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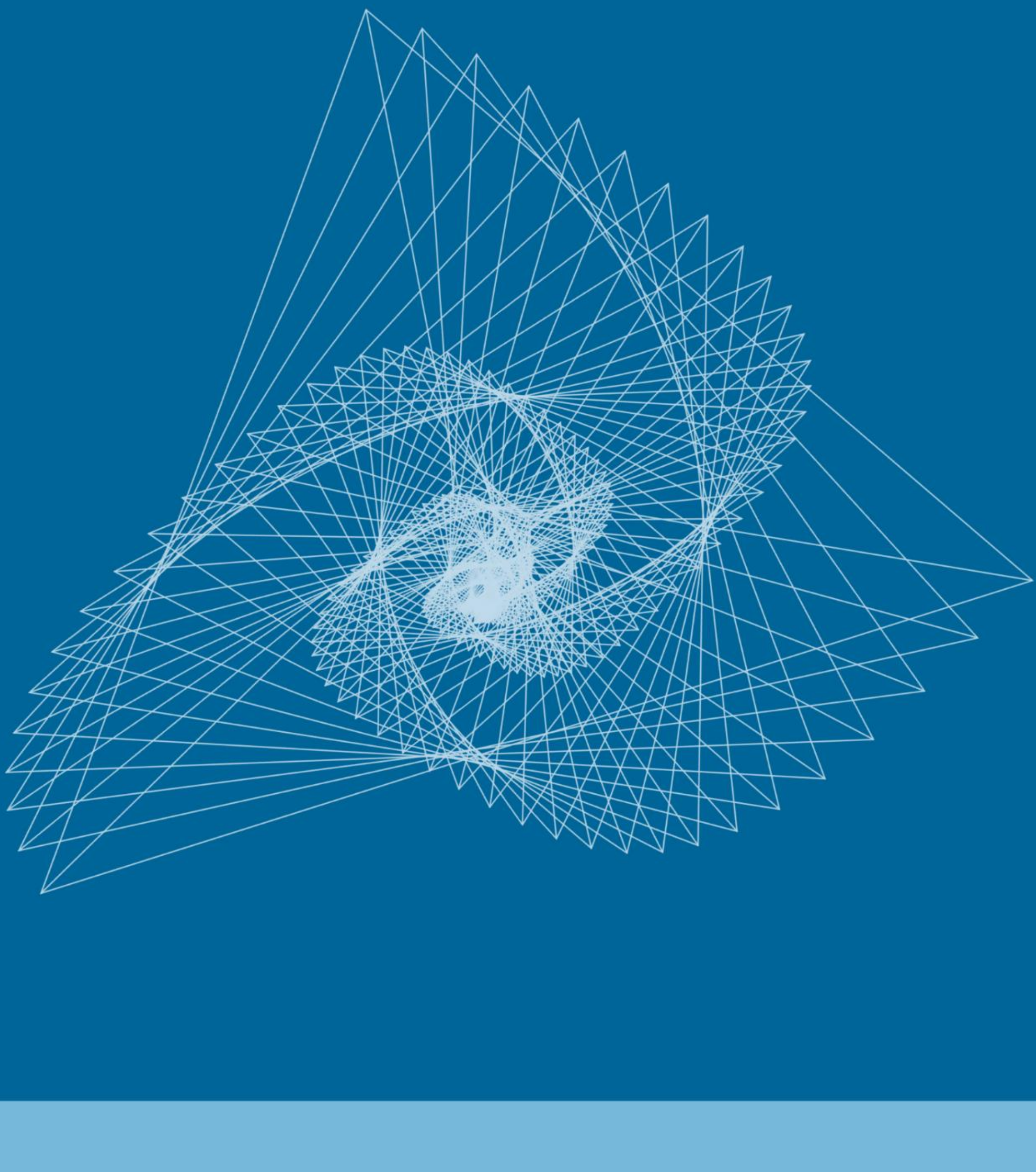

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## PART II. GENERAL TOPICS

# THE 42D INTERNATIONAL S.B. STECHKIN'S WORKSHOP-CONFERENCE ON FUNCTION THEORY

**Vitalii V. Arestov**[†] **and Vitalii I. Berdyshev**[††]

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**Abstract:** The paper is devoted to the description of the history and results of the 42d International S.B.Stechkin's Workshop on function theory, held in August 2017 in the Ilmen Nature Reserve near the town of Miass, Chelyabinsk region.

**Key words:** The 42d International S.B. Stechkin's Workshop on function theory.

From the sixties of the XX century onwards, at the Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences and at A.M. Gor'kii Ural State University, a powerful scientific school on function theory has been formed, which works intensively till nowadays. The founder of this school was Professor S.B. Stechkin—the organizer of the Institute and the professor of the University, who passed away in 1995. A number of leading world experts in the theory of functions and operators have grown up in this school: academicians V.I. Berdyshev and S.V. Konyagin, corresponding member of the Russian Academy of Sciences Yu.N. Subbotin, doctors of science, professors A.R. Alimov, N.Yu. Antonov, V.V. Arestov, A.G. Babenko, V.M. Badkov, N.I. Chernykh, V.I. Ivanov, L.V. Taikov, S.A. Telyakovskii, I.G. Tsar'kov, A.Yu. Shadrin, V.T. Shevaldin, A.S. Shvedov, V.A. Yudin, and dozens of candidates of science. Many of them at present have positions both in the Institute and in the University. The scientific school has a high reputation in the world. With the purpose of discussing research results and ways of further scientific studies, in the beginning of the 1970th, annual summer scientific workshops-conferences on function theory and approximation theory were organized. The organizer and the all-time leader of most of them was Professor S.B. Stechkin, this tradition continues to the present. During these workshops-conferences, not only new scientific results are presented but also open problems of function theory and approximation theory and possible approaches to their solution as well as forthcoming dissertations are discussed. The duration of these workshops allow their participants to deliver as complete presentation of their research as they reasonably need. Talks are accompanied with numerous revealing questions and remarks of other participants, which are traditionally welcome. The atmosphere is friendly and homelike. Talks are usually given under the open-air on a clearing in the wood. In addition to participants from Ekaterinburg (from Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences and from Ural Federal University), workshop-conferences traditionally accept leading scientists and their students from Moscow (form Moscow State University, V.A. Steklov Mathematical Institute, and other institutions), Novosibirsk, Ozersk, Saratov, Tula, and from other cities in Russia and abroad (Azerbaijan, China, Kazakhstan, Ukrain, and others). A distinctive feature of Workshop-2017 was a considerable number of new young participants which inspires hope for preserving the traditions and inimitable spirit of Stechkin's School. Detailed historical reviews on past workshops can be found in [1–3].

Figure 1. The opening ceremony of S.B. Stechkin's Workshop (August 2, 2017)

The 42d International S.B. Stechkin's Workshop-Conference on function theory was held on August 1–10, 2017. It was organized by the Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences and Ural Federal University (Ekaterinburg). The conference venue was the shore of Ilmen lake in the Ilmen Nature Reserve near the city of Miass, Chelyabinsk region.

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**Programming Committee**: V.V. Arestov (chairman), N.Yu. Antonov, and M.V. Deikalova (Ekaterinburg, Russia).

The Workshop-Conference was attended by 35 scientists from Moscow, Ekaterinburg, and former Soviet republics: Kazakhstan, Tadjikistan, Turkmenistan, and Ukraine, including one academician of the Russian Academy of Sciences, 8 doctors of science, 14 candidates of science, 9 students and undergraduates, and 3 post-graduate students. They delivered 34 research talks on basic topics of modern function theory and approximation theory, and on applications of approximation methods to solving problems in other areas of mathematics:

– general problems of function theory;

– best approximation of functions and operators;

– extremal problems of function theory and approximation theory;

– modern approximation methods: splines, wavelets, and their application to problems of data compression and medicine;

– problems of navigation by geodesic fields;

– geometric problems of approximation theory;

– numerical analysis.

Several selected papers presented at the Workshop are published in this issue of the journal.

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# APPROXIMATION OF THE DIFFERENTIATION OPERATOR ON THE CLASS OF FUNCTIONS ANALYTIC IN AN ANNULUS[1]

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**Abstract:** In the class of functions analytic in the annulus $C_r := \{z \in \mathbb{C} : r < |z| < 1\}$ with bounded $L^p$-norms on the unit circle, we study the problem of the best approximation of the operator taking the nontangential limit boundary values of a function on the circle $\Gamma_r$ of radius $r$ to values of the derivative of the function on the circle $\Gamma_\rho$ of radius $\rho$, $r < \rho < 1$, by bounded linear operators from $L^p(\Gamma_r)$ to $L^p(\Gamma_\rho)$ with norms not exceeding a number $N$. A solution of the problem has been obtained in the case when $N$ belongs to the union of a sequence of intervals. The related problem of optimal recovery of the derivative of a function from boundary values of the function on $\Gamma_\rho$ given with an error has been solved.

**Key words:** Best approximation of operators, Optimal recovery, Analytic functions.

## Introduction

The paper is devoted to studying a number of related extremal problems for the differentiation operator on the class of functions analytic in an annulus. Similar problems for the analytic continuation operator and for the differentiation operator on the class of functions analytic in a strip were solved earlier in [1] and [2], respectively. In the present paper, we follow the notation and use some auxiliary statements from [1, 2].

Let $C_r := \{z \in \mathbb{C} : r < |z| < 1\}$ be the annulus centered at the origin of internal radius $r$ and external radius 1. We denote by $A(C_r)$ the set of functions analytic in the annulus $C_r$. For a function $f \in A(C_r)$ and a number $\rho$, $r < \rho < 1$, we define the $p$-average of the function $f$ on the circle $\Gamma_\rho := \{z \in \mathbb{C} : |z| = \rho\}$ by the equality

$$\mathcal{M}^p(f,\rho) := \|f\|_{L^p(\Gamma_\rho)} = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(\rho e^{it})|^p\, dt\right)^{1/p}, & 1 \le p < \infty, \\ \max\left\{|f(\rho e^{it})| : t \in [0, 2\pi]\right\}, & p = \infty. \end{cases}$$

Let $H^p = H^p(C_r)$ be the Hardy space of functions $f \in A(C_r)$ such that

$$\sup\{\mathcal{M}^p(f,\rho) : r < \rho < 1\} < +\infty.$$

As is well known, for an arbitrary function $f \in H^p$, nontangential limit boundary values exist almost everywhere on the boundary $\Gamma_r \bigcup \Gamma_1$. We denote these values by $f(re^{it})$ and $f(e^{it})$. These functions belong to $L^p(\Gamma_r)$ and $L^p(\Gamma_1)$, respectively.

[1]This work was supported by the Russian Foundation for Basic Research (project no. 15-01-02705), the Program for State Support of Leading Scientific Schools of the Russian Federation (project no. NSh-9356.2016.1), and by the Russian Academic Excellence Project (agreement no. 02.A03.21.0006 of August 27, 2013, between the Ministry of Education and Science of the Russian Federation and Ural Federal University).

In the Hardy space $H^p$, we consider the class $Q = Q_r^p$ of functions $f$ whose boundary values on the circle $\Gamma_1$ satisfy the inequality $\mathcal{M}^p(f,1) \le 1$.

The problem of the best approximation of an unbounded linear operator by linear bounded operators on a class of elements of a Banach space appeared in 1965 in investigations of Stechkin [4]. In his 1967 paper [4], he gave a statement of the problem, presented the first principal results, and solved the problem for differentiation operators of small orders. Detailed information about studies of Stechkin's problem and related extremal problems can be found in Arestov's review paper [3]. In the present paper, we consider the problem of the best approximation of the (first-order) differentiation operator for a function on the circle $\Gamma_\rho$ by linear bounded operators on the class $Q$ of functions analytic in the annulus $C_r$. The precise statement of the problem is as follows.

**Problem 1.** Let $\mathcal{L}(N)$ be the set of linear bounded operators from $L^p(\Gamma_r)$ to $L^p(\Gamma_\rho)$ with norm $\|T\| = \|T\|_{L^p(\Gamma_r)\mapsto L^p(\Gamma_\rho)}$ not exceeding a number $N > 0$. The quantity

$$U(T) := \sup\left\{\mathcal{M}^p(f' - Tf, \rho) :\ f \in Q\right\}$$

is the deviation of an operator $T \in \mathcal{L}(N)$ from the differentiation operator on the class $Q$. The quantity

$$E(N) := \inf\left\{U(T) : T \in \mathcal{L}(N)\right\} \tag{0.1}$$

is the best approximation of the differentiation operator by the set of bounded operators $\mathcal{L}(N)$ on the class $Q$. The problem is to calculate the quantity $E(N)$ and to find an extremal operator at which the infimum in (0.1) is attained.

Problem 1 is closely interconnected with a number of extremal problems. One of them is the following problem of calculating the modulus of continuity of the differentiation operator on a class.

**Problem 2.** The function

$$\omega(\delta) = \sup\left\{\mathcal{M}^p(f', \rho) :\ f \in Q,\ \mathcal{M}^p(f, r) \le \delta\right\} \tag{0.2}$$

of real variable $\delta \in [0, +\infty)$ is called the modulus of continuity of the differentiation operator on the class Q. The problem is to calculate the quantity $\omega(\delta)$ and to find an extremal function (a sequence of functions) at which the supremum in (0.2) is attained.

Define

$$\Delta(N) := \sup\left\{\omega(\delta) - N\delta :\ \delta \ge 0\right\}, N > 0;$$

$$l(\delta) := \inf\left\{E(N) + N\delta :\ N > 0\right\}, \delta \ge 0.$$

The following statement, which connects (0.1) and (0.2), is a special case of Stechkin's theorem [5].

**Theorem A.** *The following inequalities hold:*

$$E(N) \ge \Delta(N), \quad N > 0; \tag{0.3}$$

$$\omega(\delta) \le l(\delta), \quad \delta \ge 0. \tag{0.4}$$

Definition (0.2) also implies that the sharp inequality

$$\mathcal{M}^p(f', \rho) \le \mathcal{M}^p(f, 1)\, \omega\left(\frac{\mathcal{M}^p(f, r)}{\mathcal{M}^p(f, 1)}\right)$$

is valid for functions from the space $H^p(C_r)$.

Problems of recovering values of an operator on elements of a class lying in the domain of an operator from some information about the elements of the class given with a known error arise in different areas of mathematics and have been well studied. The recovery is implemented by using some set $\mathcal{R}$ of operators. As a rule, one of the following sets of mappings is taken for $\mathcal{R}$: either the set $\mathcal{O}$ of all single-valued mappings or the set $\mathcal{B}$ of bounded operators or the set $\mathcal{L}$ of linear operators. Monograph [6] is devoted to various problems of optimal recovery, in particular, optimal recovery of derivatives on classes of analytic functions.

Problems 1 and 2 are related to the following optimal recovery problem for the derivative of a function analytic in an annulus from boundary values (on one of the boundary circles) given with an error.

**Problem 3.** For a number $\delta \geq 0$ and an operator $T \in \mathcal{R}$, define

$$\mathcal{U}(T,\delta) = \sup\left\{\mathcal{M}^p(f' - Tg, \rho) : \, f \in Q, \, g \in L^p(\Gamma_r), \, \mathcal{M}^p(f-g, r) \leq \delta\right\}.$$

Then,

$$\mathcal{E}_{\mathcal{R}}(\delta) = \inf\left\{\mathcal{U}(T,\delta) : T \in \mathcal{R}\right\} \tag{0.5}$$

is the value of the best (optimal) recovery of the differentiation operator (the derivative of an analytic function) by recovery methods $\mathcal{R}$ on functions of the class $Q$ from their boundary values on $\Gamma_r$ given with an error $\delta$. The problem is to calculate the quantity $\mathcal{E}(\delta)$ and to find an optimal recovery method, i.e., an operator at which the infimum in (0.5) is attained.

The following theorem contains a refinement of inequality (0.4); this theorem is a special case of a more general statement connecting the problem on the modulus of continuity of an operator and Stechkin's problem with optimal recovery problems (see [3]).

**Theorem B.** *The following inequalities hold:*

$$\omega(\delta) \leq \mathcal{E}_{\mathcal{O}}(\delta) \leq \mathcal{E}_{\mathcal{L}}(\delta) = \mathcal{E}_{\mathcal{B}}(\delta) \leq l(\delta), \quad \delta \geq 0. \tag{0.6}$$

## 1. Main results

We define a (convolution) operator $T_n^1 = T_n^1[\rho, r]$, $n \in \mathbb{Z}$, from $L^p(\Gamma_r)$ to $L^p(\Gamma_\rho)$ by the formula

$$(T_n^1 f)(\rho e^{ix}) = e^{-ix}\,\frac{1}{2\pi}\int_0^{2\pi} \Lambda_n^1(x-t) f(re^{it})\,dt \tag{1.1}$$

with the kernel

$$\Lambda_n^1(t) = r^{-n} e^{int}\,\lambda_n^1(t), \quad \lambda_n^1(t) = \lambda_{n,0}^1 + 2\sum_{k=1}^{\infty} \lambda_{n,k}^1 \cos kt, \tag{1.2}$$

$$\lambda_{n,0}^1 = \frac{\rho^{n-1}}{\ln r}(n\ln\rho + 1), \quad \lambda_{n,k}^1 = \rho^{n-1}\,\frac{(n+k)\rho^k - (n-k)\rho^{-k}}{r^k - r^{-k}}, \quad k \in \mathbb{N}.$$

The following two theorems are the main results of the present paper.

**Theorem 1.** *Assume that the parameter $N$ has the representation*

$$N = \frac{\rho^{n-1}\,|n\ln\rho + 1|}{r^n |\ln r|},$$

*in which $n \in \mathbb{Z}$ is such that*

$$|n| \geq \frac{\pi}{\ln^2 r}\,\sin^{-1}\left(\frac{\ln\rho}{\ln r}\pi\right). \tag{1.3}$$

*Then, quantity* (0.1) *satisfies the equality*

$$E(N) = \frac{\rho^{n-1}}{|\ln r|}\left|n \ln\frac{r}{\rho} - 1\right|.$$

*In this case, the operator* $T_n^1$ *defined by* (1.1) *and* (1.2) *is extremal in problem* (0.1).

**Theorem 2.** *Let* $\delta_n = r^n$, *where* $n \in \mathbb{Z}$ *satisfies condition* (1.3). *Then, quantities* (0.2) *and* (0.5) *satisfy the relations*

$$\omega(\delta_n) = \mathcal{E}_{\mathcal{O}}(\delta_n) = \mathcal{E}_{\mathcal{L}}(\delta_n) = \mathcal{E}_{\mathcal{B}}(\delta_n) = n\rho^{n-1}. \tag{1.4}$$

*In this case, the linear bounded operator* $T_n^1$ *defined by* (1.1) *and* (1.2) *is an optimal recovery method in problem* (0.5). *The functions* $f_n(z) = cz^n$, $|c| = 1$, *are extremal in problem* (0.2).

## 2. Auxiliary statements

In addition, we introduce a (convolution) operator $V_n^1 = V_n^1[\rho, r]$, $n \in \mathbb{Z}$, from $L^p(\Gamma_1)$ to $L^p(\Gamma_\rho)$ by the formula

$$(V_n^1 f)(\rho e^{ix}) = e^{-ix}\,\frac{1}{2\pi}\int_0^{2\pi} \mathcal{V}_n^1(x-t) f(re^{it})\,dt \tag{2.1}$$

with the kernel

$$\mathcal{V}_n^1(t) = e^{int}\,\mu_n^1(t), \quad \mu_n^1(t) = \mu_{n,0}^1 + 2\sum_{k=1}^{\infty} \mu_{n,k}^1 \cos kt, \tag{2.2}$$

$$\mu_{n,0}^1 = \frac{\rho^{n-1}}{\ln r}\left(n\ln\frac{r}{\rho} - 1\right), \quad \mu_{n,k}^1 = \rho^{n-1}\,\frac{(n+k)(\rho/r)^k - (n-k)(\rho/r)^{-k}}{r^{-k} - r^k}, \quad k \in \mathbb{N}.$$

**Lemma 1.** *For an arbitrary function* $f$ *from the class* $Q$ *and* $n \in \mathbb{Z}$, *we have the equality*

$$f'(\rho e^{ix}) = (T_n^1 f)(\rho e^{ix}) + (V_n^1 f)(\rho e^{ix}), \quad x \in [0, 2\pi]. \tag{2.3}$$

P r o o f. The function $f$ in the annulus $C_r$ is representable as the sum of the Laurent series

$$f(z) = \sum_{k=-\infty}^{+\infty} \varphi_k\, z^k, \quad z \in C_r.$$

Then, from the definitions of operators (1.1)–(1.2) and (2.1)–(2.2), we obtain the relations

$$(T_n^1 f)(\rho e^{ix}) + (V_n^1 f)(\rho e^{ix}) = \sum_{k=-\infty}^{+\infty} (\lambda_{n,k}^1 r^k + \mu_{n,k}^1)\,\varphi_{n+k}\, e^{i(n+k-1)x}.$$

Now, from the equality

$$\lambda_{n,k}^1 r^k + \mu_{n,k}^1 = (n+k)\rho^{n+k-1},$$

the assertion of Lemma 1 follows. □

**Lemma 2.** *Let a number* $n \in \mathbb{Z}$ *satisfy condition* (1.3). *Then the functions* $\lambda_n^1$ *and* $\mu_n^1$ *defined by* (1.2) *and* (2.2) *are of the same sign, which remains unchanged on the period, i.e.,* $\lambda_n^1(x)\mu_n^1(x) > 0$, $x \in [0, 2\pi]$.

P r o o f. We introduce the notation

$$g_\pm(x,y) := \frac{e^{ny}\sin\left(\dfrac{\ln\rho}{\ln r}\pi\right)}{\cosh\dfrac{x\pi}{\ln r} \pm \cos\left(\dfrac{\ln\rho}{\ln r}\pi\right)}, \quad y = \ln\rho/r.$$

For the functions $g_\pm$, the following assertion is true [2, Lemma 3]. Condition (1.3) is necessary and sufficient for the functions $\dfrac{\partial g_\pm}{\partial y}$ to maintain sign for arbitrary $x \in \mathbb{R}$ and $0 < y < \ln 1/r$. Moreover, for the functions

$$\Lambda_\pm(x) := -\pi \ln^{-1} r e^{-ny} \sum_{k=-\infty}^{+\infty} g_\pm(x + 2\pi k, y), \quad y = \ln\rho/r,$$

the following equalities hold [1, Lemma 1]:

$$\Lambda_\pm(x) = \lambda_0^\pm + 2\sum_{k=1}^{\infty} \lambda_k^\pm \cos kx,$$

$$\lambda_0^+ = \frac{\ln\rho}{\ln r}, \quad \lambda_k^+ = \frac{\rho^k - \rho^{-k}}{r^k - r^{-k}}, \quad \lambda_0^- = \frac{\ln r/\rho}{\ln r}, \quad \lambda_k^- = \frac{(\rho/r)^k - (\rho/r)^{-k}}{r^{-k} - r^k}.$$

Hence, for the functions $\lambda_n^1$ и $\mu_n^1$ defined by equalities (1.2) и (2.2), we have

$$\lambda_n^1(x) = \frac{\partial}{\partial\rho}\left(\rho^n \Lambda_+(x)\right) = -\frac{\pi r^n}{\rho\ln r} \sum_{k=-\infty}^{+\infty} \frac{\partial}{\partial y} g_+(x + 2\pi k, y),$$

$$\mu_n^1(x) = \frac{\partial}{\partial\rho}\left(\rho^n \Lambda_-(x)\right) = -\frac{\pi r^n}{\rho\ln r} \sum_{k=-\infty}^{+\infty} \frac{\partial}{\partial y} g_-(x + 2\pi k, y).$$

If $n \in \mathbb{Z}$ satisfies condition (1.3), then the right-hand sides of these equalities have the same sign, which remains unchanged on the period. Lemma 2 is proved. $\square$

**Corollary 1.** *Let $n \in \mathbb{Z}$ satisfy condition* (1.3)*. Then the equality $|\lambda_{n,0}^1| + |\mu_{n,0}^1| = n\rho^{n-1}$ holds.*

P r o o f. The proof follows from Lemma 2 and the chain of relations

$$|\lambda_{n,0}^1| + |\mu_{n,0}^1| = \left|\frac{1}{2\pi}\int_0^{2\pi} \lambda_n^1(t)\,dt\right| + \left|\frac{1}{2\pi}\int_0^{2\pi} \mu_n^1(t)\,dt\right| =$$

$$= \left|\frac{1}{2\pi}\int_0^{2\pi} \lambda_n^1(t)\,dt + \frac{1}{2\pi}\int_0^{2\pi} \mu_n^1(t)\,dt\right| = |\lambda_{n,0}^1 + \mu_{n,0}^1| = n\rho^{n-1}.$$

$\square$

**Lemma 3.** *Let $n \in \mathbb{Z}$ satisfy condition* (1.3)*. Then, for the norm and the deviations of the operator $T_n^1$ given by relations* (1.1)*, the following equalities hold:*

$$\|T_n^1\| = \frac{\rho^{n-1}\,|n\ln\rho + 1|}{r^n|\ln r|}, \tag{2.4}$$

$$U(T_n^1) = \frac{\rho^{n-1}}{|\ln r|}\left|n\ln\frac{r}{\rho} - 1\right|, \tag{2.5}$$

$$\mathcal{U}(T_n^1, r^n) = n\rho^{n-1}. \tag{2.6}$$

P r o o f. Using the definition of the operator $T_n^1$ and Lemma 2, we obtain the following upper bound for the norm:

$$\|T_n^1\| \le \|\Lambda_n^1\|_{L^1(0,2\pi)} = \left|\frac{1}{2\pi r^n}\int_0^{2\pi}\lambda_n^1(t)\,dt\right| = r^{-n}|\lambda_{n,0}^1| = \frac{\rho^{n-1}\,|n\ln\rho+1|}{r^n|\ln r|}.$$

Now equality (2.4) follows from the lower bound due to the functions $r^{-n}f_n(z) = cr^{-n}z^n$, $|c| = 1$.

From equality (2.3) of Lemma 1, we obtain the representation

$$f'(\rho e^{ix}) - (T_n^1 f)(\rho e^{ix}) = (V_n^1 f)(\rho e^{ix}), \quad x \in [0, 2\pi].$$

Then, from the definition of the deviation and taking into account that the inequality $\|f\|_{L^p(\Gamma_1)} \le 1$ holds for functions $f$ from the class $Q$ , we obtain the estimate $U(T_n^1) \le \|V_n^1\|$. Arguing as in the first part of the proof, we can obtain the equality $\|V_n^1\| = |\mu_{n,0}^1|$. To complete the proof of equality(2.5), we note that the deviation $U(T_n^1)$ and the norm of the operator $V_n^1$ are attained at the functions $f_n(z) = cz^n$, $|c| = 1$.

Finally, using the following standard reasoning, we show that equality (2.6) is true. For arbitrary functions $f \in Q$ and $g \in L^p(\Gamma_r)$, we have

$$\mathcal{M}^p(f' - T_n^1 g, \rho) \le \mathcal{M}^p(f' - T_n^1 f, \rho) + \mathcal{M}^p(T_n^1(f-g), \rho) \le U(T_n^1) + \|T_n^1\|\,\mathcal{M}^p(f-g, r).$$

Then the equalities (2.4) and (2.5) and Corollary 1 imply the upper estimate

$$\mathcal{U}(T_n^1, r^n) \le U(T_n^1) + \|T_n^1\|\,r^n = |\mu_{n,0}^1| + |\lambda_{n,0}^1| = n\rho^{n-1}.$$

To obtain a lower bound, it is sufficient to consider $f(z) = f_n(z) = cz^n$ and $g \equiv 0$. The lemma is proved. $\square$

**Lemma 4.** *For an arbitrary $n \in \mathbb{Z}$, the following inequalities hold:*

$$\omega(r^n) \ge n\rho^{n-1}, \tag{2.7}$$

$$\Delta\left(\frac{\rho^{n-1}\,|n\ln\rho+1|}{r^n|\ln r|}\right) \ge \frac{\rho^{n-1}}{|\ln r|}\left|n\ln\frac{r}{\rho} - 1\right|. \tag{2.8}$$

P r o o f. The function $f_n(z) = z^n$ belongs to the class $Q$. Then the following inequality holds:

$$\omega(r^n) \ge \mathcal{M}^p(f_n', \rho) = n\rho^{n-1}.$$

We have

$$\Delta(N) = \sup\{\omega(\delta) - N\delta \,:\, \delta \ge 0\} \ge \omega(r^n) - Nr^n \ge n\rho^{n-1} - Nr^n.$$

Substituting

$$N = \frac{\rho^{n-1}\,|n\ln\rho+1|}{r^n|\ln r|}$$

into the latter inequality and using Corollary 1, we obtain inequality (2.8). Lemma 4 is proved. $\square$

## 3. Proof of the main results

P r o o f of Theorem 1. Assume that the parameter $N$ has the representation

$$N = \frac{\rho^{n-1}\,|n\ln\rho + 1|}{r^n|\ln r|},$$

in which $n \in \mathbb{Z}$ satisfies (1.3). Combining inequalities (0.3) from Theorem A, (2.8) from Lemma 4, and equality (2.5) from Lemma 3, we obtain the chain of relations

$$\frac{\rho^{n-1}}{|\ln r|}\left|n\ln\frac{r}{\rho} - 1\right| \le \Delta(N) \le E(N) \le U(T_n^1) = \frac{\rho^{n-1}}{|\ln r|}\left|n\ln\frac{r}{\rho} - 1\right|.$$

Hence,

$$E(N) = \frac{\rho^{n-1}}{|\ln r|}\left|n\ln\frac{r}{\rho} - 1\right|.$$

This means that the operator $T_n^1$ is extremal in Problem 1. Theorem 1 is proved. □

P r o o f of Theorem 2. Let $\delta_n = r^n$, where $n \in \mathbb{Z}$ satisfies condition (1.3). Combining inequalities (0.6) from Theorem B, (2.7) from Lemma 4, and equality (2.6) from Lemma 3, we obtain the chain of relations

$$n\rho^{n-1} \le \omega(\delta_n) \le \mathcal{E}_{\mathcal{O}}(\delta_n) \le \mathcal{E}_{\mathcal{L}}(\delta_n) = \mathcal{E}_{\mathcal{B}}(\delta_n) \le \mathcal{U}(T_n^1, \delta_n) = n\rho^{n-1}.$$

Hence,

$$\omega(\delta_n) = \mathcal{E}_{\mathcal{O}}(\delta_n) \le \mathcal{E}_{\mathcal{L}}(\delta_n) = \mathcal{E}_{\mathcal{B}}(\delta_n) = n\rho^{n-1}.$$

This means that the (bounded linear) operator $T_n^1$ is extremal in Problem 3. Theorem 2 is proved. □

## 4. Generalization of the extremal operator and Theorem 1

It is proved in Lemma 2 that, if $n \in \mathbb{Z}$ satisfies condition (1.3), then the continuous $2\pi$-periodic functions $\lambda_n^1$ and $\mu_n^1$ do not vanish on $[0, 2\pi]$, more precisely, $\lambda_n^1(t)\mu_n^1(t) > 0$, $t \in [0, 2\pi]$. This means that there exists an interval $I_n$ (of positive length) defined by the equality

$$I_n = \left\{\gamma \in \mathbb{R} \,:\, (\lambda_n^1(t) + \gamma)(\mu_n^1(t) - \gamma) > 0,\, t \in [0, 2\pi]\right\}.$$

The interval $I_n = (\gamma_n^-, \gamma_n^+)$ has the boundary points

$$\gamma_n^- = \max_{t\in[0,2\pi]} \min\{-\lambda_n^1(t), \mu_n^1(t)\}, \quad \gamma_n^+ = \min_{t\in[0,2\pi]} \max\{-\lambda_n^1(t), \mu_n^1(t)\}$$

related by the inequality $\gamma_n^- < 0 < \gamma_n^+$. Let $S_n$ be the interval $[\gamma_n^-, \gamma_n^+]$.

We define a (convolution) operator $T_{n,\gamma}^1 = T_{n,\gamma}^1[\rho, r]$, $n \in \mathbb{Z}$, from $L^p(\Gamma_r)$ to $L^p(\Gamma_\rho)$ by the formula

$$(T_{n,\gamma}^1 f)(\rho e^{ix}) = e^{-ix}\,\frac{1}{2\pi}\int_0^{2\pi} \Lambda_{n,\gamma}^1(x - t) f(re^{it})\,dt \tag{4.1}$$

with the kernel

$$\Lambda_{n,\gamma}^1(t) = r^{-n}e^{int}\left(\lambda_n^1(t) + \gamma\right). \tag{4.2}$$

The following statement is a generalization of Theorem 1.

**Theorem 3.** *Assume that the parameter $N$ has the representation*

$$N = \frac{1}{r^n}\left|\frac{\rho^{n-1}(n\ln\rho + 1)}{\ln r} + \gamma\right|,$$

*in which $n \in \mathbb{Z}$ satisfies* (1.3) *and $\gamma \in S_n$. Then, quantity* (0.1) *satisfies the equality*

$$E(N) = \left|\frac{\rho^{n-1}(n\ln(r/\rho) - 1)}{\ln r} - \gamma\right|.$$

*In this case, the operator $T^1_{n,\gamma}$ defined by* (4.1) *and* (4.2) *is extremal in problem* (0.1).

P r o o f. The theorem can be proved by the scheme of the proof of Theorem 1. □

*Remark 1.* In the case when $n \in \mathbb{Z}$ satisfies (1.3) and $\gamma \in S_n$, the operators $T^1_{n,\gamma}$ defined by (4.1) and (4.2) are also extremal in Problem 3. However, these operators do not give solutions of this problem in new cases. More precisely, the equality $\mathcal{U}(T^1_{n,\gamma}, r^n) = n\rho^{n-1}$ holds.

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# ON Λ-CONVERGENCE ALMOST EVERYWHERE OF MULTIPLE TRIGONOMETRIC FOURIER SERIES[1]

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**Abstract:** We consider one type of convergence of multiple trigonometric Fourier series intermediate between the convergence over cubes and the $\lambda$-convergence for $\lambda > 1$. The well-known result on the almost everywhere convergence over cubes of Fourier series of functions from the class $L(\ln^+ L)^d \ln^+ \ln^+ \ln^+ L([0, 2\pi)^d)$ has been generalized to the case of the $\Lambda$-convergence for some sequences $\Lambda$.

**Key words:** Trigonometric Fourier series, Rectangular partial sums, Convergence almost everywhere.

Suppose that $d$ is a natural number, $\mathbb{T}^d = [-\pi, \pi)^d$ is a $d$-dimensional torus, and $\varphi\colon\ [0, +\infty) \to [0, +\infty)$ is a nondecreasing function. Let $\varphi(L)(\mathbb{T}^d)$ be the set of all Lebesgue measurable real-valued functions $f$ on the torus $\mathbb{T}^d$ such that

$$\int\limits_{\mathbb{T}^d} \varphi(|f(\mathbf{t})|)d\mathbf{t} < \infty.$$

Let $f \in L(\mathbb{T}^d)$, $\mathbf{k} = (k^1, k^2, \ldots, k^d) \in \mathbb{Z}^d$, $\mathbf{x} = (x^1, x^2, \ldots, x^d) \in \mathbb{R}^d$, and $\mathbf{kx} = k^1x^1 + k^2x^2 + \ldots + k^dx^d$. Denote by

$$c_{\mathbf{k}} = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{T}^d} f(\mathbf{t})e^{-i\mathbf{kt}}\, d\mathbf{t}$$

the $\mathbf{k}$th Fourier coefficient of the function $f$ and by

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} e^{i\mathbf{kx}} \qquad (1)$$

the multiple trigonometric Fourier series of the function $f$.

Let $\mathbf{n} = (n^1, n^2, \ldots, n^d)$ be a vector with nonnegative integer coordinates, and let $S_{\mathbf{n}}(f, \mathbf{x})$ be the $\mathbf{n}$th rectangular partial sum of series (1):

$$S_{\mathbf{n}}(f, \mathbf{x}) = \sum_{\mathbf{k}=(k^1,\ldots,k^d)\colon\, |k^j|\le n^j,\, 1\le j\le d} c_{\mathbf{k}} e^{i\mathbf{kx}}.$$

Denote by mes$E$ the Lebesgue measure of a set $E$ and let $\ln^+ u = \ln(u + e)$, $u \ge 0$.

In 1915, in the case $d = 1$, N.N. Luzin (see [1]) suggested that the trigonometric Fourier series of any function from $L^2(\mathbb{T})$ converges almost everywhere. A.N. Kolmogorov [2] constructed an example of a function $F \in L(\mathbb{T})$ whose trigonometric series diverges almost everywhere and, later on [3], of a function from $L(\mathbb{T})$ with the Fourier series divergent everywhere on $\mathbb{T}$. L. Carleson [4] proved that Luzin's conjecture is true: if $f \in L^2(\mathbb{T})$, then the Fourier series of the function $f$ converges almost

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everywhere. R. Hunt [5] generalized the statement about the almost everywhere convergence of the Fourier series to the class $L(\ln^+ L)^2(\mathbb{T})$, particularly, to $L^p(\mathbb{T})$ with $p > 1$. P. Sjölin [6] generalized it to the wider class $L(\ln^+ L)(\ln^+ \ln^+ L)(\mathbb{T})$. In [7], the author showed that the condition $f \in L(\ln^+ L)(\ln^+ \ln^+ \ln^+ L)(\mathbb{T})$ is also sufficient for the almost everywhere convergence of the Fourier series of the function $f$. At present, the best negative result in this direction belongs to S.V. Konyagin [8]: if a function $\varphi(u)$ satisfies the condition $\varphi(u) = o(u\sqrt{\ln u/\ln\ln u})$ as $u \to +\infty$, then, in the class $\varphi(L)(\mathbb{T})$, there exists a function with the Fourier series divergent everywhere on $\mathbb{T}$.

Let us now consider the case $d \geq 2$, i.e., the case of multiple Fourier series. Let $\lambda \geq 1$. A multiple Fourier series of a function $f$ is called $\lambda$-convergent at a point $\mathbf{x} \in \mathbb{T}^d$ if there exists a limit

$$\lim_{\min\{n^j:1\leq j\leq d\}\to+\infty} S_{\mathbf{n}}(f, \mathbf{x})$$

considered only for vectors $\mathbf{n} = (n^1, n^2, \ldots, n^d)$ such that $1/\lambda \leq n^i/n^j \leq \lambda, \quad 1 \leq i, j \leq d$. The $\lambda$-convergence is called the convergence over cubes (the convergence over squares for $d = 2$) in the case $\lambda = 1$ and the Pringsheim convergence in the case $\lambda = +\infty$, i. e., in the case without any restrictions on the relation between coordinates of vectors $\mathbf{n}$.

N.R. Tevzadze [9] proved that, if $f \in L^2(\mathbb{T}^2)$, then the Fourier series of the function $f$ converges over cubes almost everywhere. Ch. Fefferman [10] generalized this result to functions from $L^p(\mathbb{T}^d)$, $p > 1$, $d \geq 2$. P. Sjölin [11] showed that, if a function $f$ is from the class $L(\ln^+ L)^d(\ln^+ \ln^+ L)(\mathbb{T}^d)$, $d \geq 2$, then its Fourier series converges over cubes almost everywhere. The author [12] (see also [13]) proved the almost everywhere convergence over cubes of Fourier series of functions from the class $L(\ln^+ L)^d(\ln^+ \ln^+ \ln^+ L)(\mathbb{T}^d)$. The best current result concerning the divergence over cubes on a set of positive measure of multiple Fourier series of functions from $\varphi(L)(\mathbb{T}^d)$, $d \geq 2$, belongs to S.V. Konyagin [14]: for any function $\varphi(u) = o(u(\ln u)^{d-1} \ln\ln u)$ as $u \to +\infty$, there exists a function $F \in \varphi(L)(\mathbb{T}^d)$ with the Fourier series divergent over cubes everywhere.

On the other hand, Ch. Fefferman [15] constructed an example of a continuous function of two variables, i. e., a function from $C(\mathbb{T}^2)$ whose Fourier series diverges in the Pringsheim sense everywhere on $\mathbb{T}^2$. M. Bakhbukh and E.M. Nikishin [16] proved that there exists $F \in C(\mathbb{T}^2)$ such that its modulus of continuity satisfies the condition $\omega(F, \delta) = O\left(\ln^{-1}(1/\delta)\right)$ as $\delta \to +0$ and its Fourier series diverges in the Pringsheim sense almost everywhere. A.N. Bakhvalov [17] established that, for $m \in \mathbb{N}$ and any $\lambda > 1$, there is a function $F \in C(\mathbb{T}^{2m})$ such that the Fourier series of $F$ is $\lambda$-divergent everywhere and the modulus of continuity of $F$ satisfies the condition

$$\omega(F, \delta) = O\left(\ln^{-m}(1/\delta)\right), \quad \delta \to +0. \tag{2}$$

Later on, Bakhvalov [18] proved the existence of a function $F \in C(\mathbb{T}^{2m})$ satisfying condition (2) and such that its Fourier series is $\lambda$-divergent for all $\lambda > 1$ simultaneously.

Let $\Lambda = \{\lambda_\nu\}_{\nu=1}^{\infty}$ be a nonincreasing sequence of positive numbers. Assume that

$$\Omega_\Lambda = \left\{\mathbf{n} = (n^1, n^2, \ldots, n^d) \in \mathbb{N}^d : \frac{1}{1+\lambda_{n^i}} \leq \frac{n^i}{n^j} \leq 1 + \lambda_{n^j}, \quad 1 \leq i, j \leq d\right\}.$$

We will say that a multiple Fourier series of a function $f \in L(\mathbb{T}^d)$ is $\Lambda$-convergent at a point $\mathbf{x} \in \mathbb{T}^d$ if there exists a limit

$$\lim_{\mathbf{n}\in\Omega_\Lambda,\, \min\{n^j:1\leq j\leq d\}\to\infty} S_{\mathbf{n}}(f, \mathbf{x}).$$

Let us note that, if $\lambda_\nu \equiv \lambda - 1$ for some $\lambda > 1$, then the condition of $\Lambda$-convergence turns into the condition of $\lambda$-convergence defined above. And if $\lambda_\nu \to 0$ as $\nu \to \infty$, then the condition of $\Lambda$-convergence is weaker than the condition of $\lambda$-convergence for any $\lambda > 1$.

The author proved [19] that, if a sequence $\Lambda = \{\lambda_\nu\}_{\nu=1}^\infty$ satisfies the condition $\ln^2 \lambda_\nu = o(\ln \nu)$ as $\nu \to \infty$, then there exists a function $F \in C(\mathbb{T}^2)$ such that its Fourier series is $\Lambda$-divergent almost everywhere on $\mathbb{T}^2$.

In the present paper, we obtain the following statement that strengthens the result of [12].

**Theorem 1.** *Assume that a nonincreasing sequence of positive numbers $\Lambda = \{\lambda_\nu\}_{\nu=1}^\infty$ satisfies the condition*

$$\lambda_\nu = O\left(\frac{1}{\nu}\right) \tag{3}$$

*and a function $\varphi\colon\ [0,+\infty) \to [0,+\infty)$ is convex on $[0,+\infty)$ and such that $\varphi(0) = 0$, $\varphi(u)u^{-1}$ increases on $[u_0,+\infty)$, and $\varphi(u)u^{-1-\delta}$ decreases on $[u_0,+\infty)$ for some $u_0 \ge 0$ and any $\delta > 0$. Assume that the trigonometric Fourier series of any function $g \in \varphi(L)(\mathbb{T})$ converges almost everywhere on $\mathbb{T}$. Then, for any $d \ge 2$, the Fourier series of any function $f$ from the class $\varphi(L)(\ln^+ L)^{d-1}(\mathbb{T}^d)$ is $\Lambda$-convergent almost everywhere on $\mathbb{T}^d$.*

Theorem 1 and the result of paper [7] imply the following statement.

**Theorem 2.** *Let a nonincreasing sequence of positive numbers $\Lambda = \{\lambda_\nu\}_{\nu=1}^\infty$ satisfy condition (3), $d \ge 2$. Then the Fourier series of any function $f$ from the class*

$$L(\ln^+ L)^d(\ln^+ \ln^+ \ln^+ L)(\mathbb{T}^d)$$

*is $\Lambda$-convergent almost everywhere on $\mathbb{T}^d$.*

P r o o f of Theorem 1. Let a sequence $\Lambda = \{\lambda_\nu\}_{\nu=1}^\infty$ and a function $\varphi$ satisfy the conditions of the theorem. Let $\varphi_d(u) = \varphi(u)(\ln^+ u)^{d-1}$ for short. Without loss of generality, we can consider only functions $\varphi_d$ such that the functions $\varphi_d(\sqrt{u})$ are concave on $[0,+\infty)$. Otherwise, we can consider the functions $\varphi_d(u + a_d) - b_d$ (with appropriate constants $a_d$ and $b_d$) instead of $\varphi_d$. The corresponding class $\varphi_d(L)(\mathbb{T}^d)$ will be the same in this case.

Denote by $S_n(f,\mathbf{x})$ the $n$th cubic partial sum of the Fourier series of the function $f$:

$$S_n(f,\mathbf{x}) = S_{\mathbf{n}}(f,\mathbf{x}), \qquad \text{where} \quad \mathbf{n} = (n,\dots,n).$$

Suppose that

$$M(f,\mathbf{x}) = \sup_{n\in\mathbb{N}} |S_n(f,\mathbf{x})|,$$

$$M_\Lambda(f,\mathbf{x}) = \sup_{\mathbf{n}\in\Omega_\Lambda} |S_{\mathbf{n}}(f,\mathbf{x})|.$$

Under the conditions of the theorem (see [12, formula (3.1) and Lemma 3]), there are constants $K_d > 0$ and $y_d \ge 0$ such that

$$\operatorname{mes}\Big\{\mathbf{x} \in \mathbb{T}^d : M(f,\mathbf{x}) > y\Big\} \le \frac{K_d}{y}\bigg(\int\limits_{\mathbb{T}^d} \varphi_d(|f(\mathbf{x})|)\,d\mathbf{x} + 1\bigg), \quad y > y_d, \quad f \in \varphi_d(L)(\mathbb{T}^d). \tag{4}$$

Using (4), we will prove that, for every $y > y_d$ and $f \in \varphi_d(L)(\mathbb{T}^d)$,

$$\operatorname{mes}\Big\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(f,\mathbf{x}) > y\Big\} \le \frac{A_d}{y}\bigg(\int\limits_{\mathbb{T}^d} \varphi_d(|f(\mathbf{x})|)\,d\mathbf{x} + 1\bigg) \tag{5}$$

and, for every $f \in \varphi_{d+1}(L)(\mathbb{T}^d)$,

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f,\mathbf{x})d\mathbf{x} \le B_d\Big(\int\limits_{\mathbb{T}^d} \varphi_{d+1}(|f(\mathbf{x})|)\,d\mathbf{x} + 1\Big), \tag{6}$$

where $A_d$ is independent of $f$ and $y$; $B_d$ is independent of $f$.

The proof is by induction on $d$. Consider the base case, i. e., $d = 1$: statement (5) immediately follows from (4) because $M(f,\mathbf{x}) = M_\Lambda(f,\mathbf{x})$ in the one-dimensional case. Similarly, (6) is a consequence of [5, Theorem 2].

Let $d \ge 2$. Suppose that statements (5) and (6) hold for $d-1$ and let us show that the same is true for $d$.

