account that the expression under integral is uniformly constrained and that $O(e^{-\gamma/\sqrt{\varepsilon}}) = O(\varepsilon^\alpha)$ as $\varepsilon \to 0$ for any $\alpha > 0$, we obtain from (2.4) and (2.8)

$$-l_\varepsilon = \nabla\sigma\left(e^{A_0T}x^0 + O(\varepsilon) + O(\sqrt{\varepsilon}) + \int\limits_{\sqrt{\varepsilon}}^T e^{A_0t}B_0\frac{\big(B_0^*e^{A_0^*t} + O(\varepsilon)\big)l_\varepsilon}{S\big(\big\|\big(B_0^*e^{A_0^*t} + O(\varepsilon)\big)l_\varepsilon\big\|\big)}dt\right). \quad (2.9)$$

Let $\bar{l}$ be a partial limit of the vectors $\{l_\varepsilon\}$ as $\varepsilon \to +0$, i.e. $l_{\varepsilon_k} \to \bar{l}$ for a certain $\{\varepsilon_k\}$ so that $\varepsilon_k \to +0$. Going to the limit as $k \to \infty$ in (2.9) we obtain that $\bar{l}$ is the solution of (2.6). Because of the uniqueness of such a solution we have $\bar{l} = l_0$. □

The main problem for (2.1) is to find the complete asymptotic expansion in powers of small parameter ε of the optimal control, optimal values of the quality index, and the optimal process. Formulas (2.5) and (1.5) show that if one manages to gain the complete asymptotic expansion of vector l_ε, which determines the optimal control in problem (2.1), this vector can be used for the asymptotic expansions of the above values as well.

3. Construction of complete asymptotic expansion of vector l_ε for an optimal control problem with fast and slow variables

Consider a partial case of problem (2.1):

$$\begin{cases} \dot{x}_\varepsilon = y_\varepsilon, \quad t \in [0,T], \quad \|u\| \leqslant 1, \\ \varepsilon\dot{y}_\varepsilon = -y_\varepsilon + u, \quad x_\varepsilon(0) = x^0, \quad y_\varepsilon(0) = y^0, \\ J(u) := \frac{1}{2}\|x_\varepsilon(T)\|^2 + \int\limits_0^T \|u(t)\|^2\,dt \to \min, \end{cases} \quad (3.1)$$

where x_ε, y_ε, $u \in \mathbb{R}^n$.

Problem (3.1) simulates a motion of a material point of small mass $\varepsilon > 0$ with the coefficient of the medium resistance equals to 1 in the space $\mathbb{R}^n$ under action of the constrained control force $u(t)$.

Here $A_{11} = 0$, $A_{12} = I$, $A_{21} = 0$, $A_{22} = -I$, $B_1 = 0$, $B_2 = I$, and 0 and I are the zero and the identity matrices of dimensional $n \times n$, respectively. For the limit problem we have $A_0 = 0$, $B_0 = I$ and thus, Conditions 2 and 3 are valid.

Calculating $e^{\mathcal{A}_\varepsilon t}$ and $\nabla(\frac{1}{2}\|x_\varepsilon(T)\|^2)$, we obtain

$$\mathcal{W}_\varepsilon^{11}(t) = I, \quad \mathcal{W}_\varepsilon^{12}(t) = \varepsilon(1-e^{-t/\varepsilon})I, \quad \mathcal{W}_\varepsilon^{21}(t) = 0, \quad \mathcal{W}_\varepsilon^{22}(t) = e^{-t/\varepsilon}I, \quad \nabla\Big(\frac{1}{2}\|x_\varepsilon(T)\|^2\Big) = x_\varepsilon(T).$$

Therefore, equations (2.4) and (2.6) for l_ε and l_0 take the form

$$-l_\varepsilon = x^0 + \varepsilon(1 - e^{-T/\varepsilon})y^0 + \int\limits_0^T \frac{(1 - e^{-t/\varepsilon})^2 l_\varepsilon}{S\big(\|(1 - e^{-t/\varepsilon})l_\varepsilon\|\big)}dt, \quad -l_0 = x^0 + T\frac{l_0}{S(\|l_0\|)}. \quad (3.2)$$

If the vector-function $f_\varepsilon(t)$ is such that $f_\varepsilon(t) = O(\varepsilon^\alpha)$ as $\varepsilon \to 0$ for any $\alpha > 0$ uniformly with respect to $t \in [0;T]$ then instead of $f_\varepsilon(t)$ we will write $\mathbb{O}$. In particular, $e^{-\gamma T/\varepsilon} = \mathbb{O}$.

From (3.2) we obtain

$$\begin{aligned} &1.\ \|x^0\| < T+2 \Longrightarrow l_0 = -\frac{2}{2+T}\,x^0 \qquad \text{and} \qquad \|l_0\| < 2,\\ &2.\ \|x^0\| > T+2 \Longrightarrow l_0 = -\frac{\|x^0\|-T}{\|x^0\|}\,x^0 \quad \text{and} \qquad \|l_0\| > 2. \end{aligned} \tag{3.3}$$

1. Consider first the case: $\|x^0\| < T+2$.

By virtue of (3.3) and Theorem 1 the inequality $\|l_\varepsilon\| < 2$ is valid for any sufficiently small ε. Taking into account that $(1-e^{-t/\varepsilon}) \leqslant 1$ at any $t \geqslant 0$ and $\varepsilon > 0$, from (3.2) we obtain for l_ε the equation

$$-l_\varepsilon = x^0 + \varepsilon y^0 + \mathbb{O} + \frac{1}{2}\int_0^T (1-e^{-t/\varepsilon})^2\,dt\,l_\varepsilon. \tag{3.4}$$

Calculating the integral $\int_0^T (1-e^{-t/\varepsilon})^2\,dt \ = T - 3/(2\varepsilon) + \mathbb{O}$, from (3.4) we find

$$l_\varepsilon = -\frac{4(x^0+\varepsilon y^0+\mathbb{O})}{4+2T-3\varepsilon}.$$

It follows from this representation that l_ε is expanded in the asymptotic series in powers of ε.

Statement 3. Let $\|x^0\| < T+2$. Then the vector l_ε which determines the optimal control in problem (3.1), is expanded as $\varepsilon \to 0$ in the power asymptotic series

$$l_\varepsilon \overset{as}{=} l_0 + \sum_{k=1}^\infty \varepsilon^k l_k, \text{ where, in particular, } l_1 = -\frac{3l_0+4y^0}{4+2T}.$$

2. Now consider the case: $\|x^0\| > T+2$.

By virtue of (3.3) and Theorem 1, the inequality $\|l_\varepsilon\| < 2$ is valid for all sufficiently small ε. Since for a fixed ε the function $(1-e^{-t/\varepsilon})\|l_\varepsilon\|$ increases monotonically from 0 at $t=0$ into $(1-e^{-t/\varepsilon})\|l_\varepsilon\|$ at $t=T$ (which for sufficiently small ε gives the inequality $(1-e^{-t/\varepsilon})\|l_\varepsilon\| > 2$), there is the unique $t_{1,\varepsilon} \in (0;T)$ such that $(1-e^{-t_{1,\varepsilon}/\varepsilon})\|l_\varepsilon\| = 2$, or

$$(1-e^{-t_{1,\varepsilon}/\varepsilon})\|l_\varepsilon\| = 2, \qquad t_{1,\varepsilon} = -\varepsilon\ln\Big(1-\frac{2}{\|l_\varepsilon\|}\Big). \tag{3.5}$$

Therefore, the equation (3.2) takes the form

$$-l_\varepsilon = x^0 + \varepsilon(1-e^{-T/\varepsilon})y^0 + \frac{1}{2}\int_0^{t_{1,\varepsilon}}\Big(1-e^{-t/\varepsilon}\Big)^2 dt\,l_\varepsilon + \int_{t_{1,\varepsilon}}^T \Big(1-e^{-t/\varepsilon}\Big)\,dt\frac{l_\varepsilon}{\|l_\varepsilon\|}. \tag{3.6}$$

Calculating the integrals in (3.6) and transposing $(-l_\varepsilon)$ into the right part, we obtain

$$\begin{aligned} 0 = F(\varepsilon, l_\varepsilon) :=\, & l_\varepsilon + x^0 + \varepsilon(1-e^{-T/\varepsilon})y^0 - \varepsilon\Big(\frac{1}{\|l_\varepsilon\|} + \frac{1}{\|l_\varepsilon\|^2} + \frac{1}{2}\ln\Big(1-\frac{2}{\|l_\varepsilon\|}\Big)\Big)l_\varepsilon \\ & + \Big(T + \varepsilon\ln\Big(1-\frac{2}{\|l_\varepsilon\|}\Big) + \varepsilon e^{-T/\varepsilon} - \varepsilon + \varepsilon\frac{2}{\|l_\varepsilon\|}\Big)\frac{l_\varepsilon}{\|l_\varepsilon\|}. \end{aligned} \tag{3.7}$$

Theorem 2. Let $\|x^0\| > T + 2$. Then the vector l_ε which determines the optimal control in problem (3.1) is expanded into a power asymptotic series (for $\varepsilon \to 0$)

$$l_\varepsilon \overset{as}{=} l_0 + \sum_{k=1}^{\infty} \varepsilon^k l_k.$$

P r o o f. Consider the equation $0 = F(\varepsilon, l)$, where $F(\cdot,\cdot)$ is defined in (3.7). Additionally predetermine $e^{-T/\varepsilon}$ at the point $\varepsilon = 0$ as zero. Then we obtain that $0 = F(0, l_0)$ and $F(\cdot,\cdot)$ is infinitely differentiable in ε and l in a certain neighborhood of the point $(0; l_0)$. Since

$$\mathcal{F}\rho := \frac{\partial F(\varepsilon, l)}{\partial l}\Big|_{\varepsilon=0, l=l_0} \rho = \rho + \frac{\|l_0\|^2 \rho - \langle l_0, \rho\rangle l_0}{\|l_0\|^3} T,$$

then operator $\mathcal{F}$ is continuously reversible and

$$\mathcal{F}^{-1} g = \left(g + \frac{T\langle l_0, g\rangle l_0}{\|l_0\|^3} \right) \frac{\|l_0\|}{T + \|l_0\|}. \tag{3.8}$$

In this way, the theorem of implicitly specified function is applicable, which means that l_ε (as a function of ε) is infinitely differentiable in ε for all small ε and, therefore, l_ε is expanded into the asymptotic series. The coefficients of this series can be found via the standard procedure: substituting the series into the equation (3.7), expanding values dependent on ε into asymptotic series in power of ε, and equaling terms of the same order of smallness with respect to ε, we obtain an equation of the $\mathcal{F} l_k = g_k$ with the known right parts. Then, by the formula (3.8) we find l_k.

In particular, for l_1 we obtain the equation

$$\mathcal{F} l_1 = g_1 := -x^0 - y^0 + \left(\frac{1}{\|l_0\|} + \frac{1}{\|l_0\|^2} + \frac{1}{2} \ln\left(1 - \frac{2}{\|l_0\|}\right)\right) l_0 - \left(\ln\left(1 - \frac{2}{\|l_0\|}\right) - 1 + \frac{2}{\|l_0\|}\right) \frac{l_0}{\|l_0\|}.$$

Hence, by virtue of (3.8) we obtain

$$l_1 = \left(g_1 + \frac{T\langle l_0, g_1\rangle l_0}{\|l_0\|^3} \right) \frac{\|l_0\|}{T + \|l_0\|}.$$

□

4. Remarks

1. Both in the first and the second cases under consideration, from (3.2), (3.5) and asymptotic expansion of l_ε, the asymptotic expansions of both the quality index and optimal control as well as optimal state of the system are conventionally obtained. With this, the asymptotic expansions of the optimal control and optimal state of the system will be exponentially decreasing boundary layers in the neighborhood of point $t = 0$. Moreover, if $t \geqslant \varepsilon^\beta$ and $\beta \in (0; 1)$, the optimal control $u^o(t)$ is a constant plus the asymptotic zero.

2. It follows from the formula (3.7) that l_ε lies in the subspace Π created by vectors x^0 and y^0. Therefore, for all $t \in [0; T]$ and $u^o_\varepsilon(t)$, $x_\varepsilon(t)$ and $y_\varepsilon(t)$ lie in the same subspace Π. In this way, the problem (3.1) is equivalent to the corresponding two-dimensional problem.

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CALIBRATION RELATIONS FOR ANALOGUES OF THE BASIS SPLINES WITH UNIFORM NODES[1]

Valerii T. Shevaldin

N.N. Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences;
Ural Federal University, Ekaterinburg, Russia,
Valerii.Shevaldin@imm.uran.ru

Abstract: The paper deals with generalized linear and parabolic B–splines with the uniform nodes constructed by means only one function $\varphi(x)$. For such splines in this paper conditions have been found that guarantee satisfaction of two–scale relations.

Key words: B-spline, uniform nodes, two-scale relations.

Introduction

In contemporary mathematics, various generalizations of the polynomial spline–functions regularly appear. Besides the well–known $\mathcal{L}$-splines (see, for example [1]), let us note the source-representative splines [2], the Rvachev functions [3], the Leontiev splines [4], the Kvasov iso–geometric splines [5], the Demyanovich B_φ–splines [6], and so on.

Recently, the Author [7] suggested another generalization of a known construction of the parabolic basis spline (of the B–spline) with the uniform nodes; this spline is constructed by means of only one function $\varphi \in C^1[-h,h]$ $(h>0)$.

In [7], the approximative and form–retaining properties of the local non-interpolating splines were investigated. These ones are linear combinations of shifts of the suggested B–splines. As particular cases, there were considered examples of exponential, elliptic, and hyperbolic local splines with arbitrary collocation of nodes.

It is well known that the polynomial splines have played important role in development of the wavelet theory (see, e.g., [8–10]). Namely, in constructing the wavelet decompositions of the space $L^2(\mathbb{R})$, embeddedness of spaces $\{V_j\}_{j=-\infty}^{\infty}$ on refining meshes is used. This embeddedness follows from presence of scaling (multiple–scaled) relations (see [8, § 4.3])) for the basis functions.

But note that not each basis function $B(x)$ satisfies the general scaling equation of the form

$$B(x)=\sum_{j\in\mathbb{Z}} h_j B(2x-jh) \qquad (x\in\mathbb{R}),$$

and finding such functions $B(x)$ is a complicated problem.

In the present paper, conditions on the function φ are given that guarantee implementation of analogues for the two–scaled relations for the generalized parabolic B–splines from work [7] (all necessary definitions are given below). Moreover, an analogous problem is considered for the generalized linear B–splines and corresponding examples are given.

It is worthy to note that we have obtained these results without application of the harmonic analysis techniques.

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1. Generalized parabolic B–splines

Let $h > 0$ and $C = C[a, b]$ be the space of continuous functions given on the segment $[a, b]$ with usual definition of the norm
$$\|f\|_C = \max_{x\in[a,b]} |f(x)|.$$
Fix the function φ given on the segment $[-2h, 2h]$ and satisfying the following conditions:
$$\varphi' \in C[-2h, 2h], \qquad \varphi(-x) = \varphi(x) \quad (x \in [0, 2h]), \qquad \varphi(0) = \varphi'(0) = 0. \tag{1.1}$$
The B–spline corresponding to this function φ (see [7]) is described by the formula
$$B_{h,2}(x) = m(h) \begin{cases} \varphi(x), & x \in [0, h], \\ 2\varphi(h) - \varphi(x-h) - \varphi(x-2h), & x \in [h, 2h], \\ \varphi(3h - x), & x \in [2h, 3h], \\ 0, & x \notin [0, 3h]. \end{cases} \tag{1.2}$$
Here, $m = m(h) > 0$ is the normalizing multiplier.

In the classic case, the normalized parabolic B–spline with the uniform nodes $0,\ h,\ 2h\ \ 3h$ (see [11]) is obtained from this definition if to set $\varphi(x) = x^2$ and $m(h) = 1/(2h^2)$.

Note evident properties of the function $B_{h,2}(x)$ that follow from conditions (1.1):
$$\operatorname{supp} B_{h,2}(x) = [0, 3h], \qquad B'_{h,2} \in C(\mathbb{R}), \qquad B_{h,2}(3h - x) = B_{h,2}(x),$$
(*i.e.*, the function $B_{h,2}$ is even w.r.t. the middle carrier point $x = (3h)/2$). If to suppose another condition to be satisfied that the function $\varphi(x)$ does not decrease on $[0, h]$, then the graph of $B_{h,2}(x)$ will have the form of a symmetric "cap" (w.r.t. the point $x = (3h)/2$) as a the parabolic B–spline with the uniform nodes.

In [7], for such functions φ, the local splines of the following form were investigated
$$S(x) = S(f, x) = \sum_{j\in\mathbb{Z}} y_j B_{h,2}\Big(x + \frac{3h}{2} - jh\Big),$$
where $y_j = f(jh),\ f :\ \mathbb{R} \to \mathbb{R}$. It has been proved that these splines locally satisfy to the property of retaining the original data y_j (of the 1–monotonicity type) in the following sense: if $y_{l-1} \le y_l \le y_{l+1}$ $(l \in \mathbb{Z})$, then the spline $S(x)$ does not decrease on the segment $[(l-1/2)h,\ (l+1/2)h]$ $(l \in \mathbb{Z})$.

Together with the function $B_{h,2}(x)$, consider the function
$$B_{2h,2}(x) = m(2h) \begin{cases} \varphi(x), & x \in [0, 2h], \\ 2\varphi(2h) - \varphi(x-2h) - \varphi(x-4h), & x \in [2h, 4h], \\ \varphi(6h - x), & x \in [4h, 6h], \\ 0, & x \notin [0, 6h] \end{cases}$$
that is obtained from he function $B_{h,2}$ by formal substitution of the parameter h by the $2h$ one. It is evident that, in a general case, the graph of this function can not be obtained from the graph of the function $B_{h,2}$ by two-times extension along the horizontal axis as it happened un the classic polynomial case (see [8, § 4.3]). This is since nowhere the demand of homogeneity property of the function φ is imposed. But this is the key reasoning in the described constructing. So, in the subsequent investigation of the wavelets on the basis of these basis functions, the embeddedness

of corresponding subspaces $\{V_j\}_{j=-\infty}^{\infty}$ on the refining meshes must be understood in some other sense.

In this paper, we are searching for an answer on the following question. For what functions φ satisfying conditions (1.1), there exist real numbers A_1, A_2, A_3, and A_4 such that for any $x \in \mathbb{R}$ the equality holds

$$B_{2h}(x) = A_1 B_{h,2}(x) + A_2 B_{h,2}(x-h) + A_3 B_{h,2}(x-2h) + A_4 B_{h,2}(x-3h)? \tag{1.3}$$

We call this equation the *scaling* (two–scaled) relation for the generalized B–spline that is determined by formula (1.2). In the subsequent formulas, the expression o/o is supposed to be equal to 1.

Theorem 1. *Let the function* φ *satisfy conditions* (1.1). *Then equality* (1.3) *holds iff there exists such a number* $\lambda \in \mathbb{R}$, *for which the following equalities hold*:

$$\lambda = \frac{\varphi(t+h) - 2\varphi(h) + \varphi(t-h) + \varphi(t)}{\varphi(t)} = \frac{\varphi(t-2h) - 2\varphi(h) + \varphi(t-h) + \varphi(t)}{\varphi(t-h)} =$$

$$= \frac{2\varphi(2h) - \varphi(t-2h) - \varphi(t) - \varphi(t-h)}{2\varphi(h) - \varphi(t-h)} = \frac{2\varphi(2h) - \varphi(t-h) - \varphi(t+h) - \varphi(t)}{2\varphi(h) - \varphi(t)}, \tag{1.4}$$

$$0 \le t \le h.$$

P r o o f. By virtue of symmetry of the generalized B–spline w.r.t. the middle of the segment carrier, it is possible to think that $A_1 = A_4$ and $A_2 = A_3$.