First, let us prove the validity of (5). Let $\mathbf{n} = (n^1, n^2, \dots, n^d) \in \Omega_\Lambda$. According to (3), there is an absolute constant $C > 0$ such that $\lambda_\nu \nu \le C$ for all natural numbers $\nu$. Combining this with the definition of $\Omega_\Lambda$, we obtain that, for all $i, j \in \{1, 2, \dots, d\}$,

$$|n^i - n^j| \le C. \tag{7}$$

Recall that, if $\mathbf{n} = (n^1, n^2, \dots, n^d)$, then the following representation holds for the $\mathbf{n}$th rectangular partial sum of the Fourier series of the function $f$:

$$S_{\mathbf{n}}(f,\mathbf{x}) = \frac{1}{\pi^d}\int\limits_{\mathbb{T}^d}\prod_{j=1}^{d} D_{n^j}(t^j) f(x^1+t^1,\dots,x^d+t^d)\,dt^1\dots dt^d, \tag{8}$$

where $D_n(t) = \sin((n+1/2)t)/(2\sin(t/2))$ is the one-dimensional Dirichlet kernel of order $n$. Let us add to and subtract from the $d$-dimensional Dirichlet kernel $\prod\limits_{j=1}^{d} D_{n^j}(t^j)$ of order $\mathbf{n}$ the sum

$$\sum_{k=2}^{d}\Big(\prod_{j=1}^{k} D_{n^1}(t^j)\prod_{j=k+1}^{d} D_{n^j}(t^j)\Big)$$

(here and in what follows, we suppose that all products $\prod$ with an upper index less than a lower one are equal to 1). Rearranging the terms, we obtain

$$\prod_{j=1}^{d} D_{n^j}(t^j) = \sum_{k=1}^{d-1}\Big(\prod_{j=1}^{k} D_{n^1}(t^j)\prod_{j=k+1}^{d} D_{n^j}(t^j) - \prod_{j=1}^{k+1} D_{n^1}(t^j)\prod_{j=k+2}^{d} D_{n^j}(t^j)\Big) + \prod_{j=1}^{d} D_{n^1}(t^j) =$$
$$= \sum_{k=2}^{d}\Big(\prod_{j=1}^{k-1} D_{n^1}(t^j)\prod_{j=k+1}^{d} D_{n^j}(t^j)\Big(D_{n^k}(t^k) - D_{n^1}(t^k)\Big)\Big) + \prod_{j=1}^{d} D_{n^1}(t^j).$$

From this and (8), it follows that

$$S_{\mathbf{n}}(f,\mathbf{x}) = \sum_{k=2}^{d} \frac{1}{\pi^d} \int\limits_{\mathbb{T}^d} \Bigg( \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) \Big( D_{n^k}(t^k) - D_{n^1}(t^k) \Big) \Bigg) \times$$

$$\times f(x^1+t^1,\dots,x^d+t^d)\,dt^1\dots dt^d + \frac{1}{\pi^d} \int\limits_{\mathbb{T}^d} \prod_{j=1}^{d} D_{n^1}(t^j) f(x^1+t^1,\dots,x^d+t^d)\,dt^1\dots dt^d =$$

$$= \sum_{k=2}^{d} \frac{1}{\pi^d} \int\limits_{\mathbb{T}} \Big( D_{n^k}(t^k) - D_{n^1}(t^k) \Big) \times$$

$$\times \Bigg( \int\limits_{\mathbb{T}^{d-1}} \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) f(x^1+t^1,\dots,x^d+t^d)\,dt^1\dots dt^{k-1}dt^{k+1}\dots dt^d \Bigg)\,dt^k + S_{n^1}(f,\mathbf{x}). \tag{9}$$

Note that the latter term on the right hand side of (9) is the $n^1$th cubic partial sum of the Fourier series of the function $f$. By (7), for all $k \in \{2,3,\dots,d\}$ and $t \in \mathbb{T}$, we have $|D_{n^k}(t) - D_{n^1}(t)| \le C$. Combining this with (9), we obtain

$$|S_{\mathbf{n}}(f,\mathbf{x})| \le \sum_{k=2}^{d} \frac{C}{\pi^d} \int\limits_{\mathbb{T}} \Bigg| \int\limits_{\mathbb{T}^{d-1}} \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) \times$$

$$\times f(x^1+t^1,\dots,x^{k-1}+t^{k-1},t^k,x^{k+1}+t^{k+1},\dots,x^d+t^d)\,dt^1\dots dt^{k-1}dt^{k+1}\dots dt^d \Bigg|\,dt^k + |S_{n^1}(f,\mathbf{x})|.$$

Applying the definitions of $M_\Lambda(f,\mathbf{x})$ and $M(f,\mathbf{x})$, from the latter estimate, we obtain

$$M_\Lambda(f,\mathbf{x}) \le M(f,\mathbf{x}) + \frac{C}{\pi} \sum_{k=2}^{d} \int\limits_{\mathbb{T}} \sup_{\mathbf{n}=(n^1,n^2,\cdots,n^d)\in\Omega_\Lambda} \Bigg| \frac{1}{\pi^{d-1}} \int\limits_{\mathbb{T}^{d-1}} \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) \times$$

$$\times f(x^1+t^1,\dots,x^{k-1}+t^{k-1},t^k,x^{k+1}+t^{k+1},\dots,x^d+t^d)\,dt^1\dots dt^{k-1}dt^{k+1}\dots dt^d \Bigg|\,dt^k = \tag{10}$$

$$= M(f,\mathbf{x}) + \frac{C}{\pi} \sum_{k=2}^{d} M_k(f,\mathbf{x}),$$

where $M_k(f,\mathbf{x})$ denotes the $k$th term of the sum on the left hand side of the equality in (10). Let $k \in \{2,3,\dots,d\}$. Consider $M_k(f,\mathbf{x})$. Denote by $g_{k,t^k}$ the function of $d-1$ variables that can be obtained from the function $f$ by fixing the $k$th variable $t^k$:

$$g_{k,t^k}(t^1,\dots,t^{k-1},t^{k+1},\dots,t^d) = f(t^1,\dots,t^{k-1},t^k,t^{k+1},\dots,t^d), \quad (t^1,\dots,t^{k-1},t^{k+1},\dots,t^d) \in \mathbb{T}^{d-1}.$$

Define $\Omega'_\Lambda$ as the set of $\mathbf{m}_k = (m^1,\dots,m^{k-1},m^{k+1},\dots,m^d) \in \mathbb{N}^{d-1}$ such that $\mathbf{m} = (m^1,\dots,m^d) \in \Omega_\Lambda$. Note that, in view of the invariance of $\Omega_\Lambda$ with respect to a rearrangement of variables, the set $\Omega'_\Lambda$ is independent of $k$. Suppose that $\mathbf{n}'_k = (n^1,\dots,n^1,n^{k+1},\dots,n^d) \in \mathbb{N}^{d-1}$. Then

$$\frac{1}{\pi^{d-1}} \int\limits_{\mathbb{T}^{d-1}} \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) \times$$

$$\times f(x^1+t^1,\dots,x^{k-1}+t^{k-1},t^k,x^{k+1}+t^{k+1},\dots,x^d+t^d)\,dt^1\dots dt^{k-1}dt^{k+1}\dots dt^d =$$

$$= S_{\mathbf{n}'_k}\left(g_{k,t^k}, (x^1,\dots,x^{k-1},x^{k+1},\dots,x^d)\right)$$

and

$$M_k(f,\mathbf{x}) = \int\limits_{\mathbb{T}} \sup_{\mathbf{n}'_k\in\Omega'_\Lambda} \left| S_{\mathbf{n}'_k}\left(g_{k,x^k}, (x^1,\dots,x^{k-1},x^{k+1},\dots,x^d)\right)\right| \, dx^k.$$

Further,

$$\operatorname{mes}\left\{\mathbf{x}\in\mathbb{T}^d : M_k(f,\mathbf{x})>y\right\} = 2\pi \operatorname{mes}\left\{(x^1,\dots,x^{k-1},x^{k+1},\dots,x^d)\in\mathbb{T}^{d-1} : M_k(f,\mathbf{x})>y\right\} \le$$
$$\le \frac{2\pi}{y}\int\limits_{\mathbb{T}^{d-1}} M_k(f,\mathbf{x})\,dx^1\dots dx^{k-1}dx^{k+1}\dots dx^d =$$
$$= \frac{2\pi}{y}\int\limits_{\mathbb{T}^d} \sup_{\mathbf{n}'_k\in\Omega'_\Lambda} \left| S_{\mathbf{n}'_k}\left(g_{k,x^k}, (x^1,\dots,x^{k-1},x^{k+1},\dots,x^d)\right)\right| \, d\mathbf{x} =$$
$$= \frac{2\pi}{y}\int\limits_{\mathbb{T}}\left(\int\limits_{\mathbb{T}^{d-1}} \sup_{\mathbf{n}'_k\in\Omega'_\Lambda} \left| S_{\mathbf{n}'_k}\left(g_{k,x^k}, (x^1,\dots,x^{k-1},x^{k+1},\dots,x^d)\right)\right| dx^1\dots dx^{k-1}dx^{k+1}\dots dx^d\right) dx^k. \tag{11}$$

From this, applying the induction hypothesis (more precisely, statement (6) for the dimension $d-1$) to the inner integral on the right hand part of (11), we obtain

$$\operatorname{mes}\left\{\mathbf{x}\in\mathbb{T}^d : M_k(f,\mathbf{x})>y\right\} \le \frac{2\pi}{y}\int\limits_{\mathbb{T}}\left(B_{d-1}\int\limits_{\mathbb{T}^{d-1}} \varphi_d(|f(\mathbf{x})|)\,dx^1\dots dx^{k-1}dx^{k+1}\dots dx^d + 1\right) dx^k \le$$
$$\le \frac{(2\pi)^2 B_{d-1}}{y}\left(\int\limits_{\mathbb{T}^d}\varphi_d(|f(\mathbf{x})|)\,d\mathbf{x}+1\right). \tag{12}$$

According to (10),

$$\left\{\mathbf{x}\in\mathbb{T}^d : M_\Lambda(f,\mathbf{x})>y\right\} \subset \left\{\mathbf{x}\in\mathbb{T}^d : M(f,\mathbf{x})>\frac{y}{2}\right\}\bigcup\left(\bigcup_{k=2}^{d}\left\{\mathbf{x}\in\mathbb{T}^d : M_k(f,\mathbf{x})>\frac{\pi y}{2(d-1)C}\right\}\right). \tag{13}$$

Combining (13), (4) and (12), we obtain (5) with the constant $A_d = 2K_d + 8\pi(d-1)^2 B_{d-1}C$.

Now, we only need to prove the validity of statement (6). To this end, let us use statement (5) proved above.

From (5), it follows that the majorant $M_\Lambda(f,\mathbf{x})$ is finite almost everywhere on $\mathbb{T}^d$ for all $f\in\varphi_d(L)(\mathbb{T}^d)$, in particular, for all $f\in L^2(T^d)$. Applying Stein's theorem on limits of sequences of operators [20, Theorem 1], we see that the operator $M_\Lambda(f,\cdot)$ is of weak type $(2,2)$, i.e., there is a constant $A_d^2>0$ such that, for all $y>0$ and $f\in L^2(T^d)$,

$$\operatorname{mes}\left\{\mathbf{x}\in\mathbb{T}^d : M_\Lambda(f,\mathbf{x})>y\right\} \le \frac{A_d^2}{y^2}\int\limits_{\mathbb{T}^d}|f(\mathbf{x})|^2\,d\mathbf{x}. \tag{14}$$

Similarly, from [20, Theorem 3], we can obtain the following refinement of statement (5): there is a constant $\bar{A}_d>0$ such that, for all $y\ge \bar{y}_d/2=\bar{A}_d$ and $f\in\varphi_d(L)(\mathbb{T}^d)$,

$$\operatorname{mes}\left\{\mathbf{x}\in\mathbb{T}^d : M_\Lambda(f,\mathbf{x})>y\right\} \le \int\limits_{\mathbb{T}^d}\varphi_d\left(\frac{\bar{A}_d|f(\mathbf{x})|}{y}\right)\,d\mathbf{x} \le \frac{\bar{A}_d}{y}\int\limits_{\mathbb{T}^d}\varphi_d(|f(\mathbf{x})|)\,d\mathbf{x}. \tag{15}$$

Further, let $f \in \varphi_d(L)(\mathbb{T}^d)$ and $y > 0$. Suppose that

$$g(x) = g_y(x) = \begin{cases} f(x), & |f(x)| > y, \\ 0, & |f(x)| \le y; \end{cases} \qquad h(x) = h_y(x) = f(x) - g(x).$$

Define $\lambda_f(y) = \operatorname{mes}\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(f, \mathbf{x}) > y\}$. Then

$$\lambda_f(y) \le \operatorname{mes}\left\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(g, \mathbf{x}) > y/2\right\} + \operatorname{mes}\left\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(h, \mathbf{x}) > y/2\right\} = \lambda_g(y/2) + \lambda_h(y/2).$$

From this, using the equality

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x})\, d\mathbf{x} = -\int\limits_0^\infty y\, d\lambda_f(y) = \int\limits_0^\infty \lambda_f(y)\, dy$$

(see, for example, [21, Chapter 1, § 13, formula (13.6)]), we obtain

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x})\, d\mathbf{x} \le \bar{y}_d (2\pi)^d + \int\limits_{\bar{y}_d}^\infty \lambda_f(y)\, dy \le \bar{y}_d (2\pi)^d + \int\limits_{\bar{y}_d}^\infty \lambda_g\left(\frac{y}{2}\right) dy + \int\limits_{\bar{y}_d}^\infty \lambda_h\left(\frac{y}{2}\right) dy. \tag{16}$$

Taking into account that $g \in \varphi_d(L)(\mathbb{T}^d)$ and $h \in L^\infty(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$ and applying estimate (15) to $\lambda_g(y/2)$ and estimate (14) to $\lambda_h(y/2)$, from (16), we obtain

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x})\, d\mathbf{x} \le \bar{y}_d (2\pi)^d + 2\bar{A}_d \int\limits_{\bar{y}_d}^\infty \left( \frac{1}{y} \int\limits_{\mathbb{T}^d} \varphi_d(|g(\mathbf{t})|)\, d\mathbf{t} \right) dy + 4A_d^2 \int\limits_{\bar{y}_d}^\infty \left( \frac{1}{y^2} \int\limits_{\mathbb{T}^d} |h(\mathbf{t})|^2\, d\mathbf{t} \right) dy =$$

$$= \bar{y}_d (2\pi)^d + 2\bar{A}_d \int\limits_{\bar{y}_d}^\infty \left( \frac{1}{y} \int\limits_{\{\mathbf{t} \in \mathbb{T}^d:\ |f(\mathbf{t})| > y\}} \varphi_d(|f(\mathbf{t})|)\, d\mathbf{t} \right) dy + 4A_d^2 \int\limits_{\bar{y}_d}^\infty \left( \frac{1}{y^2} \int\limits_{\{\mathbf{t} \in \mathbb{T}^d:\ |f(\mathbf{t})| \le y\}} |f(\mathbf{t})|^2\, d\mathbf{t} \right) dy. \tag{17}$$

Applying Fubibi's theorem to the integrals on the right hand side of (17), we conclude that

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x})\, d\mathbf{x} \le 2\bar{A}_d \int\limits_{\{\mathbf{t} \in \mathbb{T}^d:\ |f(\mathbf{t})| > \bar{y}_d\}} \varphi_d(|f(\mathbf{t})|) \left( \int\limits_{\bar{y}_d}^{|f(\mathbf{t})|} \frac{dy}{y} \right) d\mathbf{t} +$$

$$+ 4A_d^2 \int\limits_{\mathbb{T}^d} |f(\mathbf{t})|^2 \left( \int\limits_{|f(\mathbf{t})|}^\infty \frac{dy}{y^2} \right) d\mathbf{t} + \bar{y}_d (2\pi)^d,$$

hence, statement (6) follows easily.

Finally, the $\Lambda$-convergence of the Fourier series of an arbitrary function from the class $\varphi_d(L)(\mathbb{T}^d)$ can be obtained from (5) by means of standard arguments (see, for example, [12, Lemma 3]). Theorem 1 is proved. □

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# A CHARACTERIZATION OF EXTREMAL ELEMENTS IN SOME LINEAR PROBLEMS[1]

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**Abstract:** We give a characterization of elements of a subspace of a complex Banach space with the property that the norm of a bounded linear functional on the subspace is attained at those elements. In particular, we discuss properties of polynomials that are extremal in sharp pointwise Nikol'skii inequalities for algebraic polynomials in a weighted $L_q$-space on a finite or infinite interval.

**Key words:** Complex Banach space, Bounded linear functional on a subspace, Algebraic polynomial, Pointwise Nikol'skii inequality.

## 1. Bounded linear functionals in complex Banach spaces

### 1.1. Introduction. Statement of the problem

Let $X = X_{\mathbb{C}}$ be a complex Banach space (more precisely, a Banach space over the field $\mathbb{C}$ of complex numbers), let $S(X)$ be its unit sphere, and let $X^* = X^*_{\mathbb{C}}$ be the dual space of $X$, i.e., the space of complex-valued bounded linear (over the field $\mathbb{C}$ of complex numbers) functionals $F$ on $X$ with the norm

$$\|F\|_{X^*} = \sup\{|F(x)|\colon x \in X,\ \|x\|_X = 1\}.$$

Let $P$ be a (closed) subspace of $X$, and let $\psi$ be a bounded linear functional on $P$. We denote by

$$D(\psi; P) = \sup\{|\psi(p)|\colon p \in P,\ \|p\|_X = 1\} \tag{1.1}$$

the norm of the functional $\psi$ on the subspace $P$. In what follows, we assume that $\psi \not\equiv 0$, so that $D(\psi; P) > 0$. The value $D(\psi; P)$ is the smallest possible (the best) constant in the inequality

$$|\psi(p)| \le D(\psi; P)\|p\|_X, \quad p \in P. \tag{1.2}$$

Nonzero elements $p$ of the subspace $P$ with the property that inequality (1.2) turns into an equality for them (if such elements exist) will be called extremal elements in this inequality. Elements $p$ of the unit sphere $S(P) = S(X) \cap P$ of the subspace $P$ that solve problem (1.1), i.e., those with the property that the supremum in (1.1) is attained at $p$, will be called extremal elements in problem (1.1). We will use the same terminology also in other similar situations. It is clear that an element $\varrho \in P$ is extremal in inequality (1.2) if and only if the element $\varrho/\|\varrho\|_X$ is extremal in problem (1.1). In this sense, extremal elements in problem (1.1) and inequality (1.2) coincide. The aim of this papers is exactly to characterize extremal elements in inequality (1.2) or in problem (1.1), which is the same.

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On the set

$$P[1](\psi) = \{p \in P \colon \psi(p) = 1\} \tag{1.3}$$

of elements of $P$ where the functional $\psi$ takes the value 1, we consider the value

$$\Delta(\psi; P) = \inf\{\|p\|_X : p \in P[1](\psi)\} \tag{1.4}$$

which is the least deviation of class (1.3) from zero in $X$. It is clear that $\Delta(\psi; P) = 1/D(\psi; P)$. Moreover, extremal elements in problem (1.4) and inequality (1.2) coincide. More precisely, each extremal element of problem (1.4) is extremal in (1.2); conversely, if $\varrho$ is an extremal element of inequality (1.2), then $\varrho/\psi(\varrho)$ is extremal in (1.4). Thus, determining the sharp constant in inequality (1.2) is equivalent to determining the least deviation (1.4) of class (1.3) from zero.

Value (1.4) can be interpreted as the best approximation of an arbitrary element $\rho \in P[1](\psi)$ in the space $X$ by the annihilator

$$P(\psi) = P[0](\psi) = \{p \in P : \ \psi(p) = 0\} \tag{1.5}$$

of the functional $\psi$ in $P$, namely,

$$\Delta(\psi; P) = \inf\{\|\rho - p\|_X : p \in P(\psi)\}. \tag{1.6}$$

There is a rich theory developed to study problems of type (1.4) in real Banach spaces. This theory is based on arguments of duality; see, e.g., [13, Ch. 2]. In order to use this approach in the complex case, however, one needs in addition to discuss some questions of geometry of complex spaces.

In papers [1–4] coauthored by the author of the present paper, the authors studied the Nikol'skii inequality between the uniform norm of a polynomial and its norm in the space $L_q^v = L_q^v(\mathbb{I})$ with a weight $v$ and $1 \le q < \infty$ on the set of algebraic polynomials $\mathscr{P}_n$ of degree at most $n \ge 1$ on a finite or infinite interval $\mathbb{I}$. One of the steps in these investigations was the study of the sharp inequality

$$|p_n(z_0)| \le D\, \|p_n\|_{L_q^v}, \quad p_n \in \mathscr{P}_n, \tag{1.7}$$

for an end point $z_0$ of the interval $\mathbb{I}$. Inequality (1.7) is a special case of (1.2) for $X = L_q^v(\mathbb{I})$, $P = \mathscr{P}_n$, and $\psi(p_n) = p_n(z_0)$. Results of [1–4] related to inequality (1.7) motivated the author to consider problem (1.2).

### 1.2. Main result

We will study problem (1.1) under the following two assumptions.

(R) *Assume that the norm of any bounded linear functional $\phi$ on $P$, i.e., $\phi \in P^*$, is attained at some point $p = p(\phi) \in P$.*

According to James' theorem [11] (see also [10, p. 643], [17, Ch. 1, Sect. 2, Corollary 2.4]), this property is equivalent to the reflexivity of the space $P$. Note that property (R) is fulfilled if the subspace $P$ is finite-dimensional.

If a functional $F \in X^*$, $F \ne 0$, attains its norm at an element $x \in X$, $x \ne 0$, and if $F(x) > 0$, or—which is the same in this case—if

$$F(x) = \|F\|_{X^*}\, \|x\|_X, \tag{1.8}$$

we will say that the functional $F$ possesses the $N$-property at the element $x$, or, shortly, the $N[x]$-property. By the complex variant of the Hahn–Banach Theorem (cf. [8, Ch. II, Sect. 3, Theorem 11] or [12, Ch. III, Sect. 5.4]), a functional with this property always exists. However, it may be not unique. In a complex Banach space, a functional $F \in X^*$ is called a supporting functional at a point $x$ (or, more precisely, a supporting or a tangent functional at a point $x$ to the sphere $S_{\|x\|}(X)$ of radius $\|x\|$ with center at 0), if its real part $f = \operatorname{Re} F$ is a real supporting (tangent) functional, see, e.g., [8, Ch. V, Sect. 9.4]. Indeed, for a functional $F \in X^*$ in a complex Banach space, the properties that the functional possesses the $N$-property at a point $x$ and that its real part is a supporting functional are equivalent; we will discuss this below in Section 1.3. Starting from this point, we will interpret the $N[x]$-property of a functional $F \in X^*$ as a property of the functional $F \in X^*$ to be a supporting functional at the point $x$.

A point $x \in S(X)$ is called a smooth point of the sphere $S(X)$ if there exists only one supporting functional at $x$. If every point of the unit sphere of a space is a smooth point, then the space is called smooth. For details concerning smooth points of the unit sphere and, in general, of convex closed sets in real Banach spaces see, e.g., [7, Ch. I, Sect. 2, Theorems 1, 2] and [6, Ch. VII, Sect. 2]. The smoothness in complex Banach spaces has some special features; it will be discusses in Section 1.3 below.

The second assumption is the following one.

(Γ) *Assume that all points of the unit sphere $S(P) = S(X) \cap P$ of the subspace $P$ are smooth points of the unit sphere $S(X)$ of the space $X$.*

Taking into account that problem (1.1) has the interpretation (1.4) in terms of approximations, one may expect the following result.

**Theorem 1.** *Assume that a Banach space $X$ and its subspace $P$ satisfy properties* (R) *and* (Γ). *Then the norm of a bounded linear functional $\psi$ on $P$ is attained at an element $\varrho \in S(P)$ if and only if the supporting functional $F = F[\varrho] \in X^*$ of the element $\varrho \in P$ vanishes on the set* (1.5), *i.e.,*

$$F[\varrho](s) = 0 \quad \text{for all} \quad s \in P(\psi). \tag{1.9}$$

Under the assumptions of the theorem, an extremal element with the property that the norm of the functional $\psi$ on $P$ is attained at it always exists but it is not necessarily unique; see the example after the proof of Theorem 4 in Section 2.2. To ensure the uniqueness of the extremal element, one needs additional restrictions on the problem. For example, if the space $X$ is strictly normed then the extremal element is unique for every (bounded linear) functional on every subspace.

In the first section of the present paper, Theorem 1 will be proved and discussed. In the second section, Theorem 1 will be applied to obtain a corresponding statement for the pointwise inequality

$$|p_n(z)| \le D(z)\, \|p_n\|_{L^v_q(\mathbb{I})}, \quad p_n \in \mathscr{P}_n,$$

where $z \in \mathbb{I}$. In papers [1–4], extremal polynomials of inequality (1.7) (in the case when $z$ is an end point of an interval) were characterized in terms that are formally different from those of Theorem 1. We will show in Section 2.2 that, in fact, Theorem 2 from [3] and its analogs from [1, 2, 4] follow from Theorem 1.

We will prove Theorem 1 using the natural arguments in terms of duality. However, the fact that $X$ is a complex Banach space causes additional difficulties. In particular, one needs to first discuss the smoothness property for points of the unit sphere $S(X)$ of a (complex) Banach space $X$.

### 1.3. Smoothness in complex Banach spaces

In real Banach spaces, a smooth point of the unit sphere can be for example characterized by the fact that the norm of the space is Gateaux differentiable at this point. For a real Banach space, the strict convexity of the dual space is a sufficient condition for the smoothness of the original space. The inverse statement does not hold in the general case. The smoothness of a space implies the strict convexity of the dual space only for reflexive spaces. Details on these topics can be found, e.g., in [7, Ch. I, Sect. 2, Theorems 1, 2] and [6, Ch. VII, Sect. 2]. In this section, we discuss smoothness in complex Banach spaces. The author neither claims that the results are novel nor that the ideas are original.

Let $X = X_{\mathbb{C}}$ be a complex Banach space. We also may consider this space as a real Banach space $X = X_{\mathbb{R}}$, i.e., a linear space over the field $\mathbb{R}$ of real numbers. Let $X^*_{\mathbb{R}}$ be the corresponding dual (real) Banach space, i.e., the space of real-valued bounded linear (over the field $\mathbb{R}$ of real numbers) functionals on $X_{\mathbb{R}}$.

The following statement is not new, cf. [8, Ch. II, Sect. 3, Theorem 11] or [9, Ch. 10, Sect. 1, Lemma 1.1]. We will give it here in the form we need in what follows. Moreover, it is useful for our purposes to give a proof of this statement.

**Lemma 1.** *The formula*

$$F(x) = f(x) - if(ix), \quad x \in X, \tag{1.10}$$

*where $F \in X^*_{\mathbb{C}}$ and $f \in X^*_{\mathbb{R}}$, sets a one-to-one correspondence between the spaces $X^*_{\mathbb{C}}$ and $X^*_{\mathbb{R}}$. Moreover, mapping* (1.10) *is an isometry, i.e.,*

$$\|F\|_{X^*_{\mathbb{C}}} = \|f\|_{X^*_{\mathbb{R}}}. \tag{1.11}$$

P r o o f. For a complex functional $F \in X^*_{\mathbb{C}}$, we consider its real part $f = \operatorname{Re} F$; it is a functional from $X^*_{\mathbb{R}}$. The functional $F$ is uniquely determined by $f = \operatorname{Re} F$ by means of formula (1.10). Indeed, define $g = -\operatorname{Im} F$, then $F(x) = f(x) - ig(x),\ x \in X$. By the (complex) homogeneity of the functional $F$, we have $F(x) = -iF(ix) = -if(ix) + g(ix),\ x \in X$. Consequently, $g(x) = f(ix)$, which proves representation (1.10).

Conversely, let $f \in X^*_{\mathbb{R}}$. Consider a (complex) functional $F$ given by formula (1.10). Obviously, $F$ is additive. Next we will show that it is (complex) homogeneous. For a point $x \in X$ and a number $\zeta = \alpha + i\beta \in \mathbb{C}$, we have

$$F(\zeta x) = F((\alpha + i\beta)x) = f((\alpha + i\beta)x) - if(i(\alpha + i\beta)x) =$$
$$= \alpha f(x) + \beta f(ix) - i\alpha f(ix) + i\beta f(x) = (\alpha + i\beta)f(x) + (\beta - i\alpha)f(ix) =$$
$$= (\alpha + i\beta)(f(x) - if(ix)) = \zeta F(x).$$

Thus, we see that functional (1.10) is homogeneous.

It follows that formula (1.10) sets a one-to-one correspondence between the complex and the real dual spaces $X^*_{\mathbb{C}}$ and $X^*_{\mathbb{R}}$, respectively.

Now we show that (1.10) is an isometry, i.e., property (1.11) holds. The inequality $\|f\|_{X^*_{\mathbb{R}}} \le \|F\|_{X^*_{\mathbb{C}}}$ is obvious. Further on, for an arbitrary point $x \in X$ and real $\theta$, we have

$$e^{i\theta}F(x) = F\left(e^{i\theta}x\right) = f\left(e^{i\theta}x\right) - if\left(ie^{i\theta}x\right).$$

In particular, for $\theta = -\arg(F(x))$, the latter equality takes the form

$$|F(x)| = f\left(e^{i\theta}x\right) - if\left(ie^{i\theta}x\right) = f\left(e^{i\theta}x\right).$$

Consequently, $|F(x)| \le \|f\|_{X^*_{\mathbb{R}}}\|x\|,\ x \in X$, and therefore the estimate $\|F\|_{X^*_{\mathbb{C}}} \le \|f\|_{X^*_{\mathbb{R}}}$ holds. Thus, (1.11) holds. This proves the lemma. □

All further statements in this section are in fact consequences of Lemma 1.

**Lemma 2.** *A complex functional $F \in X^*_{\mathbb{C}}$ attains its norm at a point $x \in S(X)$ and $F(x) > 0$ if and only if its real part $f = \operatorname{Re} F$ has the same properties.*

P r o o f. Suppose $F$ is as described in the lemma. By (1.8) and (1.11), we have

$$\|F\|_{X^*_{\mathbb{C}}} = \|f\|_{X^*_{\mathbb{R}}} = F(x) = f(x).$$

Consequently, $f$ has the same properties as $F$. Conversely, suppose $f$ has the described properties. Then, by (1.11), we have

$$f(x) \le \sqrt{(f(x))^2 + (f(ix))^2} = |F(x)| \le \|F\|_{X^*_{\mathbb{C}}} = \|f\|_{X^*_{\mathbb{R}}}.$$

Consequently, $f(x)=F(x)=\|F\|_{X^*_{\mathbb{C}}}$; hence, $F(x)>0$. Thus, $F$ has the described properties, too. □

As we have mentioned above, a functional $F \in X^*$ in a complex Banach space is called a supporting functional at a point $x$ (to the sphere $S_{\|x\|}(X)$ of radius $\|x\|$ with center at 0) if its real part $\operatorname{Re} F$ is a (real) supporting functional, cf. [8, Ch. V, Sect. 9.4]. Due to Lemma 2, the $N$-property of the functional $F \in X^*$ at a point $x$ is equivalent to the property that $F \in X^*$ is a supporting functional at this point.

**Theorem 2.** *Assume that the space $X^* = X^*_{\mathbb{C}}$ of complex bounded linear functionals in a complex Banach space $X$ is strictly convex. Then $X$ is smooth.*

P r o o f. Recall that a Banach space is called strongly convex if its unit sphere does not contain any non-degenerate segments, see, e.g. [8, Ch. V, Sect. 11.7]. As we have mentioned above, the statement of the theorem is well-known for real Banach spaces, cf. [7, Ch. I, Sect. 2, Theorems 1 and 2], [6, Ch. VII, Sect. 2].

Using Lemma 1, it is not difficult to see that $X^*_{\mathbb{C}}$ is strongly convex if and only if $X^*_{\mathbb{R}}$ is. Thus, under the assumptions of the theorem, the space $X_{\mathbb{R}}$ is smooth. This means that, at any point $x \in S(X)$, there is only one real bounded linear functional $f$ whose norm is equal to 1 and is attained at $x$, with $f(x) > 0$. By Lemma 1, this implies that, at every point $x \in S(X)$, there is only one functional $F \in X^*_{\mathbb{C}}$ with the unit norm and with the $N$-property at the point $x$. But this means that the space $X = X_{\mathbb{C}}$ is smooth. □

### 1.4. Proof of Theorem 1

Theorem 1 follows from the two auxiliary statements proved below. In what follows, we will suppose without loss of generality that all supporting functionals $F = F[x]$ at points $x \in X$, $x \ne 0$, have the norm $\|F\|_{X^*} = 1$.

#### 1.4.1. Auxiliary statements

**Lemma 3.** *Assume that a Banach space $X$ and its subspace $P$ satisfy properties* (R) *and* ($\Gamma$)*. Let $\varrho \in P$, $\varrho \not\equiv 0$, be an extremal element of problem* (1.1)*, and let $F = F[\varrho] \in S(X^*)$ be the supporting functional at the element $\varrho \in P$. Then the following representation holds*:

$$\psi(p) = \gamma(P)F[\varrho](p), \quad p \in P, \tag{1.12}$$

*where $\gamma(P)$ is a constant with the property $|\gamma(P)| = D(\psi, P)$.*

P r o o f. The functional $\psi$ is a bounded linear functional on the space $P$ endowed with the norm $\|\cdot\|_X$, and the norm of this functional on $P$ is equal to (1.1). By the Hahn–Banach theorem (cf. [8, Ch. II, Sect. 3, Theorem 11] or [12, Ch. III, Sect. 5.4]), the functional $\psi$ can be extended to a functional on the whole space $X$ with the same norm; we denote this extension by $\Psi$.

Since the functional $\Psi$ is an extension of the functional $\psi$ from $P$ to $X$ with the same norm, the norm of the functional $\Psi$ in the space $X$ is attained at an extremal element $\varrho \in P$ of problem (1.1). By property $(\Gamma)$, the functional $\Psi$ differs from the functional $F = F[\varrho] \in X^*$ only by a constant factor $\gamma(P)$:

$$\Psi(p) = \gamma(P)F[\varrho](p), \quad p \in X.$$

In particular, (1.12) holds. Taking $p = \varrho$ in (1.12), we see that $|\gamma(P)| = D(\psi, P)$. This proves representation (1.12). □

**Lemma 4.** *Assume that a Banach space $X$ and its subspace $P$ satisfy properties* (R) *and* $(\Gamma)$. *If an element $\varrho \in P$, $\varrho \not\equiv 0$, or, more precisely, the supporting functional $F = F[\varrho] \in S(X^*)$ at the element $\varrho$ has property* (1.9)*, then $\varrho$ is an extremal element of problem* (1.1).

P r o o f. Assume that an element $\varrho \in P$, $\varrho \not\equiv 0$, has property (1.9); without loss of generality, we may assume that $\|\varrho\|_X = 1$. We consider the linear functional on the set $P$ defined by the formula

$$\Psi_0(p) = F[\varrho](p). \tag{1.13}$$

For any $p \in P$, the element $s = \psi(\varrho)p - \psi(p)\varrho$ belongs to the set $P(\psi)$. Due to (1.9), we have

$$\psi(\varrho)\Psi_0(p) - \psi(p)\Psi_0(\varrho) = 0. \tag{1.14}$$

By (1.8) and (1.13), we have $\Psi_0(\varrho) = F[\varrho](p) = 1$. Thus, (1.14) can be rewritten as

$$\psi(p) = \psi(\varrho)\,\Psi_0(p), \quad p \in P. \tag{1.15}$$

We conclude that

$$|\psi(p)| = |\psi(\varrho)|\,|\Psi_0(p)| \le |\psi(\varrho)|\,\|p\|.$$

Consequently, $D(\psi, P) \le |\psi(\varrho)|$. Since $\|\varrho\|_X = 1$, we have $D(\psi, P) \ge |\psi(\varrho)|$. It follows that $D(\psi, P) = |\psi(\varrho)|$ and the element $\varrho$ is extremal in problem (1.1). □

### 1.4.2. Proof of Theorem 1

Formula (1.12) implies that an extremal element of problem (1.1) has property (1.9). According to Lemma 4, the inverse statement holds. This proves Theorem 1. □

## 2. Bounded linear functionals on the set of algebraic polynomials in spaces $L_q^\upsilon$, $1 \le q < \infty$

Assume that $\mathbb{I}$ is a finite or infinite closed interval of the real line and $\upsilon$ is a nonnegative function that is integrable and almost everywhere nonzero on $\mathbb{I}$; we will call such functions weights on $\mathbb{I}$. Denote by $L_q = L_q^\upsilon(\mathbb{I})$, $1 \le q < \infty$, the space of (complex-valued) measurable functions $f$ on $\mathbb{I}$ such that the product $|f|^q\upsilon$ is integrable on $\mathbb{I}$; this is a Banach space with the norm

$$\|f\|_{L_q^\upsilon(\mathbb{I})} = \left(\int_{\mathbb{I}} |f(x)|^q \upsilon(x)\,dx\right)^{1/q}, \quad f \in L_q^\upsilon(\mathbb{I}).$$

For $q = \infty$, we assume that $L^{\upsilon}_{\infty}(\mathbb{I})$ is the space $L_\infty = L_\infty(\mathbb{I})$ of essentially bounded functions on $\mathbb{I}$ with the norm
$$\|f\|_{L_\infty(\mathbb{I})} = \operatorname{ess\,sup}\{|f(t)|\colon t \in \mathbb{I}\}.$$

Let $\mathscr{P}_n = \mathscr{P}_n(\mathbb{C})$ for $n \geq 0$ be the set of algebraic polynomials (in one variable) of degree at most $n$ with complex coefficients. We will assume that $\mathscr{P}_n \subset L^{\upsilon}_q(\mathbb{I})$; this condition is equivalent to the fact that the function $1 + |x|^n$ belongs to the space $L^{\upsilon}_q(\mathbb{I})$.

### 2.1. Arbitrary bounded linear functionals on the space of algebraic polynomials

Assume that $\psi$ is a linear functional on $\mathscr{P}_n$. Since $\mathscr{P}_n$ is finite-dimensional, the functional $\psi$ on $\mathscr{P}_n$ is bounded and its norm
$$D(\psi; \mathscr{P}_n)_q = \sup\{|\psi(p)|\colon p \in \mathscr{P}_n,\ \|p\|_{L^{\upsilon}_q(\mathbb{I})} = 1\} \tag{2.1}$$
is attained at a certain polynomial $\varrho_n = \varrho_{\psi,\mathscr{P}_n,q} \in \mathscr{P}_n$ with the property
$$\|\varrho_{\psi,\mathscr{P}_n,q}\|_{L_q(\mathbb{I},\upsilon)} = 1.$$
In the study of extremal problems for polynomials, it is an important fact that the value $D_n(\psi) = D(\psi; \mathscr{P}_n)_q$ is the smallest possible (the best) constant in the inequality
$$|\psi(p)| \leq D_n(\psi)\|p\|_{L^{\upsilon}_q(\mathbb{I})}, \quad p \in \mathscr{P}_n. \tag{2.2}$$
Inequality (2.2) turns into an equality at the polynomial $\varrho_n$, i.e., $\varrho_n$ is extremal in (2.2). It is clear that the polynomial $c\varrho_n$ with an arbitrary constant $c \in \mathbb{C}$ is also extremal in (2.2). If all extremal polynomials in inequality (2.2) have the form $c\varrho_n,\ c \in \mathbb{C}$, we say that $\varrho_n$ is the *unique* extremal polynomial of inequality (2.2) (or of problem (2.1)). In what follows, we assume that $\psi \not\equiv 0$; this is equivalent to the fact that $|\psi(\varrho_n)| = D(\psi; \mathscr{P}_n) > 0$.

Consider the annihilator
$$\mathscr{P}_n(\psi) = \{p \in \mathscr{P}_n\colon \psi(p) = 0\} \tag{2.3}$$
of the functional $\psi$ in the set $\mathscr{P}_n$. This set is a subspace of $\mathscr{P}_n$ of codimension 1. This subspace is formed by polynomials of the form
$$s = p - \frac{\psi(p)}{\psi(\varrho_n)}\varrho_n, \quad p \in \mathscr{P}_n.$$

**Theorem 3.** *Let $1 \leq q < \infty$. A polynomial $\varrho_n = \varrho_{\psi,\mathscr{P}_n,q} \in \mathscr{P}_n$ which is extremal in inequality* (2.2) *exists. A polynomial $\varrho_n \in \mathscr{P}_n$ is extremal if and only if*
$$\int_{\mathbb{I}} s(x)\upsilon(x)|\varrho_n(x)|^{q-1}\operatorname{sign}\varrho_n(x)dx = 0 \quad \textit{for all} \quad s \in \mathscr{P}_n(\psi). \tag{2.4}$$
*In the case when $1 < q < \infty$, this extremal polynomial is unique* (*up to a constant factor*).

P r o o f. We check that all assumptions of Theorem 1 are fulfilled under the assumptions of Theorem 3. The set $\mathscr{P}_n = \mathscr{P}_n(\mathbb{C})$ of algebraic polynomials of degree at most $n$ is a finite-dimensional subspace of $L^{\upsilon}_q(\mathbb{I})$. This guarantees that property (R) holds.

Now let us verify property ($\Gamma$). We start with the case $q = 1$. The dual space of $L = L^{\upsilon}_1(\mathbb{I})$ is the space $L_\infty = L_\infty(\mathbb{I})$ of essentially bounded functions on $\mathbb{I}$. A functional $\Phi \in X^*$ has the representation
$$\Phi(f) = \int_{\mathbb{I}} f(t)\overline{\phi}(t)\upsilon(t)dt, \quad f \in L^{\upsilon}_1(\mathbb{I}), \tag{2.5}$$
where $\phi \in L_\infty(\mathbb{I})$ and $\|\Phi\|_{L^*} = \|\phi\|_{L_\infty(\mathbb{I})}$.

For a pair of functions $\phi \in L_\infty(\mathbb{I})$ and $f \in L_1^\upsilon(\mathbb{I})$, the inequality

$$\left|\int_{\mathbb{I}} f(t)\overline{\phi}(t)\upsilon(t)dt\right| \le \|\phi\|_{L_\infty(\mathbb{I})}\|f\|_{L_1^\upsilon(\mathbb{I})}$$

turns into an equality if and only if the following three conditions hold:

(1) the set

$$\mathbb{I}(\phi) = \{t \in \mathbb{I}: \ |\phi(t)| = \|\phi\|_{L_\infty(\mathbb{I})}\}$$

where the absolute value of the function $\phi$ takes its maximum has a positive measure;

(2) the function $f$ vanishes almost everywhere outside the set $\mathbb{I}(\phi)$;

(3) the product $f\overline{\phi}$ has the same sign almost everywhere on the set

$$\Theta f = \{t \in \mathbb{I}\colon f(t) \neq 0\}.$$

Taking into account these observations, it is not difficult to conclude that a supporting functional of a function $f \in S(L_1^\upsilon(\mathbb{I}))$ has the form (2.5), where the function $\phi$ satisfies the following conditions: $\phi = \operatorname{sign} f$ almost everywhere on $\Theta f$ and $|\phi| \le 1$ almost everywhere outside $\Theta f$.

Consequently, a function $f \in S(L_1^\upsilon(\mathbb{I}))$ is a smooth point of the unit sphere of the space $L_1^\upsilon(\mathbb{I})$ if and only if $f$ is nonzero almost everywhere on $\mathbb{I}$; the supporting functional in this case has the form (2.5) with the function $\phi = \operatorname{sign} f$. In particular, this property holds in the case if $f$ is an algebraic polynomial. Thus, under the assumptions of Theorem 3 for $q = 1$, property $(\Gamma)$ holds.

For $1 < q < \infty$, the dual space of $L_q = L_q^\upsilon(\mathbb{I})$ is $L_{q'} = L_{q'}^\upsilon(\mathbb{I})$, $1/q + 1/q' = 1$. The space $L_{q'}^\upsilon(\mathbb{I})$ with $1 < q' < \infty$ is uniformly convex; hence, the space $L_q^\upsilon(\mathbb{I})$ is smooth.

Thus, we have shown that all assumptions of Theorem 1 are fulfilled under the assumptions of Theorem 3. Thus, also the statement of Theorem 1 holds. For $1 < q < \infty$, the space $L_q^\upsilon(\mathbb{I})$ is uniformly smooth, hence, the extremal polynomial in inequality (2.2) is unique. This proves the theorem. □

### 2.2. Pointwise Nikol'skii inequality for algebraic polynomials on an interval

Let $u$ be another, this time continuous weight on $\mathbb{I}$. Along with $L_q^\upsilon(\mathbb{I})$, we consider the space $C = C(\mathbb{I}, u)$ of complex-valued continuous functions $f$ such that the product $fu$ is bounded on $\mathbb{I}$, endowed the (uniform weighted) norm

$$\|f\|_{C(\mathbb{I},u)} = \sup\{|f(x)u(x)|\colon x \in \mathbb{I}\}.$$

We will assume that $\mathscr{P}_n$ is contained not only in $L_q^\upsilon(\mathbb{I})$ but also in $C(\mathbb{I}, u)$; the latter is equivalent to the fact that the function $u(x)(1 + |x|^n)$ is bounded on $\mathbb{I}$.

Denote by $M(n) = M(n, u, \upsilon)_q$ the best (the smallest possible) constant in the inequality

$$\|p\|_{C(\mathbb{I},u)} \le M(n)\,\|p\|_{L_q^\upsilon(\mathbb{I})}, \quad p \in \mathscr{P}_n, \tag{2.6}$$

on the set $\mathscr{P}_n$. Inequality (2.6) is a special case of an inequality between different metrics, or the Nikol'skii inequality. Such inequalities appeared for the first time in Nikol'skii's paper [15] and, shortly after that, in a paper by Szegő and Zygmund [18]. Similar inequalities and, more generally, inequalities between the uniform norm and weighted integral norms of algebraic and trigonometric polynomials and their derivatives have been studied over a period of more than 150 years, starting with the works of Chebyshev and his students—the Markov brothers. Further information and references on this topic can be found, e.g., in monographs [5, 14] and papers [2, 3, 16].

Along with (2.6), we consider the pointwise inequality

$$|p_n(z)| \le D_n[z]\,\|p_n\|_{L_q^\upsilon(\mathbb{I})}, \quad p_n \in \mathscr{P}_n, \tag{2.7}$$

with the smallest possible constant $D_n[z] = D(n, \upsilon, q; z)$ for points $z \in \mathbb{I}$. Such inequalities are of independent interest, but they are also important in connection with inequality (2.6) since

$$M(n) = \sup\{D_n[z]\, u(z) \colon z \in \mathbb{I}\}.$$

In a number of important cases, the product $D_n[z]\, u(z)$ takes its maximal value with respect to $z \in \mathbb{I}$ at an end point of the interval $\mathbb{I}$; see, e.g., [2–4, 16] and the references therein.

In the setup we consider in this section, (1.6) and (1.3) take the form

$$\Delta_n[z] = \inf\{\|p_n\|_{L_q^\upsilon(\mathbb{I})} \colon p_n \in \mathscr{P}_n[z]\}, \tag{2.8}$$

$$\mathscr{P}_n[z] = \{p_n \in \mathscr{P}_n \colon p_n(z) = 1\}.$$

**Theorem 4.** *For $1 \le q < \infty$, the following is true for an extremal polynomial in inequality (2.7).*

(1) *An extremal polynomial $\varrho_n$ in inequality (2.7) exists, it has real coefficients, all its roots are real, and its degree is at least $n-1$. In the case when $1 < q < \infty$, the extremal polynomial is unique.*

(2) *A polynomial $\varrho_n \in \mathscr{P}_n$ is extremal in inequality (2.7) if and only if*

$$\int_{\mathbb{I}} p_{n-1}(x)(x-z)\upsilon(x)|\varrho_n(x)|^{q-1}\mathrm{sign}\,\varrho_n(x)dx = 0 \quad \textit{for all} \quad p_{n-1} \in \mathscr{P}_{n-1}. \tag{2.9}$$

P r o o f. Inequality (2.7) is a special case of inequality (2.2) for the functional $\psi(p) = p(z)$, $p \in \mathscr{P}_n$. In this case, set (2.3) is formed by polynomials of the form $s(x) = (x-z)p_{n-1}(x)$, $p_{n-1} \in \mathscr{P}_{n-1}$. Therefore, condition (2.4) for an extremal polynomial $\varrho_n$ in inequality (2.7) takes the form (2.9). Thus, the second statement of Theorem 4 is proved. Without loss of generality, we may assume that $\varrho_n(z) = 1$; for, consider $\varrho_n/\varrho_n(z)$ instead of the polynomial $\varrho_n$, if necessary.

The polynomial $\varrho_n$ is also a solution of problem (2.8). We will study some properties of the polynomial $\varrho_n$ using this fact. In general, the coefficients $\{c_k\}_{k=0}^n$ of the polynomial $\varrho_n$ are complex, namely, $c_k = a_k + ib_k$, $a_k, b_k \in \mathbb{R}$. We write $\varrho_n$ in the form $\varrho_n = u_n + iv_n$, where

$$u_n(x) = (\mathrm{Re}\,\varrho_n)(x) = \sum_{k=0}^n a_k x^k, \quad v_n(x) = (\mathrm{Im}\,\varrho_n)(x) = \sum_{k=0}^n b_k x^k$$

are real polynomials (on $\mathbb{R}$). Obviously, $u_n(z) = \varrho_n(z) = 1$; hence, $u_n \in \mathscr{P}_n[z]$. If $b_k \ne 0$ for at least one $k$, $0 \le k \le n$, then the strict inequality $|u_n(x)| < |\varrho_n(x)|$ holds for all $x \in \mathbb{I}$ except for zeros of the polynomial $v_n$. Consequently, the strict inequality $\|u_n\|_{L_q^\upsilon(\mathbb{I})} < \|\varrho_n{}_n\|_{L_q^\upsilon(\mathbb{I})}$ holds for the norms of these polynomials. The latter is a contradiction to the fact that the polynomial $\varrho_n$ is extremal in (2.8). This proves that the coefficients of the polynomial $\varrho_n$ are real.

Assume that the polynomial $\varrho_n$ has a zero $\zeta$ which is not real. Since the polynomial $\varrho_n$ is real, we conclude that $\overline{\zeta}$ is also a zero of $\varrho_n$. Consequently, $\varrho_n(x) = q_{n-2}(x)|x-\zeta|^2$, where $q_{n-2}$ is a polynomial of degree at most $n-2$. The polynomial $p_{n-1}(x) = q_{n-2}(x)(x-z)$ has degree at most $n-1$. The left-hand side of (2.9) is positive for this polynomial:

$$\int_{\mathbb{I}} p_{n-1}(x)(x-z)\upsilon(x)|\varrho_n(x)|^{q-1}\mathrm{sign}\,\varrho_n(x)dx =$$
$$= \int_{\mathbb{I}} (x-z)^2\upsilon(x)|q_{n-2}(x)|^q|x-\zeta|^{2(q-1)}\mathrm{sign}\,q_{n-2}(x)dx > 0.$$

This contradicts property (2.9). Thus, the polynomial $\varrho_n$ can have only real zeros.