Consider equality (1.3) as an equation w.r.t. the coefficients A_1, A_2, A_3 and A_4 on each segment $[0,h]$, $[h,2h], \ldots, [5h,6h]$. We obtain that

$$A_1 = A_4 = \frac{m(2h)}{m(h)},$$

$$A_2 = A_3 = \frac{m(2h)}{m(h)} \frac{[\varphi(t+h) - 2\varphi(h) + \varphi(t-h) + \varphi(t)]}{\varphi(t)} =$$

$$= \frac{m(2h)}{m(h)} \frac{[\varphi(2h-t) - 2\varphi(h) + \varphi(t-h) + \varphi(t)]}{\varphi(t-h)} =$$

$$= \frac{m(2h)}{m(h)} \frac{[2\varphi(2h) - \varphi(t-2h) - \varphi(t) - \varphi(t-h)]}{2\varphi(h) - \varphi(t-h)} =$$

$$= \frac{m(2h)}{m(h)} \frac{[2\varphi(2h) - \varphi(t-h) - \varphi(t+h) - \varphi(t)]}{2\varphi(h) - \varphi(t)}.$$

□

Examples. Give examples of three functions φ that satisfy equalities (1.4). In the sequel for simplicity, we put $m(2h) = m(h)$.

Example 1. Let $\varphi(x) = x^2$ (the parabolic splines). Then $A_1 = A_4 = 1,\ A_2 = A_3 = 3$ are the binomial coefficients from [8, formula 4.3.4].

Example 2. Let $\varphi(x) = \cosh(\beta x) - 1$ ($\beta > 0$) (the exponential splines corresponding to the linear differential operator of the third order with the form $\mathcal{L}_3 = \mathcal{L}_3(D) = D(D^2 - \beta^2)$, where D is the differentiation symbol). Then $A_1 = A_4 = 1,\ A_2 = A_3 = 1 + 2\cosh\beta h$.

Example 3. Let $\varphi(x) = 1 - \cos\alpha x$ $(\alpha > 0)$, *i.e.*, be the trigonometric splines corresponding to the linear differential operator of the third order with the form $\mathcal{L}_3 = \mathcal{L}_3(D) = D(D^2 + \alpha^2)$. Then $A_1 = A_4 = 1,\ A_2 = A_3 = 1 + 2\cos\alpha h$.

In connection with the latter two examples, note the Author's work [12]. There the scaling relations are constructed for the B-$\mathcal{L}$–splines (of an arbitrary order) in more generalized form than in (1.3).

2. Generalized linear B–splines

The scheme suggested for obtaining the two–scaled relations can be expanded onto the generalized linear B–splines.

Let the function φ be given on the segment $[0, 2h]$ and satisfy the following conditions:

$$\varphi \in C[0, 2h], \qquad \varphi(0) = 0. \tag{2.1}$$

The generalized linear B–spline is described by the formula

$$B_{h,1}(x) = m(h) \begin{cases} \varphi(x), & x \in [0, h], \\ \varphi(2h - x), & x \in [h, 2h], \\ 0, & x \notin [0, 2h]. \end{cases} \tag{2.2}$$

Here, $m(h) > 0$ is the normalizing multiplier. If to put $\varphi(x) = x$ and $m(h) = 1/h$, then formula (2.2) defines the normalized linear B–spline (see, for example [11]).

It is evident that $\operatorname{supp} B_{h,1} = [0, 2h],\ B_{h,1} \in C(\mathbb{R}),\ B_{h,1}(2h - x) = B_{h,1}(x)\ \ (x \in [0, h])$. Also, consider the function

$$B_{2h,1}(x) = m(2h) \begin{cases} \varphi(x), & x \in [0, 2h], \\ \varphi(4h - x), & x \in [2h, 4h], \\ 0, & x \notin [0, 4h] \end{cases}$$

that was obtained by formal substitution of the parameter h by $2h$ one in the function $B_{h,1}$. We are interested in the question: for what φ the equality holds

$$B_{2h,1}(x) = C_1 B_{h,1}(x) + C_2 B_{h,1}(x - h) + C_3 B_{h,1}(x - 2h) \qquad (x \in \mathbb{R}), \tag{2.3}$$

where $C_1,\ C_2$, and C_3 are some real numbers?

Theorem 2. *Let the function φ satisfy conditions* (2.1). *Then equality* (2.3) *holds iff there exists such a number $\lambda \in \mathbb{R}$ for which the following equalities hold:*

$$\lambda = \frac{\varphi(t + h) - \varphi(h - t)}{\varphi(t)} = \frac{\varphi(2h - t) - \varphi(t)}{\varphi(h - t)} \qquad (0 \le t \le h). \tag{2.4}$$

P r o o f. It is similar to one of Theorem 1. Under this,

$$C_1 = C_3 = \frac{m(2h)}{m(h)},$$

$$C_2 = \frac{m(2h)}{m(h)} \frac{[\varphi(t + h) - \varphi(h - t)]}{\varphi(t)} = \frac{m(2h)}{m(h)} \frac{[\varphi(2h - t) - \varphi(t)]}{\varphi(h - t)}.$$

$\square$

Examples. As in the previous paragraph, it is possible to give examples of three functions φ satisfying equalities (2.4). Again for simplicity, we put $m(2h) = m(h)$.

Example 4. Let $\varphi(x) = x$ (the linear splines). Then $C_1 = C_3 = 1,\ C_2 = 2$ are the binomial coefficients from [8, formula 4.3.4].

Example 5. Let $\varphi(x) = \sinh \beta x$ $(\beta > 0)$, *i.e.*, be the exponential splines of the second order corresponding to the linear differential operator of the form $\mathcal{L}_2 = \mathcal{L}_2(D) = D^2 - \beta^2$. Then $C_1 = C_3 = 1,\ C_2 = 2\cosh \beta h$.

Example 6. Let $\varphi(x) = \sinh \alpha x$ $(\alpha > 0)$, *i.e.*, be the trigonometric splines of the second order corresponding to the linear differential operator of the form $\mathcal{L}_2 = \mathcal{L}_2(D) = D^2 + \alpha^2$. Then $C_1 = C_3 = 1,\ C_2 = 2\cos \alpha h$.

3. Conclusion

It would be interesting to construct examples of other functions φ satisfying relations (1.4) or (2.4). But it is not clear, how using only one function φ, it is possible to construct analogues of the polynomial B–splines of more high degrees, *i.e.*, to derive formules of the type (1.2).

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APPROXIMATION BY LOCAL PARABOLIC SPLINES CONSTRUCTED ON THE BASIS OF INTERPOLATION IN THE MEAN[1]

Elena V. Strelkova

N.N. Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences;
Ural Federal University, Ekaterinburg, Russia,
shevaldina@r66.ru

Abstract: The paper deals with approximative and form–retaining properties of the local parabolic splines of the form $S(x)=\sum\limits_j y_jB_2(x-jh),\ (h>0)$, where B_2 is a normalized parabolic spline with the uniform nodes and functionals $y_j=y_j(f)$ are given for an arbitrary function f defined on $\mathbb{R}$ by means of the equalities

$$y_j=\frac{1}{h_1}\int\limits_{\frac{-h_1}{2}}^{\frac{h_1}{2}} f(jh+t)dt \quad (j\in\mathbb{Z}).$$

On the class W^2_∞ of functions under $0<h_1\le 2h$, the approximation error value is calculated exactly for the case of approximation by such splines in the uniform metrics.

Key words: Local parabolic splines, Approximation, Mean.

Introduction

In the function approximation theory, the local polynomial splines of the order r and of minimal defect are usually constructed as linear combinations of the corresponding B-splines $B_{r,j}(x)$. For a function f from the class of continuous ones, the local polynomial spline $S(x)=S(f,x)$ is defined as follows:

$$S(x)=\sum_j b_j(f)B_{r,j}(x), \tag{0.1}$$

where $b_j(f)$ is the sequence of linear continuous functionals, whose choice determines the form of the approximation.

As the functionals $b_j(f)$, one chooses the linear combinations of the function values and its derivatives at the mesh nodes or its divided differences.

The most simple and convenient (in computation) version of this choice is $b_j(f)=f(x_j)$ (here, x_j are the nodes of the spline S mesh). It leads to the well known local spline (see, for example, [1–4]):

$$S(x)=\sum_j f(x_j)B_{r,j}(x). \tag{0.2}$$

In formula (0.2) instead of x_j, the arithmetic mean is often used that is calculated over all nodes of the B–spline $B_{r,j}(x)$ ([1–3]).

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A spline (constructed in such a way) is not an interpolation one. But in the case $r = 2$ (*i.e.*, of the parabolic splines), it is a continuously differentiable function on the whole number axis $\mathbb{R}$ and possesses both the form–retaining and extremal features [5].

But if the function f is not continuous but only integrable, it is not natural to consider aspects of this function approximation taking into account only its values at the mesh nodes. It is so since its values at separate points are not essential for functions of such a type. In such case, one uses the interpolation on the average. Aspects of existence, uniqueness, and approximative and extremal properties of such splines were investigated in works [6–8]. (Generalizations onto the L-splines see, also, on [9, 10]).