Finally, let us check that the exact degree of the polynomial $\varrho_n$ is $n$ or $n-1$. Indeed, if $\varrho_n$ has degree at most $n-2$, then the polynomial $p_{n-1}(x) = (x-z)\varrho_n(x)$ has degree at most $n-1$. The integral on the left-hand side of (2.9) is positive for this polynomial. This contradicts property (2.9). The theorem is proved. □

*Example.* Consider the special case of problem (2.7) in the space $L = L(-1,1)$ of functions that are integrable over the interval $I = [-1,1]$ with the unit weight, with $n = 1$ and $z = 0$. In other words, we are interested in the sharp inequality

$$|p(0)| \le D\|p\|_L, \quad p \in \mathscr{P}_1. \tag{2.10}$$

It is easy to verify that we have the formula

$$p(0) = \frac{1}{2}\int_{-1}^{1} p(t)dt, \quad p \in \mathscr{P}_1.$$

Using this formula, it is straightforward that the best constant in inequality (2.10) is $D = 1/2$ and that every polynomial of constant sign on $(-1,1)$ is extremal. Thus, an extremal polynomial in inequality (2.7) may be not unique, may have (real) zeros outside the interval $\mathbb{I}$, and may have the exact degree $n-1$.

For an end point $z$ of the interval $\mathbb{I}$, we are able to derive more information about the properties of extremal polynomials in inequality (2.7) from Theorem 4. In this case, the product $(x-z)\upsilon(x)$ on the left-hand side of (2.9) has constant sign on $\mathbb{I}$. Therefore, using property (2.9), it is not difficult to see that an extremal polynomial $\varrho_n$ has degree exactly $n$, all $n$ zeros of this polynomial are simple and lie in the interior of the interval $\mathbb{I}$. Property (2.9) implies also that the extremal polynomial $\varrho_n$ is unique for all $1 \le q < \infty$. Indeed, let $\varrho_n$ and $\eta_n$ be two polynomials that solve problem (2.8). The same property is true for their half-sum $(\varrho_n + \eta_n)/2$; therefore, we have $\|\varrho_n + \eta_n\|_{L_q^\upsilon} = \|\varrho_n\|_{L_q^\upsilon} + \|\eta_n\|_{L_q^\upsilon}$. For $1 < q < \infty$, it follows immediately that $\eta_n = \varrho_n$. For $q = 1$, it only follows that the polynomials $\eta_n$ and $\varrho_n$ have the same sign almost everywhere on $\mathbb{I}$. But the zeros of these polynomials are simple and lie in the interior of the interval $\mathbb{I}$; therefore, the polynomials $\eta_n$ and $\varrho_n$ have the same set of zeros and, hence, it follows that these polynomials coincide in the case when $q = 1$, too.

For a given weight $\upsilon$ and a given point $z \in \mathbb{I}$, we define the weight

$$w(x) = |x-z|\,\upsilon(x) \tag{2.11}$$

on the interval $\mathbb{I}$. We denote by $\varrho_n^* = \varrho_{n,w,q}^*$ the polynomial of degree $n \ge 1$ with the unit leading coefficient that deviates the least from zero in the space $L_q^w = L_q^w(\mathbb{I})$, i.e., is a solution of the problem

$$\min\{\|p_n\|_{L_q^w} : p_n \in \mathscr{P}_n^1\} = \|\varrho_n^*\|_{L_q^w}$$

on the set $\mathscr{P}_n^1$ of polynomials of degree $n$ with the leading coefficient equal to 1.

The polynomial $\varrho_n^*$ can be characterized by the property that the function $|\varrho_n^*|^{q-1}\operatorname{sign}\varrho_n^*$ is orthogonal to the space $\mathscr{P}_{n-1}$ (see, for example, [13, Ch. 3, Sect. 3.3, Theorems 3.3.1, 3.3.2]), i.e.,

$$\int_{\mathbb{I}} w(x)\,p_{n-1}(x)|\varrho_n^*(x)|^{q-1}\operatorname{sign}\varrho_n^*(x)\,dx = 0, \quad p_{n-1} \in \mathscr{P}_{n-1}.$$

This property coincides with property (2.9). Therefore, the polynomials $\varrho_n$ and $\varrho_n^*$ differ only by a constant factor. Thus, the following statement holds.

**Corollary 1.** *Let $z$ be an end point of the interval $\mathbb{I}$, $1 \le q < \infty$, and $n \ge 1$. The polynomial $\varrho_n^*$ of degree $n$ with the unit leading coefficient that deviates the least from zero in the space $L_q^w$ with weight* (2.11) *is the unique extremal polynomial in inequality* (2.7).

Special cases of this statement are given in [1, Theorem 1], [2, Theorem 2], [3, Theorem 2], [4, Theorem 3]; they have been proved there by means of other arguments.

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# A NEW ALGORITHM FOR ANALYSIS OF EXPERIMENTAL MÖSSBAUER SPECTRA

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**Abstract:** A new approach to analyze the nuclear gamma resonance (NGR) spectra is presented and justified in the paper. The algorithm successively spots the Lorentz lines in the experimental spectrum by a certain optimization procedures. In Mössbauer spectroscopy, the primary analysis is based on the representation of the transmission integral of an experimental spectrum by the sum of Lorentzians. In the general case, a number of lines and values of parameters in Lorentzians are unknown. The problem is to find them. In practice, before the experimental data processing, one elaborates a model of the Mössbauer spectrum. Such a model is usually based on some additional information. Taking into account physical restrictions, one forms the shape of the lines which are close to the normalized experimental Mössbauer spectrum. This is done by choosing the remaining free parameters of the model. However, this approach does not guarantee a proper model. A reasonable way to construct a structural NGR spectrum decomposition should be based on its model-free analysis. Some model-free methods of the NGR spectra analysis have been implemented in a number of known algorithms. Each of these methods is useful but has a limited range of application. In fact, the previously known algorithms did not react to hardly noticeable primary features of the experimental spectrum, but identify the dominant components only. In the proposed approach, the difference between the experimental spectrum and the known already determined part of the spectral structure defines the next Lorentzian. This method is effective for isolation of fine details of the spectrum, although it requires a well-elaborated algorithmic procedure presented in this paper.

**Key words:** Nuclear gamma resonance (NGR) spectra.

## Introduction

In Mössbauer spectroscopy, the primary analysis of the NGR spectrum structure is based on the representation of the transmission integral of an experimental spectrum $f(x)$ by the following sum of Lorentzians:

$$f(x)=\sum_{s=1}^{n}\frac{A_s}{1+\left(\dfrac{x-x_s}{b_s}\right)^2},\quad A_s>0,\quad b_s>0,\quad x\in\mathbb{R}, \tag{1}$$

where $x_s$ is the position of the maximum of the $s$th Lorentz line on the velocity scale, and $2b_s$ is its width. In the general case, the number $n$ of lines and the values of parameters $A_s$, $x_s$, and $b_s$ are unknown. The problem is to find them for a given function $f(x)$. The Mössbauer spectrum is always measured in some bounded velocity range. The maximal number of lines in the spectrum is limited by the number of probable positions of the Mössbauer atom in the crystal lattice and by the nature of changes of the nuclear energy levels (e.g., isomer shift, quadropole splitting, or hyperfine splitting).

For most of metallic alloys and oxide compounds, the evaluation of the structural parameters of the Mössbauer spectrum is a mathematically difficult nonlinear problem.

In practice, before the experimental data processing, one elaborates a model of the Mössbauer spectrum. Such a model is usually based on some additional information on the concentration, the structural state, and the charge state of Mössbauer atoms in the material. Then, taking into account physical restrictions, one forms the shape of the lines which are close (in the sense of the minimum of $\chi^2$-criterion) to the normalized experimental Mössbauer spectrum. This is done by choosing the remaining free parameters of the model. However, this approach does not guarantee a proper, physically approved model. A reasonable way to construct a structural NGR spectrum decomposition should be based on its model-free analysis. This can be done directly by solving problem (1) and by further improving the model based on the results of other methods [1].

Some model-free methods of the NGR spectra analysis have been implemented in algorithms for the density distribution of hyperfine fields, for the density distribution of isomer shifts with lines in the Lorentz or Gauss forms [2], and for filtering and reducing noises [3]. Each of these methods is useful but has a limited range of application. In fact, the previously known algorithms did not react to hardly noticeable primary features of the spectrum, but identify the dominant components only. This is caused by the least squares methods which are applied to the whole experimental spectrum without any analysis of its details. In the presented approach, the difference between the experimental spectrum and the known part of the spectral structure defines the next Lorentzian. This method is effective for isolation of fine details of the spectrum, although it requires (see [4]) a well-elaborated algorithmic procedure presented in this paper.

## 1. Results and discussion

Below we give a detailed description and a proof of the new algorithm for the Mössbauer spectra analysis. Let

$$\varphi_s(x) = \frac{A_s b_s^2}{b_s^2 + (x - x_s)^2},$$
$$\varphi(x) = \varphi(x, b) = \frac{b}{b^2 + x^2}, \tag{2}$$

and let

$$\Phi_s(t) = [\varphi * \varphi_s]$$

be the convolution of the functions $\varphi_s(x)$ and $\varphi(x)$. Then

$$\Phi_s(t) = \Phi_s(t, b) = \frac{A_s b_s \pi (b + b_s)}{(b + b_s)^2 + (t - x_s)^2}, \quad s = 1, 2, \ldots, n \tag{3}$$

and

$$\max_t \Big| \frac{\partial^r \Phi_s(t, b)}{\partial b^r} \Big| = \frac{\pi A_s b_s r!}{(b + b_s)^{r+1}} \quad s = 1, 2, \ldots, n, \quad r \in \mathbb{N}, \tag{4}$$

where the maximum is attained for $t = x_s$ only. Equality (4) follows easily from the following formula:

$$\Big(\frac{\partial}{\partial p}\Big)^r \Big(\frac{p}{p^2 + \xi^2}\Big) = \Big(\frac{\partial}{\partial p}\Big)^r \Big(\frac{1}{2(p - i\xi)} + \frac{1}{2(p + i\xi)}\Big) = \frac{(-1)^r}{2} \Big(\frac{r!}{(p - i\xi)^{r+1}} + \frac{r!}{(p + i\xi)^{r+1}}\Big).$$

Indeed, this equality implies the relations

$$\Big| \Big(\frac{\partial}{\partial p}\Big)^r \Big(\frac{p}{p^2 + \xi^2}\Big) \Big| = \Big| \frac{(-1)^r}{2} \Big(\frac{r!}{(p - i\xi)^{r+1}} + \frac{r!}{(p + i\xi)^{r+1}}\Big) \Big|$$
$$= \Big| \frac{(-1)^r (r)!}{2} \Big(\frac{1}{\rho^{r+1} \exp(-i\varphi(r+1))} + \frac{1}{\rho^{r+1} \exp(i\varphi(r+1))}\Big) \Big|$$

$$= \Big|\frac{(-1)^r (r)!}{2} \cdot \frac{\exp(i\varphi(r+1)) + \exp(-i\varphi(r+1))}{\rho^{r+1}\exp(i\varphi(r+1) - i\varphi(r+1))}\Big|$$

$$= \Big|\frac{(-1)^r (r)!}{2} \cdot \frac{\cos((r+1)\varphi) + i\sin((r+1)\varphi) + \cos((r+1)\varphi) - i\sin((r+1)\varphi)}{\rho^{r+1}}\Big|$$

$$= \Big|\frac{(-1)^r (r)! \cos((r+1)\varphi)}{\rho^{r+1}}\Big| = \Big|\frac{(-1)^r (r)! \cos((r+1)\varphi)}{(p^2+\xi^2)^{(r+1)/2}}\Big| \le \frac{r!}{p^{r+1}},$$

where $p > 0$, $\rho = (p^2+\xi^2)^{1/2}$, and $\varphi = \arg(p+i\xi)$ ($\varphi = 0$ for $\xi = 0$), and the equality is attained only at the point $\xi = 0$.

Furthermore, we assume that

$$0 < b_1 < b_2 < \cdots < b_n, \quad a > 0, \quad b > 0 \tag{5}$$

and consider only derivatives of even order when the absolute value sign in (4) can be omitted, since

$$\Phi_s(t) = \Phi_s(t,b) = \frac{A_s b_s \pi p}{p^2+\xi^2},$$

where $p = b + b_s > 0$, and $\xi = t - x_s$.

**Assertion.** *Under assumption* (5), *the convolution of the functions* $f(x)$ *in* (1) *and* $\varphi(x) = \varphi(x,b)$ *in* (2) *has the following asymptotic behavior as a positive integer* $k$ *tends to infinity*:

$$\begin{aligned} &\frac{(b+a)^{2k+1}}{\pi a(2k)!}\max_t \Big(\frac{\partial}{\partial b}\Big)^{2k}[\varphi * f] \\ &= \frac{(b+a)^{2k+1}}{a}\cdot\frac{A_1 b_1}{(b+b_1)^{2k+1}}[1+o(1)] \to \begin{cases} \infty, & \textit{for } \ a > b_1, \\ A_1, & \textit{for } \ a = b_1, \\ 0, & \textit{for } \ 0 < a < b_1. \end{cases} \end{aligned} \tag{6}$$

Indeed, from (1), (3), and (4), it follows that

$$\frac{(b+a)^{2k+1}}{\pi a(2k)!}\max_t \Big(\frac{\partial}{\partial b}\Big)^{2k}[\varphi * f](t) = \max_t \sum_{s=1}^{n} \frac{(b+a)^{2k+1}}{\pi a(2k)!}\Big(\frac{\partial}{\partial b}\Big)^{2k}\Phi_s(t,b). \tag{7}$$

Therefore,

$$\begin{aligned} &\frac{b_1 A_1}{a}\Big(\frac{b+a}{b+b_1}\Big)^{2k+1}\Big[1 - \sum_{s=2}^{n}\frac{A_s b_s}{A_1 b_1}\Big(\frac{b+b_1}{b+b_s}\Big)^{2k+1}\Big] \\ &\le \frac{(b+a)^{2k+1}}{\pi a(2k)!}\max_t\Big(\frac{\partial}{\partial b}\Big)^{2k}[\varphi * f](t) \\ &\le \frac{(b+a)^{2k+1}}{a}\Big[\frac{Ab_1}{(b+b_1)^{2k}} + \sum_{s=2}^{n}\frac{A_s b_s}{(b+b_1)^{2k+1}}\Big] \\ &= \frac{b_1 A_1}{a}\Big(\frac{b+a}{b+b_1}\Big)^{2k+1}\Big[1 + \sum_{s=2}^{n}\frac{A_s b_s}{A_1 b_1}\Big(\frac{b+b_1}{b+b_s}\Big)^{2k+1}\Big]. \end{aligned} \tag{8}$$

Now, restrictions (5) and inequalities (8) imply (6).

**Remarks.** By the assertion, for large $k$, the left-hand side of (6) is close to the maximum of the first term ($s = 1$) of the sum in (7) attained by (4) for $s = 1$ at the point $t = x_1$. Hence, if $k$ is sufficiently large, we can approximate the value $x_1$ by a maximum point of the left-hand side of (7) which actually coincides with the left-hand side of (6).

## 2. Numerical algorithm

Suppose that the function $f(x)$ has a unique local maximum, i.e., there is only one point $x_1$ such that

$$f(x_1) = \max_x f(x).$$

We set

$$A_1 = f(x_1)$$

and find the parameter $b_1$ and points

$$\widetilde{x}_1 = x_1 - b_1,$$

and

$$\widetilde{x}_2 = x_1 + b_1$$

from the condition

$$\max_x \Big| f(x) - \frac{A_1 b_1^2}{b_1^2 + (x - x_1)^2} \Big| = \inf_c \max_x \Big| f(x) - \frac{A_1 c^2}{c^2 + (x - x_1)^2} \Big|.$$

Alternatively, they can be found by keeping the value of the half-width of $f(x)$ and using the following simple formulas:

$$f(\widetilde{x}_l) = \frac{1}{2} A_1 \quad (l = 1, 2),$$

$$x_1 = \frac{1}{2}(\widetilde{x}_1 + \widetilde{x}_2),$$

$$b_1 = \frac{1}{2}(\widetilde{x}_2 - \widetilde{x}_1) \;\text{ for }\; \widetilde{x}_2 > \widetilde{x}_1.$$

If the difference $f(x) - \varphi_1(x)$ is small enough, i.e., is comparable with the accuracy of the evaluation of the function $f(x)$, the algorithm terminates by setting $f(x) = \varphi_1(x)$.

Otherwise, or, if there are several points of local maxima of $f(x)$, a different process is applied. This process is based on the asymptotic behavior of (6) and is described below.

For arbitrary $a > 0$, and $b > 0$, and for sufficiently large $k$, it is needed to find a point of maximum of the left-hand side of (6) and the maximum value. Further, let us consider this point as an approximate value for $x_1$. After that values $a_1^{(1)}$, and $a_2^{(1)}$ and a positive integer $k^{(1)}$ are found such that the left-hand side of (6) is greater than $A = \max_x f(x)$ for $k = k^{(1)}$ and $a = a_2^{(1)}$ and is smaller than $\varepsilon$ for $k = k^{(1)}$ and $a = a_1^{(1)}$, where $\varepsilon$ is an admissible error for computing $A_1$ from the right-hand side of (6). Next, the segment $[a_1^{(1)}, a_2^{(1)}]$ is divided into two equal parts. One of these parts, which satisfies assumptions analogous to those for $[a_1^{(1)}, a_2^{(1)}]$ for some $k^{(2)}$ ($\geq k^{(1)}$), is taken as the next segment $[a_1^{(2)}, a_2^{(2)}]$. After several iterations, we eventually obtain a number $k^{(\nu)}$ and a segment $[a_1^{(\nu)}, a_2^{(\nu)}]$, whose length is less than a given error $\delta > 0$ $\big(a_2^{(\nu)} - a_1^{(\nu)} = (a_2^{(1)} - a_1^{(1)})/2^\nu < \delta\big)$. Since $b_1 \in [a_1^{(l)}, a_2^{(l)}]$ $(l = 1, 2, \ldots, \nu)$, we can set

$$b_1 \approx a^{(\nu)} = \frac{a_1^{(\nu)} + a_2^{(\nu)}}{2}$$

with error at most $\delta$.

As follows from assertions (6)–(8), the equality

$$\frac{(b + a)^{2\nu+1}}{\pi a (2\nu)!} \max_t \frac{\partial^{2\nu}}{\partial b^{2\nu}} [\varphi * f] = \frac{A_1 b_1 (b + a)^{2\nu+1}}{a (b + b_1)^{2\nu+1}} \Big[ 1 + \theta \sum_{s=2}^{n} \frac{A_s b_s}{A_1 b_1} \Big( \frac{b + b_1}{b + b_s} \Big)^{2\nu+1} \Big] \tag{9}$$

holds for all sufficiently large $\nu$, where

$$\theta = \theta_\nu, \qquad |\theta| < 1,$$

$$a = (a_1^{(\nu)} + a_2^{(\nu)})/2 = b_1 + \delta, \quad 0 \le \delta < c/2^\nu, \quad c = a_2^{(1)} - a_1^{(1)}.$$

Taking the logarithm of both sides of equality (9) implies that

$$\begin{aligned} &\ln\Big[\frac{(b+a)^{2\nu+1}}{\pi a(2\nu)!}\max_t \frac{\partial^{2\nu}}{\partial b^{2\nu}}[\varphi * f]\Big] \\ &= \ln\frac{b_1}{a} + (2\nu+1)\ln\frac{b+a}{b+b_1} + \ln\Big[1 + \theta\sum_{s=2}^{n}\frac{A_s b_s}{A_1 b_1}\Big(\frac{b+b_1}{b+b_s}\Big)^{2\nu+1}\Big] + \ln A_1. \end{aligned} \tag{10}$$

Hence,

$$\Big|\ln\frac{b_1}{a}\Big| = \Big|\ln\frac{b_1 + O(\frac{1}{2^\nu})}{b_1}\Big| = \Big|\ln\Big(1 + O\Big(\frac{1}{2^\nu}\Big)\Big)\Big| = O\Big(\frac{1}{2^\nu}\Big),$$

$$\Big|(2\nu+1)\ln\frac{b+a}{b+b_1}\Big| = (2\nu+1)\Big|\ln\Big(1 + O\Big(\frac{1}{2^\nu}\Big)\Big)\Big| = O\Big(\frac{\nu}{2^\nu}\Big),$$

$$\Big|\ln\Big[1 + \theta_\nu\sum_{s=2}^{n}\frac{A_s b_s}{A_1 b_1}\Big(\frac{b+b_1}{b+b_s}\Big)^{2\nu+1}\Big]\Big| \le l|\theta_\nu|\sum_{s=2}^{n}\frac{A_s b_s}{A_1 b_1}\Big(\frac{b+b_1}{b+b_s}\Big)^{2\nu+1} \le C_n\Big(\frac{b+b_1}{b+b_2}\Big)^{2\nu+1} = C_n q^{2\nu+1}.$$

Therefore, these values are small for large $\nu$, and then the approximate equality

$$A_1 \approx A_1^{(\nu)} = \frac{(b+a_\nu)^{2\nu+1}}{\pi a_\nu(2\nu)!}\max_t\Big(\frac{\partial^{2\nu}}{(\partial b)^{2\nu}}[\varphi * f](t)\Big)$$

holds. Once the parameters $x_1$, $b_1$, and $A_1$ are found, we can introduce the function

$$f_1(x) = f(x) - \varphi_1(x) = f(x) - \frac{A_1 b_1^2}{b_1^2 + (x-x_1)^2},$$

repeat the same operations for this function as we did for $f(x)$, and find $x_2, b_2$, and $A_2$. Then these iterations must be continued until the number $A_{n+1}$ becomes small enough.

Let us remark that, although the algorithm uses the differentiation of the convolution of the experimental function $f(x)$ with the function $\varphi(x,b)$, first, one can differentiate the integrand and then compute the convolution $\Big[\Big(\partial^r\varphi/\partial b^r\Big) * f\Big]$. This order is preferable, because the numerical integration is a more regular operation than the incorrect numerical differentiation, and the derivatives of $\varphi(x,b)$ are expressed analytically.

Let us now write explicitly the positive (see (9)) argument of the logarithm in the left-hand side of (10) whose maximum is to be found. Starting, as above, from the formula

$$\frac{b}{b^2+x^2} = \frac{1}{2}\Big(\frac{1}{b-ix} + \frac{1}{b+ix}\Big),$$

we can to conclude that

$$\begin{aligned}
&\frac{(b+a)^{2\nu+1}}{\pi a(2\nu)!}\max_t \frac{\partial^{2\nu}}{(\partial b)^{2\nu}}(\varphi * f)(t)\\
&= \frac{(b+a)^{2\nu+1}}{2\pi a}\max_t \frac{\partial^{2\nu}}{(\partial b)^{2\nu}}\int\limits_{-\infty}^{\infty} f(t-x)\frac{b}{b^2+x^2}\,dx\\
&= \frac{(b+a)^{2\nu+1}}{2\pi a}\max_t \int\limits_{-\infty}^{\infty} f(t-x)\Big[\frac{1}{(b-ix)^{2\nu+1}}+\frac{1}{(b+ix)^{2\nu+1}}\Big]\,dx\\
&= \frac{(b+a)^{2\nu+1}}{\pi a(2\nu)!}\max_t \int\limits_{-\infty}^{\infty} f(t-x)\Big[\frac{(b+ix)^{2\nu+1}+(b-ix)^{2\nu+1}}{(b^2+x^2)^{2\nu+1}}\Big]\,dx\\
&= \frac{(b+a)^{2\nu+1}}{\pi a}\max_t \int\limits_{-\infty}^{\infty} f(t-x)\Big[\frac{1}{(b^2+x^2)^{2\nu+1}}\Big]\sum_{s=0}^{[\frac{2\nu+1}{2}]}(-1)^s C_{k+1}^{2s}x^{2s}b^{k+1-2s}\,dx.
\end{aligned} \tag{11}$$

Let us now describe a different way for computing $b_1$ and $A_1$ which is somewhat simpler, but involves a larger number of calculations of maxima. Let

$$H_r(t) = H_r(t,b) = \frac{\partial^r}{\partial b^r}\int\limits_{-\infty}^{\infty} f(t-x)\frac{b\,dx}{b^2+x^2}.$$

If $r = 2k-1$ and the conditions of the assertion are satisfied, then, similar to (6), the following relations can be derived for $H_{2k-1}$ as $k \to \infty$:

$$\frac{(b+a)^{2k}}{\pi a(2k-1)!}\max_t \Big[(-1)\frac{\partial^{2k-1}}{(\partial b)^{2k-1}}[\varphi * f](t)\Big] = \frac{(b+a)^{2k}}{a}\cdot\frac{A_1 b_1}{(b+b_1)^{2k}}(1+o(1)).$$

Combining this with (6), one can find approximate values for $b_1$ and $A_1$ (in this order) from the formulas

$$b_1 = 2k\frac{\max_t |H_{2k-1}(t)|}{\max_t H_{2k}(t)} - b + o(1),$$

$$A_1 = \frac{(b+b_1)^{2k+1}\max_t H_{2k}(t)}{\pi b_1(2k)!} + o(1).$$

By choosing large $k$, all small values denoted as $o(1)$ can be neglected.

## 3. Conclusion

In this paper, the new mathematical method is described for analysis of experimental data obtained for Mössbauer spectroscopy. This method allows to find the spectral decomposition of the integral transmission as a finite sum of Lorentzians with the accuracy of calculation of their number and determinations of their parameters according to a given accuracy of experimental data.

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# ON THE BEST APPROXIMATION OF THE INFINITESIMAL GENERATOR OF A CONTRACTION SEMIGROUP IN A HILBERT SPACE[1]

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**Abstract:** Let $A$ be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space $H$. We give an upper estimate for the best approximation of the operator $A$ by bounded linear operators with a prescribed norm in the space $H$ on the class $Q_2 = \{x \in \mathcal{D}(A^2) : \|A^2x\| \le 1\}$, where $\mathcal{D}(A^2)$ denotes the domain of $A^2$.

**Key words:** Contraction semigroup, Infinitesimal generator, Stechkin's problem.

## 1. Introduction

Let $H$ be a Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, and let $A$ be the infinitesimal generator of a strongly continuous contraction semigroup in $H$. For the definition and properties of the infinitesimal generator of a semigroup in a Banach space see, e.g., [6, §14.2]. Note that a strongly continuous contraction semigroup is also called a contraction semigroup of the class $C_0$ ([8, 9]). For an operator $F$ on the space $H$, $\mathcal{D}(F)$ denotes the domain of $F$. We denote by $I$ the identity operator.

In this paper, we study the so-called Stechkin's problem of the best approximation of the operator $A$ by bounded linear operators with a prescribed norm on the class of elements $x \in \mathcal{D}(A^2)$ such that $\|A^2x\| \le 1$. We give an upper estimate for the best approximation of the operator $A$.

The problem we consider is a special case of the general problem of the best approximation of an unbounded operator by linear bounded ones on a certain class of elements in a Banach space. This problem first appeared in Stechkin's work in 1965–1967 [11]. The problem was studied by a number of authors (see surveys [1], [2], monograph [4], paper [3], and the bibliography therein).

Stechkin formulated this problem in a general setting as follows. Let $X$, $Y$ be two Banach spaces, let $A$ be a linear operator (in general, unbounded) from $X$ to $Y$, and let $Q \subseteq \mathcal{D}(A)$ be a certain class of elements from the domain $\mathcal{D}(A)$ of the operator $A$. We denote by $\mathscr{B}(N)$ the set of

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linear bounded operators from $X$ to $Y$ with the norm $\|T\|_{X\to Y} \le N$. The best approximation of the operator $A$ by linear bounded operators $T \in \mathscr{B}(N)$ on the class $Q$ is

$$E_N(A;Q) = \inf\{U(A,T,Q) : T \in \mathscr{B}(N)\},$$

where

$$U(A,T,Q) = \sup\{\|Ax - Tx\|_Y : x \in Q\}$$

is the deviation of the operator $T$ from the operator $A$ on the class $Q$.

One of the most important cases of the problem formulated above is when the class $Q$ is defined in the following way. Let $Z$ be a Banach space and $B$ be a linear operator from $X$ to $Z$ such that $\mathcal{D}(B) \subseteq \mathcal{D}(A)$. The class $Q$ is then defined as $Q = \{x \in X \ : \|Bx\|_Z \le 1\}$.

Stechkin [11] suggested an estimate from below for the best approximation $E_N(A;Q)$ in terms of the modulus of continuity of the operator $A$ on the class $Q$ defined by

$$\Phi(\delta) = \sup\{\|Ax\|_Y : x \in Q,\ \|x\|_X \le \delta\}, \quad \delta > 0.$$

Namely, Stechkin showed that

$$E_N(A;Q) \ge \sup\{\Phi(\delta) - N\delta :\ \delta > 0\}. \tag{1.1}$$

In particular, when $B = A^n$, the problem $E_N(A^k;Q)$ turned out to be closely connected to the exact constants in the Kolmogorov-type inequalities of the form

$$\|A^k x\| \le C\|x\|^{\frac{n-k}{n}}\|A^n x\|^{\frac{k}{n}}, \quad x \in \mathcal{D}(A^n), \tag{1.2}$$

with $n, k \in \mathbb{N}$, $0 < k < n$, and a certain constant $C$ that depends on $n$ and $k$.

If $A$ is the differentiation operator, inequalities (1.2) are inequalities between the norms of the derivatives of a function. Such inequalities have been studied by a large number of authors (see [1], [2], [4] and the bibliography therein). Here we only mention that Hardy, Littlewood and Pólya [7, Chapter VII, §7.8] obtained the exact inequality

$$\|f'\|^2 \le 2\|f\|\|f''\| \tag{1.3}$$

in the space $L_2(0,\infty)$ on the class of functions $f \in L_2(0,\infty)$ such that $f'$ is locally absolutely continuous on $(0,\infty)$, and $f'' \in L_2(0,\infty)$.

In 1971, Kato [9] proved the following result which can be considered as a generalization of (1.3). Let $A$ be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space $H$. Then

$$\|Ax\|^2 \le 2\|x\|\|A^2x\|, \quad x \in \mathcal{D}(A^2).$$

In this paper, we study Stechkin's problem of the best approximation of the infinitesimal generator $A$ of a strongly continuous contraction semigroup by bounded linear operators on the class

$$Q_2 = \{x \in \mathcal{D}(A^2) :\ \|A^2x\| \le 1\} \tag{1.4}$$

in a Hilbert space. Namely, we estimate

$$E_N(A;Q_2) = \inf\{U(T) :\ T \in \mathscr{B}(N)\}, \tag{1.5}$$

where

$$U(T) = U(A,T,Q_2) = \sup\{\|Ax - Tx\| :\ x \in Q_2\}. \tag{1.6}$$

## 2. The main result

The main result of the paper is the following statement.

**Theorem 1.** *The best approximation* (1.5) *of the infinitesimal generator $A$ of a strongly continuous contraction semigroup in a Hilbert space on the class $Q_2$ defined in* (1.4) *satisfies the inequality*

$$E_N(A; Q_2) \le \frac{1}{N}.$$

It is known that the infinitesimal generator $A$ of a strongly continuous contraction semigroup in a Banach space possesses the following properties:

1) The domain $\mathcal{D}(A)$ of the operator $A$ is dense (see, e.g., [6, Lemma 14.5, p. 411]).

2) The resolvent set $\rho(A)$ of the operator $A$ contains the right half-plane $\{\lambda \in \mathbb{C} |\ \Re\lambda > 0\}$. Moreover, $\|(A - \lambda I)^{-1}\| \le (\Re\lambda)^{-1}$ for all $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$ (e.g., [6, Theorem 14.7, p. 412]).

Furthermore, if $A$ is the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space, we have additionally:

3) The operator $A$ is upper semibounded, with the upper bound 0, i.e.,

$$\Re(Ax, x) \le 0$$

for $x \in \mathcal{D}(A)$ [6, Lemma 14.9, p. 416].

The following lemma is not new. However, we will formulate and prove it for the sake of completeness.

**Lemma 1.** *Let $A$ be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space $H$ and $c > 0$. Then the operator*

$$B_c = (cI + A)(cI - A)^{-1}$$

*is densely defined and bounded* (*and thus can be extended to the whole space $H$ by continuity*). *Moreover,*

$$\|B_c\| \le 1.$$

*Remark.* The operator $B_c$ is the Cayley transform of the operator $A$ in the terminology of Kato [9], see also [10, p. 545].

P r o o f. Since $c > 0$, the operator $(cI - A)^{-1}$ is defined everywhere on $H$ and bounded. Since $A$ is the infinitesimal generator of a strongly continuous contraction semigroup, the operator $-A$ is $m$-accretive (see [10, Chapter IX, §1.4 as well as Problem 1.18, both p. 485]). Therefore, the domain $\mathcal{D}(A)$ of the operator $A$ is equal to the range $\mathcal{R}((cI - A)^{-1})$ of the operator $(cI - A)^{-1}$ which is dense in $H$ (see [10, Chapter V, §3.10, p. 279]). Thus, $B_c$ is densely defined.

Now we estimate the norm of $B_c$. For $x \in \mathcal{D}(A)$ we have

$$\|cx + Ax\|^2 = c^2\|x\|^2 + \|Ax\|^2 + 2c\Re(Ax, x),$$

$$\|cx - Ax\|^2 = c^2\|x\|^2 + \|Ax\|^2 - 2c\Re(Ax, x).$$

It follows immediately that

$$\|(cI + A)x\| \leq \|(cI - A)x\|. \tag{2.1}$$

Now take $y \in \mathcal{D}((cI - A)^{-1})$. Applying (2.1) to $x = (cI - A)^{-1}y \in \mathcal{D}(A)$, we obtain

$$\|(cI + A)(cI - A)^{-1}y\| \leq \|y\|,$$

and thus $\|B_c\| \leq 1$. □

Now we are ready to prove Theorem 1.

P r o o f. We will construct a concrete approximating operator $T$ in problem (1.5) and estimate its norm and its deviation (1.6) from the operator $A$ on the class $Q_2$.

Note that all the operators we consider commute on the set $\mathcal{D}(A^2)$.

The restriction of the operator $A$ to the set $\mathcal{D}(A^2)$ (which we will denote by the same symbol) can be represented as

$$A = \frac{N}{2}(B_N - I) - \frac{1}{2N}(B_N + I)A^2.$$

Put $T : H \to H$,

$$T = \frac{N}{2}(B_N - I).$$

Then, for the restriction of the operator $A - T$ to $\mathcal{D}(A^2)$, we have

$$A - T = -\frac{1}{2N}(B_N + I)A^2.$$

We estimate the norm of the operator $T$ as follows:

$$\|T\| = \frac{N}{2}\|B_N - I\| \leq \frac{N}{2}(\|B_N\| + \|I\|) = N. \tag{2.2}$$

For the deviation $U(T)$ of the operator $T$ from the operator $A$, we obtain that

$$U(T) = \sup_{x \in Q_2} \|(A - T)x\| \leq \sup_{x \in Q_2} \frac{1}{2N}\|B_N + I\| \cdot \|A^2 x\| \leq \frac{1}{N}. \tag{2.3}$$

It follows immediately from (2.2) and (2.3) that

$$E_N(A; Q_2) \leq U(T) \leq \frac{1}{N}.$$

□

## 3. Approximation of the differentiation operator in the space $L_2(0, \infty)$

An important concrete case of problem (1.5) is the problem of the best approximation of the differentiation operator $Df = f'$ by bounded linear operators in the Hilbert space $L_2(0, \infty)$ of real-valued functions whose squares are integrable on $(0, \infty)$ on the class $Q^{(2)}$ defined as follows:

$Q^{(2)}$ is the class of functions $f \in L_2(0,\infty)$ such that $f'$ is locally absolutely continuous on $[0,\infty)$, $f'' \in L_2(0,\infty)$, and $\|f''\| \le 1$. Problem (1.5) takes in this case the form

$$E_N(D; Q^{(2)}) = \inf_{T \in \mathscr{B}(N)} \sup_{f \in Q^{(2)}} \|f' - Tf\|. \tag{3.1}$$

It took about 20 years of research to solve the problem completely. Stechkin's inequality (1.1) and inequality (1.3) of Hardy, Littlewood and Pólya provide the lower bound

$$E_N(D; Q^{(2)}) \ge \frac{1}{2N}.$$

One of the first upper bounds for (3.1)

$$E_N(D; Q^{(2)}) \le \frac{1}{\sqrt{3}N}$$

was obtained by using a concrete approximating operator by the first named author in 1996 [5]. Problem (3.1) was fully solved only in 2014 by Arestov and the second named author [3] . Namely, they showed that

$$E_N(D; Q^{(2)}) = \frac{1}{2N}.$$

In this section, we discuss what the statement of Theorem 1 means in the concrete case (3.1) of problem (1.5). The approximating operator $T$ used in Theorem 1 is

$$T = \frac{N}{2}(B_N - I) = NA(NI - A)^{-1}. \tag{3.2}$$

Below we will describe this operator in the special case. We consider and calculate its norm $\|T\|$ and its deviation $U(T)$ from the operator $A = D$ on the class $Q^{(2)}$.

It is not difficult to see that the operator $T$ in the concrete case can be represented as follows. Let $W$ be the class of functions $y \in L_2(0,\infty)$ such that $y$ is locally absolutely continuous on $[0,\infty)$ and $y' \in L_2(0,\infty)$. For $f \in L_2(0,\infty)$, we consider the differential equation

$$-y' + Ny = f, \quad y \in W. \tag{3.3}$$

For each function $f \in L_2(0,\infty)$, equation (3.3) has a unique solution which is a real-valued function from $L_2(0,\infty)$. The operator $T$ is defined as

$$Tf = Ny', \tag{3.4}$$

where $y$ is the solution of the differential equation (3.3).

Integrating by parts and taking into account that $\lim\limits_{t\to\infty} y(t) = 0$, we obtain (see [3] for details) that

$$\|f\|^2 = \int_0^\infty (-y'(t) + Ny(t))^2 dt = \int_0^\infty (y'(t))^2 dt + N^2 \int_0^\infty (y(t))^2 dt + Ny^2(0).$$

It follows from (3.4) that $\|Tf\|^2 = N^2 \int_0^\infty (y'(t))^2 dt$. Thus, we immediately obtain

$$\|Tf\|^2 \le N^2 \|f\|^2, \tag{3.5}$$

which gives the estimate $\|T\| \le N$. Now we show that indeed $\|T\| = N$. Consider the family of functions $y_K = e^{-Kt}$, $K > 0$. Let $f_K$ be the corresponding right-hand side of equation (3.3). Take an arbitrary $0 < \alpha < 1$. We have

$$\alpha N^2 \|f_K\|^2 - \|Tf_K\|^2 = \alpha N^2 \int_0^\infty (-y'_K(t) + N y_K(t))^2 dt - N^2 \int_0^\infty (y'_K(t))^2 dt$$
$$= \frac{N^2}{2K}(\alpha(K+N)^2 - K^2).$$

This expression is negative for all $0 < \alpha < \dfrac{K^2}{(N+K)^2}$ which yields $\|Tf_K\|^2 > \alpha N^2 \|f_K\|^2$. Letting $K$ go to infinity (with fixed $N$) we let $\alpha$ approach 1, and thus obtain $\|T\| \ge N$. Consequently, $\|T\| = N$.

Note that inequality (3.5) is a strict inequality if $y \ne 0$ and, consequently, $f \ne 0$. In other words, the norm of the operator $T$ is not attained.

It can be shown similarly that the norm of the operator $V = -\dfrac{1}{2N}(B_N + I)$ is equal to $1/N$. Since the domain $\mathcal{D}(D^2)$ of the operator $D^2$ is dense in $L_2(0,\infty)$, it follows that the deviation of the operator $T$ from the differentiation operator $D$ on the class $Q^{(2)}$ is equal to $1/N$.

Thus, the approximating operator (3.2) gives the estimate $E_N(D; Q^{(2)}) \le \dfrac{1}{N}$ in the general case (1.5) as well as in the concrete case (3.1).

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# DIVERGENCE OF THE FOURIER SERIES OF CONTINUOUS FUNCTIONS WITH A RESTRICTION ON THE FRACTALITY OF THEIR GRAPHS[1]

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**Abstract:** We consider certain classes of functions with a restriction on the fractality of their graphs. Modifying Lebesgue's example, we construct continuous functions from these classes whose Fourier series diverge at one point, i.e. the Fourier series of continuous functions from this classes do not converge everywhere.

**Key words:** Trigonometric Fourier series, Fractality, Divergence at one point, Continuous functions.

Let $f$ be a $2\pi$-periodic integrable function, and let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx), \tag{1}$$

where

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos kt\;dt,\quad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin kt\;dt,$$

be the trigonometric Fourier series of the function $f$. Denote by $S_n(f,x)$ the $n$th partial sum of (1). It is known (see [1, Ch. 1, Sect. 39]) that if $f$ has bounded variation on the period ($f\in BV$), then its Fourier series converges everywhere on $\mathbb{R}$, and if, in addition, $f$ is continuous on $\mathbb{R}$, then the Fourier series converges to $f$ uniformly on $\mathbb{R}$. Salem [2] (see also [1, Ch. 4, Sect. 5]) considered the classes $BV_p$ of functions of bounded $p$-variation and proved that if $f\in BV_p$, then the Fourier series of $f$ also converges everywhere on $\mathbb{R}$. (Further generalizations of these results see in [3]).

The author [4] studied relations between the classes $BV_p$ and classes of continuous functions with a restriction on the fractality of their graphs.

**Definition 1.** Let $f\colon\mathbb{R}\to\mathbb{R}$ be a bounded $2\pi$-periodic function. By the modulus of fractality of the function $f$, we call the function $\nu(f,\varepsilon)$ which, for all $\varepsilon>0$, gives the minimal number of closed squares with sides of length $\varepsilon$ parallel to the coordinate axes that cover the graph of the function $f$ on $[-\pi,\pi]$.

**Definition 2.** Let $\mu\colon(0,+\infty)\to\mathbb{R}$ be a nonincreasing continuous function such that $\lim\limits_{\varepsilon\to 0}\mu(\varepsilon)=+\infty$. We define the functional class

$$F^{\mu}:=\{f\in C_{2\pi}:\nu(f,\varepsilon)=O(\mu(\varepsilon))\}.$$

In the case $\mu(\varepsilon)=1/\varepsilon^{\alpha}$, where $1\leqslant\alpha\leqslant 2$, we will write $F_{\alpha}$ instead of $F^{1/\varepsilon^{\alpha}}$.

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The following statements were proved in [4]:

$$BV = BV_1 = F_1 \quad [4, \text{Theorem 1}]; \tag{2}$$

$$BV_p \subset F_{2-1/p}, \quad p > 1 \quad [4, \text{Theorem 2}].$$

The latter is unimprovable; i.e., $BV_{p+\varepsilon} \not\subseteq F_{2-1/p}$ for all $\varepsilon > 0$.

In the present paper, we study the pointwise behavior of the Fourier series of continuous functions from $F^\mu$.

**Theorem 1.** *Let $\mu\colon (0,+\infty) \to \mathbb{R}$ be a nonincreasing continuous function, let $\varepsilon\mu(\varepsilon)$ be a nonincreasing function, and let*

$$\lim_{\varepsilon\to+0} \varepsilon\mu(\varepsilon) = +\infty. \tag{3}$$

*Then there exists a continuous function $F^\mu$ whose Fourier series does not converge everywhere.*

P r o o f of Theorem 1. We will require that

$$\varepsilon^{-1} < \mu(\pi\varepsilon) \leqslant 2\varepsilon^{-\frac{3}{2}}, \quad \varepsilon \in (0,1]. \tag{4}$$

By (3), the former inequality holds on an interval $(0,\delta)$ and, changing the function $\mu$ on the interval $(\frac{\delta}{2}, 1)$, we will obtain the same class $F^\mu$. The latter inequality can only reduce the class $F^\mu$. Thus, if we find a required function in the narrower class, it will belong to the wider class immediately.

To obtain a function $f \in F^\mu$ with divergent Fourier series, we modify Lebesgue's example from [1, Ch. 1, Sect. 46]. We start with defining an increasing sequence of natural numbers $\{a_k\}$ as follows. Let $a_0 = 1$. Suppose that the first $k$ elements $a_0, a_1, \dots, a_{k-1}$ have been already defined.

From inequalities (4), it follows that

$$\frac{a_{k-1}^2}{a_{k-1}} < 3\mu\left(\frac{\pi}{a_{k-1}}\right)$$

and, for $b \geqslant (6a_{k-1})^2$,

$$\frac{b^2}{a_{k-1}} \geqslant 3\mu\left(\frac{\pi}{b}\right).$$

Then, by continuity, there exists the smallest number $a$ such that

$$\frac{a^2}{a_{k-1}} = 3\mu\left(\frac{\pi}{a}\right).$$

As $a_k$, we take the largest integer such that $a_k \leqslant a$ and the fraction $a_k/a_{k-1}$ is integer. It is not hard to understand that $a_k$ belongs to $[a - a_{k-1}, a]$, and, in view of the inequalities

$$\frac{a_k}{a_{k-1}} \geqslant \frac{a - a_{k-1}}{a_{k-1}} = 3\mu\left(\frac{\pi}{a}\right)\frac{1}{a} - 1 \geqslant 2, \tag{5}$$

we conclude that $a_k > a_{k-1}$.

The definition of $a_k$ implies the inequality

$$\frac{1}{\varepsilon^2 a_{k-1}} \leqslant 3\mu\big(\pi\varepsilon\big), \quad \varepsilon \in \left[\frac{1}{a_k}, 1\right]. \tag{6}$$

The definition of $a_k$, inequalities (5), and condition (3) imply that

$$\frac{a_k}{a_{k-1}} \to +\infty, \quad k \to +\infty. \tag{7}$$

Consider the half-open intervals

$$I_k = \left(\frac{\pi}{a_k}, \frac{\pi}{a_{k-1}}\right], \qquad k \in \mathbb{N}.$$

Let $\{k_i\}_{i=0}^{\infty}$, $k_0 = 1$, be an increasing sequence, on which, in what follows, two additional conditions will be imposed. Let

$$c_k = \begin{cases} \sqrt{\dfrac{1}{\ln a_k/a_{k-1}}}, & k \in \{k_i\}_{i=0}^{\infty}; \\ 0, & k \notin \{k_i\}_{i=0}^{\infty}. \end{cases}$$

Finally, we define the function $f$ on the interval $[-\pi, \pi]$:

$$f(x) = c_k \sin a_k x, \qquad x \in I_k,$$
$$f(0) = 0,$$
$$f(-x) = f(x).$$

We extend the function $f$ to $\mathbb{R}$ periodically. The resulting function is continuous on each $I_k$ and, since $a_k/a_{k-1}$ is integer, is continuous and vanishes at the points $\pm\pi/a_k$. Thus, the function $f$ is continuous on $[-\pi, \pi]$.

Since $f$ has only a finite number of maxima and minima on $[\delta, \pi]$, $\delta > 0$, it has bounded variation on this interval (and on $[-\pi, -\delta]$ as well). Thus, its Fourier series converges at every $x \in [-\pi, \pi] \backslash \{0\}$.

Consider now the sequence of partial sums of the Fourier series of $f$ at the point $x = 0$. As is known [1, Ch. 1, Sect. 32, formula (32.5)], for the function $f$, we have

$$S_k(x, f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \frac{\sin kt}{t} dt + o(1);$$

hence, for $x = 0$,

$$S_k(0, f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin kt}{t} dt + o(1).$$

The function $f$ is even; therefore,

$$S_k(0, f) = \frac{2}{\pi} \int_{0}^{\pi} f(t) \frac{\sin kt}{t} dt + o(1).$$

Let us show that, after an appropriate choice of $\{k_i\}$,

$$J_i = \int_0^{\pi} f(t) \frac{\sin a_{k_i} t}{t} dt \to +\infty, \quad i \to +\infty.$$

Then $S_{a_{k_i}}(0, f) \to +\infty$ as $i \to +\infty$, i.e., the Fourier series of $f$ diverges at $x = 0$.