Let two real numbers $h > 0$ and $h_1 > 0$ be given. For a function integrable on the whole number axis $f : \mathbb{R} \to \mathbb{R}$ assume

$$b_j(f) = y_j = \begin{cases} \dfrac{1}{h_1} \displaystyle\int\limits_{-\frac{h_1}{2}}^{\frac{h_1}{2}} f(jh + t)dt, & h_1 > 0, \\ f(jh), & h_1 = 0. \end{cases} \tag{0.3}$$

Let $B_{2,0}(x)$ be a normalized parabolic B–spline (see, for example, [1]) with nodes $-\frac{3h}{2}, -\frac{h}{2}, \frac{h}{2}, \frac{3h}{2}$, and $B_{2,j}(x) = B_{2,0}(x - jh)$.

Let, also, $W^2_\infty = W^2_\infty(\mathbb{X}) = \{f : \ f' \in AC, \ \|f''\|_\infty \le 1\}$ be a class of functions given on the set $\mathbb{X}$ ($\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = [a, b]$).

Here, AC is the class of locally absolute continuous functions $\| \cdot \|_\infty = \| \cdot \|_{L_\infty(\mathbb{X})}$, $L_\infty(\mathbb{X})$ is the class of the functions essentially restricted on $\mathbb{X}$ with the usual definition of the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{X}} |f(x)|.$$

In the present work, we investigate in details aspects of approximation of smooth functions f by the local parabolic splines of the form (0.1) (*i.e.*, for $r = 2$) with the choice of the functionals $b_j(f)$ in the form (0.3).

1. Properties of the spline S

On the axis $\mathbb{R}$ (on both its sides), consider the following mesh of nodes: $\cdots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \cdots$; and let $x_j = jh, \ h = x_{j+1} - x_j, \ x_{j+1/2} = x_j + \frac{h}{2}$ $(j \in \mathbb{Z})$.

For $x_j \le x \le x_{j+1/2}$ from formulas (0.1) and (0.3), it follows

$$\begin{aligned} S(x) = y_{j-1} \cdot \frac{(x - x_{j+1/2})^2}{2h^2} + y_j \cdot \left(\frac{x_{j+1} - x}{h} - \frac{(x - x_{j+1/2})^2}{h^2} \right) \\ + y_{j+1} \cdot \left(\frac{(x - x_{j+1/2})^2}{2h^2} + \frac{x - x_j}{h} \right), \end{aligned} \tag{1.1}$$

and for $x_{j+1/2} \le x \le x_{j+1}$

$$\begin{aligned} S(x) = y_j \cdot \left(\frac{(x - x_{j+1/2})^2}{2h^2} + \frac{x_{j+1} - x}{h} \right) + y_{j+1} \cdot \left(\frac{x - x_j}{h} - \frac{(x - x_{j+1/2})^2}{h^2} \right) \\ + y_{j+2} \cdot \frac{(x - x_{j+1/2})^2}{2h^2}. \end{aligned} \tag{1.2}$$

To the function $f(x) \in W^2_\infty(\mathbb{R})$, we put in correspondence the parabolic spline $S(x) = S(f, x)$ (see (1.1)–(1.2)), where the functionals y_j are defined by formula (0.3).

For $h_1 = 0$, the form-retaining and approximation properties of such splines were investigated in [5]; so, we shall consider the case $h_1 > 0$.

Denote by A_j the interval $\Big(x_j - \dfrac{h_1}{2}; x_j + \dfrac{h_1}{2}\Big)$, $j \in \mathbb{Z}$.

Theorem 1. *The local spline $S(x)$ defined by formulas* (1.1) − (1.2), *possesses the following properties*:
1) *locally inherits the sign of the original function f in the sense that*
  a) if $f(x) \ge 0$ (≤ 0) for $x \in A_{j-1} \bigcup A_j \bigcup A_{j+1}$, then $S(x) \ge 0$ (≤ 0) for $x_j \le x \le x_{j+1/2}$ ($j \in \mathbb{Z}$);
  b) if $f(x) \ge 0$ (≤ 0) for $x \in A_j \bigcup A_{j+1} \bigcup A_{j+2}$, then $S(x) \ge 0$ (≤ 0) for $x_{j+1/2} \le x \le x_{j+1}$ ($j \in \mathbb{Z}$);
2) *locally inherits the monotonicity property of the original function f, namely,*
  a) if the function $f(x)$ does not decrease (does not increase) in the interval $\Big(x_{j-1} - \dfrac{h_1}{2}; x_{j+1} + \dfrac{h_1}{2}\Big)$, then the spline $S(x)$ does not decrease (does not increase) in the interval $(x_j; x_{j+1/2})$ ($j \in \mathbb{Z}$);
  b) if the function $f(x)$ does not decrease (does not increase) in the interval $\Big(x_j - \dfrac{h_1}{2}; x_{j+2} + \dfrac{h_1}{2}\Big)$, then the spline $S(x)$ does not decrease (does not increase) in the interval $(x_{j+1/2}; x_{j+1})$ ($j \in \mathbb{Z}$).

P r o o f. For **the point 1)**, the proof follows directly from non-negativity of the B–spline $B_{2,0}(x)$, formula (0.2), and non-negativity of y_j for $x \in A_j$.

Point 2a). From condition of point 2) of Theorem 1 and definition of y_j it follows $y_{j+1} \ge y_j \ge y_{j-1}$. By differentiation of the right-hand side of equations (1.1), we obtain that for $x \in \big(x_j; x_{j+1/2}\big)$ the derivative

$$S'(x) = \frac{y_{j+1} - y_j}{h} + \frac{x - x_{j+1/2}}{h^2} \cdot (y_{j+1} - 2y_j + y_{j+1})$$

of the spline S in this interval is a linear function in the variable x. So, to prove point 2), it is sufficient to verify that for $y_{j+1} \ge y_j \ge y_{j-1}$, inequalities $S'(x_j + 0) \ge 0$, $S'(x_{j+1/2} - 0) \ge 0$ hold. Validity of these inequalities follows from the formulas

$$S'(x_j + 0) = \frac{y_{j+1} - y_{j-1}}{2h},$$
$$S'(x_{j+1/2} - 0) = \frac{y_{j+1} - y_j}{h};$$

and point 2a) of Theorem 1 is proved.

Point 2b). From conditions of point 2b) of Theorem 1 and definition of y_j, it follows $y_{j+2} \ge y_{j+1} \ge y_j$. By differentiation of the right-hand side of equality (1.2), we obtain that for $x \in \big(x_{j+1/2}; x_j\big)$, the derivative

$$S'(x) = \frac{y_{j+1} - y_j}{h} + \frac{x - x_{j+1/2}}{h^2} \cdot (y_{j+2} - 2y_{j+1} + y_j)$$

of the spline S in this interval is a linear function in the variable x. So, to prove point b), it is sufficient to verify that for $y_{j+2} \ge y_{j+1} \ge y_j$, the equalities $S'(x_{j+1/2} + 0) \ge 0$, $S'(x_{j+1} - 0) \ge 0$ hold. But this follows from the formulae

$$S'(x_{j+1/2} + 0) = \frac{y_{j+1} - y_j}{h},$$
$$S'(x_{j+1} - 0) = \frac{y_{j+2} - y_j}{2h}.$$

□

Before formulation of further statements, obtain the integral representation for the difference $S(x) - f(x)$ in the interval $[x_j; x_{j+1}]$ for $0 < h_1 \leq 2h$.

Let for the beginning $x \in [x_j; x_{j+1/2}]$. Then, by the Taylor formula for the function $f(x) \in W^2_\infty(\mathbb{R})$ we have

$$f(x) = f(x_j) + f'(x_j)(x - x_j) + \int\limits_{x_j}^{x} (x - t) f''(t)dt. \tag{1.3}$$

Using (1.3) and definition y_j (see (0.3)), we derive

$$y_{j-1} = f(x_j) - f'(x_j)h + \frac{1}{h_1} \int\limits_{-\frac{h_1}{2}}^{\frac{h_1}{2}} ds \int\limits_{x_j}^{s+x_{j-1}} (x_{j-1} + s - t) f''(t)dt,$$

$$y_j = f(x_j) + \frac{1}{h_1} \int\limits_{-\frac{h_1}{2}}^{\frac{h_1}{2}} ds \int\limits_{x_j}^{s+x_j} (x_j + s - t) f''(t)dt,$$

$$y_{j+1} = f(x_j) + f'(x_j)h + \frac{1}{h_1} \int\limits_{-\frac{h_1}{2}}^{\frac{h_1}{2}} ds \int\limits_{x_j}^{s+x_{j+1}} (x_{j+1} + s - t) f''(t)dt.$$

Therefore, from (1.1), we obtain

$$\begin{aligned} S(x) = \frac{(x - x_{j+1/2})^2}{2h^2} \left[f(x_j) - f'(x_j)h + \frac{1}{h_1} \int\limits_{-\frac{h_1}{2}}^{\frac{h_1}{2}} ds \int\limits_{x_j}^{s+x_{j-1}} (x_{j-1} + s - t) f''(t)dt \right] \\ + \left(\frac{x_{j+1} - x}{h} - \frac{(x - x_{j+1/2})^2}{h^2} \right) \left[f(x_j) + \frac{1}{h_1} \int\limits_{-\frac{h_1}{2}}^{\frac{h_1}{2}} ds \int\limits_{x_j}^{s+x_j} (x_j + s - t) f''(t)dt \right] \\ + \left(\frac{(x - x_{j+1/2})^2}{2h^2} + \frac{x - x_j}{h} \right) \left[f(x_j) + f'(x_j)h + \frac{1}{h_1} \int\limits_{-\frac{h_1}{2}}^{\frac{h_1}{2}} ds \int\limits_{x_j}^{s+x_{j+1}} (x_{j+1} + s - t) f''(t)dt \right]. \end{aligned} \tag{1.4}$$

Taking into account that $0 < h_1 \leq 2h$, change the integration order in the integrals entering into representation (1.4). Hence, we obtain

$$
\begin{aligned}
S(x) = {} & \frac{(x-x_{j+1/2})^2}{2h^2}\Big\{ f(x_j) - f'(x_j)h + \frac{1}{h_1}\Big[\int\limits_{x_{j-1}-\frac{h_1}{2}}^{x_{j-1}+\frac{h_1}{2}} \frac{1}{2} f''(t)\Big(t - x_{j-1} + \frac{h_1}{2}\Big)^2 dt \\
& + \int\limits_{x_{j-1}+\frac{h_1}{2}}^{x_j} f''(t) h_1 (t - x_{j-1}) dt \Big]\Big\} + \Big(\frac{x_{j+1}-x}{h} - \frac{(x-x_{j+1/2})^2}{h^2}\Big) \\
& \times \Big\{ f(x_j) + \frac{1}{h_1}\Big[\int\limits_{x_j-\frac{h_1}{2}}^{x_j} \frac{1}{2} f''(t)\Big(t - x_j + \frac{h_1}{2}\Big)^2 dt + \int\limits_{x_j}^{x_j+\frac{h_1}{2}} \frac{1}{2} f''(t)\Big(t - x_j - \frac{h_1}{2}\Big)^2 dt \Big]\Big\} \\
& + \Big(\frac{(x-x_{j+1/2})^2}{2h^2} + \frac{x-x_j}{h}\Big)\Big\{ f(x_j) + f'(x_j)h + \frac{1}{h_1}\Big[\int\limits_{x_j}^{x_{j+1}-\frac{h_1}{2}} f''(t) h_1 (x_{j+1} - t) dt \\
& + \int\limits_{x_{j+1}-\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}} \frac{1}{2} f''(t)\Big(t - x_{j+1} - \frac{h_1}{2}\Big)^2 dt \Big]\Big\}, \quad x \in [x_j; x_{j+1/2}].
\end{aligned} \tag{1.5}
$$

By virtue of symmetry of formulas for $S(x)$ w.r.t. the middle $x_{j+1/2}$ of the interval $[x_j; x_{j+1}]$, we obtain the following similar representation of $S(x)$ for $x \in [x_{j+1/2}, x_{j+1}]$:

$$
\begin{aligned}
S(x) = {} & \Big(\frac{(x-x_{j+1/2})^2}{2h^2} + \frac{x_{j+1}-x}{h}\Big)\Big\{ f(x_j) + \frac{1}{h_1}\Big[\int\limits_{x_j-\frac{h_1}{2}}^{x_j} \frac{1}{2} f''(t)\Big(t - x_j + \frac{h_1}{2}\Big)^2 dt \\
& + \int\limits_{x_j}^{x_j+\frac{h_1}{2}} \frac{1}{2} f''(t)\Big(t - x_j - \frac{h_1}{2}\Big)^2 dt \Big]\Big\} + \Big(\frac{x-x_j}{h} - \frac{(x-x_{j+1/2})^2}{h^2}\Big)\Big\{ f(x_j) + f'(x_j)h \\
& + \frac{1}{h_1}\Big[\int\limits_{x_j}^{x_{j+1}-\frac{h_1}{2}} f''(t) h_1 (x_{j+1} - t) dt + \int\limits_{x_{j+1}-\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}} \frac{1}{2} f''(t)\Big(t - x_{j+1} - \frac{h_1}{2}\Big)^2 dt \Big]\Big\} \\
& + \frac{(x-x_{j+1/2})^2}{2h^2}\Big\{ f(x_j) + f'(x_j)2h + \frac{1}{h_1}\Big[\int\limits_{x_j}^{x_{j+2}-\frac{h_1}{2}} f''(t) h_1 (x_{j+2} - t) dt \\
& + \int\limits_{x_{j+2}-\frac{h_1}{2}}^{x_{j+2}+\frac{h_1}{2}} \frac{1}{2} f''(t)\Big(t - x_{j+2} - \frac{h_1}{2}\Big)^2 dt \Big]\Big\}, \quad x \in [x_{j+1/2}, x_{j+1}].
\end{aligned}
$$

$\square$

Theorem 2. *A local spline $S(x)$ defined by formula (1.1) – (1.2), for $0 < h_1 \le 2h$, possesses the following properties:*

1) *inherits locally the convexity property of the original function* f, *namely,*

a) *if the function* $f(x)$ *is down- (upper) convex in the interval* $\Big(x_{j-1}-\dfrac{h_1}{2};x_{j+1}+\dfrac{h_1}{2}\Big)$, *then the spline* $S(x)$ *is the down- (upper-) convex function in the interval* $(x_j;x_{j+1/2})$ $(j\in\mathbb{Z})$;

b) *if the function* $f(x)$ *is down- (upper-) convex in the interval* $\Big(x_j-\dfrac{h_1}{2};x_{j+2}+\dfrac{h_1}{2}\Big)$, *then the spline* $S(x)$ *is the down- (upper-) convex function in the interval* $(x_{j+1/2};x_{j+1})$ $(j\in\mathbb{Z})$.

2) a) *for any function* $f\in W^2_\infty\Big[x_{j-1}-\dfrac{h_1}{2};x_{j+1}+\dfrac{h_1}{2}\Big]$, *the exact inequality holds*

$$|\ S''(x)\ |\le 1,\quad x\in\big(x_j;x_{j+1/2}\big)\,,$$

and, moreover, for all $x\in\big(x_j;x_{j+1/2}\big)$, *the inequality sign is provided by the function* $f(x)=\dfrac{x^2}{2}$;

b) *for any function* $f\in W^2_\infty\Big[x_j-\dfrac{h_1}{2};x_{j+2}+\dfrac{h_1}{2}\Big]$, *the exact inequality holds*

$$|\ S''(x)\ |\le 1,\quad x\in\big(x_{j+1/2};x_{j+1}\big)\,,$$

and, moreover, for all $x\in\big(x_{j+1/2};x_{j+1}\big)$, *the inequality sign is provided by the function* $f(x)=\dfrac{x^2}{2}$.

P r o o f. To prove 1a), it is necessary to verify that if

$$f''(x)\ge 0\ (\le 0)\quad\text{for}\quad x\in\Big(x_{j-1}-\frac{h_1}{2};x_{j+1}+\frac{h_1}{2}\Big),$$

then $S''(x)\ge 0\ (\le 0)$ for $x\in\big(x_j;x_{j+1/2}\big)$.

By the twice differentiation of the function $S(x)$, we obtain from formula (1.5)

$$\begin{aligned}S''(x)=&\int\limits_{x_{j-1}-\frac{h_1}{2}}^{x_{j-1}+\frac{h_1}{2}}f''(t)C_1(t)dt+\int\limits_{x_{j-1}+\frac{h_1}{2}}^{x_j}f''(t)C_2(t)dt+\int\limits_{x_j-\frac{h_1}{2}}^{x_j}f''(t)C_3(t)dt\\&+\int\limits_{x_j}^{x_j+\frac{h_1}{2}}f''(t)C_4(t)dt+\int\limits_{x_j}^{x_{j+1}-\frac{h_1}{2}}f''(t)C_5(t)dt+\int\limits_{x_{j+1}-\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}}f''(t)C_6(t)dt,\end{aligned}\tag{1.6}$$

where

$$C_1(t)=\frac{1}{2h^2h_1}\Big(t-x_{j-1}+\frac{h_1}{2}\Big)^2,\quad C_2(t)=\frac{1}{h^2}(t-x_{j-1}),\quad C_3(t)=-\frac{1}{h^2h_1}\Big(t-x_j+\frac{h_1}{2}\Big)^2,$$
$$C_4(t)=-\frac{1}{h^2h_1}\Big(t-x_j-\frac{h_1}{2}\Big)^2,\quad C_5(t)=\frac{1}{h^2}(x_{j+1}-t),\quad C_6(t)=\frac{1}{2h^2h_1}\Big(t-x_{j+1}-\frac{h_1}{2}\Big)^2.$$

Divide the further proof into two cases: 1) $0<h_1\le h$, and 2) $h<h_1\le 2h$.

Case 1). Let $0<h_1\le h$. Under this, the function $S''(x)$ is transformed to the form

$$\begin{aligned}S''(x)=&\int\limits_{x_{j-1}-\frac{h_1}{2}}^{x_{j-1}+\frac{h_1}{2}}f''(t)C_1(t)dt+\int\limits_{x_{j-1}+\frac{h_1}{2}}^{x_j-\frac{h_1}{2}}f''(t)C_2(t)dt+\int\limits_{x_j-\frac{h_1}{2}}^{x_j}f''(t)(C_2(t)+C_3(t))dt\\&+\int\limits_{x_j}^{x_j+\frac{h_1}{2}}f''(t)(C_4(t)+C_5(t))dt+\int\limits_{x_j+\frac{h_1}{2}}^{x_{j+1}-\frac{h_1}{2}}f''(t)C_5(t)dt+\int\limits_{x_{j+1}-\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}}f''(t)C_6(t)dt.\end{aligned}\tag{1.7}$$

From the definitions of $C_j(t)$ $(j=\overline{1,6})$, it follows that $C_1(t)\geq 0$ for $t\in\Big[x_{j-1}-\dfrac{h_1}{2};x_{j-1}+\dfrac{h_1}{2}\Big]$, $C_2(t)\geq 0$ for $t\in\Big[x_{j-1}+\dfrac{h_1}{2};x_j-\dfrac{h_1}{2}\Big]$, $C_5(t)\geq 0$ for $t\in\Big[x_j+\dfrac{h_1}{2};x_{j+1}-\dfrac{h_1}{2}\Big]$, and $C_6(t)\geq 0$ for $t\in\Big[x_{j+1}-\dfrac{h_1}{2};x_{j+1}+\dfrac{h_1}{2}\Big]$.

Now it remains to investigate the quadratic trinomials $C_2(t)+C_3(t)$ for $t\in\Big[x_j-\dfrac{h_1}{2};x_j\Big]$ and $C_4(t)+C_5(t)$ for $t\in\Big[x_j;x_j+\dfrac{h_1}{2}\Big]$.