To estimate $J_i$, we divide it into three terms:

$$J_i = \int_0^{\pi/a_{k_i}} f(t) \frac{\sin a_{k_i} t}{t} dt + \int_{\pi/a_{k_i}}^{\pi/a_{k_i-1}} f(t) \frac{\sin a_{k_i} t}{t} dt + \int_{\pi/a_{k_i-1}}^{\pi} f(t) \frac{\sin a_{k_i} t}{t} dt = J_i' + J_i'' + J_i'''. \quad (8)$$

We have

$$\left| \frac{\sin a_{k_i} t}{t} \right| \leqslant a_{k_i}.$$

Hence,

$$|J_i'| \leqslant \max_{0 \leqslant t \leqslant \pi/a_{k_i}} |f(t)| a_{k_i} \frac{\pi}{a_{k_i}} = \pi c_{k_{i+1}} = o(1). \quad (9)$$

Suppose that $k_1, \dots, k_{i-1}$ have been already defined. Then the function $f(t)/t$ is defined, bounded, and continuous on $\bigl(\pi/a_{k_i-1}, \pi\bigr]$. Extending this function by zero to $[-\pi, \pi]$ and assuming that $k_i$ are large enough (this is the first of two conditions on $k_i$), we can make the Fourier coefficient $a_{k_i}$ of the obtained function small enough; more precisely,

$$|J_i'''| = \left| \int_{\pi/a_{k_i-1}}^{\pi} \frac{f(t)}{t} \sin a_{k_i} t \; dt \right| \leqslant \frac{1}{i}. \tag{10}$$

It remains to estimate $J_i''$. We have

$$\begin{aligned} J_i'' &= \int_{\pi/a_{k_i}}^{\pi/a_{k_i-1}} c_{k_i} \sin a_{k_i} t \frac{\sin a_{k_i} t}{t} dt = \frac{c_{k_i}}{2} \int_{\pi/a_{k_i}}^{\pi/a_{k_i-1}} \frac{1 - \cos 2a_{k_i} t}{t} dt \\ &= \frac{c_{k_i}}{2} \ln \frac{a_{k_i}}{a_{k_i-1}} - \frac{c_{k_i}}{2} \int_{\pi/a_{k_i}}^{\pi/a_{k_i-1}} \frac{\cos 2a_{k_i} t}{t} dt. \end{aligned}$$

According to the second mean value theorem, taking into account that the function $1/t$ is positive and monotone, we find that

$$\left| \int_{\pi/a_{k_i}}^{\pi/a_{k_i-1}} \frac{\cos 2a_{k_i} t}{t} dt \right| \leqslant \frac{a_{k_i}}{\pi} \left| \int_{\pi/a_{k_i}}^{\xi} \cos 2a_{k_i} t \; dt \right| \leqslant \frac{a_{k_i}}{\pi} \frac{2}{2a_{k_i}} = \frac{1}{\pi}.$$

Thus,

$$J_i'' = \frac{c_{k_i}}{2} \ln \frac{a_{k_i}}{a_{k_i-1}} + o(1). \tag{11}$$

Combining (8), (9), (10), and (11), and taking into account (7), we conclude that

$$J_i = \frac{c_{k_i}}{2} \ln \frac{a_{k_i}}{a_{k_i-1}} + o(1) = \frac{1}{2} \sqrt{\ln \frac{a_{k_i}}{a_{k_i-1}}} + o(1) \to +\infty.$$

Let us now estimate the modulus of fractality $\nu(f, \varepsilon)$. Denote by $\nu(f, \varepsilon)_{[a,b]}$ the minimal number of squares with sides of length $\varepsilon$ parallel to the coordinate axes that cover the graph of the function $f$ on $[a, b]$.

If $k_1, \dots, k_{i-1}$ have been already defined, then the function $f$ is defined on the interval $[\pi/a_{k_{i-1}}, \pi]$ and has bounded variation; hence, by (2),

$$\nu(f, \varepsilon)_{\left[\pi/a_{k_{i-1}}, \pi\right]} = O\left(\frac{1}{\varepsilon}\right).$$

Condition (3) allows us to take $k_i$ such that, for $\pi\varepsilon \in (0, \, \pi/a_{k_i}]$,

$$\nu(f, \pi\varepsilon)_{\left[\pi/a_{k_{i-1}}, \pi\right]} \leqslant \mu(\pi\varepsilon). \tag{12}$$

This is the second condition on $k_i$.

Let $0 < \varepsilon \leqslant 1$. Then there exists $i \in \mathbb{N}$ such that $\varepsilon \in [1/a_{k_{i+1}}, 1/a_{k_i}]$. Let us prove the inequality $\nu(f, \pi\varepsilon) \leqslant C\mu(\pi\varepsilon)$ with some constant $C$. It follows from what is proved above that the required inequality holds for the covering of the graph on $[\pi/a_{k_{i-1}}, \, \pi]$. The inequality also holds for the intervals $[\pi/a_{k_{i+1}-1}, \, \pi/a_{k_i}]$ and $[\pi/a_{k_i-1}, \, \pi/a_{k_{i-1}}]$ where $f$ is identically zero; hence,

$$\nu(f, \pi\varepsilon)_{\left[\pi/a_{k_{i+1}-1}, \pi/a_{k_i}\right]} + \nu(f, \pi\varepsilon)_{\left[\pi/a_{k_i-1}, \pi/a_{k_{i-1}}\right]} \leqslant \frac{\pi}{\varepsilon}. \tag{13}$$

Covering the whole rectangle $\left[0, \pi/a_{k_{i+1}-1}\right] \times [-c_{k_i}, c_{k_i}]$ and using (6), we can obtain the estimate

$$\nu(f, \pi\varepsilon)_{\left[0,\, \pi/a_{k_{i+1}-1}\right]} \leqslant \left\lceil \frac{\pi}{a_{k_{i+1}-1}\pi\varepsilon} \right\rceil \left\lceil \frac{2c_{k_i}}{\pi\varepsilon} \right\rceil \leqslant \frac{8}{a_{k_{i+1}-1}\pi\varepsilon^2} \leqslant \frac{24}{\pi}\mu\big(\pi\varepsilon\big); \tag{14}$$

here and in what follows, $\lceil x \rceil$ stands for the rounding of $x$ upward.

It remains to cover the graph on the interval $\left[\pi/a_{k_i},\, \pi/a_{k_i-1}\right]$ where $f(x) = c_{k_i} \sin a_{k_i} x$. We can divide this interval into $N_i = 2a_{k_i}/a_{k_i-1} - 2$ intervals of monotonicity of $f$: $\left[\pi/a_{k_i} + \pi(n-1)/2a_{k_i}, \pi/a_{k_i} + \pi n/2a_{k_i}\right]$, $n = 1, \ldots, N_i$. Let us show that, to cover the graph of $f$ on each of these intervals, we need at most $8/\pi\varepsilon$ squares. Using the definition of the length of a curve, we can show that the length of the graph of $f$ on these intervals is at most $\pi/2a_{k_i} + 2c_{k_i}$. Squares with sides of length $\pi\varepsilon$ can cover the graph of a monotone function of length at least $\pi\varepsilon$. Hence,

$$\nu(f, \pi\varepsilon)_{\left[\pi/a_{k_i}+\pi(n-1)/2a_{k_i},\, \pi/a_{k_i}+\pi n/2a_{k_i}\right]} \leqslant \left\lceil \left(\frac{\pi}{2a_{k_i}} + 2c_{k_i}\right)\frac{1}{\pi\varepsilon} \right\rceil \leqslant \frac{8}{\pi\varepsilon}.$$

From (6) and the monotonicity of $\varepsilon\mu(\varepsilon)$, we obtain

$$\nu(f, \pi\varepsilon)_{\left[\pi/a_{k_i},\, \pi/a_{k_i-1}\right]} \leqslant \frac{4a_{k_i}}{\pi\varepsilon a_{k_i-1}} \leqslant \frac{12\mu\left(\dfrac{\pi}{a_{k_i}}\right)}{\pi\varepsilon a_{k_i}} = \frac{12\mu\left(\dfrac{\pi}{a_{k_i}}\right)\dfrac{\pi}{a_{k_i}}\mu(\pi\varepsilon)}{\pi^2\varepsilon\mu(\pi\varepsilon)} \leqslant \frac{12}{\pi}\mu(\pi\varepsilon). \tag{15}$$

Finally, by (12), (13), (14), and (15), we obtain the following estimate for the modulus of fractality of $f$:

$$\begin{aligned}\nu(f, \pi\varepsilon) \leqslant 2\nu(f, \pi\varepsilon)_{[0,\pi]} &\leqslant 2\bigg(\nu(f, \pi\varepsilon)_{\left[0,\, \pi/a_{k_{i+1}-1}\right]} + \nu(f, \pi\varepsilon)_{\left[\pi/a_{k_{i+1}-1},\, \pi/a_{k_i}\right]} \\ &+\nu(f, \pi\varepsilon)_{\left[\pi/a_{k_i},\, \pi/a_{k_i-1}\right]} + \nu(f, \pi\varepsilon)_{\left[\pi/a_{k_i-1},\, \pi/a_{k_{i-1}}\right]} + \nu(f, \pi\varepsilon)_{\left[\pi/a_{k_{i-1}},\, \pi\right]}\bigg) \\ &\leqslant 2\left(\frac{24}{\pi}\mu\big(\pi\varepsilon\big) + \frac{\pi}{\varepsilon} + \frac{12}{\pi}\mu(\pi\varepsilon) + \mu(\pi\varepsilon)\right) = O\big(\mu(\pi\varepsilon)\big),\end{aligned}$$

i.e., $f \in F^\mu$.

The theorem is proved.

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# CONVERGENCE OF SOLUTIONS OF BILATERAL PROBLEMS IN VARIABLE DOMAINS AND RELATED QUESTIONS

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**Abstract:** We discuss some results on the convergence of minimizers and minimum values of integral and more general functionals on sets of functions defined by bilateral constraints in variable domains. We consider the case of regular constraints, i.e., constraints lying in the corresponding Sobolev space, and the case where the lower constraint is zero and the upper constraint is an arbitrary nonnegative function. The first case concerns a larger class of integrands and requires the positivity almost everywhere of the difference between the upper and lower constraints. In the second case, this requirement is absent. Moreover, in the latter case, the exhaustion condition of an $n$-dimensional domain by a sequence of $n$-dimensional domains plays an important role. We give a series of results involving this condition. In particular, using the exhaustion condition, we prove a certain convergence of sets of functions defined by bilateral (generally irregular) constraints in variable domains.

**Key words:** Integral functional, Bilateral problem, Minimizer, Minimum value, Γ-convergence of functionals, Strong connectedness of spaces, $\mathcal{H}$-convergence of sets, Exhaustion condition.

## Introduction

This paper is mainly based on the talk given by the author at the International S.B. Stechkin Summer Workshop-Conference on Function Theory, Miass, Russia, August 1–10, 2017.

The problems considered in the paper are related to the following general problem. Let $\{W_s\}$ be a sequence of Banach spaces, and let, for every $s \in \mathbb{N}$, $\mathcal{I}_s : W_s \to \mathbb{R}$ and $V_s \subset W_s$, $V_s \neq \varnothing$. Let, for every $s \in \mathbb{N}$, $u_s$ be a minimizer of $\mathcal{I}_s$ on $V_s$. The questions are, what are general conditions under which the sequence $\{u_s\}$ converges in a certain sense to an element and this limit element minimizes a functional $\mathcal{I}$ on a set $V$, and how are the functional $\mathcal{I}$ and the set $V$ related to the sequences $\{\mathcal{I}_s\}$ and $\{V_s\}$? Problems of this kind are studied in the framework of homogenization theory. There is a special kind of convergence of functionals that helps to solve the mentioned problems. This is the Γ-convergence. There are many works devoted to the study of this convergence. The Γ-convergence of functionals with the same domain of definition was studied, for instance, in [1–3]. In the simplest case, the definition of Γ-convergence is as follows.

**Definition 1.** Let, for every $s \in \mathbb{N}$, $f_s : \mathbb{R} \to \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$. We say that the sequence $\{f_s\}$ Γ-converges to the function $f$ if the following conditions are satisfied:

(a) for every $x \in \mathbb{R}$, there exists a sequence $\{y_s\} \subset \mathbb{R}$ such that $y_s \to x$ and $f_s(y_s) \to f(x)$;

(b) for every $x \in \mathbb{R}$ and every sequence $\{x_s\} \subset \mathbb{R}$ such that $x_s \to x$, we have the inequality $\liminf\limits_{s\to\infty} f_s(x_s) \geqslant f(x)$.

The Γ-convergence of ordinary real functions and functionals defined on Banach spaces has some interesting properties that distinguish it from other kinds of convergence of the corresponding mappings. Among various properties of the Γ-convergence, we only mention its variational property that describes the relation of this convergence of functionals to the convergence of their minimizers and minimum values. A simple version of the variational property of the Γ-convergence is the following proposition.

**Proposition 1.** *Let, for every $s \in \mathbb{N}$, $f_s : \mathbb{R} \to \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$. Assume that the sequence $\{f_s\}$ Γ-converges to the function $f$. Let, for every $s \in \mathbb{N}$, $x_s$ be a minimizer of $f_s$ on $\mathbb{R}$. Assume that $x_s \to x$. Then $x$ minimizes $f$ on $\mathbb{R}$ and $f_s(x_s) \to f(x)$.*

P r o o f. Since $x_s \to x$, by condition (b) in Definition 1, we have

$$\liminf_{s\to\infty} f_s(x_s) \geqslant f(x). \tag{1}$$

Now, let $y \in \mathbb{R}$. By virtue of condition (a) in Definition 1, there exists a sequence $\{y_s\} \subset \mathbb{R}$ such that

$$f_s(y_s) \to f(y). \tag{2}$$

Since, for every $s \in \mathbb{N}$, $x_s$ minimizes $f_s$ on $\mathbb{R}$, we have

$$\forall s \in \mathbb{N}, \quad f_s(x_s) \leqslant f_s(y_s). \tag{3}$$

Relations (2) and (3) imply that

$$\limsup_{s\to\infty} f_s(x_s) \leqslant f(y). \tag{4}$$

From (1) and (4), we derive that $x$ minimizes $f$ on $\mathbb{R}$ and $f_s(x_s) \to f(x)$. We note that the latter limit relation follows from inequality (1) and from inequality (4) with $y = x$. □

Here, we have restricted ourselves only to a simplest version of the variational property of the Γ-convergence, having shown how both conditions (a) and (b) in Definition 1 work. The considered case is very simple not only due the fact that we dealt with functions defined on $\mathbb{R}$ but also because of the assumption that the minimizers of these functions are global. In the case of minimizers on sets defined by certain constraints, the situation is more complicated, and not always the "global" Γ-convergence (i.e., the convergence of the kind described in Definition 1 with a Γ-realizing sequence $\{y_s\}$ taken in the whole corresponding space) can be used for the study of the convergence of such minimizers.

There are analogues of the above definition of Γ-convergence for functionals defined on a Banach space (in particular, on a Lebesgue or Sobolev space). In this connection, see, for instance, [2, 4]. The notion of Γ-convergence of functionals with varying domain of definition (in particular, of functionals $\mathcal{I}_s : W^{m,p}(\Omega_s) \to \mathbb{R}$ with taking into account the structure of domains $\Omega_s$) was introduced and studied, for instance, in [5–7].

Next, note that, in the study of the convergence of minimizers $u_s$ of functionals $\mathcal{I}_s : W_s \to \mathbb{R}$, a connection of the spaces $W_s$ with a space $W$ plays an important role. Often, this connection is expressed as the requirement that there exists a sequence of operators $l_s : W_s \to W$ with certain properties. In particular, these properties should provide the following property: for every sequence $v_s \in W_s$ such that $\sup\limits_{s\in\mathbb{N}} \|v_s\|_{W_s} < +\infty$, the sequence $\{l_s v_s\}$ is bounded in $W$. Under appropriate and in some sense natural conditions on the functionals $\mathcal{I}_s$, for the sequence of minimizers $u_s \in W_s$ of the functionals $\mathcal{I}_s$, the inequality $\sup\limits_{s\in\mathbb{N}} \|u_s\|_{W_s} < +\infty$ holds. Therefore, if there exists a sequence $l_s : W_s \to W$ with the above mentioned property, then the sequence $\{l_s u_s\}$ is bounded. Consequently, if the space $W$ is reflexive, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and an element $u \in W$ such that $l_{s_j} u_{s_j} \to u$ weakly in $W$. Actually, this is the first step in the study of the convergence of the sequence of minimizers $u_s \in W_s$ of the functionals $\mathcal{I}_s$. The described idea with the operators $l_s$ is realized in the justification of the results stated below for functionals defined on the Sobolev spaces $W^{1,p}(\Omega_s)$, where $\{\Omega_s\}$ is a sequence of domains contained in a bounded domain $\Omega$ of $\mathbb{R}^n$. Essentially, the mentioned idea goes back to [8]. In this connection, see also [5–7].

The main content of this paper is organized as follows. In Section 1, we state the initial assumptions and the necessary definitions. In Section 2, we present our results on the convergence of minimizers and minimum values of integral and more general functionals on sets of functions defined by bilateral constraints in variable domains. We consider the case of regular constraints, i.e., constraints lying in the corresponding Sobolev space (see [9]), and the case where the lower constraint is zero and the upper constraint is an arbitrary nonnegative function (in this connection, see [10]). In both cases, a certain connection of the spaces $W^{1,p}(\Omega_s)$ with the space $W^{1,p}(\Omega)$ and the $\Gamma$-convergence of functionals defined on the spaces $W^{1,p}(\Omega_s)$ to a functional defined on $W^{1,p}(\Omega)$ are essentially used. At the same time, some other conditions on the involved domains, integrands, and constraints are also important for our convergence results. On the whole, the conditions providing these results are discussed in Section 3, where a special attention is paid to the so-called exhaustion condition of the domain $\Omega$ by the domains $\Omega_s$. This condition is the requirement that, for every increasing sequence $\{m_j\} \subset \mathbb{N}$, the measure of the union of all the domains $\Omega_{m_j}$ is equal to the measure of the domain $\Omega$. We also consider the notion of $\mathcal{H}$-convergence of sequences of sets $U_s \subset W^{1,p}(\Omega_s)$ to a set $U \subset W^{1,p}(\Omega)$ and show the importance of the exhaustion condition for the $\mathcal{H}$-convergence of sets of functions defined by irregular bilateral constraints.

## 1. Assumptions and definitions

Let $n \in \mathbb{N}$, $n \geqslant 2$, let $\Omega$ be a bounded domain of $\mathbb{R}^n$, and let $p > 1$. Let $\{\Omega_s\}$ be a sequence of domains of $\mathbb{R}^n$ contained in $\Omega$.

It is easy to see that if $v \in W^{1,p}(\Omega)$ and $s \in \mathbb{N}$, then $v|_{\Omega_s} \in W^{1,p}(\Omega_s)$.

**Definition 2.** If $s \in \mathbb{N}$, then $q_s : W^{1,p}(\Omega) \to W^{1,p}(\Omega_s)$ is the mapping such that, for every function $v \in W^{1,p}(\Omega)$, we have $q_s v = v|_{\Omega_s}$.

**Definition 3.** We say that the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$ if there exists a sequence of linear continuous operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that:

(a) the sequence of norms $\|l_s\|$ is bounded;

(b) for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, we have $q_s(l_s v) = v$ a.e. in $\Omega_s$.

The prototype of the notion in Definition 3 is the condition of strong connectedness of $n$-dimensional domains introduced in [8].

**Definition 4.** Let, for every $s \in \mathbb{N}$, $I_s : W^{1,p}(\Omega_s) \to \mathbb{R}$, and let $I : W^{1,p}(\Omega) \to \mathbb{R}$. We say that the sequence $\{I_s\}$ $\Gamma$-converges to the functional $I$ if the following conditions are satisfied:

(a) for every function $v \in W^{1,p}(\Omega)$, there exists a sequence $w_s \in W^{1,p}(\Omega_s)$ such that $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and $I_s(w_s) \to I(v)$;

(b) for every function $v \in W^{1,p}(\Omega)$ and for every sequence $v_s \in W^{1,p}(\Omega_s)$ such that $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$, we have $\liminf\limits_{s\to\infty} I_s(v_s) \geqslant I(v)$.

Next, let $c_1, c_2 > 0$, and let, for every $s \in \mathbb{N}$, $\mu_s \in L^1(\Omega_s)$ and $\mu_s \geqslant 0$ in $\Omega_s$. We assume that the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$ is bounded.

Let, for every $s \in \mathbb{N}$, $f_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the following conditions: for every $\xi \in \mathbb{R}^n$, the function $f_s(\cdot, \xi)$ is measurable on $\Omega_s$; for almost every $x \in \Omega_s$, the function $f_s(x, \cdot)$ is convex on $\mathbb{R}^n$; for almost every $x \in \Omega_s$ and for every $\xi \in \mathbb{R}^n$, we have

$$c_1|\xi|^p - \mu_s(x) \leqslant f_s(x,\xi) \leqslant c_2|\xi|^p + \mu_s(x). \tag{5}$$

In view of the assumptions on the functions $f_s$ and $\mu_s$, for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, the function $f_s(x, \nabla v)$ is summable on $\Omega_s$.

**Definition 5.** If $s \in \mathbb{N}$, then $F_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ is the functional such that, for every function $v \in W^{1,p}(\Omega_s)$, we have

$$F_s(v) = \int\limits_{\Omega_s} f_s(x, \nabla v)dx.$$

By virtue of the conditions on the functions $f_s$, for every $s \in \mathbb{N}$, the functional $F_s$ is convex and locally bounded. Therefore, for every $s \in \mathbb{N}$, the functional $F_s$ is weakly lower semicontinuous.

Let $c_3, c_4 > 0$, and let, for every $s \in \mathbb{N}$, $G_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ be a weakly continuous functional. We assume that, for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$,

$$G_s(v) \geqslant c_3 \|v\|^p_{L^p(\Omega_s)} - c_4. \tag{6}$$

Obviously, for every $s \in \mathbb{N}$, the functional $F_s + G_s$ is weakly lower semicontinuous. Moreover, in view of (5) and (6) and the boundedness of the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$, there exist positive constants $c_5$ and $c_6$ such that, for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, we have

$$(F_s + G_s)(v) \geqslant c_5 \|v\|^p_{W^{1,p}(\Omega_s)} - c_6. \tag{7}$$

Thus, in view of the known results on the existence of minimizers of functionals (see, for instance, [11]), if $s \in \mathbb{N}$ and $U_s$ is a sequentially weakly closed set in $W^{1,p}(\Omega_s)$, then there exists a minimizer of the functional $F_s + G_s$ on the set $U_s$.

## 2. Variational problems with bilateral constraints

First, we consider the case of regular bilateral constraints.

Let $\varphi, \psi \in W^{1,p}(\Omega)$, and let $\varphi \leqslant \psi$ a.e. in $\Omega$. We define

$$V(\varphi, \psi) = \{v \in W^{1,p}(\Omega) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega\},$$

and let, for every $s \in \mathbb{N}$,

$$V_s(\varphi, \psi) = \{v \in W^{1,p}(\Omega_s) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega_s\}.$$

It is easy to see that the set $V(\varphi, \psi)$ is nonempty, closed, and convex. Similarly, for every $s \in \mathbb{N}$, the set $V_s(\varphi, \psi)$ is nonempty, closed, and convex.

Clearly, for every $s \in \mathbb{N}$, there exists a function belonging to the set $V_s(\varphi, \psi)$ and minimizing the functional $F_s + G_s$ on this set.

**Theorem 1.** *Assume that the following conditions are satisfied*:

($*_1$) *the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact*;

($*_2$) *the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$*;

($*_3$) *for every sequence of measurable sets $H_s \subset \Omega_s$ such that $\operatorname{meas} H_s \to 0$, we have*

$$\int\limits_{H_s} \mu_s \, dx \to 0;$$

($*_4$) *the sequence $\{F_s\}$ $\Gamma$-converges to a functional $F : W^{1,p}(\Omega) \to \mathbb{R}$*;

($*_5$) *there exists a functional $G : W^{1,p}(\Omega) \to \mathbb{R}$ such that, for every function $v \in W^{1,p}(\Omega)$ and for every sequence $v_s \in W^{1,p}(\Omega_s)$ with the property $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$, we have $G_s(v_s) \to G(v)$*;

($*_6$) $\psi - \varphi > 0$ *a.e. in* $\Omega$.

*Let, for every* $s \in \mathbb{N}$, $u_s$ *be a function in* $V_s(\phi, \psi)$ *minimizing the functional* $F_s + G_s$ *on the set* $V_s(\varphi, \psi)$. *Then there exist an increasing sequence* $\{s_j\} \subset \mathbb{N}$ *and a function* $u \in V(\varphi, \psi)$ *such that* $u$ *minimizes the functional* $F + G$ *on the set* $V(\varphi, \psi)$, $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$, *and* $(F_{s_j} + G_{s_j})(u_{s_j}) \to (F + G)(u)$.

Essentially, a similar result was obtained in [12] but under stronger assumptions on the functionals $F_s$ and $G_s$ and under the condition $\psi - \varphi \geqslant \alpha$ a.e. in $\Omega$, where $\alpha > 0$. In this connection, see also [13, Theorem 2.9].

Concerning the proof of Theorem 1, we note the following. First, using operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ described in Definition 3 and defining the functions $\tilde{u}_s = \min\{\max\{l_s u_s, \varphi\}, \psi\}$, we find that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in W^{1,p}(\Omega)$ such that $\tilde{u}_{s_j} \to u$ strongly in $L^p(\Omega)$ and almost everywhere in $\Omega$. Then we obtain the inclusion $u \in V(\varphi, \psi)$, the limit relation $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$, and, by virtue of conditions ($*_4$) and ($*_5$) of Theorem 1, the inequality $\liminf\limits_{s\to\infty}(F_{s_j} + G_{s_j})(u_{s_j}) \geqslant (F + G)(u)$. The next and most important step is to establish, for every function $v \in V(\varphi, \psi)$, the existence of a sequence $w_s \in V_s(\varphi, \psi)$ with the following properties: $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and

$$\limsup_{s\to\infty} F_s(w_s) \leqslant F(v). \tag{8}$$

The construction of such a sequence involves the function $v$ and a $\Gamma$-realizing sequence $\{v_s\}$ for $v$, i.e., a sequence $v_s \in W^{1,p}(\Omega_s)$ such that $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and $F_s(v_s) \to F(v)$, which exists in view of condition ($*_4$) of Theorem 1. Moreover, it involves the difference $\psi - \varphi$. Using the limit relation $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and condition ($*_6$) of Theorem 1, we find that, for a sequence $\{\sigma_s\} \subset (0, 1]$ converging to 0, $\operatorname{meas}\{|v_s - q_s v| \geqslant \sigma_s q_s(\psi - \varphi)\} \to 0$. This is a key moment in the proof of inequality (8). For further details leading to the required properties of the function $u$, see [9, Section 2].

We now proceed to the case of irregular bilateral constraints. More precisely, we consider the case where the lower constraint is zero and the upper constraint is an arbitrary nonnegative function. Thus, in contrast to the previous case, the upper constraint can be irregular and both constraints can coincide on a set of positive measure. This is due to an additional condition on the domains $\Omega_s$ and a stronger condition on the functions $\mu_s$ as compared to condition ($*_3$) of Theorem 1.

Let $\psi : \Omega \to \overline{\mathbb{R}}$ and $\psi \geqslant 0$ a.e. in $\Omega$. We define

$$V(\psi) = \{v \in W^{1,p}(\Omega) : 0 \leqslant v \leqslant \psi \text{ a.e. in } \Omega\},$$

and let, for every $s \in \mathbb{N}$,

$$V_s(\psi) = \{v \in W^{1,p}(\Omega_s) : 0 \leqslant v \leqslant \psi \text{ a.e. in } \Omega_s\}.$$

It is easy to see that the set $V(\psi)$ is nonempty, closed, and convex. Moreover, for every $s \in \mathbb{N}$, the set $V_s(\psi)$ is nonempty, closed, and convex.

Obviously, for every $s \in \mathbb{N}$, there exists a function belonging to the set $V_s(\psi)$ and minimizing the functional $F_s + G_s$ on this set.

**Theorem 2.** *Assume that conditions* ($*_1$), ($*_2$), ($*_4$), *and* ($*_5$) *of Theorem* 1 *are satisfied. In addition, suppose that the following conditions are satisfied*:

($*'$) *for every increasing sequence* $\{m_j\} \subset \mathbb{N}$, *we have* $\operatorname{meas}\Big(\Omega \setminus \bigcup\limits_{j=1}^{\infty} \Omega_{m_j}\Big) = 0$;

($*''$) $\|\mu_s\|_{L^1(\Omega_s)} \to 0$;

*Let, for every* $s \in \mathbb{N}$, $u_s$ *be a function in* $V_s(\psi)$ *minimizing the functional* $F_s + G_s$ *on the set* $V_s(\psi)$. *Then there exist an increasing sequence* $\{s_j\} \subset \mathbb{N}$ *and a function* $u \in V(\psi)$ *such that* $u$ *minimizes the functional* $F + G$ *on the set* $V(\psi)$, $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$, *and* $(F_{s_j} + G_{s_j})(u_{s_j}) \to (F + G)(u)$.

As for the proof of Theorem 2, we give the following remarks. Since, in general, the function $\psi$ is irregular, we cannot use functions like the above functions $\tilde{u}_s$ in the proof of Theorem 1. Therefore, using operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ described in Definition 3, first, we find that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in W^{1,p}(\Omega)$ such that $l_{s_j} u_{s_j} \to u$ strongly in $L^p(\Omega)$ and almost everywhere in $\Omega$. Then, to prove that $u \in V(\psi)$, along with the inclusions $u_s \in V_s(\psi)$, we use condition ($*'$) of Theorem 2 which effectively works in this situation. Similarly to the proof of Theorem 1, the most important step in the proof of Theorem 2 is to establish, for every function $v \in V(\psi)$, the existence of a sequence $w_s \in V_s(\psi)$ such that $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and inequality (8) holds. The construction of such a sequence involves the function $v$ and a $\Gamma$-realizing sequence $\{v_s\}$ for $v$ but does not involve the constraint $\psi$. To prove inequality (8), we essentially use condition ($*''$) of Theorem 2 and the fact that $\operatorname{meas}(\{|v_s - q_s v| \geqslant \sigma_s q_s v\} \cap \{v > 0\}) \to 0$, where $\{\sigma_s\}$ is a sequence in $[0, 1)$ such that $\sigma_s \to 0$. For details, see the proof of Theorem 3.1 in [10].

The next result describes a situation where we have the convergence of the whole sequence of minimizers and of the whole sequence of minimum values.

**Theorem 3.** *Assume that conditions* ($*_1$), ($*_2$), ($*_4$), *and* ($*_5$) *of Theorem* 1 *are satisfied, and the functional* $G$ *is strictly convex on the set* $V(\psi)$. *In addition, suppose that conditions* ($*'$) *and* ($*''$) *of Theorem* 2 *are satisfied. Let, for every* $s \in \mathbb{N}$, $u_s$ *be a function in* $V_s(\psi)$ *minimizing the functional* $F_s + G_s$ *on the set* $V_s(\psi)$. *Then there exists a unique function* $u \in V(\psi)$ *minimizing the functional* $F + G$ *on the set* $V(\psi)$ *and the following relations hold*: $\|u_s - q_s u\|_{L^p(\Omega_s)} \to 0$ *and* $(F_s + G_s)(u_s) \to (F + G)(u)$.

## 3. Comments to the conditions of Theorems 1–3

As is known (see, for instance, [14, Chapter 6]), condition ($*_1$) of Theorem 1 is satisfied if $\Omega$ is a Lipschitz domain. In particular, bounded convex domains are Lipschitz domains. A more general requirement guaranteeing the fulfillment of condition ($*_1$) is that $\Omega$ is an extension domain (see, for instance, [15, Chapter 1]).

Condition ($*_2$) of Theorem 1 is satisfied, in particular, if the domains $\Omega_s$ have a certain perforated structure. In this regard, see, for instance, [16, Section 2].

As far as conditions ($*_3$) and ($*_4$) of Theorem 1 are concerned, we note the following. In the case where the functions $\mu_s$ take a constant value independent of $s$, theorems on conditions for the $\Gamma$-convergence of the integral functionals $F_s$ with the integrands $f_s$ satisfying condition (5) follow from the results of [17, 18], where the $\Gamma$-convergence of integral functionals defined on the spaces $W^{m,p}(\Omega_s)$ with an arbitrary $m \in \mathbb{N}$ was studied. In this case, the sequence $\{F_s\}$ $\Gamma$-converges to an integral functional defined on the space $W^{1,p}(\Omega)$, in particular, if the domains $\Omega_s$ have a periodic perforated structure and all the integrands $f_s$ coincide with the same integrand having a certain regularity (see [17]). Obviously, in the specified case for the functions $\mu_s$, the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$ is bounded and condition ($*_3$) of Theorem 1 is satisfied. In the more general case where $\mu_s \in L^1(\Omega_s)$ and $\mu_s \geqslant 0$ in $\Omega_s$ for every $s \in \mathbb{N}$ and, in addition, the inequality

$$\limsup_{s\to\infty} \int\limits_{Q\cap\Omega_s} \mu_s \, dx \leqslant \int\limits_{Q\cap\Omega} \mu \, dx \tag{9}$$

holds for a function $\mu \in L^1(\Omega)$, $\mu \geqslant 0$ in $\Omega$, and for every open cube $Q$ of $\mathbb{R}^n$, a theorem on the $\Gamma$-compactness of the sequence $\{F_s\}$ can be proved similarly to the corresponding results in [19, 20]. Obviously, in this case, the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$ is bounded. We also note that there are examples of sequences of nonnegative functions $\mu_s \in L^1(\Omega_s)$ for which condition (9) and condition $(*_3)$ of Theorem 1 are satisfied but there is no function $\mu_* : \Omega \to \mathbb{R}$ such that, for every $s \in \mathbb{N}$, $\mu_s \leqslant \mu_*$ a.e. in $\Omega_s$. Such examples can be given with the use of the functions constructed in [21].

In connection with condition $(*_5)$ of Theorem 1, we give the following example.

*Example 1.* Let $a \in L^{p/(p-1)}(\Omega)$. Let $\beta_1 \in (0,1)$, let $\beta_2 > 0$, and let $\Phi : [0,+\infty) \to \mathbb{R}$ be a continuous function such that

$$\forall\, \eta \in [0,+\infty), \quad |\Phi(\eta)| \leqslant \beta_1|\eta|^p + \beta_2. \tag{10}$$

For every $s \in \mathbb{N}$, we define the functional $G_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ by

$$G_s(v) = \int\limits_{\Omega_s} \{|v|^p + av\}dx + \Phi(\|v\|_{L^p(\Omega_s)}), \quad v \in W^{1,p}(\Omega_s).$$

In view of (10), for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, inequality (6) holds with constants $c_3$ and $c_4$ depending only on $p$, $\beta_1$, $\beta_2$, and $\|a\|_{L^{p/(p-1)}(\Omega)}$. We also note that if conditions $(*_1)$ and $(*_2)$ of Theorem 1 are satisfied, then, for every $s \in \mathbb{N}$, the functional $G_s$ is weakly continuous. Next, assume that the following condition is satisfied:

$(*)$ there exists a nonnegative bounded measurable function $b : \Omega \to \mathbb{R}$ such that, for every open cube $Q \subset \Omega$, we have $\operatorname{meas}(Q \cap \Omega_s) \to \int_Q b\, dx$.

Now, let $G : W^{1,p}(\Omega) \to \mathbb{R}$ be the functional such that, for every function $v \in W^{1,p}(\Omega)$, we have

$$G(v) = \int\limits_{\Omega} b\{|v|^p + av\}dx + \Phi(\|b^{1/p}v\|_{L^p(\Omega)}). \tag{11}$$

Using condition $(*)$ and the continuity of the function $\Phi$, we find that, for the sequence of functionals $G_s$, condition $(*_5)$ of Theorem 1 is satisfied.

We remark that if the domain $\Omega$ is Lipschitz and the domains $\Omega_s$ have a certain periodically perforated structure, then conditions $(*_1)$ and $(*_2)$ of Theorem 1 are satisfied along with condition $(*)$ in which the function $b$ takes a constant positive value. Obviously, for such a function $b$, the functional $G$ defined by (11) is strictly convex if the function $\Phi$ is nondecreasing and convex.

We emphasize the importance of condition $(*_6)$ of Theorem 1 for its conclusion. In [9], we gave an example where all the conditions of Theorem 1 are satisfied except for condition $(*_6)$ but the conclusion of this theorem does not hold on the whole. We note that, in this example, for an arbitrary pre-assigned positive $\varepsilon$, the measure of the set where the lower and upper constraints coincide does not exceed $\varepsilon$. Here is a simple example where condition $(*_6)$ of Theorem 1 is satisfied.

*Example 2.* Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, and let, for every $x \in \Omega$, we have $\varphi(x) = 0$ and $\psi(x) = |x|^2(1-|x|^2)$. In view of these assumptions, we have $\varphi, \psi \in \overset{\circ}{W}{}^{1,p}(\Omega)$ and $\varphi \leqslant \psi$ in $\Omega$. In addition, for every $x \in \Omega \setminus \{0\}$, $(\psi - \varphi)(x) > 0$. Thus, condition $(*_6)$ of Theorem 1 is satisfied. We observe that, in the case considered here, we have $V(\varphi,\psi) = \{v \in \overset{\circ}{W}{}^{1,p}(\Omega) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega\}$. Hence, for $p = 2$, the set $V(\varphi,\psi)$ has the same form as the set defined by bilateral constraints in [22]. We also note that if $\omega$ is a domain of $\mathbb{R}^n$ such that $\overline{\omega} \subset \Omega$ and the origin is contained in $\omega$, then there is no number $\delta^\omega > 0$ such that $\psi - \varphi \geqslant \delta^\omega$ a.e. in $\omega$. We remark in this connection that it was shown in [22] that the $G$-convergence of a sequence of linear continuous divergence operators $A_s : \overset{\circ}{W}{}^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ to an operator $A : \overset{\circ}{W}{}^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ of the same form implies the weak convergence of solutions of variational inequalities with the operators $A_s$ and the set of constraints $K(\psi_1, \psi_2) = \{v \in \overset{\circ}{W}{}^{1,2}(\Omega) : \psi_1 \leqslant v \leqslant \psi_2 \text{ a.e. in } \Omega\}$ to the solution of the corresponding variational inequality with the operator $A$ and the same set of constraints. At the same time, it was assumed in [22] that $\psi_1, \psi_2 \in L^2(\Omega)$ and, for every subdomain $\omega \subset\subset \Omega$, there exist a number $\delta^\omega > 0$ and functions $\psi_1^\omega, \psi_2^\omega \in \overset{\circ}{W}{}^{1,2}(\Omega)$ such that $\psi_1 \leqslant \psi_1^\omega \leqslant \psi_2^\omega \leqslant \psi_2$ in $\Omega$ and $\psi_2^\omega - \psi_1^\omega \geqslant \delta^\omega$ in $\omega$. Obviously, the functions $\varphi$ and $\psi$ defined at the beginning of this example do not satisfy the assumption given in [22].

We now discuss condition $(*')$ of Theorem 2. This condition is essential for the conclusion of Theorem 2. In [10], we construct an example where all the conditions of Theorem 2 are satisfied except for condition $(*')$ but the conclusion of this theorem does not hold. We call condition $(*')$ of Theorem 2 the exhaustion condition of the domain $\Omega$ by the domains $\Omega_s$. This condition plays an important role in the study of the convergence of solutions of variational problems with irregular unilateral and bilateral constraints in variable domains. In this regard, in addition to the present paper, see [23, 24]. We used the same exhaustion condition earlier in [6] for the investigation of both a convergence of sets in variable Sobolev spaces and the coercivity of the $\Gamma$-limit of functionals defined on these spaces. Below, we show how such questions are solved for sequences of sets $U_s \subset W^{1,p}(\Omega_s)$ and the functionals $F_s + G_s$. Before we do this, let us give some useful results.

**Proposition 2.** *Condition* $(*')$ *of Theorem* 2 *is equivalent to the following condition*:

$$\text{if } v \in L^1(\Omega) \text{ and } \liminf_{s\to\infty} \int\limits_{\Omega_s} |v|dx = 0, \text{ then } v = 0 \text{ a.e. in } \Omega. \tag{12}$$

P r o o f. Assume that condition $(*')$ of Theorem 2 is satisfied. Let $v \in L^1(\Omega)$, and let

$$\liminf_{s\to\infty} \int\limits_{\Omega_s} |v|dx = 0.$$

Fixing an arbitrary $\varepsilon > 0$, we find that there exists an increasing sequence $\{s_j\} \subset \mathbb{N}$ such that

$$\forall j \in \mathbb{N}, \quad \int\limits_{\Omega_{s_j}} |v|dx \leqslant \frac{\varepsilon}{2^j}. \tag{13}$$

Setting $\Omega' = \bigcup\limits_{j=1}^{\infty} \Omega_{s_j}$, by condition $(*')$ of Theorem 2, we have $\operatorname{meas}(\Omega \setminus \Omega') = 0$. Then

$$\int\limits_{\Omega} |v|dx = \int\limits_{\Omega'} |v|dx \leqslant \sum_{j=1}^{\infty} \int\limits_{\Omega_{s_j}} |v|dx.$$

This and (13) imply that

$$\int\limits_{\Omega} |v|dx \leqslant \varepsilon.$$

Hence, in view of the arbitrariness of $\varepsilon$, we conclude that $v = 0$ a.e. in $\Omega$. Thus, condition (12) is satisfied.

Conversely, assume that condition (12) is satisfied. Let $\{m_j\}$ be an increasing sequence in $\mathbb{N}$. Setting $E_0 = \Omega \setminus \bigcup_{j=1}^{\infty} \Omega_{m_j}$, we suppose that $\operatorname{meas} E_0 > 0$. Let $\chi : \Omega \to \mathbb{R}$ be the characteristic function of the set $E_0$. Obviously, $\chi \in L^1(\Omega)$ and $\int_{\Omega_{m_j}} \chi\, dx = 0$ for every $j \in \mathbb{N}$. Therefore,

$$\liminf_{s\to\infty} \int_{\Omega_s} \chi\, dx = 0.$$

Then, by condition (12), we have $\chi = 0$ a.e. in $\Omega$. Hence, there exists a set $E \subset \Omega$ of measure zero such that, for every $x \in \Omega \setminus E$, we have $\chi(x) = 0$. Then, fixing $x \in E_0 \setminus E$, we obtain $\chi(x) = 0$. On the other hand, by the definition of the function $\chi$, we have $\chi(x) = 1$. The obtained contradiction proves that $\operatorname{meas} E_0 = 0$. Thus, condition $(*')$ of Theorem 2 is satisfied. □

**Proposition 3.** *Let condition $(*')$ of Theorem 2 be satisfied. Then the following condition is satisfied:*

$$\text{if } v \in W^{1,p}(\Omega) \text{ and } \liminf_{s\to\infty} \|q_s v\|_{L^p(\Omega_s)} = 0, \text{ then } v = 0 \text{ a.e. in } \Omega. \tag{14}$$

P r o o f. Let $v \in W^{1,p}(\Omega)$ and $\liminf_{s\to\infty} \|q_s v\|_{L^p(\Omega_s)} = 0$. Setting $w = |v|^p$, we have

$$w \in L^1(\Omega), \qquad \liminf_{s\to\infty} \int_{\Omega_s} w\, dx = 0. \tag{15}$$

Since, by assumption, condition $(*')$ of Theorem 2 is satisfied, we deduce from Proposition 2 that condition (12) is satisfied. The latter condition along with (15) implies that $w = 0$ a.e. in $\Omega$. Hence, $v = 0$ a.e. in $\Omega$. Thus, condition (14) is satisfied. □

**Proposition 4.** *Let condition $(*_1)$ of Theorem 1 be satisfied, and assume that there exists a sequence of linear continuous operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that the sequence of norms $\|l_s\|$ is bounded and, for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, we have $q_s(l_s v) = v$ a.e. in $\Omega_s$. Let, for every $s \in \mathbb{N}$, $w_s \in W^{1,p}(\Omega_s)$. Assume that the sequence of norms $\|w_s\|_{W^{1,p}(\Omega_s)}$ is bounded. Then there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $w \in W^{1,p}(\Omega)$ such that $l_{s_j} w_{s_j} \to w$ weakly in $W^{1,p}(\Omega)$, $l_{s_j} w_{s_j} \to w$ a.e. in $\Omega$, and $\|w_{s_j} - q_{s_j} w\|_{L^p(\Omega_{s_j})} \to 0$.*

P r o o f. The properties of the operators $l_s$ along with the boundedness of the sequence of norms $\|w_s\|_{W^{1,p}(\Omega_s)}$ imply that the sequence $\{l_s w_s\}$ is bounded in $W^{1,p}(\Omega)$ and

$$\forall s \in \mathbb{N}, \quad q_s(l_s w_s) = w_s \text{ a.e. in } \Omega_s. \tag{16}$$

Since the space $W^{1,p}(\Omega)$ is reflexive and the sequence $\{l_s w_s\}$ is bounded in $W^{1,p}(\Omega)$, there exist an increasing sequence $\{\bar{s}_k\} \subset \mathbb{N}$ and a function $w \in W^{1,p}(\Omega)$ such that $l_{\bar{s}_k} w_{\bar{s}_k} \to w$ weakly in $W^{1,p}(\Omega)$. Hence, by condition $(*_1)$ of Theorem 1, we have $l_{\bar{s}_k} w_{\bar{s}_k} \to w$ strongly in $L^p(\Omega)$. Therefore, there exists an increasing sequence $\{s_j\} \subset \{\bar{s}_k\}$ such that $l_{s_j} w_{s_j} \to w$ a.e. in $\Omega$. It is clear that $l_{s_j} w_{s_j} \to w$ weakly in $W^{1,p}(\Omega)$ and $l_{s_j} w_{s_j} \to w$ strongly in $L^p(\Omega)$. The latter convergence along with (16) implies that $\|w_{s_j} - q_{s_j} w\|_{L^p(\Omega_{s_j})} \to 0$. □

**Proposition 5.** *Let condition* $(*_1)$ *of Theorem* 1 *be satisfied, and assume that there exists a sequence of linear continuous operators* $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ *such that the sequence of norms* $\|l_s\|$ *is bounded and, for every* $s \in \mathbb{N}$ *and for every* $v \in W^{1,p}(\Omega_s)$, *we have* $q_s(l_s v) = v$ *a.e. in* $\Omega_s$. *In addition, assume that condition* $(*')$ *of Theorem* 2 *is satisfied. Let, for every* $s \in \mathbb{N}$, $w_s \in W^{1,p}(\Omega_s)$, *and let* $w \in W^{1,p}(\Omega)$. *Assume that the sequence of norms* $\|w_s\|_{W^{1,p}(\Omega_s)}$ *is bounded and* $\|w_s - q_s w\|_{L^p(\Omega_s)} \to 0$. *Then* $l_s w_s \to w$ *weakly in* $W^{1,p}(\Omega)$.

P r o o f. The properties of the operators $l_s$ imply that the sequence $\{l_s w_s\}$ is bounded in $W^{1,p}(\Omega)$ and

$$\forall s \in \mathbb{N}, \quad q_s(l_s w_s) = w_s \text{ a.e. in } \Omega_s. \tag{17}$$

Assume that the sequence $\{l_s w_s\}$ does not converge weakly to $w$ in $W^{1,p}(\Omega)$. Then there exist a functional $g \in (W^{1,p}(\Omega))^*$, a number $\varepsilon > 0$, and an increasing sequence $\{\bar{s}_k\} \subset \mathbb{N}$ such that

$$\forall k \in \mathbb{N}, \quad |\langle g, l_{\bar{s}_k} w_{\bar{s}_k}\rangle - \langle g, w\rangle| > \varepsilon. \tag{18}$$

Since the space $W^{1,p}(\Omega)$ is reflexive and the sequence $\{l_s w_s\}$ is bounded in $W^{1,p}(\Omega)$, there exist an increasing sequence $\{s_j\} \subset \{\bar{s}_k\}$ and a function $w_0 \in W^{1,p}(\Omega)$ such that

$$l_{s_j} w_{s_j} \to w_0 \text{ weakly in } W^{1,p}(\Omega). \tag{19}$$

Hence, by condition $(*_1)$ of Theorem 1, we have $l_{s_j} w_{s_j} \to w_0$ strongly in $L^p(\Omega)$. Then, in view of (17), we have $\|w_{s_j} - q_{s_j} w_0\|_{L^p(\Omega_{s_j})} \to 0$. This and the assumption that $\|w_s - q_s w\|_{L^p(\Omega_s)} \to 0$ imply that $\|q_{s_j}(w - w_0)\|_{L^p(\Omega_{s_j})} \to 0$. Consequently, $\liminf\limits_{s\to\infty} \|q_s(w - w_0)\|_{L^p(\Omega_s)} = 0$. From this equality, condition $(*')$ of Theorem 2, and Proposition 3, we derive that $w = w_0$ a.e. in $\Omega$. Then, in view of (19), we have $l_{s_j} w_{s_j} \to w$ weakly in $W^{1,p}(\Omega)$. However, this contradicts (18). The obtained contradiction proves that $l_s w_s \to w$ weakly in $W^{1,p}(\Omega)$. ☐

The following definition essentially is a particular case of Definition 5 in [6].

**Definition 6.** Let, for every $s \in \mathbb{N}$, $U_s$ be a nonempty set in $W^{1,p}(\Omega_s)$, and let $U$ be a nonempty set in $W^{1,p}(\Omega)$. We say that the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set $U$ if the following conditions are satisfied:

(a) for every function $v \in U$, there exists a sequence $w_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|w_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$;

(b) for every sequence $v_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $v \in U$ such that $\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0$.