Non-negativity of the functions $C_2(t)+C_3(t)$ for $t\in\Big[x_j-\dfrac{h_1}{2};x_j\Big]$ and $C_4(t)+C_5(t)$ for $t\in\Big[x_j;x_j+\dfrac{h_1}{2}\Big]$ follows from the fact that the branches of corresponding parabolas are down-directed and their values (at the ends of the intervals under investigation) are non-negative. Namely,

$$C_2(x_j)+C_3(x_j)=C_4(x_j)+C_5(x_j)=\frac{4h-h_1}{4h^2}\geq 0,$$

$$C_2(x_j-\frac{h_1}{2})+C_3(x_j-\frac{h_1}{2})=C_4(x_j+\frac{h_1}{2})+C_5(x_j+\frac{h_1}{2})=\frac{2h-h_1}{2h^2}\geq 0.$$

From the statements proved, representation (1.7), and the condition $f''(t)\geq 0$ for $t\in\Big(x_{j-1}-\dfrac{h_1}{2};x_{j+1}+\dfrac{h_1}{2}\Big)$, it follows that $S''(x)\geq 0$ for $x\in(x_j;x_{j+1/2})$.

Case 2). Let $h<h_1\leq 2h$. Under this, the function $S''(x)$ is transformed to the form

$$S''(x)=\int\limits_{x_{j-1}-\frac{h_1}{2}}^{x_j-\frac{h_1}{2}} f''(t)C_1(t)dt+\int\limits_{x_j-\frac{h_1}{2}}^{x_{j-1}+\frac{h_1}{2}} f''(t)(C_1(t)+C_3(t))dt+\int\limits_{x_{j-1}+\frac{h_1}{2}}^{x_j} f''(t)(C_2(t)+C_3(t))dt$$
$$+\int\limits_{x_j}^{x_{j+1}-\frac{h_1}{2}} f''(t)(C_4(t)+C_5(t))dt+\int\limits_{x_{j+1}-\frac{h_1}{2}}^{x_j+\frac{h_1}{2}} f''(t)(C_4(t)+C_6(t))dt+\int\limits_{x_j+\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}} f''(t)C_6(t)dt. \quad (1.8)$$

From the definitions of $C_j(t)$ $(j=\overline{1,6})$, it follows that $C_1(t)\geq 0$ for $t\in\Big[x_{j-1}-\dfrac{h_1}{2};x_j-\dfrac{h_1}{2}\Big]$ and $C_6(t)\geq 0$ for $t\in\Big[x_j+\dfrac{h_1}{2};x_{j+1}+\dfrac{h_1}{2}\Big]$.
Elementary calculations show that

$$C_1\Big(x_j-\frac{h_1}{2}\Big)+C_3\Big(x_j-\frac{h_1}{2}\Big)=C_4\Big(x_j+\frac{h_1}{2}\Big)+C_6\Big(x_j+\frac{h_1}{2}\Big)=\frac{1}{2h_1}\geq 0,$$

$$C_1\Big(x_{j-1}+\frac{h_1}{2}\Big)+C_3\Big(x_{j-1}+\frac{h_1}{2}\Big)=C_4\Big(x_{j+1}+\frac{h_1}{2}\Big)+C_6\Big(x_{j+1}+\frac{h_1}{2}\Big)=\frac{2h^2-(h_1-2h)^2}{2h_1h^2}\geq 0,$$

$$C_2(x_j)+C_3(x_j)=C_4(x_j)+C_5(x_j)=\frac{1}{h^2}\Big(h-\frac{h_1}{4}\Big)\geq 0,$$

$$C_2\Big(x_{j-1}+\frac{h_1}{2}\Big)+C_3\Big(x_{j-1}+\frac{h_1}{2}\Big)=C_4\Big(x_{j+1}-\frac{h_1}{2}\Big)+C_5\Big(x_{j+1}-\frac{h_1}{2}\Big)=\frac{2h^2-(h_1-2h)^2}{2h_1h^2}\geq 0.$$

Since the quadratic trinomials $C_1(t)+C_3(t)$, $C_2(t)+C_3(t)$, $C_4(t)+C_5(t)$, and $C_4(t)+C_6(t)$ have negative leading coefficients and their values at the ends of the corresponding intervals are non-negative, we have

$$
\begin{gathered}
C_1(t)+C_3(t)\geq 0 \ \text{ for } \ t\in\Big[x_j-\frac{h_1}{2};x_{j-1}+\frac{h_1}{2}\Big],\\
C_2(t)+C_3(t)\geq 0 \ \text{ for } \ t\in\Big[x_{j-1}+\frac{h_1}{2};x_j\Big],\\
C_4(t)+C_5(t)\geq 0 \ \text{ for } \ t\in\Big[x_j;x_{j+1}-\frac{h_1}{2}\Big],\\
C_4(t)+C_6(t)\geq 0 \ \text{ for } \ t\in\Big[x_{j+1}-\frac{h_1}{2};x_j+\frac{h_1}{2}\Big].
\end{gathered}
$$

From the statements proved, representation (1.8), and the condition $f''(t)\geq 0$ for $t\in\Big(x_{j-1}-\frac{h_1}{2};x_{j+1}+\frac{h_1}{2}\Big)$, it follows that $S''(x)\geq 0$ for $x\in(x_j;x_{j+1/2})$.

Proof of point 1b) one-to-one repeats the reasonings mentioned in the prof of point 1a) after substitution the variable $x-x_j$ by the $x_{j+1}-x$ one.

Point 2a). Estimate $|S''(x)|$ for $x\in[x_j;x_{j+1/2}]$. Under $0<h_1\leq h$ by (1.7) for any function $f\in W^2_\infty\Big[x_{j-1}-\frac{h_1}{2};x_{j+1}+\frac{h_1}{2}\Big]$, the value $|S''(x)|$ is estimated from above by the sum of integrals, and this sum is equal to 1. Namely,

$$
\begin{aligned}
|S''(x)|\leq \int\limits_{x_{j-1}-\frac{h_1}{2}}^{x_{j-1}+\frac{h_1}{2}} C_1(t)dt+\int\limits_{x_{j-1}+\frac{h_1}{2}}^{x_j-\frac{h_1}{2}} C_2(t)dt+\int\limits_{x_j-\frac{h_1}{2}}^{x_j}(C_2(t)+C_3(t))dt\\
+\int\limits_{x_j}^{x_j+\frac{h_1}{2}}(C_4(t)+C_5(t))dt+\int\limits_{x_j+\frac{h_1}{2}}^{x_{j+1}-\frac{h_1}{2}} C_5(t)dt+\int\limits_{x_{j+1}-\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}} C_6(t)dt=1,\quad x\in[x_j;x_{j+1/2}].
\end{aligned}
$$

Similarly, for $h<h_1\leq 2h$ from (1.7), we derive that for any function
$f\in W^2_\infty\Big[x_j-\frac{h_1}{2};x_{j+2}+\frac{h_1}{2}\Big]$ for $x\in[x_j;x_{j+1/2}]$, the following inequality holds:

$$
\begin{aligned}
|S''(x)|\leq \int\limits_{x_{j-1}-\frac{h_1}{2}}^{x_j-\frac{h_1}{2}} C_1(t)dt+\int\limits_{x_j-\frac{h_1}{2}}^{x_{j-1}+\frac{h_1}{2}}(C_1(t)+C_3(t))dt+\int\limits_{x_{j-1}+\frac{h_1}{2}}^{x_j}(C_2(t)+C_3(t))dt\\
+\int\limits_{x_j}^{x_{j+1}-\frac{h_1}{2}}(C_4(t)+C_5(t))dt+\int\limits_{x_{j+1}-\frac{h_1}{2}}^{x_j+\frac{h_1}{2}}(C_4(t)+C_6(t))dt+\int\limits_{x_j+\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}} C_6(t)dt=1.
\end{aligned}
$$

From the proofs considered above, it follows that in both cases the function that realizes exact equality in the inequalities proved is the function $f(x)=\frac{x^2}{2}$.

Point 2b). The proof follows directly from the function $S(x)$ symmetry w.r.t. the middle point $x_{j+1/2}$ of the interval $[x_j;x_{j+1}]$.

$\square$

2. Estimations of approximation errors

Theorem 3. *For* $0 \le h_1 \le 2h$, *the following equality holds*:

$$\sup_{f\in W^2_\infty(\mathbb{R})} \|f-S\|_\infty = \frac{h^2}{8}+\frac{h_1^2}{24}.$$

P r o o f. Consider for $x \in [x_j; x_{j+1/2}]$ non-integral terms in the function $f \in W^2_\infty\Big[x_{j-1} - \dfrac{h_1}{2}; x_{j+1}+\dfrac{h_1}{2}\Big]$ in representation (1.3) and in the function $S(x)$ in representation (1.5). Note that they coincide since

$$\frac{(x-x_{j+1/2})^2}{2h^2}(f(x_j)-f'(x_j)h)+\Big(\frac{x_{j+1}-x}{h}-\frac{(x-x_{j+1/2})^2}{h^2}\Big)f(x_j)$$
$$+\Big(\frac{(x-x_{j+1/2})^2}{2h^2}+\frac{x-x_j}{h}\Big)(f(x_j)+f'(x_j)h)=f(x_j)+f'(x_j).$$

Taking this into account, we have for any $x \in [x_j; x_{j+1/2}]$

$$S(x)-f(x)=\frac{(x-x_{j+1/2})^2}{2h^2}\frac{1}{h_1}\Big[\int\limits_{x_{j-1}-\frac{h_1}{2}}^{x_{j-1}+\frac{h_1}{2}}\frac{1}{2}f''(t)\Big(t-x_{j-1}+\frac{h_1}{2}\Big)^2dt+\int\limits_{x_{j-1}+\frac{h_1}{2}}^{x_j}f''(t)h_1(t-x_{j-1})dt\Big]$$
$$+\Big(\frac{x_{j+1}-x}{h}-\frac{(x-x_{j+1/2})^2}{h^2}\Big)\frac{1}{h_1}\Big[\int\limits_{x_j-\frac{h_1}{2}}^{x_j}\frac{1}{2}f''(t)\Big(t-x_j+\frac{h_1}{2}\Big)^2dt+\int\limits_{x_j}^{x_j+\frac{h_1}{2}}\frac{1}{2}f''(t)\Big(t-x_j-\frac{h_1}{2}\Big)^2dt\Big]$$
$$+\Big(\frac{(x-x_{j+1/2})^2}{2h^2}+\frac{x-x_j}{h}\Big)\frac{1}{h_1}\Big[\int\limits_{x_j}^{x_{j+1}-\frac{h_1}{2}}f''(t)h_1(x_{j+1}-t)dt+\int\limits_{x_{j+1}-\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}}\frac{1}{2}f''(t)\Big(t-x_{j+1}-\frac{h_1}{2}\Big)^2dt\Big]$$
$$-\int\limits_{x_j}^{x}f''(t)(x-t)dt. \tag{2.1}$$

So, for any $x\in[x_j;x_{j+1/2}]$, the following equality holds:

$$S(x)-f(x)=\int\limits_{x_{j-1}-\frac{h_1}{2}}^{x_{j-1}+\frac{h_1}{2}}f''(t)K_1(x,t)dt+\int\limits_{x_{j-1}+\frac{h_1}{2}}^{x_j}f''(t)K_2(x,t)dt$$
$$+\int\limits_{x_j-\frac{h_1}{2}}^{x_j}f''(t)K_3(x,t)dt+\int\limits_{x_j}^{x_j+\frac{h_1}{2}}f''(t)K_4(x,t)dt+\int\limits_{x_j}^{x_{j+1}-\frac{h_1}{2}}f''(t)K_5(x,t)dt \tag{2.2}$$
$$+\int\limits_{x_{j+1}-\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}}f''(t)K_6(x,t)dt-\int\limits_{x_j}^{x}f''(t)K_7(x,t)dt,$$

where

$$K_1(x,t) = \frac{(x - x_{j+1/2})^2}{4h^2 h_1}\Big(t - x_{j-1} + \frac{h_1}{2}\Big)^2,$$

$$K_2(x,t) = \frac{(x - x_{j+1/2})^2}{2h^2}(t - x_{j-1}),$$

$$K_3(x,t) = \left(\frac{x_{j+1} - x}{h} - \frac{(x - x_{j+1/2})^2}{h^2}\right)\frac{1}{2h_1}\left(t - x_j + \frac{h_1}{2}\right)^2,$$

$$K_4(x,t) = \left(\frac{x_{j+1} - x}{h} - \frac{(x - x_{j+1/2})^2}{h^2}\right)\frac{1}{2h_1}\left(t - x_j - \frac{h_1}{2}\right)^2,$$

$$K_5(x,t) = \left(\frac{(x - x_{j+1/2})^2}{2h^2} + \frac{x - x_j}{h}\right)(x_{j+1} - t),$$

$$K_6(x,t) = \left(\frac{(x - x_{j+1/2})^2}{2h^2} + \frac{x - x_j}{h}\right)\frac{1}{2h_1}\left(t - x_{j+1} - \frac{h_1}{2}\right)^2,$$

$$K_7(x,t) = t - x.$$

Further proof for $x \in [x_j; x_{j+1/2}]$ is divided into two cases: 1) $0 < h_1 \le h$ and 2) $h < h_1 \le 2h$.

Case 1). Let $0 < h_1 \le h$. Under this, the difference $S(x) - f(x)$ for $x \in \left[x_j, x_j + \dfrac{h_1}{2}\right]$ is transformed to the form

$$S(x) - f(x) = \int\limits_{x_{j-1}-\frac{h_1}{2}}^{x_{j-1}+\frac{h_1}{2}} f''(t)K_1(x,t)dt + \int\limits_{x_{j-1}+\frac{h_1}{2}}^{x_j-\frac{h_1}{2}} f''(t)K_2(x,t)dt$$

$$+ \int\limits_{x_j-\frac{h_1}{2}}^{x_j} f''(t)(K_2(x,t) + K_3(x,t))dt + \int\limits_{x_j}^{x} f''(t)(K_4(x,t) + K_5(x,t) + K_7(x,t))dt \tag{2.3}$$

$$+ \int\limits_{x}^{x_j+\frac{h_1}{2}} f''(t)(K_4(x,t) + K_5(x,t))dt + \int\limits_{x_j+\frac{h_1}{2}}^{x_{j+1}-\frac{h_1}{2}} f''(t)K_5(x,t)dt + \int\limits_{x_{j+1}-\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}} f''(t)K_6(x,t)dt.$$

But for $x \in \left[x_j + \dfrac{h_1}{2}, x_{j+1/2}\right]$, its form is

$$S(x) - f(x) = \int\limits_{x_{j-1}-\frac{h_1}{2}}^{x_{j-1}+\frac{h_1}{2}} f''(t)K_1(x,t)dt + \int\limits_{x_{j-1}+\frac{h_1}{2}}^{x_j-\frac{h_1}{2}} f''(t)K_2(x,t)dt$$

$$+ \int\limits_{x_j-\frac{h_1}{2}}^{x_j} f''(t)(K_2(x,t) + K_3(x,t))dt + \int\limits_{x_j}^{x_j+\frac{h_1}{2}} f''(t)(K_4(x,t) + K_5(x,t) + K_7(x,t))dt \tag{2.4}$$

$$+ \int\limits_{x_j+\frac{h_1}{2}}^{x} f''(t)(K_5(x,t) + K_7(x,t))dt + \int\limits_{x}^{x_{j+1}-\frac{h_1}{2}} f''(t)K_5(x,t)dt + \int\limits_{x_{j+1}-\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}} f''(t)K_6(x,t)dt.$$

To obtain the estimation value on the function class $W^2_\infty = W^2_\infty(\mathbb{R})$, we shall prove that

under $x \in [x_j, x_{j+1/2}]$: $K_1(x,t) \geq 0$ for $t \in \left[x_{j-1} - \frac{h_1}{2}; x_{j-1} + \frac{h_1}{2}\right]$, $K_2(x,t) \geq 0$ for $t \in \left[x_{j-1} + \frac{h_1}{2}; x_j - \frac{h_1}{2}\right]$, $K_2(x,t) + K_3(x,t) \geq 0$ for $t \in \left[x_j - \frac{h_1}{2}; x_j\right]$, $K_6(x,t) \geq 0$ for $t \in \left[x_{j+1} - \frac{h_1}{2}; x_{j+1} + \frac{h_1}{2}\right]$;

under $x \in \left[x_j, x_j + \frac{h_1}{2}\right]$: $K_4(x,t) + K_5(x,t) + K_7(x,t) \geq 0$ for $t \in [x_j; x]$, $K_4(x,t) + K_5(x,t) \geq 0$ for $t \in \left[x; x_j + \frac{h_1}{2}\right]$, $K_5(x,t) \geq 0$ for $t \in \left[x_j + \frac{h_1}{2}; x_{j+1} - \frac{h_1}{2}\right]$;

under $x \in \left[x_j + \frac{h_1}{2}, x_{j+1/2}\right]$: $K_4(x,t) + K_5(x,t) + K_7(x,t) \geq 0$ for $t \in [x_j; x_j + \frac{h_1}{2}]$, $K_5(x,t) + K_7(x,t) \geq 0$ for $t \in \left[x_j + \frac{h_1}{2}; x\right]$, $K_5(x,t) \geq 0$ for $t \in \left[x; x_{j+1} - \frac{h_1}{2}\right]$.

All these inequalities (except only two) immediately follow from definitions of the functions $K_j(x,t)$ $(j = \overline{1,7})$. So, it is only necessary to verify that $K_4(x,t) + K_5(x,t) + K_7(x,t) \geq 0$ for $x \in \left[x_j, x_j + \frac{h_1}{2}\right]$, $t \in [x_j; x]$, for $x \in \left[x_j + \frac{h_1}{2}, x_{j+1/2}\right]$, $t \in \left[x_j; x_j + \frac{h_1}{2}\right]$ and $K_5(x,t) + K_7(x,t) \geq 0$ for $x \in \left[x_j + \frac{h_1}{2}, x_{j+1/2}\right]$, $t \in \left[x_j + \frac{h_1}{2}; x\right]$.

The function $K_4(x,t) + K_5(x,t) + K_7(x,t)$ is the quadratic trinomial in the variable t with the positive leading coefficient; at the ends of the intervals $[x_j; x]$, $\left[x_j; x_j + \frac{h_1}{2}\right]$ this function (as one of the variable t) takes the positive values, and abscissa of the corresponding parabola apex is placed at the left from the point x_j. From this, the non-negativity of this function on the mentioned sets follows.

The function $K_5(x,t) + K_7(x,t)$ is linear in the variable t and takes non-negative values at the ends of the interval $\left[x_j + \frac{h_1}{2}; x\right]$. Hence, $K_5(x,t) + K_7(x,t) \geq 0$ in the whole interval $\left[x_j + \frac{h_1}{2}; x\right]$ for $x \in \left[x_j + \frac{h_1}{2}, x_{j+1/2}\right]$.

Taking into account the above proved facts, it follows from formulas (2.3) and (2.4) that to obtain the estimate from above for the value $|S(x) - f(x)|$ (for these formulas) in the class $W^2_\infty(\mathbb{R})$ and, hence, in formula (2.1), the function $f''(t)$ can be substituted by 1 in (2.3) and (2.4).