**Proposition 6.** *Let condition* $(*')$ *of Theorem* 2 *be satisfied. Then a sequence of nonempty sets* $U_s \subset W^{1,p}(\Omega_s)$ *may* $\mathcal{H}$*-converge to only one nonempty set* $U \subset W^{1,p}(\Omega)$.

P r o o f. Assume that a sequence of nonempty sets $U_s \subset W^{1,p}(\Omega_s)$ $\mathcal{H}$-converges to nonempty sets $U \subset W^{1,p}(\Omega)$ and $V \subset W^{1,p}(\Omega)$. Let $w \in U$. Since the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set $U$, there exists a sequence $w_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|w_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and $\|w_s - q_s w\|_{L^p(\Omega_s)} \to 0$. Since the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set $V$, for the sequence $\{w_s\}$, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $v \in V$ such that $\|w_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0$. This convergence along with the convergence $\|w_s - q_s w\|_{L^p(\Omega_s)} \to 0$ implies that $\|q_{s_j}(v - w)\|_{L^p(\Omega_{s_j})} \to 0$. Then, taking into account condition $(*')$ of Theorem 2 and Proposition 3, we find that $w = v$ a.e. in $\Omega$. Therefore, in view of the inclusion $v \in V$, we have $w \in V$. Consequently, $U \subset V$. In the same way, we prove that $V \subset U$. Thus, $U = V$. $\square$

*Remark 1.* In the proof of Proposition 6, concerning the considered sets in $W^{1,p}(\Omega)$, we implicitly assumed that functions equivalent to elements of these sets belong to the same sets.

**Proposition 7.** *Assume that the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact and the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$. Then the sequence $\{W^{1,p}(\Omega_s)\}$ $\mathcal{H}$-converges to the set $W^{1,p}(\Omega)$.*

P r o o f. Let $v \in W^{1,p}(\Omega)$. For every $s \in \mathbb{N}$, we set $w_s = q_s v$. Obviously, for every $s \in \mathbb{N}$, we have $w_s \in W^{1,p}(\Omega_s)$. It is also easy to see that $\sup\limits_{s\in\mathbb{N}} \|w_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$. Next, taking a sequence $v_s \in W^{1,p}(\Omega_s)$ such that $\sup\limits_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$, in view of the assumptions of this proposition, we deduce from Proposition 4 that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a functon $v \in W^{1,p}(\Omega)$ such that $\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0$. Now, by Definition 6, we conclude that the sequence $\{W^{1,p}(\Omega_s)\}$ $\mathcal{H}$-converges to the set $W^{1,p}(\Omega)$. $\square$

We note that condition $(*')$ of Theorem 2 is essential for the conclusion of Proposition 6. This is justified by the following simple example.

*Example 3.* Assume that $\Omega$ is a Lipschitz domain. Then the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact. Let $B$ be a closed ball in $\mathbb{R}^n$ such that $B \subset \Omega$, and assume that, for every $s \in \mathbb{N}$, $\Omega_s = \Omega \setminus B$. In view of the known extension results for Sobolev spaces (see, for instance, [25, Theorem 7.25]), there exists a linear continuous operator $l : W^{1,p}(\Omega \setminus B) \to W^{1,p}(\Omega)$ such that, for every function $v \in W^{1,p}(\Omega \setminus B)$, we have $lv = v$ in $\Omega \setminus B$. Setting, for every $s \in \mathbb{N}$, $l_s = l$, we find that the sequence $\{l_s\}$ has all the properties described in Definition 3. Therefore, the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$. Thus, Proposition 7 implies that the sequence $\{W^{1,p}(\Omega_s)\}$ $\mathcal{H}$-converges to the set $W^{1,p}(\Omega)$. Now, let $y$ and $r$ be the center and the radius of the ball $B$, respectively, and let $B_0 = \{x \in \mathbb{R}^n : |x - y| \leqslant r/2\}$. We define

$$U = \{v \in W^{1,p}(\Omega) : v = 0 \text{ a.e. in } B_0\}.$$

It is easy to see that, for every function $v \in U$, there exists a sequence $w_s \in W^{1,p}(\Omega_s)$ such that $\sup\limits_{s\in\mathbb{N}} \|w_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$. Next, we fix an arbitrary sequence $v_s \in W^{1,p}(\Omega_s)$ such that $\sup\limits_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$. Since the sequence $\{W^{1,p}(\Omega_s)\}$ $\mathcal{H}$-converges to the set $W^{1,p}(\Omega)$, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $v \in W^{1,p}(\Omega)$ such that

$$\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0. \tag{20}$$

Let $\varphi$ be a function in $C_0^\infty(\Omega)$ such that $0 \leqslant \varphi \leqslant 1$ in $\Omega$, $\varphi = 1$ in $B_0$, and $\varphi = 0$ in $\Omega \setminus B$. We have $v\varphi \in W^{1,p}(\Omega)$. Then, since $\varphi = 1$ in $B_0$, we have $v - v\varphi \in U$. Moreover, taking into account that $\varphi = 0$ in $\Omega \setminus B$, we derive from (20) that $\|v_{s_j} - q_{s_j}(v - v\varphi)\|_{L^p(\Omega_{s_j})} \to 0$. Now, we conclude that the sequence $\{W^{1,p}(\Omega_s)\}$ $\mathcal{H}$-converges to the set $U$. Obviously, $U \neq W^{1,p}(\Omega)$. It remains to observe that $\Omega \setminus \bigcup\limits_{s=1}^{\infty} \Omega_s = B$. Hence, $\operatorname{meas}\Big(\Omega \setminus \bigcup\limits_{s=1}^{\infty} \Omega_s\Big) > 0$. Consequently, condition $(*')$ of Theorem 2 is not satisfied.

We now proceed to a more delicate question on the $\mathcal{H}$-convergence of sets defined by bilateral constraints.

**Proposition 8.** *Assume that conditions* $(*_1)$ *and* $(*_2)$ *of Theorem* 1 *and condition* $(*')$ *of Theorem* 2 *are satisfied. Let* $\varphi, \psi : \Omega \to \overline{\mathbb{R}}$, *and let* $\varphi \leqslant \psi$ *a.e. in* $\Omega$. *Let, for every* $s \in \mathbb{N}$, $U_s = \{v \in W^{1,p}(\Omega_s) : \varphi \leqslant v \leqslant \psi$ *a.e. in* $\Omega_s\}$, *and let* $U = \{v \in W^{1,p}(\Omega) : \varphi \leqslant v \leqslant \psi$ *a.e. in* $\Omega\}$. *Assume that the set* $U$ *is nonempty. Then the sequence* $\{U_s\}$ $\mathcal{H}$*-converges to the set* $U$.

P r o o f. Let $v \in U$. For every $s \in \mathbb{N}$, we set $w_s = q_s v$. Obviously, for every $s \in \mathbb{N}$, we have $w_s \in U_s$. It is also easy to see that $\sup\limits_{s\in\mathbb{N}} \|w_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$.

Next, we fix an arbitrary sequence $v_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$. Since condition $(*_2)$ of Theorem 1 is satisfied, there exists a sequence of linear continuous operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that the sequence of norms $\|l_s\|$ is bounded and, for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, we have $q_s(l_s v) = v$ a.e. in $\Omega_s$. Then, taking into account that condition $(*_1)$ of Theorem 1 is satisfied, we derive from Proposition 4 that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $w \in W^{1,p}(\Omega)$ such that $l_{s_j} v_{s_j} \to w$ a.e. in $\Omega$ and $\|v_{s_j} - q_{s_j} w\|_{L^p(\Omega_{s_j})} \to 0$. Let us show that $\varphi \leqslant w \leqslant \psi$ a.e. in $\Omega$. Since, for every $s \in \mathbb{N}$, we have $v_s \in U_s$, there exists a set $E' \subset \Omega$ of measure zero such that, for every $s \in \mathbb{N}$ and for every $x \in \Omega_s \setminus E'$, we have $\varphi(x) \leqslant v_s(x) \leqslant \psi(x)$. In addition, by the properties of the operators $l_s$, there exists a set $E'' \subset \Omega$ of measure zero such that, for every $s \in \mathbb{N}$ and for every $x \in \Omega_s \setminus E''$, we have $(l_s v_s)(x) = v_s(x)$. It is clear that

$$s \in \mathbb{N},\ x \in \Omega_s \setminus (E' \cup E'') \Longrightarrow \varphi(x) \leqslant (l_s v_s)(x) \leqslant \psi(x). \tag{21}$$

Since $l_{s_j} v_{s_j} \to w$ a.e. in $\Omega$, there exists a set $E''' \subset \Omega$ of measure zero such that

$$\forall x \in \Omega \setminus E''', \qquad (l_{s_j} v_{s_j})(x) \to w(x). \tag{22}$$

Next, for every $k \in \mathbb{N}$, we set $E^{(k)} = \Omega \setminus \bigcup\limits_{j=k}^{\infty} \Omega_{s_j}$. In view of condition $(*')$ of Theorem 2, for every $k \in \mathbb{N}$, we have $\operatorname{meas} E^{(k)} = 0$. Therefore, setting $E = \bigcup\limits_{k=1}^{\infty} E^{(k)}$, we have $\operatorname{meas} E = 0$. Now, let $x \in \Omega \setminus (E' \cup E'' \cup E''' \cup E)$. We fix an arbitrary $\varepsilon > 0$. Since $x \in \Omega \setminus E'''$, by (22), we have $(l_{s_j} v_{s_j})(x) \to w(x)$. Consequently, there exists $k \in \mathbb{N}$ such that

$$j \in \mathbb{N},\ j \geqslant k \implies |(l_{s_j} v_{s_j})(x) - w(x)| \leqslant \varepsilon. \tag{23}$$

Since $x \in \Omega \setminus E$, there exists $j \in \mathbb{N}$, $j \geqslant k$, such that $x \in \Omega_{s_j}$. Then we derive from (21) and (23) that $\varphi(x) - \varepsilon \leqslant w(x) \leqslant \psi(x) + \varepsilon$. Hence, in view of the arbitrariness of $\varepsilon$, we obtain the inequality $\varphi(x) \leqslant w(x) \leqslant \psi(x)$. Therefore, $\varphi \leqslant w \leqslant \psi$ a.e. in $\Omega$. Then $w \in U$. Thus, we have established that, for every sequence $v_s \in U_s$ such that $\sup\limits_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $w \in U$ such that $\|v_{s_j} - q_{s_j} w\|_{L^p(\Omega_{s_j})} \to 0$.

We now conclude that the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set $U$. □

We note that condition $(*')$ of Theorem 2 is essential for the conclusion of Proposition 8. This is justified by the following example.

*Example 4.* Assume that the domain $\Omega$ and the sequence of domains $\Omega_s$ are the same as in Example 3. Then conditions $(*_1)$ and $(*_2)$ of Theorem 1 are satisfied but condition $(*')$ of Theorem 2 is not satisfied. Let $\varphi : \Omega \to \overline{\mathbb{R}}$ be the function such that, for every $x \in \Omega$, $\varphi(x) = 0$. Moreover, let $\psi : \Omega \to \overline{\mathbb{R}}$ be the function such that

$$\psi(x) = \begin{cases} 0 & \text{if } x \in B, \\ 1 & \text{if } x \in \Omega \setminus B. \end{cases}$$

Obviously, $\varphi \leqslant \psi$ in $\Omega$. Let, for every $s \in \mathbb{N}$, $U_s = \{v \in W^{1,p}(\Omega_s) : \varphi \leqslant v \leqslant \psi$ a.e. in $\Omega_s\}$, and let $U = \{v \in W^{1,p}(\Omega) : \varphi \leqslant v \leqslant \psi$ a.e. in $\Omega\}$. Clearly, the set $U$ is nonempty. Thus, all the conditions of Proposition 8 are satisfied except for condition $(*')$ of Theorem 2. At the same time, the sequence $\{U_s\}$ does not $\mathcal{H}$-converge to the set $U$. In fact, suppose that the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set $U$. Then, taking the sequence $v_s \in W^{1,p}(\Omega_s)$ such that, for every $s \in \mathbb{N}$, $v_s = 1$ in $\Omega_s$, we find that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $v \in U$ such that $\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0$. Hence, $v = 1$ a.e. in $\Omega \setminus B$. Therefore, $v - 1 \in \overset{\circ}{W}{}^{1,p}(\Omega)$. Moreover, since $v \in U$, we have $v = 0$ a.e. in $B$. Thus, $|\nabla v| = 0$ a.e. in $\Omega$. Then, fixing a number $r$ such that $1 < r < \min\{p, n\}$ and taking into account that $v - 1 \in \overset{\circ}{W}{}^{1,r}(\Omega)$, we apply the corresponding Sobolev inequality for the function $v - 1$ and find that $v = 1$ a.e. in $\Omega$. However, this contradicts the fact that $v = 0$ a.e. in $B$. The obtained contradiction proves that the sequence $\{U_s\}$ does not $\mathcal{H}$-converge to the set $U$.

Although, in the general case, condition $(*')$ of Theorem 2 is essential for the $\mathcal{H}$-convergence of sets defined by bilateral constraints, in the case of regular constraints, this condition does not play any role for the $\mathcal{H}$-convergence of the corresponding sets. We demonstrate this by proving the following result.

**Proposition 9.** *Assume that conditions* $(*_1)$ *and* $(*_2)$ *of Theorem* 1 *are satisfied. Let* $\varphi, \psi \in W^{1,p}(\Omega)$, *and let* $\varphi \leqslant \psi$ *a.e. in* $\Omega$. *Let, for every* $s \in \mathbb{N}$, $U_s = \{v \in W^{1,p}(\Omega_s) : \varphi \leqslant v \leqslant \psi$ *a.e. in* $\Omega_s\}$, *and let* $U = \{v \in W^{1,p}(\Omega) : \varphi \leqslant v \leqslant \psi$ *a.e. in* $\Omega\}$. *Then the sequence* $\{U_s\}$ $\mathcal{H}$*-converges to the set* $U$.

P r o o f. As in the proof of Proposition 8, we establish that, for every function $v \in U$, there exists a sequence $w_s \in U_s$ such that $\sup\limits_{s \in \mathbb{N}} \|w_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$.

Next, we fix an arbitrary sequence $v_s \in U_s$ such that $\sup\limits_{s \in \mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$. In view of condition $(*_2)$ of Theorem 1, there exists a sequence of linear continuous operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that the sequence of norms $\|l_s\|$ is bounded and

$$\forall s \in \mathbb{N}, \qquad q_s(l_s v_s) = v_s \ \text{ a.e. in } \ \Omega_s. \tag{24}$$

It is easy to see that the sequence $\{l_s v_s\}$ is bounded in $W^{1,p}(\Omega)$. For every $s \in \mathbb{N}$, we set

$$z_s = \min\{\max\{l_s v_s, \varphi\}, \psi\}.$$

We have $\{z_s\} \subset U$ and the sequence $\{z_s\}$ is bounded in $W^{1,p}(\Omega)$. Moreover, using (24) and the inclusions $v_s \in U_s$, we establish that

$$\forall s \in \mathbb{N}, \qquad q_s z_s = v_s \ \text{ a.e. in } \ \Omega_s. \tag{25}$$

Using the reflexivity of the space $W^{1,p}(\Omega)$, the boundedness of the sequence $\{z_s\}$ in $W^{1,p}(\Omega)$, and condition $(*_1)$ of Theorem 1, we find that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $v \in W^{1,p}(\Omega)$ such that

$$z_{s_j} \to v \ \text{ strongly in } \ L^p(\Omega) \tag{26}$$

and $z_{s_j} \to v$ a.e. in $\Omega$. The latter limit relation along with the inclusion $\{z_{s_j}\} \subset U$ implies that $v \in U$. Finally, we derive from (25) and (26) that $\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0$. Thus, we have established that, for every sequence $v_s \in U_s$ such that $\sup\limits_{s \in \mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $v \in U$ such that $\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0$.

We now conclude that the sequence $\{U_s\}$ $\mathcal{H}$-converges to the set $U$. $\square$

*Remark 2.* Concerning some notions of convergence of sets lying in the same space, see, for instance, [26, 27]. Our notion of $\mathcal{H}$-convergence of sets lying generally in variable spaces differs from the notions of convergence of sets in the sense of Kuratowski [26, Section 29] and in the sense of Mosco [27, Definition 1.1] even in the case of sets belonging to the same space.

We give one more result involving condition $(*')$ of Theorem 2.

**Proposition 10.** *Let conditions $(*_1)$, $(*_2)$, $(*_4)$, and $(*_5)$ of Theorem 1 be satisfied. In addition, let condition $(*')$ of Theorem 2 be satisfied. Then there exist positive constants $b_1$ and $b_2$ such that, for every function $v \in W^{1,p}(\Omega)$, we have $(F+G)(v) \geqslant b_1\|v\|^p_{W^{1,p}(\Omega)} - b_2$.*

P r o o f. By condition $(*_2)$ of Theorem 1, there exists a sequence of linear continuous operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that the sequence of norms $\|l_s\|$ is bounded and, for every $s \in \mathbb{N}$ and for every $v \in W^{1,p}(\Omega_s)$, we have $q_s(l_s v) = v$ a.e. in $\Omega_s$. We set $\lambda = \sup\limits_{s\in\mathbb{N}} \|l_s\|$. It is not difficult to find that $\lambda$ is a real number such that $\lambda \geqslant 1$. Next, let $v \in W^{1,p}(\Omega)$. By virtue of condition $(*_4)$ of Theorem 1, there exists a sequence $w_s \in W^{1,p}(\Omega_s)$ such that $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and $F_s(w_s) \to F(v)$. The first of these limit relations and condition $(*_5)$ of Theorem 1 imply that $G_s(w_s) \to G(v)$. Thus,

$$(F_s + G_s)(w_s) \to (F+G)(v). \tag{27}$$

In view of (7), we have

$$\forall s \in \mathbb{N}, \qquad (F_s + G_s)(w_s) \geqslant c_5\|w_s\|^p_{W^{1,p}(\Omega_s)} - c_6. \tag{28}$$

This along with (27) implies that the sequence of norms $\|w_s\|_{W^{1,p}(\Omega_s)}$ is bounded. Now, since condition $(*_1)$ of Theorem 1 and condition $(*')$ of Theorem 2 are satisfied, we deduce from Proposition 5 that $l_s w_s \to v$ weakly in $W^{1,p}(\Omega)$. Therefore,

$$\liminf_{s\to\infty} \|l_s w_s\|_{W^{1,p}(\Omega)} \geqslant \|v\|_{W^{1,p}(\Omega)}. \tag{29}$$

Moreover, we have

$$\forall s \in \mathbb{N}, \qquad \|l_s w_s\|_{W^{1,p}(\Omega)} \leqslant \lambda\|w_s\|_{W^{1,p}(\Omega_s)}. \tag{30}$$

From (27)–(30), we derive that $(F+G)(v) \geqslant c_5\lambda^{-p}\|v\|^p_{W^{1,p}(\Omega)} - c_6$. $\square$

We observe that condition $(*')$ of Theorem 2 is essential for the conclusion of Proposition 10. In this regard, see [10, Example 4.3].

We complete the exposition of the results related to condition $(*')$ of Theorem 2 with the following proposition.

**Proposition 11.** *Assume that $c > 0$ and, for every open set $H$ of $\mathbb{R}^n$ such that $H \subset \Omega$, we have $\liminf\limits_{s\to\infty} \operatorname{meas}(H \cap \Omega_s) \geqslant c \operatorname{meas} H$. Then condition $(*')$ of Theorem 2 is satisfied.*

Concerning the proof of this result, see, for instance, [10]. We also remark that the condition of Proposition 11 is satisfied in the case where the domains $\Omega_s$ have a perforated structure of the same kind as the structure of the domains considered in [16, Section 2].

Finally, we note that condition $(*'')$ of Theorem 2 is also important for the conclusion of this theorem. In this regard, see [10, Example 4.4]. Obviously, condition $(*'')$ of Theorem 2 is satisfied if all the functions $\mu_s$ are zero in the corresponding domains or if, for instance, for every $s \in \mathbb{N}$, we have $\mu_s = \alpha_s\mu|_{\Omega_s}$, where $\{\alpha_s\} \subset [0, +\infty)$, $\alpha_s \to 0$, and $\mu$ is a nonnegative function in $L^1(\Omega)$.

## 4. Conclusion

In this paper, we have formulated and have discussed some results on the convergence of sequences of minimizers and minimum values of functionals $F_s + G_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ on sets of functions defined by bilateral constraints in domains $\Omega_s$. These domains are assumed to be contained in a bounded domain $\Omega$ of $\mathbb{R}^n$. The functionals $F_s$ are integral and convex, and their integrands satisfy the bilateral estimate $c_1|\xi|^p - \mu_s(x) \leqslant f_s(x,\xi) \leqslant c_2|\xi|^p + \mu_s(x)$ for almost every $x \in \Omega_s$ and for every $\xi \in \mathbb{R}^n$, where $c_1$ and $c_2$ are positive constants and $\mu_s$ are nonnegative functions such that the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$ is bounded. The functionals $G_s$ are assumed to be weakly continuous on the corresponding Sobolev spaces. They are generally not integral and play a subordinate role.

We have considered two cases: the case of regular constraints, i.e., constraints lying in the Sobolev space $W^{1,p}(\Omega)$, and the case where the lower constraint is zero and the upper constraint is an arbitrary nonnegative function. In both cases, a certain connection of the spaces $W^{1,p}(\Omega_s)$ with the space $W^{1,p}(\Omega)$, the $\Gamma$-convergence of the functionals $F_s$, and a convergence of the functionals $G_s$ are essentially used. At the same time, each of these cases has a distinctive feature. In the first case, it is required that the difference between the upper and lower constraints be positive almost everywhere. In the second case, this requirement is absent. However, in the latter case, it is assumed that $\|\mu_s\|_{L^1(\Omega_s)} \to 0$ and it is required that the exhaustion condition of the domain $\Omega$ by the domains $\Omega_s$ be satisfied.

We have given a series of results involving the exhaustion condition. In particular, we have obtained an equivalent statement of this condition and, using it, have proved the $\mathcal{H}$-convergence of sets of functions defined by bilateral (generally irregular) constraints in the domains $\Omega_s$.


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# ON INTERPOLATION BY ALMOST TRIGONOMETRIC SPLINES[1]

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**Abstract:** The existence and uniqueness of an interpolating periodic spline defined on an equidistant mesh by the linear differential operator $\mathcal{L}_{2n+2}(D) = D^2(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)$ with $n \in \mathbb{N}$ are reproved under the final restriction on the step of the mesh. Under the same restriction, sharp estimates of the error of approximation by such interpolating periodic splines are obtained.

**Key words:** Splines, Interpolation, Approximation, Linear differential operator.

## Introduction

Let $D = d/dx$, $n \in \mathbb{N}$, and let

$$\mathcal{L}_{2n+2}(D) = D^2(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2) \qquad (0.1)$$

be the $(2n+2)$th-order linear differential operator with constant real coefficients. We denote the characteristic polynomial of $\mathcal{L}_{2n+2}(D)$ by $p_{2n+2}$, and let $T_{2n+2} = \{0, 0, \pm i, \ldots, \pm in\}$ be the set of its zeros, with each zero repeated according to its multiplicity, where $i$ is the imaginary unit. The kernel of the differential operator (0.1) is the linear space spanned by the system of functions $\{1, x, \sin x, \cos x, \ldots, \sin nx, \cos nx\}$.

Denote by $\mathbb{T}$ the circumference considered as the interval $[0, 2\pi]$ with identified ends, and let $\|\cdot\|_{L_p(\mathbb{T})} = \|\cdot\|_p$ $(1 \le p \le \infty)$ with the usual modification in the case $p = \infty$.

We associate with the differential operator $\mathcal{L}_{2n+2}(D)$ the standard class of differentiable functions

$$W_\infty(\mathcal{L}_{2n+2}) = \{f \in C^{(2n+1)}(\mathbb{T}) : \ f^{(2n+1)} \ \text{ is abs. cont.,} \ \|\mathcal{L}_{2n+2}(D)f\|_\infty \le 1\}.$$

Let $N \in \mathbb{N}$ and $h = \pi/N$. Denote by $\Delta_N = \{jh : j = 0, 1, \ldots, 2N-1\}$ the uniform mesh on $[0, 2\pi)$ which can be extended on $\mathbb{R}$ if required; $h$ is the step of the mesh.

We say that a $2\pi$-periodic function $s_{2n+2}$ is a periodic *almost trigonometric spline* with knots at the points of $\Delta_N$ if $s_{2n+2}$ satisfies the following conditions:

1) $s_{2n+2} \in C^{(2n)}(\mathbb{T})$,

2) $\mathcal{L}_{2n+2}(D)s_{2n+2}(x) = 0 \quad \forall x \in (jh, (j+1)h), \quad j \in \mathbb{Z}$.

The set of all almost trigonometric splines is denoted by $S(\mathcal{L}_{2n+2}, \Delta_N)$.

Almost trigonometric splines are a special case of the large family of $\mathcal{L}$-splines defined by linear differential operators (see [2], [3], [8], and others).

The term "almost trigonometric spline" is explained by the fact that such a spline is formed by functions which differ from trigonometric polynomials for only one addend $ax$, where $a$ is some

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constant. This term is not standard, and we use it only not to specify every time by what differential operator the considered splines are defined.

We interpolate at the knots of the mesh $\Delta_N$ by elements from $S(\mathcal{L}_{2n+2},\Delta_N)$; i.e., for every bounded $2N$-periodic sequence $y=\{y_\nu : \nu\in\mathbb{Z}\}$, $y_\nu=y_{\nu+2N}$, we consider the interpolation problem: to find $s\in S(\mathcal{L}_{2n+2},\Delta_N)$ such that $s(\nu h)=y_\nu, \nu\in\mathbb{Z}$.

For interpolation by polynomial splines, the existence, uniqueness and estimates of the error of approximation in many classes of functions are well-known (see, for instance, [1, Ch. V], [11], [12], [13], and references therein).

The existence and uniqueness of periodic interpolating $L$-splines corresponding to an arbitrary linear differential operator with constant real coefficients were established in [10]. As far as almost trigonometric splines are concerned the result in [10] means that if $N>n$, then for every bounded $2N$-periodic interpolated sequence, there exists a unique interpolating almost trigonometric spline.

In the present paper, we give another proof of this result and observe such an important feature that the inequality $N>n$ cannot be replaced by a weaker one (Theorem 1). After this, for $N>n$, we obtain a sharp estimate of the error of pointwise approximation by periodic interpolating almost trigonometric splines in the class of functions $W_\infty(\mathcal{L}_{2n+2})$ (Theorem 2).

**Theorem 1.** *If $N>n$, then, for every bounded $2N$-periodic sequence $\{y_\nu\}_{\nu\in\mathbb{Z}}$, $y_\nu=y_{\nu+2N}$, there exists a unique $s\in S(\mathcal{L}_{2n+2},\Delta_N)$ such that $s(\nu h)=y_\nu, \nu\in\mathbb{Z}$.*

*If $N\le n$, then periodic interpolating almost trigonometric spline cannot exist.*

Let $N>n$. We set

$$A_n(x)=\frac{x(x-h)}{4(n!)^2}+2\sum_{\nu=1}^{n}\frac{(-1)^\nu\sin\dfrac{\nu x}{2}\;\sin\dfrac{\nu(x-h)}{2}}{\nu^2\;(n-\nu)!\;(n+\nu)!\;\cos\dfrac{\nu h}{2}} \tag{0.2}$$

for $0\le x\le h$ and extend $A_n(x)$ to the whole real line by the equality $A_n(x+h)=-A_n(x)$ for $x\in\mathbb{R}\backslash[0,h]$.

We show that $A_n\in C^{(2n+1)}(\mathbb{T})$. In the class $W_\infty(\mathcal{L}_{2n+2})$, the deviation from the periodic interpolating almost trigonometric splines is estimated by this function.

**Theorem 2.** *If $N>n$, then, for every function $f\in W_\infty(\mathcal{L}_{2n+2})$, the inequality*

$$|f(x)-s(f)(x)|\le 2|A_n(x)| \tag{0.3}$$

*holds at any point $x\in\mathbb{R}$. The inequality turns into an equality for $f(x)=2A_n(x)$.*

For interpolation by periodic polynomial splines, inequality (0.3) was proved by Tikhomirov [12]. For $N>3^{n-1}n$, inequality (0.3) is a particular case of the author's result [3]. For periodic trigonometric splines, the corresponding result was established by Nguen [5], [6, Ch. 2, §6].

## 1. Auxiliary results

First, we study the properties of the function $A_n(x)$.

**Lemma 1.** *If $N>n$, then*

$$A_n^{(j)}(x)\mid_{x=h}=-A_n^{(j)}(x)\mid_{x=0},\quad j=1,3,\dots,2n+1,$$

*and*

$$A_n^{(j)}(x)\mid_{x=h}=A_n^{(j)}(x)\mid_{x=0}=0,\quad j=0,2,\dots,2n.$$

P r o o f. By easy calculations, we verify that $A_n(h) = A_n(0) = 0$ and $A'_n(x) \mid_{x=h} = -A'_n(x) \mid_{x=0}$. Further,

$$A''_n(x) \mid_{x=h} = A''_n(x) \mid_{x=0} = \frac{1}{2(n!)^2} - \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}}{(n-\nu)!\ (n+\nu)!}.$$

Using the known identity [7, Ch.IV, § 4.2.1, eq. 4], we obtain

$$\sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}}{(n-\nu)!\ (n+\nu)!} = \sum_{m=0}^{n-1} \frac{(-1)^{m+n-1}}{m!\ (2n-m)!} = \frac{(-1)^{n-1}}{(2n)!} \sum_{m=0}^{n-1} (-1)^m \binom{2n}{m} = \frac{1}{2(n!)^2}.$$

From this, it follows that $A''_n(x) \mid_{x=h} = A''_n(x) \mid_{x=0} = 0$.

For $j = 3, 4, \ldots, 2n+1$, we have

$$A_n^{(j)}(x) = \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1} \nu^{j-2} \cos(\nu(x-h/2) + \pi j/2)}{(n-\nu)!\ (n+\nu)!\ \cos(\nu h/2)}.$$

For $j = 2k+1$ $(k = 1, 2, \ldots, n)$, easy calculations yield

$$A_n^{(2k+1)}(x) \mid_{x=h} = -A_n^{(2k+1)}(x) \mid_{x=0} = (-1)^k \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}\ \nu^{2k-1}\ \tan(\nu h/2)}{(n-\nu)!\ (n+\nu)!}.$$

For $j = 2k$ $(k = 2, 3, \ldots, n)$, we obtain

$$A_n^{(2k)}(x) \mid_{x=h} = A_n^{(2k)}(x) \mid_{x=0} = (-1)^k \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}\ \nu^{2k-2}}{(n-\nu)!\ (n+\nu)!}$$

$$= \frac{(-1)^{n+k}}{(2n)!} \sum_{m=0}^{n-1} (-1)^m (n-m)^{2k-2} \binom{2n}{m} = 0.$$

Here, we used the identity [7, Ch.IV, § 4.2.2, eq. 34]. The lemma is proved. □

We now extend the function $A_n(x)$ from $[0, h]$ to the whole real line by setting $A_n(x+h) = -A_n(x)$. Lemma 1 gives that $A_n$ belongs to $C^{(2n+1)}(\mathbb{R})$ and is $2\pi$-periodic.

**Lemma 2.** *If $N > n$, then $\mathcal{L}_{2n+2}(D)(2A_n(x)) = \operatorname{sign}\sin Nx,\ x \in \mathbb{R}$.*

P r o o f. Let $0 \le x \le h$. Since

$$\sin\frac{\nu x}{2}\ \sin\frac{\nu(x-h)}{2} = A_\nu \cos\nu x + B_\nu \sin\nu x + C_\nu, \quad \nu = 1, 2, \ldots, n,$$

where $A_\nu, B_\nu$ and $C_\nu$ are independent of $x$, the sum on the right-hand side of (0.2) vanishes by the differential operator $D(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)$. Taking into account that the factors on the right-hand side of (0.1) can be rearranged, we obtain

$$D^2(D^2+1^2)\cdots(D^2+n^2)\left(\frac{x(x-h)}{2(n!)^2}\right) = \frac{1}{2(n!)^2}(D^2+1^2)\cdots(D^2+n^2)(x(x-h))'' = 1.$$

For $x \in \mathbb{R}\backslash[0, h]$, we use the equality $A_n(x+h) = -A_n(x)$. □

The next statement is a special case of a result proved in [9] for an arbitrary linear differential operator with constant real coefficients.

**Lemma 3.** *If $N > n$, then $x = 0$ is the unique zero of $A_n(x)$ in $[0, h)$ and this zero is simple.*

P r o o f. By Lemma 1, $A_n(0) = 0$. Moreover, the function $A_n(x)$ coincides, up to a nonzero constant, with some function $P_n(x)$ introduced in [9]. It was proved in [9] that if $N > n$, then $P_n(x)$ has a unique zero in $[0, h)$ and this zero is simple. Therefore, $A_n(x)$ has the same property. □

To prove our two theorems, we also need the periodic analog of the Rolle theorem on the relation between the number of zeros of a smooth function $\varphi(x)$ and the number of sign changes of $D(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)\varphi(x)$ on $\mathbb{T}$.

We say that a continuous function $f$ changes sign at some point $t_0$ if the inequality $f(t_0-\varepsilon)f(t_0+\varepsilon) < 0$ holds for all sufficiently small $\varepsilon > 0$. If $f$ has a jump at the point $t_0$, then, instead of $f(t_0-\varepsilon)$ and $f(t_0+\varepsilon)$, we write $\lim\limits_{t\to t_0-0} f(t)$ and $\lim\limits_{t\to t_0+0} f(t)$, respectively. Denote by $Z(f, \mathbb{T})$ the number of zeros of the function $f$ on $\mathbb{T}$, and by $\nu(f, \mathbb{T})$ the number of sign changes of $f \not\equiv 0$ on $\mathbb{T}$ (the number of sign changes of $f \equiv 0$ is not defined). The number $\nu(f, \mathbb{T})$ is always even. We denote by $G(\mathbb{T})$ the set of $2\pi$-functions of bounded variation with a finite number of jumps on the period and absolutely continuous on all intervals of continuity. We also denote by $G_m(\mathbb{T})$ the set of $2\pi$-periodic functions whose derivatives of order $m-2$ are absolutely continuous on $\mathbb{T}$ and $f^{(m-1)} \in G(\mathbb{T})$. Let $\mathcal{T}_n$ be the set of trigonometric polynomials of order at most $n$.

**Lemma 4.** *For every function $f \in G_{2n+1}(\mathbb{T})\backslash\mathcal{T}_n$, the following inequality holds:*

$$\nu\big(D(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)f, \mathbb{T}\big) \ge Z(f, \mathbb{T}).$$

This result was established by Nguen [5] (see also [6]) and is the periodic analog of the Rolle theorem for the trigonometric differential operator.

Note that the periodic analog of the Rolle theorem in the form of Lemma 4 exists not for any linear differential operator. More detailed information on some results and open problems in this area can be found in [4] and references therein.

## 2. Proofs of Theorems

We now pass directly to the proofs of Theorems 1 and 2.

P r o o f of Theorem 1. Let $N > n$. We prove that if $s \in S(\mathcal{L}_{2n+2}, \Delta_N)$ and $s(jh) = 0$ $\forall j \in \mathbb{Z}$, then $s \equiv 0$. After this, the existence and uniqueness of the interpolating periodic almost trigonometric spline for every interpolated periodic sequence is a simple consequence of the Kramer theorem for the corresponding system of linear algebraic equations.

Suppose that there exist $s_1, s_2 \in S(\mathcal{L}_{2n+2}, \Delta_N)$ such that $s_k(jh) = 0$ $\forall j \in \mathbb{Z}$ $(k = 1, 2)$ and $s_1 \not\equiv s_2$. This means that there is a point $x_* \notin \Delta_N$ such that $s_1(x_*) \ne s_2(x_*)$. Let $s_1(x_*) \ne 0$ and $C = s_2(x_*)/s_1(x_*)$. Then the function $\varphi(x) = Cs_1(x) - s_2(x)$ has the following properties:

1) $\varphi \in S(\mathcal{L}_{2n+2}, \Delta_N)$;
2) $\varphi(jh) = 0, \quad j = 0, 1, \ldots, 2N-1$;
3) $\varphi(x_*) = 0$.

Thus, $\varphi(x)$ has at least $2N+1$ zeros on the period. From Lemma 4, we have

$$\nu(D(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)\varphi, \mathbb{T}) \ge 2N+1.$$

But $D(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)\varphi(x)$ is a piecewise constant function with possible jumps at the points of the mesh $\Delta_N$. Therefore, this function cannot change sign more than $2N$ times on $\mathbb{T}$. We have a contradiction from which it easily follows that $s_1 \equiv s_2 \equiv 0$.

The inequality $N > n$ cannot be replaced by a weaker one. Indeed, if $N = n$, then the function $\sin nx$ interpolates the sequence $y \equiv 0$ at the points of $\Delta_N$. This function lies in the kernel of the linear differential operator (0.1) and can be interpreted as an element of the space $S(\mathcal{L}_{2n+2}, \Delta_N)$. Theorem 1 is proved. □

P r o o f of Theorem 2 is based on the ideas of [12]. Let $N > n$. Suppose that (0.3) fails; i.e., there exist a point $x_* \in [0, 2\pi)$ and a function $f \in W_\infty(\mathcal{L}_{2n+2})$ such that the inequality

$$|f(x_*) - s(f)(x_*)| > 2|A_n(x_*)|$$

holds. Define $\delta(x) = f(x) - s(f)(x)$. This function is zero at the points of the mesh $\Delta_N$. According to Lemma 3, the function $A_n(x)$ vanishes at the same points. From these facts, we have $x_* \notin \Delta_N$. Therefore, there is a number $\lambda$, $0 < |\lambda| < 1$, such that $\lambda\delta(x_*) = 2A_n(x_*)$.

We now introduce the function $\Delta(x) = \lambda\delta(x) - 2A_n(x)$. It is zero at all points of the set $\Delta_N \cup \{x_*\}$ and possibly also at some other points. Therefore $Z(\Delta(x), \mathbb{T}) \geq 2N + 1$. It is clear that $\Delta \in G_{2n+1}(\mathbb{T}) \backslash \mathcal{T}_n$. We denote $\mathcal{L}_{2n+1}(D) = D(D^2 + 1^2)(D^2 + 2^2) \cdots (D^2 + n^2)$, apply Lemma 4, and obtain

$$\nu\left(\mathcal{L}_{2n+1}(D)\Delta(x), \mathbb{T}\right) \geq 2N + 1. \tag{2.1}$$

From (0.1) and the definition of almost trigonometric splines, we have the equalities $\mathcal{L}_{2n+1}(D)\delta(x) = \mathcal{L}_{2n+1}(D)f(x) - c_j$ on every interval $[jh, (j+1)h)$, $j = 0, 1, \ldots, 2N-1$, where $c_j$ are constants. Using the Lagrange finite increments formula and the inequality $|\lambda| < 1$, we obtain

$$|\mathcal{L}_{2n+1}(D)(\lambda\delta(t')) - \mathcal{L}_{2n+1}(D)(\lambda\delta(t''))| < |\mathcal{L}_{2n+1}(D)f(t') - \mathcal{L}_{2n+1}(D)f(t'')|$$

$$= |\mathcal{L}_{2n+2}(D)f(\xi)| \cdot |t' - t''| \ \leq |t' - t''|$$

on an arbitrary subinterval $[t', t''] \subset [jh, (j+1)h)$ for every interpolated function of our class. From (0.2), it follows that $\mathcal{L}_{2n+1}(D)(2A_n(x)) = x - h/2$ $\forall x \in [0, h)$. Hence, $|t' - t''| = |\mathcal{L}_{2n+1}(D)(2A_n(t')) - \mathcal{L}_{2n+1}(D)(2A_n(t''))|$. Thus,

$$|\mathcal{L}_{2n+1}(D)(\lambda\delta(t')) - \mathcal{L}_{2n+1}(D)(\lambda\delta(t''))| < |\mathcal{L}_{2n+1}(D)(2A_n(t')) - \mathcal{L}_{2n+1}(D)(2A_n(t''))|.$$

It is easy to see that if $|a| < |b|$, then $\operatorname{sign}(b - a) = \operatorname{sign} b$. Applying this fact, we come to the conclusion that the function $\mathcal{L}_{2n+1}(D)\Delta(x)$ changes sign no more than once in every interval $[jh, (j+1)h)$. If $\mathcal{L}_{2n+1}(D)\Delta(x)$ changes sign at the point $jh$ (this is possible if the function is discontinuous at $jh$), then $\mathcal{L}_{2n+1}(D)\Delta(x)$ preserves sign in one of two adjacent intervals $((j-1)h, jh)$ or $(jh, (j+1)h)$. Thus, we arrive at the inequality

$$\nu\left(\mathcal{L}_{2n+1}(D)\Delta(x), \mathbb{T}\right) \leq 2N.$$

The obtained inequality contradicts to (2.1). The simple observation that inequality (0.3) turns into an equality for $f = 2A_n$ completes the proof. □

**Corollary 1.** *If $N > n$, then*

$$\sup_{f \in W_\infty(\mathcal{L}_{2n+2})} \|f - s(f)\|_p = 2\|A_n\|_p, \quad 1 \leq p < \infty,$$

*and*

$$\sup_{f \in W_\infty(\mathcal{L}_{2n+2})} \|f - s(f)\|_\infty = \left| \frac{h^2}{8(n!)^2} + 4 \sum_{\nu=1}^{n} \frac{(-1)^\nu \sin^2 \dfrac{\nu h}{4}}{\nu^2 \ (n-\nu)! \ (n+\nu)! \ \cos \dfrac{\nu h}{2}} \right|.$$

We now consider separately the case $n = 1$, i.e., the case of $\mathcal{L}$-splines corresponding to the differential operator $\mathcal{L}_4(D) = D^2(D^2 + 1)$. They belong piecewise to the space $span\{1, t, \sin t, \cos t\}$. These splines generalize the well-known cubic splines and have many applications in numerical analysis for the shape preserving approximation, the description of curves and their parametrization, and other problems (see, for instance, [14], [15], [16], and references therein). In particular (see [15]) these splines are attractive from a geometrical point of view, because they are able to provide parameterizations of conic sections with respect to the arc length so that equally spaced points in the parameter domain correspond to equally spaced points on the described curve.

The restriction on the grid step is the least strong here: $h \le \pi/2$, and the "minimal" equidistant grid on the period is $\{0, \pi/2, \pi, 3\pi/2\}$. Theorem 1 gives the existence and uniqueness of spline interpolants for $N \ge 2$. According to Corollary 1, the error of approximation in the class $W_\infty(\mathcal{L}_4)$ is

$$\sup_{f \in W_\infty(\mathcal{L}_4)} \|f - s(f)\|_\infty = \left| 1 + \frac{\pi^2}{8N^2} - \frac{1}{\cos \dfrac{\pi}{2N}} \right|.$$

## 3. Conclusion

We established that, for $2\pi$-periodic $\mathcal{L}$-splines corresponding to the differential operator (0.1) on the equidistant mesh with the step $h = \pi/N$, the restriction $N > n$ provides the existence and uniqueness of the $\mathcal{L}$-spline interpolant as well as the exact estimates of the error of approximation. This restriction is final, i.e., cannot be replaced by a weaker one.

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# $\mathcal{K}$-FUNCTIONALS AND EXACT VALUES OF $n$-WIDTHS IN THE BERGMAN SPACE

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**Abstract:** In this paper, we consider the problem of mean-square approximation of complex variables functions which are regular in the unit disk of the complex plane. We obtain sharp estimates of the value of the best approximation by algebraic polynomials in terms of $\mathcal{K}$-functionals. Exact values of some widths of the specified class of functions are calculated.

**Key words:** Bergman space, Best mean-square approximation, $\mathcal{K}$-functional, $n$-width.

## Introduction and preliminary facts

We consider the problem of mean-square approximation by Fourier sums of complex functions $f$ which are regular in a simply connected domain $\mathcal{D}\subset\mathbb{C}$ and belong to the space $L_2:=L_2(\mathcal{D})$ with the finite norm

$$\|f\|:=\|f\|_{L_2(\mathcal{D})}=\left(\frac{1}{\pi}\iint_{(\mathcal{D})}|f(z)|^2d\sigma\right)^{1/2},$$

where the integral is understood in the Lebesgue sense and $d\sigma$ is an element of area.

The study of the mean-square approximation of functions in the domain $\mathcal{D}\subset\mathbb{C}$ is closely related to the theory of orthogonal functions. A sequence of complex functions $\{\varphi_k(z)\}$ $(k=0,1,2,...)$ is called an orthogonal system on the domain $\mathcal{D}$ if

$$\frac{1}{\pi}\iint_{(\mathcal{D})}\varphi_k(z)\overline{\varphi_l(z)}d\sigma=0,\qquad k\neq l.$$

Such a sequence of functions is called orthonormal system if

$$\frac{1}{\pi}\iint_{(\mathcal{D})}\varphi_k(z)\overline{\varphi_l(z)}d\sigma=\delta_{k,l},$$

where $\delta_{k,l}=0,\;\;k\neq l$, and $\delta_{k,k}=1,\;\;k\in\mathbb{N}$. If $f\in L_2$, then the numbers

$$a_k(f)=\frac{1}{\pi}\iint_{(\mathcal{D})}f(z)\overline{\varphi_k(z)}d\sigma \tag{1}$$

are called the Fourier coefficients of the function $f$ with respect to the orthonormal system $\{\varphi_k(z)\}$ $(k=0,1,2,...)$. We associate with a given function $f$ its Fourier series with respect to the specified orthogonal system:

$$f(z)\sim\sum_{k=0}^{\infty}a_k(f)\varphi_k(z). \tag{2}$$

Let

$$S_{n-1}(f,z)=\sum_{k=0}^{n-1}a_k(f)\varphi_k(z)$$

be the partial sum of order $n$ of the series (2). We form a linear combination of the first $n$ functions of the system $\{\varphi_k(z)\}$:

$$p_{n-1}(z) = \sum_{k=0}^{n-1} d_k \varphi_k(z),$$

where $d_k \in \mathbb{C}$ are arbitrary complex coefficients. We call this linear combination a generalized polynomial. It is well known (see, for example, [1], p.263) that

$$\begin{aligned} E_{n-1}(f) &= \inf\{\|f - p_{n-1}\| : \ d_k \in \mathbb{C}\} \\ &= \|f - S_{n-1}(f)\| = \left\{\sum_{k=n}^{\infty} |a_k(f)|^2\right\}^{1/2}, \end{aligned} \tag{3}$$

where $a_k(f)$ are the Fourier coefficients of the function $f$ defined by (1).

In the case of the mean approximation of complex functions in a simply connected domain $\mathcal{D} \subset \mathbb{C}$ by Fourier series with respect to an orthogonal system of functions $\{\varphi_k(z)\}_{k=0}^{\infty}$ on $\mathcal{D}$, the problem of finding the exact constant in the Jackson-Stechkin inequality was studied in [2]. Recall that Jackson-Stechkin inequalities are inequalities in which the value of the best approximation of a function by a finite dimensional subspace of a given normed space is estimated by the modulus of smoothness of the function itself or some its derivative. In this paper, we use the same methods as in [2, 3, 5, 15].

We study in more detail the case where $\mathcal{D}$ is the unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$. In this case, it is clear that the system of functions $\varphi_k(z) = z^k (k = 0, 1, 2, ...)$ is orthogonal in the disk $U$:

$$\frac{1}{\pi}\iint_{(U)} \varphi_k(z)\overline{\varphi_l(z)} d\sigma = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} r^{k+l+1} e^{i(k-l)t} dr dt = 0, \quad k \neq l.$$

However, this system is not orthonormal, since

$$\frac{1}{\pi}\iint_{(U)} |\varphi_k(z)|^2 d\sigma = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} r^{2k+1} dr dt = \frac{1}{k+1}.$$

Therefore, the system of functions $\varphi_k^*(z) = \sqrt{k+1} z^k$ $(k = 0, 1, 2, ...)$ is orthonormal. We denote by $\mathcal{A}(U)$ the set of all functions $f$ analytic in $U$. The Maclaurin series of such a function has the form

$$f(z) = \sum_{k=0}^{\infty} c_k(f) z^k, \tag{4}$$

where $c_k(f)$ are the Maclaurin coefficients of $f$. We note that

$$\|f\|^2 = \sum_{k=0}^{\infty} \frac{|c_k(f)|^2}{k+1}, \quad E_{n-1}^2(f) = \sum_{k=n}^{\infty} \frac{|c_k(f)|^2}{k+1}. \tag{5}$$

It was proved in the monograph [1] that the Fourier series of a function $f$ with respect to the orthonormal system $\varphi_k^*(z) = \sqrt{k+1} z^k$, $k = 0, 1, 2, ...,$ coincides with the series (4) for $f \in \mathcal{A}(U)$; i.e.,

$$f(z) = \sum_{k=0}^{\infty} a_k(f) \varphi_k^*(z) = \sum_{k=0}^{\infty} c_k(f) z^k. \tag{6}$$

Therefore, the series (6) can be differentiated term by term any number of times and, according to the Weierstrass theorem [6, p.107], for any $r \in \mathbb{N}$, we get

$$f^{(r)}(z) = \sum_{k=r}^{\infty} c_k(f) k(k-1)\cdots(k-r+1) z^{k-r} := \sum_{k=r}^{\infty} \alpha_{k,r} c_k(f) z^{k-r}, \tag{7}$$

where

$$\alpha_{k,r} := k(k-1)\cdots(k-r+1), \quad k\in\mathbb{N}, \quad r\in\mathbb{Z}_+, \quad k\ge r.$$

We denote by $L_2^{(r)} := L_2^{(r)}(U)$ ($L_2^{(0)} := L_2(U)$) the class of all functions $f \in L_2$ such that $f^{(r)} \in L_2$ ($r \in \mathbb{Z}_+$, $f^{(0)} \equiv f$).

## 1. Sharp estimates of the value of the best approximation by means of $\mathcal{K}$-functionals

In this section, we prove some sharp inequalities relating the value $E_{n-1}(f)$ of the best approximation of functions in the class $L_2^{(r)}$ and Peetre $\mathcal{K}$-functionals. The definition and some properties of Peetre $\mathcal{K}$-functionals are given in [7]. The direct and inverse theorems of the theory of approximation by means of $\mathcal{K}$-functionals were proved in [8, 9]. We define the $\mathcal{K}$-functional constructed by the spaces $L_2$ and $L_2^{(m)}$ as follows:

$$\mathcal{K}_m(f,t^m)_2 := \mathcal{K}\left(f,t^m;L_2;L_2^{(m)}\right) = \inf\left\{\|f-g\|_2 + t^m\cdot\|g^{(m)}\|_2 : g\in L_2^{(m)}\right\}, \tag{8}$$

where $m\in\mathbb{N}$ and $0<t\le 1$. We note that a weak equivalence of the $\mathcal{K}$-functional defined by (8) and a special generalized modulus of continuity of order $m$ was established in [8].

**Theorem 1.** *Let $n,m\in\mathbb{N}$ and $r\in\mathbb{Z}_+$ be arbitrary numbers such that $n\ge r+m$. Then the following equality holds*:

$$\sup_{\substack{f\in L_2^{(r)}\\ f\notin\mathcal{P}_r}} \frac{\sqrt{(n+1)/(n-r+1)}\cdot\alpha_{n,r}E_{n-1}(f)}{\mathcal{K}_m\left(f^{(r)}, \sqrt{\dfrac{n-r-m+1}{n-r+1}}\cdot\dfrac{1}{\alpha_{n-r,m}}\right)} = 1. \tag{9}$$

P r o o f. Using (7), we easily find that

$$E_{n-r-1}^2(f^{(r)}) = \sum_{k=n}^{\infty}\alpha_{k,r}^2\frac{|c_k(f)|^2}{k-r+1}, \quad r\in\mathbb{Z}_+. \tag{10}$$

Taking into account equality (10), we obtain

$$\begin{aligned} E_{n-1}^2(f) &= \sum_{k=n}^{\infty}\frac{|c_k(f)|^2}{k+1} = \sum_{k=n}^{\infty}\frac{k-r+1}{(k+1)\alpha_{k,r}^2}\cdot\alpha_{k,r}^2\cdot\frac{|c_k(f)|^2}{k-r+1} \\ &\le \max_{\substack{k\in\mathbb{N}\\ k\ge n}}\left\{\frac{k-r+1}{(k+1)\alpha_{k,r}^2}\right\}\cdot\sum_{k=n}^{\infty}\alpha_{k,r}^2\frac{|c_k(f)|^2}{k-r+1} \\ &= \frac{n-r+1}{n+1}\cdot\frac{1}{\alpha_{n,r}^2}\cdot E_{n-r-1}^2\left(f^{(r)}\right). \end{aligned} \tag{11}$$

Now, for an arbitrary function $f\in L_2^{(r)}$, we write

$$E_{n-1}(f) \le \sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}E_{n-r-1}\left(f^{(r)}\right) \le \sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}\|f^{(r)}-S_{n-r-1}(g)\|, \tag{12}$$

where $S_{n-r-1}(g)$ is the partial sum of order $n-r$ of the Fourier series of an arbitrary function $g \in L_2^{(m)}$. In view of (2) and (11), we get

$$\begin{aligned} \|g - S_{n-r-1}(g)\| = E_{n-r-1}(g) &\le \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} E_{n-r-m-1}\left(g^{(m)}\right) \\ &\le \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} \left\|g^{(m)}\right\|. \end{aligned} \tag{13}$$

It follows from inequalities (12) and (13) that

$$\begin{aligned} E_{n-1}(f) &\le \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \left\{\|f^{(r)} - g\| + \|g - S_{n-r-1}(g)\|\right\} \\ &\le \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \left\{\|f^{(r)} - g\| + \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} \left\|g^{(m)}\right\|\right\}. \end{aligned} \tag{14}$$

Now, we note that the left-hand side of inequality (14) does not depend on $g \in L_2^{(m)}$. Therefore, passing to the infimum over all functions $g \in L_2^{(m)}$ on the right-hand side of (14) and using the definition (8) of $\mathcal{K}$, we get

$$E_{n-1}(f) \le \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \mathcal{K}_m\left(f^{(r)}, \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right).$$

This implies the following upper bound:

$$\sup_{\substack{f \in L_2^{(r)} \\ f \notin \mathcal{P}_r}} \frac{\sqrt{(n+1)/(n-r+1)} \cdot \alpha_{n,r} E_{n-1}(f)}{\mathcal{K}_m\left(f^{(r)}, \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right)} \le 1, \tag{15}$$

where $\mathcal{P}_r$ is the subspace of complex algebraic polynomials of degree at most $r$.

To obtain a lower bound of the extremal characteristic on the left-hand side of (15), in (8), we put $f(z) := p_n(z)$, where $p_n(z)$ is an arbitrary complex algebraic polynomial in $\mathcal{P}_n$. Since the function $g(z) \equiv 0$ belongs to the class $L_2^{(m)}$, we obtain from (8) the upper bound

$$\mathcal{K}_m(p_n; t^m)_2 \le \|p_n\|.$$

Since the function $g(z) := p_n(z)$ also belongs to the class $L_2^{(m)}$, we find from (8) that

$$\mathcal{K}_m(p_n; t^m)_2 \le t^m \|p_n^{(m)}\|.$$

Thus, the last two relations imply that, for any element $p_n(z) \in \mathcal{P}_n$,

$$\mathcal{K}_m(p_n; t^m)_2 \le \min\left\{\|p_n\|; t^m \|p_n^{(m)}\|\right\}. \tag{16}$$

We consider the function $f_0(z) = z^n$. Since

$$f_0^{(r+m)} = n(n-1)\cdots(n-r+1)\cdots(n-r-m+1) z^{n-r-m} = \alpha_{n,r} \cdot \alpha_{n-r,m} z^{n-r-m},$$

according to (16), we have

$$\begin{aligned} \mathcal{K}\left(f_0^{(r)}; \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right) &\le \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} \|f_0^{(r+m)}\| \\ = \sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}} \cdot \frac{\alpha_{n,r} \cdot \alpha_{n-r,m}}{\sqrt{n-r-m+1}} &= \frac{\alpha_{n,r}}{\sqrt{n-r+1}}. \end{aligned}$$

Using the obtained inequality and the second equality in (5), we establish that

$$\begin{aligned}\sup_{\substack{f\in L_2^{(r)}\\ f\in\mathcal{P}_r}}&\frac{\sqrt{(n+1)/(n-r+1)}\cdot\alpha_{n,r}E_{n-1}(f)}{\mathcal{K}_m\left(f^{(r)},\sqrt{\dfrac{n-r-m+1}{n-r+1}}\cdot\dfrac{1}{\alpha_{n-r,m}}\right)}\\ &\geq\frac{\sqrt{(n+1)/(n-r+1)}\cdot\alpha_{n,r}E_{n-1}(f_0)}{\mathcal{K}_m\left(f_0^{(r)},\sqrt{\dfrac{n-r-m+1}{n-r+1}}\cdot\dfrac{1}{\alpha_{n-r,m}}\right)}\geq 1.\end{aligned}\tag{17}$$

We obtain equality (9) by comparing the upper bound (15) with the lower bound (17). The theorem is proved.

## 2. Exact values of $n$-widths of a class of functions

We assume that $S$ is the unit ball in the space $L_2$, $\Lambda_n\subset L_2$ is an $n$-dimensional subspace, and $\Lambda^n\subset L_2$ is a subspace of codimension $n$. Let $\mathcal{L}:L_2\to\Lambda_n$ be a continuous linear operator, let $\mathcal{L}^\perp:L_2\to\Lambda_n$ be a continuous linear projection operator, and let $\mathfrak{M}$ be a convex centrally symmetric subset of $L_2$. The quantities

$$b_n(\mathfrak{M},L_2)=\sup\left\{\sup\left\{\varepsilon>0;\ \varepsilon S\cap\Lambda_{n+1}\subset\mathfrak{M}\right\}:\Lambda_{n+1}\subset L_2\right\},$$

$$d_n(\mathfrak{M},L_2)=\inf\left\{\sup\left\{\inf\left\{\|f-g\|:g\in\Lambda_n\right\}:f\in\mathfrak{M}\right\}:\Lambda_n\subset L_2\right\},$$

$$\delta_n(\mathfrak{M},L_2)=\inf\left\{\inf\left\{\sup\left\{\|f-\mathcal{L}f\|:f\in\mathfrak{M}\right\}:\mathcal{L}L_2\subset\Lambda_n\right\}:\Lambda_n\subset L_2\right\},$$

$$d^n(\mathfrak{M},L_2)=\inf\left\{\sup\left\{\|f\|_{2,\gamma}:f\in\mathfrak{M}\cap\Lambda^n\right\}:\Lambda^n\subset L_2\right\},$$

$$\Pi_n(\mathfrak{M},L_2)=\inf\left\{\inf\left\{\sup\left\{\|f-\mathcal{L}^\perp f\|:f\in\mathfrak{M}\right\}:\mathcal{L}^\perp L_2\subset\Lambda_n\right\}:\Lambda_n\subset L_2\right\}$$

are called, respectively, the *Bernstein*, *Kolmogorov*, *linear*, *Gelfand*, and *projection* $n$-widths of the subset $\mathfrak{M}$ in the space $L_2$. These widths are monotone with respect to $n$, and the following relation holds (see, for example, [10, 11]):

$$b_n(\mathfrak{M},L_2)\leq d^n(\mathfrak{M},L_2)\leq d_n(\mathfrak{M},L_2)=\delta_n(\mathfrak{M},L_2)=\Pi_n(\mathfrak{M},L_2).\tag{18}$$

We recall (see, for example, [12, p. 25]) that a nondecreasing function $\Psi$ on $\mathbb{R}_+$ is called a $k$-majorant if the function $t^{-k}\Psi(t)$ is nonincreasing in $\mathbb{R}_+$, $\Psi(0)=0$, and $\Psi(t)\to 0$ as $t\to 0$. For $k=1$, the function $\Psi$ is simply called a majorant.

Let $W_2^{(r)}(\mathcal{K}_m,\Psi)$, $r\in\mathbb{Z}_+,m\in\mathbb{N}$, be the class of all functions $f\in L_2^{(r)}$ whose derivatives $f^{(r)}$ satisfy the condition

$$\mathcal{K}_m(f^{(r)},t^m)\leq\Psi(t^m),\quad 0<t<1.$$

In this definition, $\Psi$ is a majorant, $L_2^{(0)}\equiv L_2$, and $W_2^{(0)}(\mathcal{K}_m,\Psi)=W_2(\mathcal{K}_m,\Psi)$. For any subset $\mathfrak{M}\subset L_2$, we define

$$E_{n-1}(\mathfrak{M})_{L_2}:=\sup\left\{E_{n-1}(f):\ f\in\mathfrak{M}\right\}.$$

We note that, in the Bergman space, values of widths of some classes of analytic functions in a disk were calculated, for example, in [13–19].

**Theorem 2.** *Let $\Psi$ be the majorant defining the class $W_2^{(r)}(\mathcal{K}_m, \Psi)$, $m \in \mathbb{N}$, and $r \in \mathbb{R}_+$. Then, for any natural number $n \geq m + r$, we have*

$$\lambda_n\left(W_2^{(r)}(\mathcal{K}_m, \Psi), L_2\right) = E_{n-1}\left(W_2^{(r)}(\mathcal{K}_m, \Psi)\right) = \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right), \tag{19}$$

*where $\lambda_n(\cdot)$ is any of the $n$-widths $b_n(\cdot)$, $d_n(\cdot)$, $d^n(\cdot)$, $\delta_n(\cdot)$, and $\Pi_n(\cdot)$.*

P r o o f. Let $n$ be a natural number such that $n \geq m + r$. In view of the definition of the class $W_2^{(r)}(\mathcal{K}_m, \Psi)$, relations (15) and (18) imply that

$$\lambda_n\left(W_2^{(r)}(\mathcal{K}_m, \Psi), L_2\right) \leq d_n\left(W_2^{(r)}(\mathcal{K}_m, \Psi), L_2\right) \leq E_{n-1}\left(W_2^{(r)}(\mathcal{K}_m, \Psi)\right) \leq \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right). \tag{20}$$

To find the corresponding lower bound, in view of (18), it suffices to estimate the Bernstein $n$-width of the class $W_2^{(r)}(\mathcal{K}_m, \Psi)$. On the set $\mathcal{P}_n \cap L_2$, we define the ball

$$\mathcal{M}_{n+1} := \left\{p_n \in \mathcal{P}_n : \|p_n\| \leq \sqrt{\frac{n-r+1}{n+1}} \cdot \frac{1}{\alpha_{n,r}} \Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right)\right\}.$$

Now, we note that, in view of formula (7) and the identity $\alpha_{k,r+m} = \alpha_{k,r}\, \alpha_{k-r,m}$, for an arbitrary $p_n(z) = \sum_{k=0}^n a_k(p_n) z^k \in \mathcal{P}_n$, the following equality holds:

$$p_n^{(r+m)}(z) = \sum_{k=r+m}^{n} a_k(p_n)\alpha_{k,r+m} z^{k-r-m} := \sum_{k=r+m}^{n} a_k(p_n)\alpha_{k,r} \cdot \alpha_{k-r,m} z^{k-r-m}.$$

Hence, using the Parseval equality and the inequality $\alpha_{k,r} \leq \alpha_{n,r}, k \leq n$, we obtain the Bernstein type inequality

$$\|p_n^{(r+m)}\| = \left\{\sum_{k=r+m}^{n} |a_k(p_n)|^2 \alpha_{k,r}^2 \cdot \alpha_{k-r,m}^2\right\}^{1/2} \leq \alpha_{n,r} \cdot \alpha_{n-r,m} \|p_n\|. \tag{21}$$

By definition, for the majorant $\Psi$ and for any $0 < \tau_1 \leq \tau_2 \leq 1$, we have the inequality $\tau_1 \Psi(\tau_2) \leq \tau_2 \Psi(\tau_1)$. Therefore, for any $0 < t_1 \leq t_2 \leq 1$, setting $\tau_1 = t_1^m$ and $\tau_2 = t_2^m$, we obtain

$$t_1^{-m} \Psi(t_1^m) \geq t_2^{-m} \Psi(t_2^m). \tag{22}$$

We now show that $\mathcal{M}_{n+1} \subset W_2^{(r)}(\mathcal{K}_m, \Psi)$. Thus, we need to prove that, for any polynomial $p_n \subset \mathcal{M}_{n+1}$,

$$\mathcal{K}_m(p_n^{(r)}, t^m) \leq \Psi(t^m),\ 0 < t \leq 1.$$

Since, by assumption, $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, and $n \geq m + r$, we consider two cases:

$$0 < t \leq \left(\sqrt{\frac{n-r-m+1}{n-r+1}} \cdot \frac{1}{\alpha_{n-r,m}}\right)^{1/m}$$

and

$$\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)^{1/m}\leq t\leq 1.$$

First, assume that

$$0<t\leq\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)^{1/m}.$$

In this case, using inequality (22) with

$$t_1=t,\quad t_2=\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)^{1/m}$$

and applying (12) and (21), for any $p_n\in\mathcal{M}_{n+1}$, we obtain

$$\begin{aligned}\mathcal{K}_m(p_n^{(r)},t^m)_2&\leq t^m\cdot\|p_n^{(r+m)}\|\leq t^m\cdot\alpha_{n,r}\cdot\alpha_{n-r,m}\|p_n\|\\&\leq t^m\cdot\alpha_{n,r}\cdot\alpha_{n-r,m}\cdot\sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\\&\leq t^m\cdot\alpha_{n-r,m}\cdot\sqrt{\frac{n-r+1}{n-r-m+1}}\cdot\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\leq\Psi(t^m).\end{aligned}\tag{23}$$

Now, let

$$\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)^{1/m}\leq t\leq 1.$$

Then using (16) and the Bernstein type inequality

$$\|p_n^{(r)}\|\leq\alpha_{n,r}\cdot\|p_n\|$$

and taking into account that the majorant $\Psi$ is nondecreasing, we find that

$$\begin{aligned}\mathcal{K}_m(p_n^{(r)},t^m)_2&\leq\|p_n^{(r)}\|_2\leq\alpha_{n,r}\|p_n\|_2\\&\leq\alpha_{n,r}\sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\\&\leq\sqrt{\frac{n-r+1}{n+1}}\cdot\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\\&\leq\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right)\leq\Psi(t^m).\end{aligned}\tag{24}$$

The definition of the class $W_2^{(r)}(\mathcal{K}_m,\Psi)$ along with (23) and (24) implies that $\mathcal{M}_{n+1}\subset W_2^{(r)}(\mathcal{K}_m,\Psi)$. Then, taking into account the definition of the Bernstein $n$-width and (18), we obtain

$$\begin{aligned}&\lambda_n\left(W_2^{(r)}(\mathcal{K}_m,\Psi),L_2\right)\geq b_n\left(W_2^{(r)}(\mathcal{K}_m,\Psi),L_2\right)\\&\geq b_n(\mathcal{M}_{n+1};L_2)\geq\sqrt{\frac{n-r+1}{n+1}}\cdot\frac{1}{\alpha_{n,r}}\Psi\left(\sqrt{\frac{n-r-m+1}{n-r+1}}\cdot\frac{1}{\alpha_{n-r,m}}\right).\end{aligned}\tag{25}$$

Comparing the upper bound (20) and the lower bound (25), we get the required equality (19). The theorem is proved.

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# POSITIVE DEFINITE FUNCTIONS AND SHARP INEQUALITIES FOR PERIODIC FUNCTIONS

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**Abstract:** Let $\varphi$ be a positive definite and continuous function on $\mathbb{R}$, and let $\mu$ be the corresponding Bochner measure. For fixed $\varepsilon, \tau \in \mathbb{R}$, $\varepsilon \neq 0$, we consider a linear operator $A_{\varepsilon,\tau}$ generated by the function $\varphi$:

$$A_{\varepsilon,\tau}(f)(t) := \int_{\mathbb{R}} e^{-iu\tau} f(t+\varepsilon u) d\mu(u), \quad t \in \mathbb{R}, \quad f \in C(\mathbb{T}).$$

Let $J$ be a convex and nondecreasing function on $[0,+\infty)$. In this paper, we prove the inequalities

$$\|A_{\varepsilon,\tau}(f)\|_p \leqslant \varphi(0)\|f\|_p, \quad \int_{\mathbb{T}} J\left(|A_{\varepsilon,\tau}(f)(t)|\right) dt \le \int_{\mathbb{T}} J\left(\varphi(0)|f(t)|\right) dt$$

for $p \in [1,\infty]$ and $f \in C(\mathbb{T})$ and obtain criteria of extremal function. We study in more detail the case in which $\varepsilon = 1/n$, $n \in \mathbb{N}$, $\tau = 1$, and $\varphi(x) \equiv e^{i\beta x}\psi(x)$, where $\beta \in \mathbb{R}$ and the function $\psi$ is 2-periodic and positive definite. In turn, we consider in more detail the case where the 2-periodic function $\psi$ is constructed by means of a finite positive definite function $g$. As a particular case, we obtain the Bernstein–Szegő inequality for the derivative in the Weyl–Nagy sense of trigonometric polynomials. In one of our results, we consider the case of the family of functions $g_{1/n,h}(x) := hg(x) + (1-1/n-h)g(nx)$, where $n \in \mathbb{N}$, $n \ge 2$, $-1/n \le h \le 1-1/n$, and the function $g \in C(\mathbb{R})$ is even, nonnegative, decreasing, and convex on $(0,+\infty)$ with $\operatorname{supp} g \subset [-1,1]$. This case is related to the positive definiteness of piecewise linear functions. We also obtain some general interpolation formulas for periodic functions and trigonometric polynomials which include the known interpolation formulas of M. Riesz, of G. Szegő, and of A.I. Kozko for trigonometric polynomials.

**Key words:** Positive definite function, Trigonometric polynomial, Weyl–Nagy derivative, Bernstein–Szegő inequality, Interpolation formula.

## 1. Introduction

The role of positive definite functions in obtaining sharp inequalities for trigonometric polynomials and entire functions is well known (see, for instance, Boas [6, Ch. 11], Timan [22, Sect. 4.8], Lizorkin [13], Gorin [9], and Trigub and Belinsky [23]). For instance, the classical Bernstein inequality $\max|f'(x)| \le n \max|f(x)|$ for trigonometric polynomials of degree at most $n$ is related to the positive definiteness of the function $(1-|x|)_+$. A historical survey of such inequalities and the methods of their proof are given in the works by Gorin [9], Arestov and Glazyrina [5], Gashkov [8], and Vinogradov [25]. In the present paper, we obtain sharp inequalities for continuous periodic functions and, in particular, for trigonometric polynomials. These inequalities are related to positive definite functions. As consequences, we obtain generalizations of Bernstein–Szegő inequalities. We give criteria and descriptions of extremal functions in these inequalities.

A complex-valued function $f : \mathbb{R} \to \mathbb{C}$ is called positive definite on $\mathbb{R}$ ($f \in \Phi(\mathbb{R})$) if, for any $m \in \mathbb{N}$, any set of points $\{x_k\}_{k=1}^m \subset \mathbb{R}$, and any complex numbers $\{c_k\}_{k=1}^m \subset \mathbb{C}$, the following inequality holds:

$$\sum_{k,j=1}^{m} c_k \overline{c_j} f(x_k - x_j) \geqslant 0.$$

It is easy to verify that, for any $\beta \in \mathbb{R}$, the function $f(x) = e^{i\beta x}$ is positive definite. For a function in $\Phi(\mathbb{R})$, the continuity at zero is equivalent to the continuity on $\mathbb{R}$. If $f, g \in \Phi(\mathbb{R})$, then $|f(x)| \le f(0)$,

$\overline{f(-x)} = f(x)$, $|f(x+y)-f(x)|^2 \le 2f(0)(f(0)-\mathrm{Re}f(y))$, $x, y \in \mathbb{R}$, and $\bar{f}$, $\mathrm{Re}\, f$, $fg \in \Phi(\mathbb{R})$. In 1932, S. Bochner and, independently, A. Khinchin proved the following criterion of positive definiteness.

**Theorem 1** (Bochner–Khinchin). *The inclusion $f \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ holds if and only if there exists a finite nonnegative Borel measure $\mu$ on $\mathbb{R}$ such that*

$$f(x) = \int_{\mathbb{R}} e^{ixt} d\mu(t), \quad x \in \mathbb{R}.$$

The proof of this theorem can be found, for instance, in [2, 7, 19, 23, 24]. As a direct consequence, we obtain the following criterion of positive definiteness in terms of nonnegativity of the Fourier transform: *if $f \in C(\mathbb{R}) \cap L_1(\mathbb{R})$, then $f \in \Phi(\mathbb{R}) \iff \widehat{f}(t) \ge 0$, $t \in \mathbb{R}$, where*

$$\widehat{f}(t) := \int_{\mathbb{R}} e^{-itx} f(x) dx, \quad t \in \mathbb{R}.$$

Using this criterion, it is not difficult to see that the functions $(1-|x|)_+$, $e^{-|x|}$, and $e^{-x^2}$ are positive definite.

We denote by $C(\mathbb{T})$, $\mathbb{T} := [-\pi, \pi]$, the class of $2\pi$-periodic continuous functions $f : \mathbb{R} \to \mathbb{C}$. For $f \in C(\mathbb{T})$, we define

$$\|f\|_\infty := \sup\{|f(t)| : \ t \in \mathbb{T}\} \ \text{and} \ \|f\|_p := \left( \int_{\mathbb{T}} |f(t)|^p dt \right)^{1/p}, \quad 1 \le p < \infty.$$

Let $\varphi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, and let $\mu$ be the corresponding finite nonnegative Borel measure on $\mathbb{R}$ such that

$$\varphi(x) = \int_{\mathbb{R}} e^{ixu} d\mu(u), \quad x \in \mathbb{R}.$$

For fixed $\varepsilon, \tau \in \mathbb{R}$, $\varepsilon \ne 0$, we consider the linear operator $A_{\varepsilon,\tau}$ generated by the function $\varphi$:

$$A_{\varepsilon,\tau}(f)(t) := \int_{\mathbb{R}} e^{-iu\tau} f(t + \varepsilon u) d\mu(u), \quad t \in \mathbb{R}, \quad f \in C(\mathbb{T}). \tag{1.1}$$

The function $A_{\varepsilon,\tau}(f)(t)$ is continuous on $\mathbb{R}$ and $2\pi$-periodic. Therefore, $A_{\varepsilon,\tau} : C(\mathbb{T}) \to C(\mathbb{T})$. In this paper, we prove the inequalities

$$\|A_{\varepsilon,\tau}(f)\|_p \leqslant \varphi(0)\|f\|_p, \quad \int_{\mathbb{T}} J\left(|A_{\varepsilon,\tau}(f)(t)|\right) dt \le \int_{\mathbb{T}} J\left(\varphi(0)|f(t)|\right) dt,$$

where $1 \leqslant p \leqslant \infty$, $f \in C(\mathbb{T})$, and $J$ is a convex nondecreasing function on $[0, +\infty)$. In addition, we obtain some criteria of extremal function in these inequalities (see Theorems 2 and 4 and Remark 2). We study in more detail the case in which $\varepsilon = 1/n$, $n \in \mathbb{N}$, $\tau = 1$, and $\varphi(x) \equiv e^{i\beta x}\psi(x)$, where $\beta \in \mathbb{R}$ and $\psi$ is a 2-periodic function of the class $\Phi(\mathbb{R}) \cap C(\mathbb{R})$ (see Theorem 5 and Remarks 4 and 5). In turn, we consider in more detail the case where a 2-periodic function $\psi$ is constructed by means of a finite function $g \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ (Theorem 6). As a particular case, we obtain the Berstein–Szegő inequality for the Weyl–Nagy derivative of trigonometric polynomials (Remark 7). In Theorem 8, we consider the case of the family of functions $g_{1/n,h}(x) := hg(x) + (1 - 1/n - h)g(nx)$, where $n \in \mathbb{N}$, $n \ge 2$, $-1/n \le h \le 1 - 1/n$, and the function $g \in C(\mathbb{R})$ is even, nonnegative, decreasing, and convex on $(0, +\infty)$ with $\mathrm{supp}\, g \subset [-1, 1]$. This case is related to the positive definiteness of piecewise linear functions [15]. In Theorem 9 and Corollary 3, we obtain general interpolation formulas for periodic functions which include the known interpolation formulas of M. Riesz, of G. Szegő, and of A.I. Kozko [11] for trigonometric polynomials (see Remark 8).

## 2. Auxiliary facts of measure and integration theory

We recall some well-known facts which are used in the paper to describe extremal functions. In this section, a measure $\mu$ is a nonnegative countably additive function defined on a $\sigma$-algebra $\gamma$ with identity element $\Omega$. For $p \in (0, +\infty)$, the class $L_p(\Omega, \gamma, \mu)$ is the set of all $\gamma$-measurable functions $f : \Omega \to \mathbb{C}$ such that

$$\|f\|_p := \left( \int\limits_{\Omega} |f(u)|^p \, d\mu(u) \right)^{1/p} < +\infty.$$

The class $L_\infty(\Omega, \gamma, \mu)$ is the set of all $\gamma$-measurable functions $f : \Omega \to \mathbb{C}$ for which there exists $K = K(f) < +\infty$ such that $|f(u)| \le K$ for $\mu$-almost every $u \in \Omega$. For $f \in L_\infty(\Omega, \gamma, \mu)$, the norm is defined by the formula

$$\|f\|_\infty := \inf\{K : |f(u)| \le K \text{ for } \mu\text{-almost all } u \in \Omega\}.$$

For convenience, we assume that $L_p(\Omega, \gamma, \mu) = L_p(\Omega, \mu) = L_p(\Omega)$.

**Proposition 1.** *Let $(\Omega, \gamma, \mu)$ be a measurable space with measure. If $f \in L_1(\Omega, \mu)$, then*

$$\left| \int\limits_{\Omega} f(u) \, d\mu(u) \right| \le \int\limits_{\Omega} |f(u)| \, d\mu(u)$$

*and the inequality turns into an equality if and only if the equality $f(u) = e^{i\theta}|f(u)|$ holds for some $\theta \in \mathbb{R}$ and for $\mu$-almost all $u \in \Omega$.*

P r o o f. See, for instance, [18, Theorems 1.33 and 1.39]. Obviously, for some $\beta \in \mathbb{R}$, we have

$$\left| \int\limits_{\Omega} f(u) \, d\mu(u) \right| = e^{i\beta} \int\limits_{\Omega} f(u) \, d\mu(u) = \int\limits_{\Omega} e^{i\beta} f(u) \, d\mu(u) = \int\limits_{\Omega} \mathrm{Re}(e^{i\beta} f(u)) \, d\mu(u) \le \int\limits_{\Omega} |f(u)| \, d\mu(u)$$

and the inequality turns into an equality if and only if $\mathrm{Re}(e^{i\beta} f(u)) = |f(u)|$ for $\mu$-almost all $u \in \Omega$ or if and only if $e^{i\beta} f(u) = |f(u)|$ for $\mu$-almost all $u \in \Omega$. $\square$

**Proposition 2.** *Assume that $J$ is a convex function on $\mathbb{R}$, $(\Omega, \gamma, \mu)$ is a measurable space with finite measure, $\mu(\Omega) > 0$, and $f$ is a real-valued function in $L_1(\Omega, \mu)$. Then*

$$J\left( \frac{1}{\mu(\Omega)} \int\limits_{\Omega} f(u) \, d\mu(u) \right) \le \frac{1}{\mu(\Omega)} \int\limits_{\Omega} J(f(u)) \, d\mu(u). \tag{2.1}$$

*If the function $J$ is strictly convex at the point $\alpha = \int_\Omega f(u) \, d\mu(u)/\mu(\Omega)$, then equality in (2.1) is attained if and only if $f(u) = \alpha$ for $\mu$-almost all $u \in \Omega$.*

For a proof of this result, see, for instance, [12, Sect. 2.2].

The next proposition will be needed only in Remark 3.

**Proposition 3.** *Let $(\Omega, \gamma, \mu)$ be a measurable space with measure. Then*:

(i) *if, for some $q > 0$, we have $f \in L_p(\Omega)$ for all $p \in [q, +\infty)$ and $\liminf\limits_{p \to +\infty} \|f\|_p < +\infty$, then $f \in L_\infty(\Omega)$ and $\|f\|_\infty \le \liminf\limits_{p \to +\infty} \|f\|_p$;*

(ii) *if, for some $q > 0$, we have $f \in L_\infty(\Omega) \cap L_q(\Omega)$, then $f \in L_p(\Omega)$ for all $p \in [q, +\infty)$ and $\|f\|_\infty = \lim\limits_{p \to +\infty} \|f\|_p$.*

P r o o f. (i) We take a sequence $\{p_n\}$, $n \in \mathbb{N}$, such that $p_n > 0$, $p_n \to +\infty$, and $\|f\|_{p_n} \to c := \liminf\limits_{p\to+\infty} \|f\|_p \geq 0$. For an arbitrary $\sigma > c$, we define $\varepsilon := (\sigma - c)/2 > 0$. Then there exists a number $n(\sigma)$ such that the inequality $\|f\|_{p_n} \leq c + \varepsilon = (\sigma + c)/2 < \sigma$ holds for all $n \geqslant n(\sigma)$. The Chebyshev inequality implies that

$$\mu(\{x \in \Omega : |f(x)| \geq \sigma\}) \leq \left(\frac{\|f\|_{p_n}}{\sigma}\right)^{p_n} \to 0, \quad n \to +\infty.$$

Therefore, $|f(x)| < \sigma$ for $\mu$-almost all $x \in \Omega$ and, hence, $\|f\|_\infty \leq c$.

(ii) If $\|f\|_q = 0$, the required assertion is obvious. Let $\|f\|_q > 0$. Then, for any $p > q$, the inequality $\|f\|_p \leq \|f\|_\infty^{(p-q)/p} \|f\|_q^{q/p}$ holds. This inequality and assertion (i) yield

$$\limsup_{p\to+\infty} \|f\|_p \leq \|f\|_\infty \leq \liminf_{p\to+\infty} \|f\|_p \leq \limsup_{p\to+\infty} \|f\|_p \,.$$

□

## 3. Sharp $L_p$-inequalities for periodic functions

Equality (1.1) implies the inequality

$$|A_{\varepsilon,\tau}(f)(t)| \leq \int_{\mathbb{R}} |f(t + \varepsilon u)| d\mu(u), \quad f \in C(\mathbb{T}), \quad t \in \mathbb{R}. \tag{3.1}$$

Obviously, $\|A_{\varepsilon,\tau}(f)\|_\infty \leqslant \varphi(0)\|f\|_\infty$.

If $1 \leqslant p < \infty$, then inequality (3.1) along with the Minkowski inequality [12, Theorem 2.4] yields

$$\begin{aligned}\|A_{\varepsilon,\tau}(f)\|_p &= \left(\int_{\mathbb{T}} |A_{\varepsilon,\tau}(f)|^p\, dt\right)^{\frac{1}{p}} \leqslant \left(\int_{\mathbb{T}} \left(\int_{\mathbb{R}} |f(t+\varepsilon u)| d\mu(u)\right)^p dt\right)^{\frac{1}{p}} \\ &\leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{T}} |f(t+\varepsilon u)|^p dt\right)^{\frac{1}{p}} d\mu(u) = \varphi(0)\|f\|_p.\end{aligned}$$

It follows from the Fubini theorem that the Fourier series of the function $A_{\varepsilon,\tau}(f)(t)$ has the form

$$A_{\varepsilon,\tau}(f)(t) \sim \sum_{k\in\mathbb{Z}} \varphi(\varepsilon k - \tau) c_k(f) e^{ikt}, \quad f \in C(\mathbb{T}), \tag{3.2}$$

where $c_k(f)$ are the Fourier coefficients of the function $f$:

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt, \quad k \in \mathbb{Z}\,.$$

Let us find sufficient conditions for the equality

$$\|A_{\varepsilon,\tau}(f)\|_p = \varphi(0)\|f\|_p. \tag{3.3}$$

If $|\varphi(\varepsilon s - \tau)| = \varphi(0)$ for some $s \in \mathbb{Z}$, then equality (3.3) holds for the polynomial $f(t) = ce^{ist}$, $c \in \mathbb{C}$, since, in this case, $A_{\varepsilon,\tau}(f)(t) = \varphi(\varepsilon s - \tau) ce^{ist}$. If $\tau/\varepsilon \in \mathbb{Z}$, this condition is satisfied for $s = \tau/\varepsilon$.

If, for some $s, m \in \mathbb{Z}$, $s \neq m$, we have

$$|\varphi(\varepsilon s - \tau)| = |\varphi(\varepsilon m - \tau)| = \varphi(0), \tag{3.4}$$

then equality (3.3) holds for the polynomial $f(t) = ce^{ist} + \nu e^{imt}$, $c, \nu \in \mathbb{C}$, since, in this case,

$$A_{\varepsilon,\tau}(f)(t) = \varphi(\varepsilon s - \tau)ce^{ist} + \varphi(\varepsilon m - \tau)\nu e^{imt}.$$

We only need to take into account that, for any $\delta, \alpha \in \mathbb{R}$, the following equalities hold:

$$\left\|ce^{ist} + e^{i\delta}\nu e^{imt}\right\|_p = \left\|ce^{is(t+\alpha)} + e^{i\delta}\nu e^{im(t+\alpha)}\right\|_p = \left\|ce^{ist} + e^{i(\delta+m\alpha-s\alpha)}\nu e^{imt}\right\|_p.$$

In particular, the latter equality holds for $\alpha = \delta/(s-m)$.

If $\tau \neq 0$, $|\varphi(-2\tau)| = \varphi(0)$, $\varepsilon = \tau/n$, and $n \in \mathbb{N}$, then condition (3.4) is satisfied for $s = n$ and $m = -n$. Hence, $\|A_{\tau/n,\tau}(f)\|_p = \varphi(0)\|f\|_p$ for the polynomial $f(t) = ce^{int} + \nu e^{-int}$ with $c, \nu \in \mathbb{C}$.

Thus, we have proved the following theorem.

**Theorem 2.** *Assume that $\varphi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\tau, \varepsilon \in \mathbb{R}$, and $\varepsilon \neq 0$. Then*:

1) *the operator $A_{\varepsilon,\tau}$ acts from $C(\mathbb{T})$ to $C(\mathbb{T})$, is a multiplier, and satisfies the inequality*

$$\|A_{\varepsilon,\tau}(f)\|_p \leqslant \varphi(0)\|f\|_p, \quad 1 \leqslant p \leqslant \infty, \quad f \in C(\mathbb{T}); \tag{3.5}$$

2) *if, for some $s \in \mathbb{Z}$, the condition $|\varphi(\varepsilon s - \tau)| = \varphi(0)$ is satisfied, then equality in* (3.5) *is attained at the polynomials $f(t) = ce^{ist}$, $c \in \mathbb{C}$. If $\tau/\varepsilon \in \mathbb{Z}$, this condition is satisfied for $s = \tau/\varepsilon$.*

*If, for some $s, m \in \mathbb{Z}$, $s \neq m$, condition* (3.4) *is satisfied, then equality in* (3.5) *is attained at the polynomials $f(t) = ce^{ist} + \nu e^{imt}$, $c, \nu \in \mathbb{C}$.*

*If $\tau \neq 0$ and $|\varphi(-2\tau)| = \varphi(0)$, then equality in* (3.5) *for $\varepsilon = \tau/n$, $n \in \mathbb{N}$, is attained at the polynomials $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$.*

*Remark 1.* In connection with the conditions in Theorem 2, the following simple property of positive definite functions is useful: if $\varphi \in \Phi(\mathbb{R})$ and, for some $y, \delta \in \mathbb{R}$, $y \neq 0$, we have $\varphi(y) = \varphi(0)e^{i\delta y}$, then $\varphi(x) \equiv f(x)e^{i\delta x}$, where $f \in \Phi(\mathbb{R})$ and $f$ is periodic with period $|y| > 0$. Indeed, the function $f(x) \equiv \varphi(x)e^{-i\delta x}$ is the product of two positive definite functions. Therefore, $f \in \Phi(\mathbb{R})$ and, hence, for any $x \in \mathbb{R}$, we have

$$|f(x+y) - f(x)|^2 \leq 2f(0)(f(0) - \mathrm{Re} f(y)).$$

Since $f(y) = \varphi(y)e^{-i\delta y} = \varphi(0) = f(0) \geq 0$, we have $f(x+y) - f(x) = 0$ for all $x \in \mathbb{R}$. If, in addition, $\varphi \in C(\mathbb{R})$, then the Bochner measure of the function $\varphi$ is discrete and concentrated at the points $t_k = 2\pi k/|y| + \delta$, $k \in \mathbb{Z}$, and $\mu(\{t_k\}) = c_k(f) \geq 0$, $k \in \mathbb{Z}$ (see Theorem 3 below).

*Remark 2.* When $p = \infty$, inequality (3.5) turns into an equality at some function $f \in C(\mathbb{T})$ (see inequality (3.1) and Proposition 1) if and only if the equality $f(\xi + \varepsilon u) = e^{i(u\tau+\beta)}\|f\|_\infty$ holds for some $\xi, \beta \in \mathbb{R}$ and $\mu$-almost all $u \in \mathbb{R}$.

When $p = 1$, inequality (3.5) turns into an equality at some function $f \in C(\mathbb{T})$ (see inequality (3.1) and Proposition 1) if and only if, for any $t \in \mathbb{R}$, there exists a number $\beta(t) \in \mathbb{R}$ such that the equality $f(t + \varepsilon u) = e^{i(u\tau+\beta(t))}|f(t + \varepsilon u)|$ holds for $\mu$-almost all $u \in \mathbb{R}$. This implies that if a function $f \in C(\mathbb{T})$ is extremal in inequality (3.5) with $p = 1$, then any function of the form $cf(t)g(t)$, where $c \in \mathbb{C}$, $g \in C(\mathbb{T})$, and $g(t) \geq 0$ for all $t \in \mathbb{R}$, is also extremal.

When $p \in (1, \infty)$, inequality (3.5) turns into an equality at some function $f \in C(\mathbb{T})$ if and only if, for any $t \in \mathbb{R}$ and $\mu$-almost all $u \in \mathbb{R}$, the equality $f(t + \varepsilon u) = e^{iu\tau}c(t)$ holds, where

$c(t) = A_{\varepsilon,\tau}(f)(t)/\varphi(0) \in C(\mathbb{T})$ (for such $p$, see Theorem 4 below for $J(t) = t^p$).

*Remark 3.* If $1 \le p < \infty$, the class $C(\mathbb{T})$ is everywhere dense in $L_p(\mathbb{T})$ (the Lebesgue measure is taken as a measure). Therefore, inequality (3.5) implies that the multiplier $A_{\varepsilon,\tau} : C(\mathbb{T}) \to C(\mathbb{T})$ defined by formula (3.2) is extended to the multiplier $A_{\varepsilon,\tau} : L_p(\mathbb{T}) \to L_p(\mathbb{T})$, $1 \le p < \infty$, and

$$\|A_{\varepsilon,\tau}(f)\|_p \leqslant \varphi(0)\|f\|_p, \quad 1 \leqslant p < \infty, \quad f \in L_p(\mathbb{T}). \tag{3.6}$$

Hence, $A_{\varepsilon,\tau} : L_\infty(\mathbb{T}) \to L_\infty(\mathbb{T})$ and inequality (3.6) holds with $p = \infty$. We only need to use the well-known facts from measure and integration theory (see Proposition 3).

## 4. Periodic positive definite functions

The following description of periodic functions of the class $\Phi(\mathbb{R}) \cap C(\mathbb{R})$ is well known (see, for instance, [7, Theorem 1.7.5] and [10, Sect. II.1]).

**Theorem 3.** *If $\psi \in C(\mathbb{R})$ and $\psi$ is $2T$-periodic with $T > 0$, then $\psi \in \Phi(\mathbb{R})$ if and only if $c_k(\psi) \ge 0$, $k \in \mathbb{Z}$, where*

$$c_k(\psi) := \frac{1}{2T} \int_{-T}^{T} \psi(x) e^{-i\pi kx/T} dx, \quad k \in \mathbb{Z}.$$

*In this case, the function $\psi$ is expanded into the absolutely convergent Fourier series*

$$\psi(x) = \sum_{k \in \mathbb{Z}} c_k(\psi) e^{i\pi kx/T}, \quad x \in \mathbb{R}.$$

**Corollary 1.** *Assume that $f \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\operatorname{supp} f \subset [-1, 1]$, and a 2-periodic function $\psi(x)$ coincides with the function $f(x)$ for $x \in [-1, 1]$. Then $\psi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ and $\psi(x-1) = f(x-1) + f(x+1)$ for $x \in [-2, 2]$.*

P r o o f. Since $\psi(\pm 1) = f(\pm 1) = 0$, we have $\psi \in C(\mathbb{R})$ and

$$2c_k(\psi) = \int_{-1}^{1} f(x) e^{-i\pi kx} dx = \widehat{f}(\pi k) \ge 0, \quad k \in \mathbb{Z}.$$

Therefore, $\psi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$. Since $\operatorname{supp} f \subset [-1, 1]$, we obviously have

$$\psi(x-1) = \sum_{k \in \mathbb{Z}} f(x - 1 + 2k), \quad x \in \mathbb{R}.$$

Only terms with $k = 0$ and $k = 1$ remain in this sum for $x \in [-2, 2]$. □

## 5. Sharp integral inequalities for periodic functions

Let $\varphi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ and $\varphi(0) > 0$. Assume that $J$ is a convex nondecreasing function on $[0, +\infty)$. Then $J$ is continuous on $[0, +\infty)$ and can be extended to $\mathbb{R}$ with preservation of convexity (for instance, by defining $J(t) := J(0)$ for $t < 0$ or by means of the even extension).

Successively using the monotonicity and the Jensen inequality (see, for instance, [12, Sect. 2.2] or Proposition 2), for $f \in C(\mathbb{T})$, we derive from inequality (3.1) that

$$\begin{aligned} J\left(\frac{1}{\varphi(0)}|A_{\varepsilon,\tau}(f)(t)|\right) &\le J\left(\frac{1}{\varphi(0)}\int\limits_{\mathbb{R}} |f(t+\varepsilon u)| d\mu(u)\right) \\ &\le \frac{1}{\varphi(0)}\int\limits_{\mathbb{R}} J\left(|f(t+\varepsilon u)|\right) d\mu(u), \quad t \in \mathbb{R}. \end{aligned} \tag{5.1}$$

We integrate the left-hand and right-hand sides of inequality (5.1) with respect to $t \in \mathbb{T}$. Applying the Fubini theorem and taking into account the periodicity of $f$, we obtain

$$\int\limits_{\mathbb{T}} J\left(\frac{1}{\varphi(0)}|A_{\varepsilon,\tau}(f)(t)|\right) dt \le \int\limits_{\mathbb{T}} J\left(|f(t)|\right) dt.$$

In view of the arbitrariness of $f$, it is convenient to write the latter inequality in the form

$$\int\limits_{\mathbb{T}} J\left(|A_{\varepsilon,\tau}(f)(t)|\right) dt \le \int\limits_{\mathbb{T}} J\left(\varphi(0)|f(t)|\right) dt. \tag{5.2}$$

Inequality (5.2) also holds if $\varphi(0) = 0$, since, in this case, $\varphi(x) \equiv 0$ and, hence, $A_{\varepsilon,\tau}(f)(t) \equiv 0$ for any $f \in C(\mathbb{T})$. Thus, we obtain the following theorem.

**Theorem 4.** *Assume that $\varphi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\tau, \varepsilon \in \mathbb{R}$, $\varepsilon \ne 0$, and $J$ is a convex nondecreasing function on $[0, +\infty)$. Then:*

1) *The operator $A_{\varepsilon,\tau}$ generated by the function $\varphi$ by formula* (1.1) *satisfies inequality* (5.2) *for any function $f \in C(\mathbb{T})$.*

2) *If the condition $|\varphi(\varepsilon s - \tau)| = \varphi(0)$ holds for some $s \in \mathbb{Z}$, then equality in* (5.2) *is attained at the polynomials $f(t) = ce^{ist}$, $c \in \mathbb{C}$. If $\tau/\varepsilon \in \mathbb{Z}$, then this condition holds for $s = \tau/\varepsilon$.*

*If condition* (3.4) *holds for some $s, m \in \mathbb{Z}$, $s \ne m$, then equality in* (5.2) *is attained at the polynomials $f(t) = ce^{ist} + \nu e^{imt}$, $c, \nu \in \mathbb{C}$.*

*If $\tau \ne 0$, $|\varphi(-2\tau)| = \varphi(0)$, $\varepsilon = \tau/n$, and $n \in \mathbb{N}$, then equality in* (5.2) *is attained at the polynomials $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$.*

3) *If the function $J$ is strictly convex at any point of the interval $(0, +\infty)$ and $\varphi(0) > 0$, then inequality* (5.2) *turns into an equality at some function $f \in C(\mathbb{T})$ if and only if, for any $t \in \mathbb{R}$ and $\mu$-almost all $u \in \mathbb{R}$, the equality $e^{-iu\tau} f(t + \varepsilon u) = c(t)$ holds, where $c(t) = A_{\varepsilon,\tau}(f)(t)/\varphi(0) \in C(\mathbb{T})$.*

P r o o f. Only the latter statement needs to be proved. The sufficiency is obvious. Let us prove the necessity. Let inequality (5.2) turn into an equality for some function $f \in C(\mathbb{T})$. Then inequalities (5.1) turn into equalities for all $t \in \mathbb{R}$. Let

$$\alpha(t) := \frac{1}{\varphi(0)}\int\limits_{\mathbb{R}} |f(t+\varepsilon u)| d\mu(u), \quad t \in \mathbb{R}.$$

Obviously, $\alpha(t) \ge 0$ for all $t \in \mathbb{R}$. If $\alpha(t) = 0$, then $f(t+\varepsilon u) = 0$ for $\mu$-almost all $u \in \mathbb{R}$ and, in this case, $c(t) = 0$. If $\alpha(t) > 0$, then $|f(t+\varepsilon u)| = \alpha(t)$ for $\mu$-almost all $u \in \mathbb{R}$ (see Proposition 2). Since the function $J$ strictly increases on $[0, +\infty)$, inequality (3.1) also turns into an equality for all $t \in \mathbb{R}$. Therefore, for some $\beta(t) \in \mathbb{R}$ and $\mu$-almost all $u \in \mathbb{R}$, we have the equality (see Proposition 1)

$$e^{-iu\tau} f(t+\varepsilon u) = e^{i\beta(t)}|e^{-iu\tau} f(t+\varepsilon u)| = e^{i\beta(t)}\alpha(t) = c(t).$$

This implies that $A_{\varepsilon,\tau}(f)(t) = \varphi(0)c(t),\ t \in \mathbb{R}$. $\square$

For $\varepsilon = 1/n$, $n \in \mathbb{N}$, and $\tau = 1$, we can distinguish the case where the condition on the extremal function in Theorem 4 is more clear.