Put $f''(t) = 1$ and calculate for it values of integrals in the right-hand side of formula (2.1); denote this value by the symbol J. After elementary calculations, we obtain that $J = \frac{h^2}{8} + \frac{h_1^2}{24}$. It implies that the exact inequality

$$|f(x) - S(x)| \leq \frac{h^2}{8} + \frac{h_1^2}{24}$$

holds for any function $f \in W^2_\infty\left[x_{j-1} - \frac{h_1}{2}; x_{j+1} + \frac{h_1}{2}\right]$ for any $x \in [x_j; x_{j+1/2}]$. Moreover, the sign of equality is provided by the function $f(t) = \frac{t^2}{2}$ for $t \in \left[x_{j-1} - \frac{h_1}{2}; x_{j+1} + \frac{h_1}{2}\right]$.

Similarly, for the function $f \in W^2_\infty\left[x_j - \frac{h_1}{2}; x_{j+2} + \frac{h_1}{2}\right]$ for any $x \in [x_{j+1/2}; x_{j+1}]$, we derive the exact inequality

$$|f(x) - S(x)| \leq \frac{h^2}{8} + \frac{h_1^2}{24},$$

and the sign of equality is provided by the function $f(t) = \frac{t^2}{2}$ for $t \in \left[x_j - \frac{h_1}{2}; x_{j+2} + \frac{h_1}{2}\right]$.

Case 2). Let $h < h_1 \le 2h$. In this case, the difference $S(x) - f(x)$ for $x \in \left[x_j, x_{j+1} - \dfrac{h_1}{2}\right]$ is transformed to the form

$$\begin{aligned} S(x) - f(x) = &\int\limits_{x_{j-1}-\frac{h_1}{2}}^{x_j-\frac{h_1}{2}} f''(t)K_1(x,t)dt + \int\limits_{x_j-\frac{h_1}{2}}^{x_{j-1}+\frac{h_1}{2}} f''(t)(K_1(x,t)+K_3(x,t))dt \\ &+ \int\limits_{x_{j-1}+\frac{h_1}{2}}^{x_j} f''(t)(K_2(x,t)+K_3(x,t))dt + \int\limits_{x_j}^{x} f''(t)(K_4(x,t)+K_5(x,t)+K_7(x,t))dt \\ &+ \int\limits_{x}^{x_{j+1}-\frac{h_1}{2}} f''(t)(K_4(x,t)+K_5(x,t))dt + \int\limits_{x_{j+1}-\frac{h_1}{2}}^{x_j+\frac{h_1}{2}} f''(t)(K_4(x,t)+K_6(x,t))dt \\ &+ \int\limits_{x_j+\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}} f''(t)K_6(x,t)dt, \end{aligned} \tag{2.5}$$

$$\begin{aligned} S(x) - f(x) = &\int\limits_{x_{j-1}-\frac{h_1}{2}}^{x_j-\frac{h_1}{2}} f''(t)K_1(x,t)dt + \int\limits_{x_j-\frac{h_1}{2}}^{x_{j-1}+\frac{h_1}{2}} f''(t)(K_1(x,t)+K_3(x,t))dt \\ &+ \int\limits_{x_{j-1}+\frac{h_1}{2}}^{x_j} f''(t)(K_2(x,t)+K_3(x,t))dt + \int\limits_{x_j}^{x_{j+1}-\frac{h_1}{2}} f''(t)(K_4(x,t)+K_5(x,t)+K_7(x,t))dt \\ &+ \int\limits_{x_{j+1}-\frac{h_1}{2}}^{x} f''(t)(K_4(x,t)+K_6(x,t)+K_7(x,t))dt + \int\limits_{x}^{x_j+\frac{h_1}{2}} f''(t)(K_4(x,t)+K_6(x,t))dt \\ &+ \int\limits_{x_j+\frac{h_1}{2}}^{x_{j+1}+\frac{h_1}{2}} f''(t)K_6(x,t)dt. \end{aligned} \tag{2.6}$$

To obtain the error estimate in the class $W^2_\infty(\mathbb{R})$, prove that

under $x \in [x_j, x_{j+1/2}]$, the following inequalities hold: $K_1(x,t) \ge 0$ for $t \in \left[x_{j-1} - \dfrac{h_1}{2}; x_j - \dfrac{h_1}{2}\right]$, $K_1(x,t) + K_3(x,t) \ge 0$ for $t \in \left[x_j - \dfrac{h_1}{2}; x_{j-1} + \dfrac{h_1}{2}\right]$, $K_2(x,t) + K_3(x,t) \ge 0$ for $t \in \left[x_{j-1} + \dfrac{h_1}{2}; x_j\right]$, $K_6(x,t) \ge 0$ for $t \in \left[x_j + \dfrac{h_1}{2}; x_{j+1} + \dfrac{h_1}{2}\right]$;

under $x \in \left[x_j, x_{j+1} - \dfrac{h_1}{2}\right]$ these inequalities are: $K_4(x,t) + K_5(x,t) + K_7(x,t) \ge 0$ for $t \in [x_j; x]$, $K_4(x,t) + K_5(x,t) \ge 0$ for $t \in \left[x; x_{j+1} - \dfrac{h_1}{2}\right]$, $K_4(x,t) + K_6(x,t) \ge 0$ for $t \in \left[x_{j+1} - \dfrac{h_1}{2}; x_j + \dfrac{h_1}{2}\right]$;

under $x \in \left[x_{j+1} - \frac{h_1}{2}, x_{j+1/2}\right]$ the inequalities hold: $K_4(x,t) + K_5(x,t) + K_7(x,t) \geq 0$ for $t \in \left[x_j; x_{j+1} - \frac{h_1}{2}\right]$, $K_4(x,t) + K_6(x,t) = K_7(x,t) \geq 0$ for $t \in \left[x_{j+1} - \frac{h_1}{2}; x\right]$, and $K_4(x,t) + K_6(x,t) \geq 0$ for $t \in \left[x; x_j + \frac{h_1}{2}\right]$.

All these inequalities (except two ones) immediately follow from definitions of the function $K_j(x,t)$ $(j = \overline{1,7})$. It is only necessary to verify that $K_4(x,t) + K_5(x,t) + K_7(x,t) \geq 0$ under $x \in \left[x_j, x_{j+1} - \frac{h_1}{2}\right]$, $t \in [x_j; x]$ and under $x \in \left[x_{j+1} - \frac{h_1}{2}, x_{j+1/2}\right]$, $t \in \left[x_j; x_{j+1} - \frac{h_1}{2}\right]$, and $K_4(x,t) + K_6(x,t) + K_7(x,t) \geq 0$ under $x \in \left[x_{j+1} - \frac{h_1}{2}, x_{j+1/2}\right]$, $t \in \left[x_{j+1} - \frac{h_1}{2}; x\right]$.

The function $K_4(x,t) + K_5(x,t) + K_7(x,t)$ is the quadratic trinomial in the variable t with the positive leading coefficient; at the ends of the intervals $[x_j; x]$, $\left[x_j; x_{j+1} - \frac{h_1}{2}\right]$, this trinomial takes positive values, and abscissa of the corresponding parabola apex is placed at the left from the point x_j.

The function $K_4(x,t) + K_6(x,t) + K_7(x,t)$ for $x \in \left[x_{j+1} - \frac{h_1}{2}; x_{j+1/2}\right]$, $t \in \left[x_{j+1} - \frac{h_1}{2}; x\right]$ possesses the same properties. Remind that this function is also the quadratic trinomial in the variable t with the positive leading coefficient; at the ends of the interval $\left[x_{j+1} - \frac{h_1}{2}; x\right]$, this trinomial takes positive values, and abscissa of the corresponding parabola apex is placed at the left from the point $x_{j+1} - \frac{h_1}{2}$. It implies non-negativity of the considered functions in the mentioned sets.

Taking into account the above proved facts, it follows from formulas (2.5) and (2.6) that to obtain the estimate from above for the value $|S(x) - f(x)|$ (for these formulas) in the class $W^2_\infty(\mathbb{R})$ and, hence, in formula (2.1), the function $f''(t)$ can be substituted by 1.

Put $f''(t) = 1$ and calculate for it values of integrals in the right-hand side of formula (2.1); denote this value by the symbol J. After elementary calculations, we obtain that $J = \frac{h^2}{8} + \frac{h_1^2}{24}$. It implies that the exact inequality

$$|f(x) - S(x)| \leq \frac{h^2}{8} + \frac{h_1^2}{24}$$

holds for any function $f \in W^2_\infty\left[x_{j-1} - \frac{h_1}{2}; x_{j+1} + \frac{h_1}{2}\right]$ under any $x \in [x_j; x_{j+1/2}]$. Moreover, the equality sign is provided by the function $f(t) = \frac{t^2}{2}$ for $t \in \left[x_{j-1} - \frac{h_1}{2}; x_{j+1} + \frac{h_1}{2}\right]$.

Similarly, for the function $f \in W^2_\infty\left[x_j - \frac{h_1}{2}; x_{j+2} + \frac{h_1}{2}\right]$ under any $x \in [x_{j+1/2}; x_{j+1}]$, we derive the exact inequality

$$|S(x) - f(x)| \leq \frac{h^2}{8} + \frac{h_1^2}{24},$$

and the equality sign is provided by the function $f(t) = \frac{t^2}{2}$ for $t \in \left[x_j - \frac{h_1}{2}; x_{j+2} + \frac{h_1}{2}\right]$.

□

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Contact Information

16 S. Kovalevskaya str., Ekaterinburg, Russia, 620990

Phone: +7 (343) 375-34-73

Fax: +7 (343) 374-25-81

Email: secretary@umjuran.ru

Web-site: https://umjuran.ru

N.N.Krasovskii Institute of Mathematics and Mechanics
of the Ural Branch of Russian Academy of Sciences

Ural Federal University named after the first President of Russia B.N.Yeltsin

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