**Theorem 5.** *Let $\varphi(x) \equiv e^{i\beta x}\psi(x)$, where $\beta \in \mathbb{R}$, and let $\psi$ be a 2-periodic function in $\Phi(\mathbb{R}) \cap C(\mathbb{R})$. Let $J$ be a convex nondecreasing function on $[0, +\infty)$. Then the operator $A_{1/n,1}$, $n \in \mathbb{N}$, generated by the function $\varphi$ by formula* (1.1) *for $\varepsilon = 1/n$ and $\tau = 1$ satisfies the inequality*

$$\int\limits_{\mathbb{T}} J\left(|A_{1/n,1}(f)(t)|\right)\,dt \le \int\limits_{\mathbb{T}} J\left(\psi(0)|f(t)|\right)\,dt, \quad f \in C(\mathbb{T}). \tag{5.3}$$

*Inequality* (5.3) *turns into an equality, in particular, at every function $f \in C(\mathbb{T})$ whose Fourier series has the form*

$$f(t) \sim \sum_{m\in\mathbb{Z}} d_m e^{in(2m+1)t}. \tag{5.4}$$

*If the function $J$ is strictly convex at any point of the interval $(0, +\infty)$ and $\psi(0) > 0$, then inequality* (5.3) *turns into an equality at some function $f \in C(\mathbb{T})$ if and only if the functions $(-1)^s f\left(t + \frac{\pi s}{n}\right)$ are identical on $\mathbb{R}$ for all $s = 0, \dots, 2n-1$ such that $\mu_s(n, \psi) > 0$, where*

$$\mu_k(n,\psi) = \sum_{m\in\mathbb{Z}} c_{k+2nm}(\psi), \quad k \in \mathbb{Z}, \tag{5.5}$$

*and $c_k(\psi) \ge 0$, $k \in \mathbb{Z}$, are the Fourier coefficients of the function $\psi$. If, in addition, the inequalities $\mu_s(n,\psi) > 0$ and $\mu_{s+1}(n,\psi) > 0$ hold for some $s \in \mathbb{Z}$, then inequality* (5.3) *turns into an equality only at functions $f \in C(\mathbb{T})$ whose Fourier series has the form* (5.4).

P r o o f. In our case, $\varphi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ and $\varphi(0) = \psi(0)$. Therefore, inequality (5.3) follows immediately from inequality (5.2).

Since the function $\psi$ belongs to $\Phi(\mathbb{R}) \cap C(\mathbb{R})$ and is 2-periodic, its Fourier coefficients $c_k(\psi)$, $k \in \mathbb{Z}$, are nonnegative and $\psi$ is expanded into an absolutely convergent Fourier series. Then the function $\varphi$ is also expanded into an absolutely convergent series:

$$\varphi(x) = \sum_{k\in\mathbb{Z}} c_k(\psi)e^{i(\pi k+\beta)x}, \quad x \in \mathbb{R}.$$

It follows from this representation that the Bochner measure $\mu$ of the function $\varphi$ is concentrated at the points $t_k = \pi k + \beta$, $k \in \mathbb{Z}$, and $\mu(\{t_k\}) = c_k(\psi)$, $k \in \mathbb{Z}$. Therefore, for any $f \in C(\mathbb{T})$, we have

$$A_{1/n,1}(f)(t) = e^{-i\beta} \sum_{k\in\mathbb{Z}} (-1)^k f\left(t + \frac{t_k}{n}\right) c_k(\psi), \quad t \in \mathbb{R}.$$

Taking into account the periodicity of $f$, it is convenient to divide the terms in this sum into disjoint groups in which the summation index has the form $k + 2nm$ with $m \in \mathbb{Z}$ and $k = 0, \dots, 2n-1$. Then

$$A_{1/n,1}(f)(t) = e^{-i\beta} \sum_{k=0}^{2n-1} (-1)^k f\left(t + \frac{\pi k + \beta}{n}\right) \mu_k(n,\psi), \quad t \in \mathbb{R}, \tag{5.6}$$

where the numbers $\mu_k(n,\psi)$ are defined by formula (5.5). For these numbers, the following equalities hold:

$$\sum_{k=0}^{2n-1} \mu_k(n,\psi) = \sum_{k\in\mathbb{Z}} c_k(\psi) = \psi(0); \quad \mu_k(n,\psi) = \mu_{k+2n}(n,\psi), \quad k \in \mathbb{Z}. \tag{5.7}$$

If a function $f$ belongs to $C(\mathbb{T})$ and its Fourier series has the form (5.4), then, obviously, $(-1)^s f(t+\pi s/n) \equiv f(t)$ for all $s \in \mathbb{Z}$. Therefore, for such functions, we have $A_{1/n,1}(f)(t) \equiv e^{-i\beta}\psi(0) f(t+\beta/n)$ and inequality (5.3) turns into an equality.

If the function $J$ is strictly convex at any point of the interval $(0,+\infty)$ and $\psi(0) > 0$, then Theorem 4 implies that inequality (5.3) turns into an equality at some function $f \in C(\mathbb{T}) \iff$ the functions $(-1)^s f(t+(\pi s+\beta)/n)$ are identical on $\mathbb{R}$ for all $s \in \mathbb{Z}$ such that $\mu(\{t_s\}) = c_s(\psi) > 0$ $\iff$ the functions $(-1)^s f(t+\pi s/n)$ are identical on $\mathbb{R}$ for all $s = 0,\dots,2n-1$ such that $\mu_s(n,\psi) > 0$. The latter equivalence is a consequence of the following properties: (1) the functions of this family with numbers $s \in \mathbb{Z}$ and $s+2nm$, $m \in \mathbb{Z}$, are identical; (2) $c_k(\psi) \geq 0$, $\mu_k(n,\psi) \geq 0$, $k \in \mathbb{Z}$, and $\mu_k(n,\psi) > 0 \iff c_{k+2nm}(\psi) > 0$ for some $m \in \mathbb{Z}$.

Assume that inequality (5.3) turns into an equality at some function $f \in C(\mathbb{T})$. If, in addition, the inequalities $\mu_s(n,\psi) > 0$ and $\mu_{s+1}(n,\psi) > 0$ hold for some $s \in \mathbb{Z}$, then, by what has been proved,

$$(-1)^s f\left(t+\frac{\pi s}{n}\right) \equiv (-1)^{s+1} f\left(t+\frac{\pi(s+1)}{n}\right).$$

Then, for the Fourier coefficients of the function $f$, we have the equalities $c_k(f) = -e^{ik\pi/n} c_k(f)$, $k \in \mathbb{Z}$. If $c_k(f) \neq 0$ for some $k \in \mathbb{Z}$, then $k = n(2m+1)$ for some $m \in \mathbb{Z}$. This means that the Fourier series of the function $f$ has the form (5.4). The theorem is proved. □

*Remark 4.* Let $\varphi(x) \equiv e^{i\beta x}\psi(x)$, where $\beta \in \mathbb{R}$, and assume that a 2-periodic function $\psi$ belongs to $\Phi(\mathbb{R}) \cap C(\mathbb{R})$ and satisfies the inequality $\psi(0) > 0$. Then the operator $A_{1/n,1}$, $n \in \mathbb{N}$, satisfies the inequality (see (5.3) for $J(t) = t^p$, $1 \leq p < \infty$, or (3.5) for $\varepsilon = 1/n$, $\tau = 1$)

$$\|A_{1/n,1}(f)\|_p \leqslant \psi(0)\|f\|_p, \quad 1 \leqslant p \leqslant \infty, \quad f \in C(\mathbb{T}). \tag{5.8}$$

This inequality turns into an equality, for instance, at every function $f \in C(\mathbb{T})$ whose Fourier series has the form (5.4), since, for such functions, $A_{1/n,1}(f)(t) \equiv e^{-i\beta}\psi(0) f(t+\beta/n)$. When $1 < p < \infty$, only functions of the form (5.4) are extremal in inequality (5.8) if the inequalities $\mu_s(n,\psi) > 0$ and $\mu_{s+1}(n,\psi) > 0$ hold for some $s \in \mathbb{Z}$ (see Theorem 5 for $J(t) = t^p$). We state criteria for a function to be extremal when $p = \infty$ and $p = 1$. Taking into account Remark 2 and the fact that the Bochner measure $\mu$ of the function $\varphi$ is concentrated at the points $t_k = \pi k + \beta$, $k \in \mathbb{Z}$, and $\mu(\{t_k\}) = c_k(\psi) \geq 0$, $k \in \mathbb{Z}$ (see the proof of Theorem 5), we obtain:

1) When $p = \infty$, inequality (5.8) turns into an equality at some function $f \in C(\mathbb{T})$ if and only if, for some $\eta, \delta \in \mathbb{R}$, the equality

$$(-1)^s f(\eta + \pi s/n) = e^{i\delta}\|f\|_\infty \tag{5.9}$$

holds for all $s = 0,\dots,2n-1$ such that $\mu_s(n,\psi) > 0$. This condition is satisfied not only for functions of the form (5.4). For instance, for $s = 0,\dots,2n$, we set $f(\pi s/n) := (-1)^s M$ and, at the remaining points $t \in [0,2\pi]$, we define $f$ so that it is continuous on $[0,2\pi]$ with the only condition $|f(t)| \leq |M|$. For such a function $f$, inequality (5.8) with $p = \infty$ turns into an equality.

If $\mu_s(n,\psi) > 0$ for $s = 0,\dots,2n-1$, then only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal among trigonometric polynomials of degree at most $n$ for which inequality (5.8) with $p = \infty$ turns into an equality. Indeed, if $f$ is an extremal polynomial of degree at most $n$, then condition (5.9) is satisfied for $s = 0,\dots,2n-1$ and, hence, for all $s \in \mathbb{Z}$. Then one can use the arguments of [1, Sect. 84, p. 189] for entire functions in the class $B_\sigma$ with $\sigma = n$.

2) When $p = 1$, inequality (5.8) turns into an equality at some function $f \in C(\mathbb{T})$ if and only if, for any $t \in \mathbb{R}$, there exists a number $\delta(t) \in \mathbb{R}$ such that the identity

$$(-1)^s f\left(t+\frac{\pi s}{n}\right) \equiv e^{i\delta(t)}\left|f\left(t+\frac{\pi s}{n}\right)\right| \tag{5.10}$$

holds for all $s = 0, \dots, 2n-1$ such that $\mu_s(n,\psi) > 0$. This implies that if a function $f \in C(\mathbb{T})$ is extremal in inequality (5.8) with $p = 1$, then any function of the form $cf(t)g(t)$, where $c \in \mathbb{C}$, $g \in C(\mathbb{T})$, and $g(t) \geq 0$ for all $t \in \mathbb{R}$, is also extremal. In particular, functions of the form $h(t)g(t)$ are extremal if the function $h \in C(\mathbb{T})$ has the form (5.4), $g \in C(\mathbb{T})$, and $g(t) \geq 0$ for all $t \in \mathbb{R}$. In some sense, the converse statement holds: if the inequalities $\mu_s(n,\psi) > 0$ and $\mu_{s+1}(n,\psi) > 0$ hold for some $s \in \mathbb{Z}$, a function $f \in C(\mathbb{T})$ is extremal in inequality (5.8) with $p = 1$, and $f(t) \neq 0$ for almost all $t \in \mathbb{R}$ (with respect to the Lebesgue measure), then the function $h(t) := f(t)/|f(t)|$ belongs to $L_\infty(\mathbb{T})$ and has the form (5.4) (see the proof of Theorem 5).

We note the following well-known fact. If a function $f \in C(\mathbb{T})$ is extremal in inequality (5.8) with $p = 1$, then condition (5.10) implies that the function

$$g(u) := \int\limits_{\mathbb{T}} f(t+u)e^{-i\delta(t)}\,dt \in C(\mathbb{T})$$

is extremal in inequality (5.8) with $p = \infty$. Indeed, for all $s = 0, \dots, 2n-1$ such that $\mu_s(n,\psi) > 0$, we have $\|f\|_1 = (-1)^s g(\pi s/n) \leq \|g\|_\infty \leq \|f\|_1$ and, hence, $(-1)^s g(\pi s/n) = \|g\|_\infty$.

If $\mu_s(n,\psi) > 0$ for $s = 0, \dots, 2n-1$, then only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal among trigonometric polynomials of degree at most $n$ for which inequality (5.8) with $p = 1$ turns into an equality. Indeed, if $f$ is an extremal trigonometric polynomial of degree at most $n$, then condition (5.10) is satisfied for $s = 0, \dots, 2n-1$. Then one can use the Riesz interpolation formula [16, 17] (see also [28, Ch. X, Sect. 3, (3.11)])

$$f'\left(t + \frac{\pi}{2n}\right) \equiv \sum_{s=1}^{2n} (-1)^{s-1} f\left(t + \frac{\pi s}{n}\right) a_s, \quad \text{where all } a_s > 0 \text{ and } \sum_{s=1}^{2n} a_s = n,$$

which implies the equality $\|f'\|_1 = n\|f\|_1$. Therefore, $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$ (see [3, Corollary 6]).

*Remark 5.* If, in Theorem 5, the function $J$ is convex and strictly increasing on $[0, +\infty)$ and $\mu_s(n,\psi) > 0$ for all $s = 0, \dots, 2n-1$ (this implies that $\psi(0) > 0$), then only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal among trigonometric polynomials of degree at most $n$ for which inequality (5.3) turns into an equality. Indeed, if inequality (5.3) turns into an equality at some function $f \in C(\mathbb{T})$, then the corresponding inequalities (5.1) and (3.1) turn into equalities for any $t \in \mathbb{R}$ and, hence, inequality (5.8) with $p = 1$ turns into an equality at $f$. Then we need to use the last statement in Remark 4.

In conclusion of this section, we note that the integral inequalities (5.2) for the class of trigonometric polynomials and for different differential operators and Szegő compositions were studied by many authors, in particular, by A. Zygmund, V.V. Arestov, V.I. Ivanov, E.A. Storozhenko, V.G. Krotov, P. Oswald, and A.I. Kozko. In this case, not only convex functions $J$ were considered. A history of this question was described in great detail in the paper by Arestov [4].

## 6. Generalization of Bernstein–Szegő inequalities

We denote by $\mathscr{F}_n$, $n \in \mathbb{N}$, the set of trigonometric polynomials

$$f(t) := \sum_{|k| \leq n} c_k e^{ikt} = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt), \quad c_k = c_k(f) \in \mathbb{C},$$

of degree at most $n$ with coefficients in $\mathbb{C}$, where $a_k := c_k + c_{-k}$ and $b_k := i(c_k - c_{-k}),\ k \geq 0$. There are several different definitions of fractional derivative. The following operator for $r > 0$ and $\beta \in \mathbb{R}$ presumably first appeared in the paper by Sz.-Nagy [21, equality (2) for $m = 1$, $\lambda(k) = k^r$]. For $f \in \mathscr{F}_n$, we define

$$f^{(r,\beta)}(t) := \sum_{|k|\leq n} |k|^r e^{i\beta \operatorname{sign} k} c_k e^{ikt} = \sum_{k=1}^{n} k^r \left(a_k \cos(kt+\beta) + b_k \sin(kt+\beta)\right).$$

For $\beta = r\pi/2$, we obtain the Weyl derivative which, for $r \in \mathbb{N}$, coincides with the usual derivative of order $r$. Often, this operator is called the Weyl–Nagy derivative.

Let $J$ be a convex and nondecreasing function on $[0,+\infty)$. Kozko proved (see [11, Theorem 1, Corollary 1]) that if $1 \leq p \leq \infty$, then, for any $n \in \mathbb{N}$, $r \geq 1$, and $\beta \in \mathbb{R}$, the following inequalities hold:

$$\int_{\mathbb{T}} J\left(\left|f^{(r,\beta)}(f)(t)\right|\right) dt \leq \int_{\mathbb{T}} J\left(n^r |f(t)|\right) dt, \quad f \in \mathscr{F}_n, \tag{6.1}$$

$$\|f^{(r,\beta)}\|_p \leq n^r \|f\|_p, \quad f \in \mathscr{F}_n. \tag{6.2}$$

For the usual derivative, i.e., when $r = 1$ and $\beta = \pi/2$, inequality (6.2) was proved by Bernstein in the case $p = \infty$. For $r = 1$ and $\beta \in \mathbb{R}$, inequality (6.2) was obtained by Szegő [20] in the case $p = \infty$ and inequality (6.1) was proved by Zygmund [28, Ch. X, Sect. 3, (3.25)] (his proof for real polynomials is also true for polynomials in $\mathscr{F}_n$). This and the identity

$$f^{(r+1,\beta)}(t) \equiv \left(f^{(r,\beta)}(t)\right)^{(1,0)}, \quad r > 0, \quad \beta \in \mathbb{R},$$

imply the validity of inequality (6.2) for any $r \in \mathbb{N}$. Inequality (6.2) for $p = \infty$, $r \geq 1$, $\beta = -r\pi/2$, and $\beta = 0$ (the case of the Riesz derivative) was proved by Lizorkin [13, Theorems 2, 2′].

Obviously, inequalities (6.1) and (6.2) turn into equalities for the polynomials $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$. Szegő [20, p. 66] proved that, in inequality (6.2) with $p = \infty$, there are no other extremal polynomials in the case $r = 1$ and $\beta \neq q\pi$, $q \in \mathbb{Z}$ (see also arguments in [1, Sect. 84, p. 189]). If, in addition, the function $tJ'(t)$ is strictly increasing on $(0,+\infty)$, then, in inequalities (6.1) and (6.2) for $1 \leq p < \infty$, $n \in \mathbb{N}$, $r \geq 1$, and $\beta \in \mathbb{R}$, there are no other extremal polynomials at least in the following cases (see [3, Corollary 6], [5, Theorems 1,2]): (1) in the case of the usual derivative of order $r \in \mathbb{N}$; (2) $n = 1$, $r \geq 1$, and $\beta \in \mathbb{R}$ or $n \geq 2$, $r \geq \ln(2n)/\ln(n/(n-1))$, and $\beta \in \mathbb{R}$.

For $r = 1$ and $\beta \neq q\pi$, $q \in \mathbb{Z}$, in inequalities (6.2) and (6.1) (if, in addition, the function $J(t)$ is strictly increasing on $(0,+\infty)$), only polynomials of the form $f(t) = a\cos nt + b\sin nt$, $a, b \in \mathbb{R}$, are extremal in the class of real trigonometric polynomials. This result is due to Zygmund [28, Ch. X, Sect 3, (3.24), (3.25)].

Other cases in which inequality (6.2) holds, when $r < 1$ or $0 \leq p < 1$, were considered in the paper by Arestov and Glazyrina [5], where these inequalities are called Bernstein–Szegő inequalities and a complete history of such inequalities is given.

Inequalities more general than (6.1) and (6.2) are obtained from Theorem 5 under an appropriate choice of the function $\psi$. The method of construction of the function $\psi$ described below is essentially contained in the paper by Lizorkin [13].

Assume that $g \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\operatorname{supp} g \subset [-1,1]$, and $\beta \in \mathbb{R}$. We consider the auxiliary function $F(x) := g(-x)e^{-i\beta x}$, $x \in \mathbb{R}$. Obviously, $F \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ and $\operatorname{supp} F \subset [-1,1]$. Using the function $F$, we construct the 2-periodic function $\psi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ (see Corollary 1). For $x \in [-2,2]$, we have

$$\psi(x-1) = F(x-1) + F(x+1) = g(1-x)e^{-i\beta(x-1)} + g(-1-x)e^{-i\beta(x+1)}.$$

Then

$$\psi(x-1) = e^{-i\beta x}\begin{cases} g(1-|x|)e^{i\beta}, & 0 \le x \le 2; \\ g(|x|-1)e^{-i\beta}, & -2 \le x \le 0. \end{cases}$$

Taking into account that the real and imaginary parts of a positive definite function are even and odd functions, respectively, we obtain the equality $\psi(x-1) = e^{-i\beta x} e^{i\beta \operatorname{sign} x} g_0(x)$, $|x| \le 2$, where

$$g_0(x) = \operatorname{Re} g(1-|x|) + i \operatorname{sign} x \operatorname{Im} g(1-|x|), \quad |x| \le 2.$$

Obviously, the function $\varphi(x) := e^{i\beta x}\psi(x)$ belongs to $\Phi(\mathbb{R}) \cap C(\mathbb{R})$ and

$$\varphi(x-1) = e^{-i\beta} g_0(x) e^{i\beta \operatorname{sign} x} = e^{-i\beta}(\operatorname{Re} g(1-|x|) + i \operatorname{sign} x \operatorname{Im} g(1-|x|)) e^{i\beta \operatorname{sign} x}, \quad |x| \le 2. \quad (6.3)$$

Consider the operator $A_{1/n,1}$ generated by the function $\varphi$ by formula (1.1) for $\varepsilon = 1/n$ and $\tau = 1$. We can apply Theorem 5 and Remarks 4 and 5 to this operator. It should be taken into account that $\psi(0) = g(0)$ and $c_k(\psi) = \widehat{g}(-\beta - k\pi)/2$, $k \in \mathbb{Z}$. For polynomials $f \in \mathscr{F}_{2n}$, the operator $A_{1/n,1}$ has the following form (see (3.2) and (6.3)):

$$A_{1/n,1}(f)(t) \equiv e^{-i\beta} \sum_{|k| \le 2n} \left( \operatorname{Re} g\left(1 - \frac{|k|}{n}\right) + i \operatorname{sign} k \operatorname{Im} g\left(1 - \frac{|k|}{n}\right)\right) e^{i\beta \operatorname{sign} k} c_k(f) e^{ikt}.$$

We introduce one more parameter. Obviously, for any $\theta \in [-1,1]$, the function

$$g_\theta(x) := ((1+\theta)g(x) + (1-\theta)g(-x))/2 = \operatorname{Re} g(x) + i\theta \operatorname{Im} g(x), \quad x \in \mathbb{R},$$

also belongs to the class $\Phi(\mathbb{R}) \cap C(\mathbb{R})$ and $\operatorname{supp} g_\theta \subset [-1,1]$. Therefore, all the above arguments are applicable to the function $g_\theta$ as well. It should be taken into account that, for the corresponding function $\psi_\theta$, we have $\psi_\theta(0) = g_\theta(0) = g(0)$ and

$$c_k(\psi_\theta) = ((1+\theta)\widehat{g}(-\beta - k\pi) + (1-\theta)\widehat{g}(\beta + k\pi))/4, \quad k \in \mathbb{Z}.$$

For the function $\varphi_\theta(x) := e^{i\beta x}\psi_\theta(x) \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, we consider the corresponding operator $A_{1/n,1}$ with $\varepsilon = 1/n$ and $\tau = 1$ (see (1.1)). We state the results obtained in Theorem 5 and Remarks 4 and 5 for the following operator defined on polynomials $f \in \mathscr{F}_{2n}$:

$$\begin{aligned} &D_{n,\theta}^{g,\beta}(f)(t) := A_{1/n,1}(f)(t) \equiv \\ &e^{-i\beta} \sum_{|k| \le 2n} \left( \operatorname{Re} g\left(1 - \frac{|k|}{n}\right) + i\theta \operatorname{sign} k \operatorname{Im} g\left(1 - \frac{|k|}{n}\right)\right) e^{i\beta \operatorname{sign} k} c_k(f) e^{ikt}. \end{aligned} \quad (6.4)$$

**Theorem 6.** *Assume that $g \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\operatorname{supp} g \subset [-1,1]$, $g(0) > 0$, $\beta \in \mathbb{R}$, $\theta \in [-1,1]$, and $1 \le p \le \infty$. Let $J$ be a convex nondecreasing function on $[0,+\infty)$. Then:*

1) *For any $n \in \mathbb{N}$, we have*

$$\int_{\mathbb{T}} J\left(|D_{n,\theta}^{g,\beta}(f)(t)|\right) dt \le \int_{\mathbb{T}} J\left(g(0)|f(t)|\right) dt, \quad f \in \mathscr{F}_{2n}, \quad (6.5)$$

$$\|D_{n,\theta}^{g,\beta}(f)\|_p \le g(0)\|f\|_p, \quad f \in \mathscr{F}_{2n}. \quad (6.6)$$

*Inequalities (6.5) and (6.6) turn into equalities, for instance, for polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$.*

2) *If the function $J$ is strictly convex at any point of the interval $(0,+\infty)$, then inequality (6.5) or inequality (6.6) with $p \in (1,\infty)$ turns into an equality at some polynomial $f \in \mathscr{F}_{2n}$ if and only if the*

*functions* $(-1)^s f(t+\pi s/n)$ *are identical on* $\mathbb{R}$ *for all* $s=0,\dots,2n-1$ *such that* $\mu_s(n,g,\beta,\theta)>0$, *where, for* $k\in\mathbb{Z}$,

$$4\mu_k(n,g,\beta,\theta)=(1+\theta)\sum_{m\in\mathbb{Z}}\widehat{g}(-\beta-(k+2nm)\pi)+(1-\theta)\sum_{m\in\mathbb{Z}}\widehat{g}(\beta+(k+2nm)\pi). \qquad (6.7)$$

*If, in addition, for some* $s\in\mathbb{Z}$, *the inequalities* $\mu_s(n,g,\beta,\theta)>0$ *and* $\mu_{s+1}(n,g,\beta,\theta)>0$ *hold, then inequality* (6.5) *or inequality* (6.6) *with* $p\in(1,\infty)$ *turns into an equality only at the polynomials* $f(t)=ce^{int}+\nu e^{-int}$, $c,\nu\in\mathbb{C}$.

3) *When* $p=\infty$, *inequality* (6.6) *turns into an equality at some polynomial* $f\in\mathscr{F}_{2n}$ *if and only if, for some* $\eta,\delta\in\mathbb{R}$, *the equality* $(-1)^s f(\eta+\pi s/n)=e^{i\delta}\|f\|_\infty$ *holds for all* $s=0,\dots,2n-1$ *such that* $\mu_s(n,g,\beta,\theta)>0$.

*If* $\mu_s(n,g,\beta,\theta)>0$ *for* $s=0,\dots,2n-1$, *then only polynomials of the form* $f(t)=ce^{int}+\nu e^{-int}$, $c,\nu\in\mathbb{C}$, *are extremal among trigonometric polynomials of degree at most* $n$ *for which inequality* (6.6) *with* $p=\infty$ *turns into an equality.*

4) *When* $p=1$, *inequality* (6.6) *turns into an equality at some polynomial* $f\in\mathscr{F}_{2n}$ *if and only if, for any* $t\in\mathbb{R}$, *there exists a number* $\delta(t)\in\mathbb{R}$ *such that the identity* $(-1)^s f(t+\pi s/n)\equiv e^{i\delta(t)}|f(t+\pi s/n)|$ *holds for all* $s=0,\dots,2n-1$ *such that* $\mu_s(n,g,\beta,\theta)>0$.

*If a polynomial* $f\in\mathscr{F}_q$, $1\le q<2n$, *is extremal in inequality* (6.6) *with* $p=1$, *then any polynomial of the form* $cf(t)g(t)$, *where* $c\in\mathbb{C}$, $g\in\mathscr{F}_{2n-q}$, *and* $g(t)\ge 0$ *for all* $t\in\mathbb{R}$, *is also extremal. In particular, polynomials of the form* $(ce^{int}+\nu e^{-int})g(t)$, *where* $c,\nu\in\mathbb{C}$ *and* $g$ *is an arbitrary nonnegative trigonometric polynomial of degree at most* $n$, *are extremal in inequality* (6.6) *with* $p=1$.

5) *If* $\mu_s(n,g,\beta,\theta)>0$ *for all* $s=0,\dots,2n-1$ *and the function* $J$ *is strictly increasing on* $(0,+\infty)$, *then only polynomials of the form* $f(t)=ce^{int}+\nu e^{-int}$, $c,\nu\in\mathbb{C}$, *are extremal among trigonometric polynomials of degree at most* $n$ *for which inequality* (6.5) *or inequality* (6.6) *with* $p=1$ *turns into an equality.*

*Remark 6.* If $q\in\mathbb{Z}$ and $q=2nl+r$, where $l,r\in\mathbb{Z}$ and $0\le r\le 2n-1$, then

$$\mu_k(n,g,\beta+\pi q,\theta)=\begin{cases}\mu_{k+r}(n,g,\beta,\theta), & 0\le k\le 2n-1-r,\\ \mu_{k+r-2n}(n,g,\beta,\theta), & 2n-r\le k\le 2n-1,\quad r\ge 1.\end{cases}$$

*Remark 7.* Inequalities (6.1) and (6.2) follow from inequalities (6.5) and (6.6) if, for $g$, we take the function $g_r(x)=(1-|x|)_+^r$ which is positive definite for $r\ge 1$ (the Pólya property). Since $g_r(1-|x|)=|x|^r$ for $|x|\le 1$, we have $D_{n,\theta}^{g_r,\beta}(f)(t)\equiv e^{-i\beta}f^{(r,\beta)}(t)/n^r$ for any polynomial $f\in\mathscr{F}_n$, $n\in\mathbb{N}$. In our case, the values (6.7) are independent of $\theta$ and such that

$$\mu_k(n,g_r,\beta)=\sum_{m\in\mathbb{Z}}\widehat{g_r}(\beta+(k+2nm)\pi))/2,\quad k\in\mathbb{Z}\,.$$

It is well known that, for $r>1$, the Fourier transform $\widehat{g_r}(t)$ is positive for all $t\in\mathbb{R}$ (see, for instance, [27, Lemma 7, $n=\lambda=\delta=1$]). Therefore, $\mu_s(n,g_r,\beta)>0$ for all $r>1$, $\beta\in\mathbb{R}$, $n\in\mathbb{N}$, and $s\in\mathbb{Z}$.

For $r=1$, the Fourier transform of the function $g_1$ is easily calculated and is equal to $\widehat{g_1}(t)=2(1-\cos t)/t^2$. Obviously, $\widehat{g_1}(t)=0$ only for $t=2q\pi$ with $q\in\mathbb{Z}$, $q\ne 0$. Therefore, if $\beta\ne q\pi$, $q\in\mathbb{Z}$, then $\mu_s(n,g_1,\beta)>0$ for all $n\in\mathbb{N}$ and $s\in\mathbb{Z}$.

If $\beta=0$ and $n\in\mathbb{N}$, then: (1) $\mu_s(n,g_1,0)>0$ for $s=0$ and for all odd $s\in[1,2n-1]$; (2) $\mu_s(n,g_1,0)=0$ for all even $s\in[2,2n-1]$ if $n\ge 2$. In this case, the number of positive values among $\mu_s(n, g_1, 0)$, $s = 0, \ldots, 2n-1$, is $n+1$ and the remaining are zero. The latter property also holds for any $\beta = \pi q$ with $q \in \mathbb{Z}$ (see Remark 6).

Thus, only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal in inequalities (6.1) and (6.2) under conditions $(A_1)$ and $(B_1)$ or $(A_2)$ and $(B_2)$, where

$(A_1)$ $r > 1$ and $\beta \in \mathbb{R}$; or $r = 1$, $\beta \in \mathbb{R}$, and $n = 1$; or $r = 1$, $\beta \neq \pi q$, $q \in \mathbb{Z}$, and $n \geq 2$;

$(B_1)$ the function $J$ is strictly increasing on $(0, +\infty)$ for (6.1) or $1 \leq p \leq \infty$ for (6.2);

$(A_2)$ $r = 1$, $\beta = \pi q$ with $q \in \mathbb{Z}$, and $n \geq 2$;

$(B_2)$ the function $J$ is strictly convex on $(0, +\infty)$ for (6.1) or $1 < p < \infty$ for (6.2).

The case where $r = 1$, $\beta = \pi q$ with $q \in \mathbb{Z}$, $n \geq 2$, and $p = 1$ or $p = \infty$ has not been studied.

## 7. Case of piecewise linear functions

In [15], the following R.M. Trigub problem on the positive definiteness of piecewise linear functions was solved. For given $\alpha \in (0,1)$ and $h \in \mathbb{R}$, the function $f_{\alpha,h} : \mathbb{R} \to \mathbb{C}$ is defined as follows: (1) the function $f_{\alpha,h}$ is even; (2) $f_{\alpha,h}(x) = 0$ for $x > 1$, the function $f_{\alpha,h}$ is linear on each of the intervals $[0, \alpha]$ and $[\alpha, 1]$, $f_{\alpha,h}(0) = 1$, $f_{\alpha,h}(\alpha) = h$, and $f_{\alpha,h}(1) = 0$. For any fixed $\alpha \in (0, 1)$, it is required to find the set of all $h \in \mathbb{R}$ such that the piecewise linear function $f_{\alpha,h}$ is positive definite on $\mathbb{R}$. If $0 \leq h \leq 1 - \alpha$, then the continuous even function $f_{\alpha,h}(x)$ is convex on $(0, +\infty)$, $f_{\alpha,h}(+\infty) = 0$, and, hence, it is positive definite by the Pólya theorem (see, for instance, [14, Theorem 4.3.1]). A complete description of such $h \in \mathbb{R}$ is given in the following theorem.

**Theorem 7** [15]. *Let $\alpha \in (0, 1)$ and $h \in \mathbb{R}$. Then $f_{\alpha,h} \in \Phi(\mathbb{R})$ if and only if $m(\alpha) \leq h \leq 1-\alpha$, where $m(\alpha) = 0$ if $1/\alpha \notin \mathbb{N}$ and $m(\alpha) = -\alpha$ if $1/\alpha \in \mathbb{N}$.*

From Theorem 7, we obtain the following sufficient condition for the positive definiteness.

**Corollary 2.** *If a function $g \in C(\mathbb{R})$ is even, nonnegative, decreasing, and convex on $(0, +\infty)$, then, for $\alpha \in (0, 1)$, $1/\alpha \in \mathbb{N}$, and $-\alpha \leq h \leq 1-\alpha$, the function $g_{\alpha,h}(x) := hg(x)+(1-\alpha-h)g(x/\alpha)$ belongs to the class $\Phi(\mathbb{R})$.*

The nontrivial case here is when $-\alpha \leq h < 0$.

P r o o f. The function $g$ is represented in the form (see, for instance, [26])

$$g(x) = \int_0^{+\infty} (1 - |sx|)_+ \, d\mu(s), \quad x \in \mathbb{R},$$

where $\mu$ is a nonnegative finite Borel measure on $[0, +\infty)$. Obviously,

$$g_{\alpha,h}(x) = (1-\alpha) \int_0^{+\infty} f_{\alpha,h}(sx) \, d\mu(s), \quad x \in \mathbb{R}.$$

For the specified $\alpha$ and $h$, we have $f_{\alpha,h} \in \Phi(\mathbb{R})$. Hence, $g_{\alpha,h} \in \Phi(\mathbb{R})$ (see, for instance, [27, Lemma 1]). $\square$

One can use the positive definite function $g_{\alpha,h}$ given in Corollary 2 to obtain new sharp inequalities for trigonometric polynomials.

Let a function $g \in C(\mathbb{R})$ be even, nonnegative, decreasing, and convex on $(0, +\infty)$, and let $\operatorname{supp} g \subset [-1, 1]$. Assume that $n \in \mathbb{N}$, $n \ge 2$, and $-1/n \le h \le 1 - 1/n$. Let $g_{1/n,h}(x) := hg(x) + (1 - 1/n - h)g(nx)$, $x \in \mathbb{R}$. It follows from Corollary 2 that $g_{1/n,h} \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$. Since $\operatorname{supp} g \subset [-1, 1]$, we have $\operatorname{supp} g_{1/n,h} \subset [-1, 1]$. Therefore, for the function $g_{1/n,h}$, we can construct operator (6.4) which does not depend on the parameter $\theta$, since $\operatorname{Im}(g_{1/n,h}) \equiv 0$. It is not difficult to verify that, for polynomials $f \in \mathscr{F}_{2n}$, the following equality holds:

$$D_{n,0}^{g_{1/n,h},\beta}(f)(t) = hD_{n,0}^{g,\beta}(f)(t) + (1 - 1/n - h)g(0)R_n^{\beta}(f)(t), \tag{7.1}$$

where

$$R_n^{\beta}(f)(t) := e^{-i\beta} \sum_{|k|=n} e^{i\beta \operatorname{sign} k} c_k(f)e^{ikt} = \frac{e^{-i\beta}}{\pi} \int_{-\pi}^{\pi} \cos(nx - \beta)f(t + x)\, dx. \tag{7.2}$$

We note that $g_{1/n,h}(0) = (1 - 1/n)g(0)$. In addition, if $g(x) = (1 - |x|)_+^r$, $r \ge 1$, then $D_{n,\theta}^{g_r,\beta}(f)(t) \equiv e^{-i\beta} f^{(r,\beta)}(t)/n^r$ for any polynomial $f \in \mathscr{F}_n$. We write Theorem 6 for the operator (7.1) and restrict ourselves only to inequality (6.6).

**Theorem 8.** *Let a function $g \in C(\mathbb{R})$ be even, nonnegative, decreasing, and convex on $(0, +\infty)$, and let $\operatorname{supp} g \subset [-1, 1]$. Assume that $n \geqslant 2$, $-1/n \le h \le 1 - 1/n$, $\beta \in \mathbb{R}$, and $1 \leqslant p \leqslant \infty$. Then, for any polynomial $f \in \mathscr{F}_{2n}$, we have*

$$\left\| hD_{n,0}^{g,\beta}(f) + (1 - 1/n - h)g(0)R_n^{\beta}(f) \right\|_p \leqslant (1 - 1/n)g(0)\|f\|_p. \tag{7.3}$$

*If $r \ge 1$, then, for any polynomial $f \in \mathscr{F}_n$, we have*

$$\left\| hf^{(r,\beta)}/n^r + (1 - 1/n - h)e^{i\beta}R_n^{\beta}(f) \right\|_p \leqslant (1 - 1/n)\|f\|_p. \tag{7.4}$$

*Inequalities (7.3) and (7.4) turn into equalities for polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$.*

Without proof, we note that if the function $g$ in Theorem 8 is not piecewise linear on $[0, +\infty)$ with equidistant nodes, then only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal in inequality (7.3) with $p \in (1, \infty)$. When $p = 1$ or $p = \infty$, a similar conclusion holds, but for the class of trigonometric polynomials of degree at most $n$. If $r > 1$, then only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal in inequality (7.4).

## 8. Interpolation formulas for periodic functions

If the trigonometric series on the right-hand side of relation (3.2) converges uniformly on $\mathbb{T}$, then one can put the sign of equality in this relation and the obtained equality can be regarded as some interpolation formula. We explain this with the example of the following theorem.

**Theorem 9.** *Assume that $n \in \mathbb{N}$, a 2-periodic function $\psi$ belongs to $\Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\beta \in \mathbb{R}$, and the numbers $\mu_k(n, \psi)$ are defined by formula (5.5). Then the identity*

$$\sum_{k\in\mathbb{Z}} e^{i\beta k/n} \psi\left(\frac{k}{n} - 1\right) c_k(f)e^{ikt} \equiv \sum_{k=0}^{2n-1} (-1)^k f\left(t + \frac{\pi k + \beta}{n}\right) \mu_k(n, \psi) \tag{8.1}$$

*holds for any function $f \in C(\mathbb{T})$ such that the series on the left converges uniformly on $\mathbb{T}$. Moreover, $\mu_0(n, \psi) + \ldots + \mu_{2n-1}(n, \psi) = \psi(0)$, $c_k(\psi) \ge 0$, $k \in \mathbb{Z}$, $\mu_k(n, \psi) \ge 0$, $k = 0, \ldots, 2n - 1$, and $\mu_k(n, \psi) = 0$ for some $k = 0, \ldots, 2n - 1$ if and only if $c_{k+2nm}(\psi) = 0$ for all $m \in \mathbb{Z}$.*

P r o o f. Consider operator (1.1) for the function $\varphi(x) \equiv e^{i\beta x}\psi(x)$. Under the conditions of the theorem, we can put the sign of equality in relation (3.2) for $\varepsilon = 1/n$ and $\tau = 1$. Therefore, the left-hand side of equality (5.6) can be replaced by the sum of the series in (3.2). We obtain identity (8.1) with accuracy up to the factor $e^{-i\beta}$. The specified properties of the numbers $\mu_k(n,\psi)$ follow from (5.5) and (5.7). $\square$

**Corollary 3.** *Assume that $g \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$, $\operatorname{supp} g \subset [-1,1]$, $\beta \in \mathbb{R}$, and $n \in \mathbb{N}$. Then, for any polynomial $f \in \mathscr{F}_{2n}$, the following equality holds*:

$$\begin{aligned}\sum_{|k|\le 2n} \left(\operatorname{Re} g\left(1-\frac{|k|}{n}\right) + i \operatorname{sign} k \operatorname{Im} g\left(1-\frac{|k|}{n}\right)\right) e^{i\beta \operatorname{sign} k} c_k(f) e^{ikt} \\ = \sum_{k=0}^{2n-1} (-1)^k f\left(t + \frac{\pi k + \beta}{n}\right) \mu_k(n,g,\beta), \quad t \in \mathbb{R},\end{aligned} \tag{8.2}$$

*where* $\mu_k(n,g,\beta) = \sum\limits_{m\in\mathbb{Z}} \widehat{g}(-\beta - (k+2nm)\pi)/2$, $k \in \mathbb{Z}$, *and* $\sum\limits_{k=0}^{2n-1} \mu_k(n,g,\beta) = g(0)$.

P r o o f. Let $\psi$ be a 2-periodic function, and let $\psi(x) = g(-x)e^{-i\beta x}$ for $x \in [-1,1]$. Then $\psi \in \Phi(\mathbb{R}) \cap C(\mathbb{R})$ and

$$\psi(x-1) = e^{-i\beta x} e^{i\beta \operatorname{sign} x} (\operatorname{Re} g(1-|x|) + i \operatorname{sign} x \operatorname{Im} g(1-|x|)), \quad |x| \le 2.$$

It remains to take into account that $c_k(\psi) = \widehat{g}(-\beta - k\pi)/2$, $k \in \mathbb{Z}$. $\square$

*Remark 8.* We note that if, for $g$, we take the function $g_r(x) = (1-|x|)_+^r$, $r \ge 1$, then, in (8.2), we obtain the interpolation formula of A.I. Kozko [11] (and of M. Riesz and of G. Szegő for $r=1$) for the Weyl–Nagy derivative:

$$f^{(r,\beta)}(t) = n^r \sum_{k=0}^{2n-1} (-1)^k f\left(t + \frac{\pi k+\beta}{n}\right) \mu_k(n,g_r,\beta), \quad t \in \mathbb{R}, \quad f \in \mathscr{F}_n; \quad \sum_{k=0}^{2n-1} \mu_k(n,g_r,\beta) = 1,$$

where $\mu_k(n,g_r,\beta) > 0$ for all $n \in \mathbb{N}$, $k = 0,\dots,2n-1$, $\beta \in \mathbb{R}$, and $r > 1$. These coefficients are also positive for $r = 1$ if $n = 1$ and $\beta \in \mathbb{R}$ or if $n \ge 2$ and $\beta \ne q\pi$, $q \in \mathbb{Z}$. If $r = 1$, $n \ge 2$, and $\beta = \pi q$ with $q \in \mathbb{Z}$, then, the number of positive coefficients among $\mu_k(n,g_1,\beta)$, $k = 0,\dots,2n-1$, is $n+1$ and the remaining are zero (see Remark 7). For $r = 1$, these coefficients are easily calculated. Since $\widehat{g_1}(t) = 2(1-\cos t)/t^2$, we have

$$\mu_k(n,g_1,\beta) = \frac{1-(-1)^k \cos\beta}{4n^2} \sum_{m\in\mathbb{Z}} \frac{1}{\left(\dfrac{\beta+k\pi}{2n} + m\pi\right)^2} = \frac{1-(-1)^k\cos\beta}{2n^2\left(1-\cos\dfrac{\beta+k\pi}{n}\right)} > 0, \quad \beta \ne q\pi, \quad q \in \mathbb{Z},$$

For $\beta = q\pi$ with $q \in \mathbb{Z}$, we can restrict ourselves to the case $\beta = 0$ (see Remark 6): $\mu_{2k}(n,g_1,0) = 0$ for $k = 1,\dots,n-1$ (if $n \ge 2$), $\mu_0(n,g_1,0) = 1/2$, and

$$\mu_{2k-1}(n,g_1,0) = \frac{1}{n^2\left(1-\cos\dfrac{(2k-1)\pi}{n}\right)} > 0, \quad k = 1,\dots,n.$$

*Remark 9.* It is not difficult to see that all the arguments in the proof of Theorem 9 remain in force also in the case where the 2-periodic continuous function $\psi$ is expanded in an absolutely convergent Fourier series (without the assumption of nonnegativity of the Fourier coefficients $c_k(\psi)$). Therefore, the following statement holds: *Assume that a* 2*-periodic function* $\psi \in C(\mathbb{R})$ *is expanded into an absolutely convergent Fourier series and* $\beta \in \mathbb{R}$*. Then equality* (8.1) *holds for any function* $f \in C(\mathbb{T})$ *such that the series on the left in* (8.1) *converges uniformly on* $\mathbb{T}$*.*

## 9. Conclusion

In conclusion, we point out some problems which, in our opinion, have not been solved yet.

1) To prove or disprove that only polynomials of the form $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$, are extremal in the Berstein–Szegő inequality (6.2) for $r = 1$ and $\beta = 0$ (the case of the derivative of the adjoint polynomial) when $p = \infty$ or $p = 1$. When $p = \infty$, this case was distinguished in the paper by Szegő [20, p. 66]. We note that the arguments in the monographs by Zygmund [28, Ch. X, Sect. 3, (3.24)] and Akhiezer [1, Sect. 84, p. 189] corresponding to this case are not correct, since some coefficients in the interpolation formulas are zero (see [28, Ch. X, Sect. 3, (3.22)] for $\alpha = \pi/2$ and [1, Sect. 84, p. 188, (II)] for $\alpha = 0$).

2) Let $n \in \mathbb{N}$, and let, for a trigonometric polynomial $f \in \mathscr{F}_n$, condition (5.9) or (5.10) be satisfied for all integers $s = 0, \ldots, 2n-1$. Then $f(t) = ce^{int} + \nu e^{-int}$, $c, \nu \in \mathbb{C}$ (see Remark 4). The question is, which values of $s$ can be left to have the same conclusion? This is a more general problem than the previous one.

3) To prove or disprove that if, for some $s \in \mathbb{Z}$, inequalities $\mu_s(n, \psi) > 0$ and $\mu_{s+1}(n, \psi) > 0$ hold and a function $f \in C(\mathbb{T})$ is extremal in inequality (5.8) with $p = 1$, then $f(t) = h(t)g(t)$, where the function $h$ belongs to $L_\infty(\mathbb{T})$ and has the form (5.4), $g \in C(\mathbb{T})$, and $g(t) \geq 0$ for $t \in \mathbb{R}$. This is true if, in addition, $f(t) \neq 0$ for almost all $t \in \mathbb{R}$ with respect to the Lebesgue measure (see Remark 4 for the case $p = 1$).

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# SOME REPRESENTATIONS CONNECTED WITH ULTRAFILTERS AND MAXIMAL LINKED SYSTEMS

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**Abstract:** Ultrafilters and maximal linked systems (MLS) of a lattice of sets are considered. Two following variants of topological equipment are investigated: the Stone and Wallman topologies. These two variants are used both in the case of ultrafilters and for space of MLS. Under Wallman equipment, an analog of superextension is realized. Namely, the space of MLS with topology of the Wallman type is supercompact topological space. By two above-mentioned equipments a bitopological space is realized.

**Key words:** Lattice, Linked system, Ultrafilter.

## Introduction

In connection with the supercompactness property, maximal linked systems (MLS) of closed sets in a topological space (TS) are investigated (see [1–4]; in particular, we note the important statement of [3] about supercompactness of metrizable compactums). The space of «closed» MLS with topology of the Wallman type is a superextension of the initial TS.

Now, following [5], we consider more general approach. Namely, we suppose that a lattice of subsets of arbitrary nonempty set is given. Then, MLS of sets of this lattice are investigated. In particular, the lattice of closed sets in a TS can be used. Then, we obtain the above-mentioned variant of [1–4]. But, many other realizations are possible. For example, we can consider an algebra of sets as variant of the above-mentioned lattice. Note by the way, that in this case the Stone topology on the ultrafilter space is very natural. Since in many respects, MLS are similar to ultrafilters, the Stone equipment is submitted natural and for space of MLS. So, the idea of emploument of the two types of topologies arises: we keep in mind the Wallman and Stone variants.

We recall that ultrafilters were used as generalized elements in problems connected with attainability under constraints of asymptotic character (see, for example, [6–8]). Now, we seek to explore spaces which are comprehending for ultrafilters. In this article, it is established that the space of MLS is comprehending in this sense. In addition, it is logical to consider two characteristic types of topologies both for ultrafilters and for MLS. And what is more, we obtain two bitopological spaces (as a bitopological space, we consider every set equipped with two comparable topologies; in this connection, we note monograph [9]).

The case when two above-mentioned topologies coincide, we consider as degenerate. In the following, characteristic cases of such degeneracy are established (a variant of non-degenerate realization of bitopological space specified also). We indicate important types of lattices for which above-mentioned constructions are realized sufficiently understandably.

## 1. General notions and designations

We use the standard set-theoretical symbolics (quatifiers, connectives and so on); $\emptyset$ is anempty set and $\triangleq$ is the equality by definition. We call a set by a family in the case when every element of this set is a set also. We take the axiom of choice.

For every objects $x$ and $y$, we denote by $\{x;y\}$ the set containing $x$ and $y$ and not containing no other elements. If $h$ is an object, then we suppose that $\{h\} \triangleq \{h;h\}$. Of course, sets are objects. Therefore, by [10, ch. II, §3], for every objects $u$ and $v$, we suppose that $(u,v) \triangleq \big\{\{u\};\{u;v\}\big\}$ receiving the ordered pair with first element $u$ and second element $v$. If $z$ is an arbitrary ordered pair, then by $\mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$ we denote the first and second elements of $z$ respectively; of course, $z = \big(\mathrm{pr}_1(z), \mathrm{pr}_2(z)\big)$ and $\mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$ are defined uniquely.

If $X$ is a set, then by $\mathcal{P}(X)$ we denote the family of all subsets of $X$ and suppose that $\mathrm{Fin}(X)$ is the family of all finite nonempty subsets of $X$; of course. $\mathrm{Fin}(X) \subset \mathcal{P}'(X)$, where $\mathcal{P}'(X) \triangleq \mathcal{P}(X)\backslash\{\emptyset\}$ is the family of all nonempty subsets of $X$. In addition, a family can be used as $X$. For every nonempty family $\mathfrak{X}$, we suppose that

$$\begin{gathered}\{\cup\}(\mathfrak{X}) \triangleq \big\{\bigcup_{X\in\mathcal{X}} X : \mathcal{X} \in \mathcal{P}(\mathfrak{X})\big\}, \quad \{\cap\}(\mathfrak{X}) \triangleq \big\{\bigcap_{X\in\mathcal{X}} X : \mathcal{X} \in \mathcal{P}'(\mathfrak{X})\big\}, \\ \{\cup\}_\sharp(\mathfrak{X}) \triangleq \big\{\bigcup_{X\in\mathcal{K}} X : \mathcal{K} \in \mathrm{Fin}(\mathfrak{X})\big\}, \quad \{\cap\}_\sharp(\mathfrak{X}) \triangleq \big\{\bigcap_{X\in\mathcal{K}} X : \mathcal{K} \in \mathrm{Fin}(\mathfrak{X})\big\};\end{gathered} \tag{1.1}$$

of course, every family of (1.1) is contained in $\mathcal{P}\big(\bigcup_{X\in\mathfrak{X}} X\big)$ and contains $\mathfrak{X}$. For any set $\mathbb{M}$ and $\mathcal{M} \in \mathcal{P}'\big(\mathcal{P}(\mathbb{M})\big)$,

$$\mathbf{C}_{\mathbb{M}}[\mathcal{M}] \triangleq \{\mathbb{M} \setminus M : M \in \mathcal{M}\} \in \mathcal{P}'\big(\mathcal{P}(\mathbb{M})\big). \tag{1.2}$$

In addition (see (1.2)), for any set $S$ and a family $\mathcal{S} \in \mathcal{P}'\big(\mathcal{P}(S)\big)$, the equality $\mathcal{S} = \mathbf{C}_S\big[\mathbf{C}_S[\mathcal{S}]\big]$ is realized. If $\mathcal{A}$ is a nonempty family and $B$ is a set, then

$$\mathcal{A}|_B \triangleq \{A \cap B : A \in \mathcal{A}\} \in \mathcal{P}'\big(\mathcal{P}(B)\big) \tag{1.3}$$

is trace of $\mathcal{A}$ on the set $B$. Usually, in (1.3), the variant $\mathcal{A} \in \mathcal{P}'\big(\mathcal{P}(\mathbb{A})\big)$ and $B \in \mathcal{P}(\mathbb{A})$, where $\mathbb{A}$ is a set, is considered.

For any sets $A$ and $B$, by $B^A$ the set of all mappings from $A$ into $B$ is denoted. Under $f \in B^A$ and $a \in A$, by $f(a), f(a) \in B$, the value of $f$ at the point $a$ is denoted. For $f \in B^A$ and $C \in \mathcal{P}(A)$, we suppose that $f^1(C) \triangleq \{f(x) : x \in C\}$; of course, $f^1(C) \subset B$ and

$$(C \neq \emptyset) \Rightarrow \big(f^1(C) \neq \emptyset\big).$$

**Special families.** In given item, we fix a set $I$ (the case $I = \emptyset$ is not excluded). In the form of

$$\pi[I] \triangleq \{\mathcal{I} \in \mathcal{P}'\big(\mathcal{P}(I)\big) \,|\, (\emptyset \in \mathcal{I})\&(I \in \mathcal{I})\&(A \cap B \in \mathcal{I} \ \ \forall A \in \mathcal{I} \ \ \forall B \in \mathcal{I})\}, \tag{1.4}$$

we have the family of all $\pi$-systems of subsets of $I$ with «zero» and «unit». In terms of

$$(\mathrm{LAT})[I] \triangleq \big\{\mathcal{L} \in \mathcal{P}'\big(\mathcal{P}(I)\big) \,|\, (\emptyset \in \mathcal{L})\&\big(\forall A \in \mathcal{L} \ \ \forall B \in \mathcal{L} \ \ (A \cup B \in \mathcal{L})\&(A \cap B \in \mathcal{L})\big)\big\} \tag{1.5}$$

(the family of all lattices of subsets of $I$), we define (see (1.4)) the basic family

$$(\mathrm{LAT})_0[I] \triangleq \{\mathcal{I} \in (\mathrm{LAT})[I] \,|\, I \in \mathcal{I}\} = \{\mathcal{I} \in \pi[I] \,|\, A \cup B \in \mathcal{I} \ \ \forall A \in \mathcal{I} \ \ \forall B \in \mathcal{I}\} \tag{1.6}$$

of all lattices of subsets of $I$ with «zero» and «unit». In addition, by

$$(\mathrm{alg})[I] \triangleq \{\mathcal{A} \in \pi[I] | \, I \setminus A \in \mathcal{A} \ \ \forall A \in \mathcal{A}\} \tag{1.7}$$

the family of all algebras of subsets of $I$ is defined. Moreover, by

$$(\mathrm{top})[I] \triangleq \{\tau \in \pi[I] | \bigcup_{G \in \mathcal{G}} G \in \tau \ \ \forall \mathcal{G} \in \mathcal{P}'(\tau)\} = \{\tau \in (\mathrm{LAT})_0[I] | \bigcup_{G \in \mathcal{G}} G \in \tau \ \ \forall \mathcal{G} \in \mathcal{P}'(\tau)\} \tag{1.8}$$

and (analogously)

$$\begin{gathered}(\mathrm{clos})[I] \triangleq \big\{\mathcal{F} \in \mathcal{P}'\big(\mathcal{P}(I)\big) | \, (\emptyset \in \mathcal{F})\&(I \in \mathcal{F})\&(A \cup B \in \mathcal{F} \ \ \forall A \in \mathcal{F} \ \ \forall B \in \mathcal{F})\& \\ \big(\bigcap_{F \in \mathcal{F}'} F \in \mathcal{F} \ \ \forall \mathcal{F}' \in \mathcal{P}'(\mathcal{F})\big)\big\} = \big\{\mathcal{F} \in (\mathrm{LAT})_0[I] | \bigcap_{F \in \mathcal{F}'} F \in \mathcal{F} \ \ \forall \mathcal{F}' \in \mathcal{P}'(\mathcal{F})\big\}\end{gathered} \tag{1.9}$$

we define the families of all open and closed [11] topologies on $I$ respectively. So, by (1.7)– (1.9) we obtain many useful examples of lattices of the family (1.6). Yet one particular case of a lattice of subsets of $I$ is connected with $\sigma$-topologies of A.D. Alexandroff [12]:

$$(\sigma - \mathrm{top})[I] \triangleq \{\tau \in \pi[I] | \bigcup_{k \in \mathbb{N}} G_k \in \tau \ \ \forall (G_k)_{k \in \mathbb{N}} \in \tau^{\mathbb{N}}\} \subset (\mathrm{LAT})_0[I],$$

where as usually $\mathbb{N} \triangleq \{1; 2; \ldots\}$. Of course, under $\mathcal{A} \in (\mathrm{alg})[I]$, in the form of $(I, \mathcal{A})$, we obtain a measurable space with algebra of sets. If $\tau \in (\mathrm{top})[I]$, then $(I, \tau)$ is a topological space (TS). In addition, we use the notions $T_1$- and $T_2$-space (see [13, Ch.1]). Moreover, we use compactness [13, Ch.3] and other notions relating to general topology; see [13]. In particular, under $\tau \in (\mathrm{top})[I]$, by $(\tau-\mathrm{comp})[I]$ the family of all compact in $(I, \tau)$ subsets of $I$ is denoted; $(\tau-\mathrm{comp})[I] \in \mathcal{P}'\big(\mathcal{P}(I)\big)$. We note the obvious property

$$\mathcal{L} \cup \{I\} \in (\mathrm{LAT})_0[I] \quad \forall \mathcal{L} \in (\mathrm{LAT})[I]. \tag{1.10}$$

Of course, in (1.10) we have an insignificant transformation of initial lattice.

Let

$$\widetilde{\pi}^0[I] \triangleq \{\mathcal{L} \in \pi[I] | \, \forall L \in \mathcal{L} \ \ \forall x \in I \setminus L \ \ \exists \Lambda \in \mathcal{L} : \, (x \in \Lambda)\&(\Lambda \cap L = \emptyset)\}.$$

Moreover, let

$$(\mathrm{Cen})[\mathcal{L}] \triangleq \{\mathcal{Z} \in \mathcal{P}'(\mathcal{L}) | \bigcap_{Z \in \mathcal{K}} Z \neq \emptyset \ \ \forall \mathcal{K} \in \mathrm{Fin}(\mathcal{Z})\} \quad \forall \mathcal{L} \in \pi[I].$$

**Bases and subbases.** For brevity of desingnations, until end of this section, we fix a nonempty set $X$ and use (1.1). Then,

$$\begin{gathered}(\mathrm{BAS})[X] \triangleq \big\{\mathcal{B} \in \mathcal{P}'\big(\mathcal{P}(X)\big) | \, \big(X = \bigcup_{B \in \mathcal{B}} B\big)\&\big(\forall B_1 \in \mathcal{B} \ \ \forall B_2 \in \mathcal{B} \\ \forall x \in B_1 \cap B_2 \ \exists B_3 \in \mathcal{B} : \, (x \in B_3)\&(B_3 \subset B_1 \cap B_2)\big)\big\}\end{gathered} \tag{1.11}$$

is the family of all open bases of topologies on $X$. Under $\mathcal{B} \in (\mathrm{BAS})[X]$, we obtain that $\{\cup\}(\mathcal{B}) \in (\mathrm{top})[X]$. Then, for $\tau \in (\mathrm{top})[X]$

$$(\tau - \mathrm{BAS})_0[X] \triangleq \{\mathcal{B} \in (\mathrm{BAS})[X] | \, \tau = \{\cup\}(\mathcal{B})\}$$

is the family of all bases of TS $(X, \tau)$. In addition,

$$(\mathrm{p}-\mathrm{BAS})[X] \triangleq \{\mathfrak{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big) \mid \{\cap\}_\sharp(\mathfrak{X}) \in (\mathrm{BAS})[X]\} = \big\{\mathfrak{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big) \mid X = \bigcup_{\mathbb{X} \in \mathfrak{X}} \mathbb{X}\big\}$$

is the family of all open subbases of topologies on $X$. For any $\mathfrak{X} \in (\mathrm{p}-\mathrm{BAS})[X]$, we obtain that $\{\cup\}\big(\{\cap\}_\sharp(\mathfrak{X})\big) \in (\mathrm{top})[X]$. Finally, under $\tau \in (\mathrm{top})[X]$, we suppose that

$$(\mathrm{p}-\mathrm{BAS})_0[X;\tau] \triangleq \{\mathfrak{X} \in (\mathrm{p}-\mathrm{BAS})[X] \mid \{\cap\}_\sharp(\mathfrak{X}) \in (\tau-\mathrm{BAS})_0[X]\};$$

so, we obtain the family of all open subbases of TS $(X, \tau)$. It is useful to introduce one auxiliary construction of [6]:

$$(\mathrm{op}-\mathrm{BAS})_\emptyset[X] \triangleq \{\mathcal{B} \in (\mathrm{BAS})[X] \mid \emptyset \in \mathcal{B}\};$$

moreover, it is logical to consider the following family:

$$(\mathrm{p}-\mathrm{BAS})_\emptyset[X] \triangleq \{\mathfrak{X} \in (\mathrm{p}-\mathrm{BAS})[X] \mid \{\cap\}_\sharp(\mathfrak{X}) \in (\mathrm{op}-\mathrm{BAS})_\emptyset[X]\}.$$

If $\tau \in (\mathrm{top})[X]$, then we suppose that

$$(\mathrm{p}-\mathrm{BAS})^0_\emptyset[X;\tau] \triangleq \{\mathcal{X} \in (\mathrm{p}-\mathrm{BAS})_0[X;\tau] \mid \emptyset \in \{\cap\}_\sharp(\mathcal{X})\},$$

$$(\mathrm{p}-\mathrm{BAS})^0_\emptyset[X;\tau] \subset (\mathrm{p}-\mathrm{BAS})_\emptyset[X].$$

Of course, under $\mathcal{B} \in (\mathrm{BAS})[X]$, we obtain that $\mathcal{B} \cup \{\emptyset\} \in (\mathrm{op}-\mathrm{BAS})_\emptyset[X]$ and $\{\cup\}(\mathcal{B} \cup \{\emptyset\}) = \{\cup\}(\mathcal{B})$. So, we were introduce an unessential transformation of open bases; the goal of such transformation was indicated in [6, §1].

Now, we consider closed bases and subbases. Let

$$(\mathrm{cl}-\mathrm{BAS})[X] \triangleq \big\{\mathcal{B} \in \mathcal{P}'\big(\mathcal{P}(X)\big) \mid (X \in \mathcal{B})\&(\bigcap_{B \in \mathcal{B}} B = \emptyset)\&$$

$$\&\big(\forall B_1 \in \mathcal{B} \;\; \forall B_2 \in \mathcal{B} \;\; \forall x \in X \setminus (B_1 \cup B_2) \;\; \exists B_3 \in \mathcal{B} : \, (B_1 \cup B_2 \subset B_3)\&(x \notin B_3)\big)\big\};$$

so, we introduce the family of all closed bases of topologies on $X$. Of course, $\{\cap\}(\mathfrak{B}) \in (\mathrm{clos})[X]$ for $\mathfrak{B} \in (\mathrm{cl}-\mathrm{BAS})[X]$. Under $\tau \in (\mathrm{top})[X]$, we suppose that

$$(\mathrm{cl}-\mathrm{BAS})_0[X;\tau] \triangleq \{\mathcal{B} \in (\mathrm{cl}-\mathrm{BAS})[X] \mid \mathbf{C}_X[\tau] = \{\cap\}(\mathcal{B})\};$$

then, the family of all closed bases of TS $(X, \tau)$ is defined. Now, we introduce the family of all closed subbases of topologies on $X$ :

$$(\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}[X] \triangleq \{\mathcal{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big) \mid \{\cup\}_\sharp(\mathcal{X}) \in (\mathrm{cl}-\mathrm{BAS})[X]\}.$$

Respectively, in the form

$$(\mathrm{p}-\mathrm{BAS})^0_{\mathrm{cl}}[X;\tau] \triangleq \{\mathcal{X} \in (\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}[X] \mid \{\cup\}_\sharp(\mathcal{X}) \in (\mathrm{cl}-\mathrm{BAS})_0[X;\tau]\},$$

we obtain the family of all closed subbases of TS $(X, \tau)$. In addition,

$$\Big(\{\cup\}\big(\{\cap\}_\sharp(\mathfrak{S})\big) \in (\mathrm{top})[X] \;\; \forall \mathfrak{S} \in (\mathrm{p}-\mathrm{BAS})[X]\Big)\&$$
$$\Big(\{\cap\}\big(\{\cup\}_\sharp(\mathcal{S})\big) \in (\mathrm{clos})[X] \;\; \forall \mathcal{S} \in (\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}[X]\Big).$$

Recall following useful duality relations:

$$\big(\mathbf{C}_X[\mathcal{B}] \in (\mathrm{op}-\mathrm{BAS})_\emptyset[X] \ \ \forall\,\mathcal{B} \in (\mathrm{cl}-\mathrm{BAS})[X]\big)\&\big(\mathbf{C}_X[\mathfrak{B}] \in (\mathrm{cl}-\mathrm{BAS})[X] \ \ \forall\,\mathfrak{B} \in (\mathrm{op}-\mathrm{BAS})_\emptyset[X]\big). \tag{1.12}$$

We note also [6, (1.20)] and some simple corollaries of [6, (1.17)]:

$$\big(\mathbf{C}_X[\{\cap\}(\mathcal{B})] = \{\cup\}(\mathbf{C}_X[\mathcal{B}]) \in (\mathrm{top})[X] \ \ \forall\,\mathcal{B} \in (\mathrm{cl}-\mathrm{BAS})[X]\big)\&\big(\{\cap\}(\mathbf{C}_X[\mathfrak{B}]) = \mathbf{C}_X[\{\cup\}(\mathfrak{B})] \in (\mathrm{clos})[X] \ \ \forall\,\mathfrak{B} \in (\mathrm{op}-\mathrm{BAS})_\emptyset[X]\big). \tag{1.13}$$

In connection with (1.12) and (1.13), it is useful to note that under $\beta \in (\mathrm{BAS})[X]$

$$\beta \cup \{\emptyset\} \in (\mathrm{op}-\mathrm{BAS})_\emptyset[X] : \ \{\cup\}(\beta) = \{\cup\}(\beta \cup \{\emptyset\}).$$

Now, consider some analogs concerning to subbases. In particular,

$$(\mathrm{p}-\mathrm{BAS})_\emptyset[X] = \{\mathfrak{X} \in (\mathrm{p}-\mathrm{BAS})[X] \,|\, \emptyset \in \{\cap\}_\sharp(\mathfrak{X})\}. \tag{1.14}$$

In terms of (1.14), we obtain the next analog of (1.12):

$$\big(\mathbf{C}_X[\mathfrak{X}] \in (\mathrm{p}-\mathrm{BAS})_\emptyset[X] \ \ \forall\,\mathfrak{X} \in (\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}[X]\big)\& \big(\mathbf{C}_X[\mathcal{X}] \in (\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}[X] \ \ \forall\,\mathcal{X} \in (\mathrm{p}-\mathrm{BAS})_\emptyset[X]\big). \tag{1.15}$$

As a corollary, from (1.15), it follows that $\forall\,\tau \in (\mathrm{top})[X]$

$$\big(\mathbf{C}_X[\mathfrak{X}] \in (\mathrm{p}-\mathrm{BAS})^0_\emptyset[X;\tau] \ \ \forall\,\mathfrak{X} \in (\mathrm{p}-\mathrm{BAS})^0_{\mathrm{cl}}[X;\tau]\big)\& \big(\mathbf{C}_X[\mathcal{X}] \in (\mathrm{p}-\mathrm{BAS})^0_{\mathrm{cl}}[X;\tau] \ \ \forall\,\mathcal{X} \in (\mathrm{p}-\mathrm{BAS})^0_\emptyset[X;\tau]\big). \tag{1.16}$$

**A special family of lattices.** By (1.10) we can consider lattices from $(\mathrm{LAT})[X]$. Now, we introduce the family

$$(\downarrow-\mathrm{LAT})^0[X] \triangleq \big\{\mathcal{L} \in (\mathrm{LAT})[X] \,|\, (X \notin \mathcal{L})\&(\{x\} \in \mathcal{L} \ \ \forall\,x \in X)\&\big(\bigcap_{L \in \mathcal{L}'} L \in \mathcal{L} \ \ \forall\,\mathcal{L}' \in \mathcal{P}'(\mathcal{L})\big)\big\}. \tag{1.17}$$

It is possible to consider elements of (1.17) as lattices of «small» subsets of $X$. It is obvious that

$$\mathcal{L} \cup \{X\} \in (\mathrm{clos})[X] \quad \forall\,\mathcal{L} \in (\downarrow-\mathrm{LAT})^0[X]. \tag{1.18}$$

The relation (1.18) assumes an amplification. For this, we introduce

$$\big((\mathcal{D}-\mathrm{top})[X] \triangleq \{\tau \in (\mathrm{top})[X] \,|\, \{x\} \in \mathbf{C}_X[\tau] \ \ \forall\,x \in X\}\big)\& \big((\mathcal{D}-\mathrm{clos})[X] \triangleq \{\mathcal{F} \in (\mathrm{clos})[X] \,|\, \{x\} \in \mathcal{F} \ \ \forall\,x \in X\}\big); \tag{1.19}$$

of course, under $\mathbf{t} \in (\mathcal{D}-\mathrm{top})[X]$, in the form of $(X,\mathbf{t})$, we have a $T_1$-space. In addition, open and closed topologies from (1.19) are situated in the natural duality. From (1.17) and (1.19), we obtain that

$$\mathcal{L} \cup \{X\} \in (\mathcal{D}-\mathrm{clos})[X] \quad \forall\,\mathcal{L} \in (\downarrow-\mathrm{LAT})^0[X]. \tag{1.20}$$

So, under $\mathcal{L} \in (\downarrow-\mathrm{LAT})^0[X]$, we obtain that

$$\tau^0_{\mathcal{L}}[X] \triangleq \mathbf{C}_X[\mathcal{L} \cup \{X\}] = \mathbf{C}_X[\mathcal{L}] \cup \{\emptyset\} \in (\mathcal{D}-\mathrm{top})[X]$$

realizes the initial lattice $\mathcal{L} \cup \{X\}$ as the lattice $\mathbf{C}_X\big[\tau^0_{\mathcal{L}}[X]\big]$ of closed sets in $T_1$-space:

$$\mathcal{L} \cup \{X\} = \mathbf{C}_X\big[\tau^0_{\mathcal{L}}[X]\big]; \tag{1.21}$$

in addition, $(X, \tau^0_{\mathcal{L}}[X])$ is not $T_2$-space and

$$\tau^0_{\mathcal{L}}[X] \neq \mathcal{P}(X). \tag{1.22}$$

Recall that $\mathcal{L} \cup \{X\} \in (\mathrm{LAT})_0[X] \;\; \forall \mathcal{L} \in (\downarrow -\mathrm{LAT})^0[X]$. Now, we consider some examples.

*Example 1.1.* Suppose that $X$ is equipped with a pseudometric

$$\rho : \; X \times X \to [0, \infty[$$

(here, $[0, \infty[ \stackrel{\triangle}{=} \{\xi \in \mathbb{R} | \, 0 \leqslant \xi\}$, where $\mathbb{R}$ is real line); so, $(X, \rho)$ is a pseudometric space, $X \neq \emptyset$. Let

$$B_\rho(X, \varepsilon) \stackrel{\triangle}{=} \{y \in X | \, \rho(x, y) \leqslant \varepsilon\} \;\; \forall x \in X \;\; \forall \varepsilon \in [0, \infty[.$$

We suppose that

$$\mathfrak{B}^\sharp(X, \rho) \stackrel{\triangle}{=} \{H \in \mathcal{P}(X) | \, \exists x \in X \; \exists \varepsilon \in [0, \infty[: \; H \subset B_\rho(x, \varepsilon)\} =$$

$$= \{H \in \mathcal{P}(X) | \, \exists x \in X \; \exists \varepsilon \in ]0, \infty[: \; H \subset B_\rho(x, \varepsilon)\},$$

where $]0, \infty[ \stackrel{\triangle}{=} \{\xi \in \mathbb{R} | \, 0 < \xi\}$. Of course, $\mathfrak{B}^\sharp(X, \rho)$ is the family of $\rho$-bounded subsets of $X$.

We suppose that $X \notin \mathfrak{B}^\sharp(X, \rho)$. So, the pseudometric $\rho$ is unbounded (in particular, real line $\mathbb{R}$ with the metric-modulus can be used as $(X, \rho)$). Then,

$$\mathfrak{B}^\sharp(X, \rho) \in (\downarrow -\mathrm{LAT})^0[X]. \tag{1.23}$$

The proof of (1.23) is obvions (see (1.17)). We note only that $\mathfrak{B}^\sharp(X, \rho) = \{\cap\}\big(\mathfrak{B}^\sharp(X, \rho)\big)$. $\square$

*Example 1.2.* Fix a topology $\tau \in (\mathrm{top})[X]$ for which $(X, \tau)$ is a $T_2$-space (of course, $X \neq \emptyset$). We suppose that

$$X \notin (\tau - \mathrm{comp})[X].$$

So, $T_2$-space $(X, \tau)$ is noncompact. Then

$$(\tau - \mathrm{comp})[X] \in (\downarrow -\mathrm{LAT})^0[X]. \tag{1.24}$$

We consider the scheme of the proof of (1.24). In addition, we recall some known properties. So, at first, we show that

$$(\tau - \mathrm{comp})[X] \in (\mathrm{LAT})[X] \tag{1.25}$$

(we check this understandable property). We recall that $\emptyset \in (\tau - \mathrm{comp})[X]$ and

$$\{x\} \in (\tau - \mathrm{comp})[X] \quad \forall x \in X. \tag{1.26}$$

So, $(\tau - \mathrm{comp})[X] \in \mathcal{P}'\big(\mathcal{P}(X)\big)$. Let $\mathbb{A} \in (\tau - \mathrm{comp})[X]$ and $\mathbb{B} \in (\tau - \mathrm{comp})[X]$. Then $\mathbb{A} \cup \mathbb{B} \in (\tau - \mathrm{comp})[X]$ by definition of the compactness property. Consider $\mathbb{A} \cap \mathbb{B}$. By separability of $(X, \tau)$ we have that

$$(\mathbb{A} \in \mathbf{C}_X[\tau]) \& (\mathbb{B} \in \mathbf{C}_X[\tau]); \tag{1.27}$$

as a corollary, $\mathbb{A} \cap \mathbb{B} \in \mathbf{C}_X[\tau]$. If $\theta \triangleq \tau|_{\mathbb{A}}$, then by transitivity of the operation of passage to a subspace of TS we have the equality

$$\tau|_{\mathbb{A}\cap\mathbb{B}} = \theta|_{\mathbb{A}\cap\mathbb{B}}, \tag{1.28}$$

where $(\mathbb{A}, \theta)$ is a compact TS. By (1.27) $\mathbb{A} \cap \mathbb{B} \in \mathbf{C}_{\mathbb{A}}[\theta]$ and, as a corollary,

$$\mathbb{A} \cap \mathbb{B} \in (\theta - \mathrm{comp})[\mathbb{A}].$$

So, $(\mathbb{A} \cap \mathbb{B}, \theta|_{\mathbb{A}\cap\mathbb{B}})$ is a compact TS. Using (1.28), we obtain that $(\mathbb{A} \cap \mathbb{B}, \tau|_{\mathbb{A}\cap\mathbb{B}})$ is a compact TS. Therefore, $\mathbb{A} \cap \mathbb{B} \in (\tau - \mathrm{comp})[X]$. Since the choice of $\mathbb{A}$ and $\mathbb{B}$ was arbitrary, it is established that $\forall\, A \in (\tau - \mathrm{comp})[X] \;\; \forall\, B \in (\tau - \mathrm{comp})[X]$

$$\big(A \cup B \in (\tau - \mathrm{comp})[X]\big) \& \big(A \cap B \in (\tau - \mathrm{comp})[X]\big). \tag{1.29}$$

So, by (1.5) and (1.29) we obtain that $(\tau - \mathrm{comp})[X] \in (\mathrm{LAT})[X]$. We recall (1.26). Finally, let $\mathcal{T} \in \mathcal{P}'\big((\tau - \mathrm{comp})[X]\big)$. Then, in particular, $\mathcal{T} \in \mathcal{P}'\big(\mathcal{P}(X)\big)$ and we have the set

$$\mathbb{T} \triangleq \bigcap_{T\in\mathcal{T}} T \in \mathcal{P}(X). \tag{1.30}$$

By separability of $(X, \tau)$ $\mathcal{T} \subset \mathbf{C}_X[\tau]$ and (see (1.30)) $\mathbb{T} \in \mathbf{C}_X[\tau]$. In addition, $\mathcal{T} \neq \emptyset$. Choose $\mathbf{T} \in \mathcal{T}$; then $\mathbf{T} \in (\tau - \mathrm{comp})[X]$ and $\mathbb{T} \in \mathcal{P}(\mathbf{T})$. We note that $\mathbf{t} \triangleq \tau|_{\mathbf{T}} \in (\mathrm{top})[\mathbf{T}]$ and TS $(\mathbf{T}, \mathbf{t})$ is a compactum. In addition, $\mathbb{T} \in \mathbf{C}_{\mathbf{T}}[\mathbf{t}]$ (indeed, $(\mathbf{T}, \mathbf{t})$ is a closed subspace of $(X, \tau)$). As a corollary, $\mathbb{T} \in (\mathbf{t} - \mathrm{comp})[\mathbf{T}]$; therefore, $\mathbf{t}|_{\mathbb{T}}$ realizes compactum $(\mathbb{T}, \mathbf{t}|_{\mathbb{T}})$. But, by transitivity we obtain that $\mathbf{t}|_{\mathbb{T}} = \tau|_{\mathbb{T}}$. So, $(\mathbb{T}, \tau|_{\mathbb{T}})$ is compactum; as a corollary $\mathbb{T} \in (\tau - \mathrm{comp})[X]$. Since the choice of $\mathcal{T}$ was arbitrary, we establish (see (1.30)) that

$$\bigcap_{K\in\mathcal{C}} K \in (\tau - \mathrm{comp})[X] \quad \forall\, \mathcal{C} \in \mathcal{P}'\big((\tau - \mathrm{comp})[X]\big). \tag{1.31}$$

Therefore (see (1.17), (1.26), (1.29), and (1.31)), we obtain (1.24).

*Example 1.3.* Consider the case of infinite set $X$ and suppose that $(\mathrm{FIN})[X] \triangleq \mathrm{Fin}(X) \cup \{\emptyset\}$ (the family of all finite subsets of $X$.) Of course, in our case

$$X \notin (\mathrm{FIN})[X].$$

We show that $(\mathrm{FIN})[X] \in (\downarrow -\mathrm{LAT})^0[X]$. Indeed, $(\mathrm{FIN})[X] \in (\mathrm{LAT})[X]$ by obvious properties of finite sets. Moreover, $\{x\} \in (\mathrm{FIN})[X] \;\; \forall\, x \in X$. Let $\mathcal{F} \in \mathcal{P}'\big((\mathrm{FIN})[X]\big)$. Then, $\mathcal{F} \neq \emptyset$ and $\mathcal{F} \subset (\mathrm{FIN})[X]$. We choose $\mathbb{F} \in \mathcal{F}$. Then, in particular, $\mathbb{F} \in (\mathrm{FIN})[X]$. Since

$$\mathbf{F} \triangleq \bigcap_{F\in\mathcal{F}} F \subset \mathbb{F},$$

we have the obvious inclusion $\mathbf{F} \in (\mathrm{FIN})[X]$. Since the choice of $\mathcal{F}$ was arbitrary, we obtain that

$$\bigcap_{H\in\mathcal{H}} H \in (\mathrm{FIN})[X] \quad \forall\, \mathcal{H} \in \mathcal{P}'\big((\mathrm{FIN})[X]\big).$$

So, the required property $(\mathrm{FIN})[X] \in (\downarrow -\mathrm{LAT})^0[X]$ is established. Now, we note that

$$\tau^0_{(\mathrm{FIN})[X]}[X] = \mathbf{C}_X\big[(\mathrm{FIN})[X]\big] \cup \{\emptyset\} \in (\mathrm{top})[X]$$

is known cofinite topology and

$$(\mathrm{FIN})[X] \cup \{X\} = \mathbf{C}_X\big[\tau^0_{(\mathrm{FIN})[X]}[X]\big]$$

is the family of closed sets in this topology.

*Example 1.4.* Let $X$, $X \neq \emptyset$, be uncountable set. Consider the family $\omega[X]$ of all no more than countable subsets of $X$. In addition, $\omega[X] = (\mathrm{count})[X] \cup \{\emptyset\}$, where $(\mathrm{count})[X] \overset{\triangle}{=} \{f^1(\mathbb{N}) :\ f \in X^{\mathbb{N}}\}$ under $\mathbb{N} = \{1; 2; \ldots\}$ and $\widetilde{f}^1(\mathbb{N}) = \{\widetilde{f}(k) :\ k \in \mathbb{N}\}$ for $\widetilde{f} \in X^{\mathbb{N}}$. Then, $\omega[X] \in (\downarrow -\mathrm{LAT})^0[X]$. The corresponding proof is similar to previous example. □

**Coverings and linked families.** Recall that $X$ is a nonempty set. If $\mathcal{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big)$, then

$$(\mathrm{COV})[X|\,\mathcal{X}] \overset{\triangle}{=} \big\{\mathfrak{X} \in \mathcal{P}'(\mathcal{X})|\, X = \bigcup_{\mathbb{X} \in \mathfrak{X}} \mathbb{X}\big\} \tag{1.32}$$

is the family of all coverings of $X$ by sets from $\mathcal{X}$. Let

$$(\mathrm{link})[X] \overset{\triangle}{=} \{\mathcal{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big)|\, A \cap B \neq \emptyset \ \ \forall A \in \mathcal{X} \ \ \forall B \in \mathcal{X}\}. \tag{1.33}$$

Then, the family of all linked systems of subsets of $X$ is introduced. Moreover, suppose that

$$(\mathrm{link})_0[X] \overset{\triangle}{=} \{\mathcal{E} \in (\mathrm{link})[X]|\, \forall \mathcal{S} \in (\mathrm{link})[X] \ \ (\mathcal{E} \subset \mathcal{S}) \Rightarrow (\mathcal{E} = \mathcal{S})\}. \tag{1.34}$$

We obtain the family of all MLS of subsets of $X$. In the following, we consider MLS containing in a given family. So, under $\mathfrak{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big)$

$$(\mathfrak{X} - \mathrm{link})[X] \overset{\triangle}{=} \{\mathcal{E} \in (\mathrm{link})[X]|\, \mathcal{E} \subset \mathfrak{X}\} \in \mathcal{P}\big((\mathrm{link})[X]\big) \tag{1.35}$$

and by analogy with (1.34)

$$(\mathfrak{X} - \mathrm{link})_0[X] \overset{\triangle}{=} \{\mathcal{E} \in (\mathfrak{X} - \mathrm{link})[X]|\, \forall \widetilde{\mathcal{E}} \in (\mathfrak{X} - \mathrm{link})[X] \ \ (\mathcal{E} \subset \widetilde{\mathcal{E}}) \Rightarrow (\mathcal{E} = \widetilde{\mathcal{E}})\}. \tag{1.36}$$

In (1.36), we obtain the family of all MLS containing in the family $\mathfrak{X}$.

**Proposition 1.** *If* $\mathfrak{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big)$, $\mathcal{E} \in (\mathrm{link})[X]$, *and* $\mathfrak{X} \cap \mathcal{E} \neq \emptyset$, *then*

$$\mathfrak{X} \cap \mathcal{E} \in (\mathfrak{X} - \mathrm{link})[X]. \tag{1.37}$$

P r o o f. Fix $\mathfrak{X}$ and $\mathcal{E}$ with above-mentioned properties. In particular, $\mathfrak{X} \cap \mathcal{E} \in \mathcal{P}'\big(\mathcal{P}(X)\big)$. Let $U \in \mathfrak{X} \cap \mathcal{E}$ and $V \in \mathfrak{X} \cap \mathcal{E}$. Then, in particular, $U \in \mathcal{E}$ and $V \in \mathcal{E}$. By (1.33) we obtain that $U \cap V \neq \emptyset$. Since the choice of $U$ and $V$ was arbitrary, we have the property

$$\mathfrak{X} \cap \mathcal{E} \in \mathcal{P}'\big(\mathcal{P}(X)\big):\ A \cap B \neq \emptyset \ \ \forall A \in \mathfrak{X} \cap \mathcal{E} \ \ \forall B \in \mathfrak{X} \cap \mathcal{E}.$$

By (1.33) $\mathfrak{X} \cap \mathcal{E} \in (\mathrm{link})[X]$. Then (see (1.35)), (1.37) is fulfilled. □

**Supercompactness.** If $\tau \in (\mathrm{top})[X]$, then we suppose that

$$\big((\mathrm{p}, \mathrm{bin}) - \mathrm{cl}\big)[X; \tau] \overset{\triangle}{=} \big\{\mathfrak{X} \in (\mathrm{p} - \mathrm{BAS})^0_{\mathrm{cl}}[X; \tau]|\, \bigcap_{\mathbb{X} \in \mathcal{X}} \mathbb{X} \neq \emptyset \ \ \forall \mathcal{X} \in (\mathfrak{X} - \mathrm{link})[X]\big\} \tag{1.38}$$

((1.38) is the family of all closed binary subbases of TS $(X,\tau)$); it is obvious that $\forall\,\kappa\in(\mathrm{p-BAS})^0_{\mathrm{cl}}[X;\tau]$

$$\Big(\kappa\in\big((\mathrm{p,bin})-\mathrm{cl}\big)[X;\tau]\Big)\Leftrightarrow\big(\forall\,\mathfrak{C}\in(\mathrm{COV})\big[X|\,\mathbf{C}_X[\kappa]\big]\ \exists\,C_1\in\mathfrak{C}\ \exists\,C_2\in\mathfrak{C}:\ X=C_1\cup C_2\big).\quad(1.39)$$

In addition, we suppose that

$$\big((\mathbb{SC})-\mathrm{top}\big)[X]\overset{\triangle}{=}\big\{\tau\in(\mathrm{top})[X]\,|\,\big((\mathrm{p,bin})-\mathrm{cl}\big)[X;\tau]\neq\emptyset\big\};\qquad(1.40)$$

in addition, $\big((\mathbb{SC})-\mathrm{top}\big)[X]$ is the family of all supercompact topologies on $X$. Under $\tau\in\big((\mathbb{SC})-\mathrm{top})[X]$, we obtain supercompact TS $(X,\tau)$; moreover, if $(X,\tau)$ is a $T_2$-space, then $(X,\tau)$ is called supercompactum. Every supercompact TS is compact. Then, under $\tau\in\big((\mathbb{SC})-\mathrm{top})[X]$, in the form of $(X,\tau)$, we obtain (in particular) a compact TS.

## 2. Maximal linked systems and ultrafilters: general properties

In the following, a nonempty set $E$ is fixed. We consider families from $\mathcal{P}'\big(\mathcal{P}(E)\big)$. In addition, we use (1.4)– (1.10).

**Filters and ultrafiltres.** In the following, we fix $\mathcal{L}\in\pi[E]$ (later, with respect to $\mathcal{L}$, additional conditions will overlap). We consider $(E,\mathcal{L})$ as widely understood measurable space. Then,

$$\mathbb{F}^*(\mathcal{L})\overset{\triangle}{=}\{\mathcal{F}\in\mathcal{P}'(\mathcal{L}\backslash\{\emptyset\})|\,(A\cap B\in\mathcal{F}\ \forall\,A\in\mathcal{F}\ \forall\,B\in\mathcal{F})\&\big(\forall\,F\in\mathcal{F}\ \forall\,L\in\mathcal{L}\ (F\subset L)\Rightarrow(L\in\mathcal{F})\big)\}\qquad(2.1)$$

is the family of all filters of $(E,\mathcal{L})$. Maximal filters are called ultrafilters (u/f). Then

$$\begin{gathered}\mathbb{F}^*_0(\mathcal{L})\overset{\triangle}{=}\{\mathcal{U}\in\mathbb{F}^*(\mathcal{L})|\,\forall\,\mathcal{F}\in\mathbb{F}^*(\mathcal{L})\ \ (\mathcal{U}\subset\mathcal{F})\Rightarrow(\mathcal{U}=\mathcal{F})\}=\{\mathcal{U}\in\mathbb{F}^*(\mathcal{L})|\,\forall\,L\in\mathcal{L}\\ (L\cap U\neq\emptyset\ \ \forall\,U\in\mathcal{U})\Rightarrow(L\in\mathcal{U})\}=\{\mathcal{U}\in(\mathrm{Cen})[\mathcal{L}]|\,\forall\,\mathcal{V}\in(\mathrm{Cen})[\mathcal{L}]\ \ (\mathcal{U}\subset\mathcal{V})\Rightarrow(\mathcal{U}=\mathcal{V})\}\end{gathered}\qquad(2.2)$$

is the nonempty family of all u/f of $(E,\mathcal{L})$. If $x\in E$, then

$$(\mathcal{L}-\mathrm{triv})[x]\overset{\triangle}{=}\{L\in\mathcal{L}|\,x\in L\}\in\mathbb{F}^*(\mathcal{L})$$

is trivial (fixed) filter corresponding to the point $x$. It is known [14, (5.9)] that

$$\big((\mathcal{L}-\mathrm{triv})[x]\in\mathbb{F}^*_0(\mathcal{L})\ \ \forall\,x\in E\big)\Leftrightarrow(\mathcal{L}\in\widetilde{\pi}^0[E]).\qquad(2.3)$$

We suppose that $\Phi_{\mathcal{L}}(L)\overset{\triangle}{=}\{\mathcal{U}\in\mathbb{F}^*_0(\mathcal{L})|\,L\in\mathcal{U}\}\ \ \forall\,L\in\mathcal{L}$. Then, how easy check,

$$(\mathbb{UF})[E;\mathcal{L}]\overset{\triangle}{=}\{\Phi_{\mathcal{L}}(L):\ L\in\mathcal{L}\}\in\pi[\mathbb{F}^*_0(\mathcal{L})].\qquad(2.4)$$

From (1.11) and (2.4), the inclusion $(\mathbb{UF})[E;\mathcal{L}]\in(\mathrm{BAS})[\mathbb{F}^*_0(\mathcal{L})]$ follows. In addition, topology

$$\mathbf{T}^*_{\mathcal{L}}[E]\overset{\triangle}{=}\{\cup\}\big((\mathbb{UF})[E;\mathcal{L}]\big)=\{\mathbb{G}\in\mathcal{P}\big(\mathbb{F}^*_0(\mathcal{L})\big)|\,\forall\,\mathcal{U}\in\mathbb{G}\ \exists\,U\in\mathcal{U}:\ \Phi_{\mathcal{L}}(U)\subset\mathbb{G}\}\in(\mathrm{top})[\mathbb{F}^*_0(\mathcal{L})]$$

realizes [14] zero-dimensional $T_2$-space

$$\big(\mathbb{F}^*_0(\mathcal{L}),\mathbf{T}^*_{\mathcal{L}}[E]\big).\qquad(2.5)$$

Everywhere in the future, we suppose that

$$\mathcal{L}\in(\mathrm{LAT})_0[E].\qquad(2.6)$$

By (1.5) and (2.6) we obtain that $\Phi_{\mathcal{L}}(L_1 \cup L_2) \in \mathcal{P}\big(\mathbb{F}_0^*(\mathcal{L})\big)$ is defined under $L_1 \in \mathcal{L}$ and $L_2 \in \mathcal{L}$; in addition [6], $\Phi_{\mathcal{L}}(L_1 \cup L_2) = \Phi_{\mathcal{L}}(L_1) \cup \Phi_{\mathcal{L}}(L_2)$. And what is more, in our case (under (2.6))

$$(\mathbb{UF})[E;\mathcal{L}] \in (\mathrm{LAT})_0[\mathbb{F}_0^*(\mathcal{L})]. \tag{2.7}$$

*Remark 2.1.* From (2.4) and (2.7), the following singularity is noticcable: for $(\mathbb{UF})[E;\mathcal{L}]$, properties of $\mathcal{L}$ are repeated. In this connection, we recall [6, (9.6)]:

$$\big(\mathcal{L} \in (\mathrm{alg})[E]\big) \Longrightarrow \big((\mathbb{UF})[E;\mathcal{L}] \in (\mathrm{alg})[\mathbb{F}_0^*(\mathcal{L})]\big). \qquad \square$$

Returning to general case of (2.6), we note that (see [6, (6.7)])

$$(\mathbb{UF})[E;\mathcal{L}] \in (\mathrm{cl}-\mathrm{BAS})[\mathbb{F}_0^*(\mathcal{L})]; \tag{2.8}$$

(2.8) permit to define yet one topology. Indeed, by (2.8)

$$\{\cap\}\big((\mathbb{UF})[E;\mathcal{L}]\big) \in (\mathrm{clos})[\mathbb{F}_0^*(\mathcal{L})].$$

As a corollary, we obtain that

$$\mathbf{T}_{\mathcal{L}}^0[E] \triangleq \mathbf{C}_{\mathbb{F}_0^*(\mathcal{L})}\big[\{\cap\}\big((\mathbb{UF})[E;\mathcal{L}]\big)\big] \in (\mathrm{top})[\mathbb{F}_0^*(\mathcal{L})]. \tag{2.9}$$

In addition, topology (2.9) converts [6, Section 6] $\mathbb{F}_0^*(\mathcal{L})$ in a compact $T_1$-space

$$\big(\mathbb{F}_0^*(\mathcal{L}), \mathbf{T}_{\mathcal{L}}^0[E]\big). \tag{2.10}$$

We consider (2.5) as analog of Stone space and (2.10) as analog of Wallman space (the space of Wallman extension). In addition (see [15, Proposition 4.1])

$$\mathbf{T}_{\mathcal{L}}^0[E] \subset \mathbf{T}_{\mathcal{L}}^*[E]. \tag{2.11}$$

With regard to (2.11), we consider triplet

$$\big(\mathbb{F}_0^*(\mathcal{L}), \mathbf{T}_{\mathcal{L}}^0[E], \mathbf{T}_{\mathcal{L}}^*[E]\big) \tag{2.12}$$

as a bitopological space (BTS); in this connection, see [9]. We do not discuss inessential differences with constructions of [9] and follow to above-mentioned interpretation of (2.12). So,

$$(\mathbb{UF})[E;\mathcal{L}] \in (\mathrm{BAS})[\mathbb{F}_0^*(\mathcal{L})] \cap (\mathrm{cl}-\mathrm{BAS})[\mathbb{F}_0^*(\mathcal{L})] \tag{2.13}$$

generates BTS (2.12). It is useful to note the important particular case; namely, if $\mathcal{L} \in (\mathrm{alg})[E]$, then (2.5) is a zero-dimensional compactum or rather the Stone space.

**Maximal linked systems.** Now, we consider the families $(\mathcal{L}-\mathrm{link})[E]$ and $(\mathcal{L}-\mathrm{link})_0[E]$. It is obvious that $\mathbb{F}^*(\mathcal{L}) \subset (\mathcal{L}-\mathrm{link})[E]$ and

$$\mathbb{F}_0^*(\mathcal{L}) \subset (\mathcal{L}-\mathrm{link})_0[E]. \tag{2.14}$$

Moreover, easy to check that

$$(\mathcal{L}-\mathrm{link})_0[E] = \{\mathcal{E} \in (\mathcal{L}-\mathrm{link})[E] \,|\, \forall\, L \in \mathcal{L} \ \ (L \cap \Sigma \neq \emptyset \ \ \forall\, \Sigma \in \mathcal{E}) \Longrightarrow (L \in \mathcal{E})\} \tag{2.15}$$

(we use the maximality property). With employment of the Zorn lemma, we obtain that

$$\forall\, \mathcal{E}_1 \in (\mathcal{L}-\mathrm{link})[E] \ \exists\, \mathcal{E}_2 \in (\mathcal{L}-\mathrm{link})_0[E]: \ \mathcal{E}_1 \subset \mathcal{E}_2. \tag{2.16}$$

Finally, we note the following corollary of maximality of MLS: $\forall\, \mathcal{E} \in (\mathcal{L}-\mathrm{link})_0[E] \ \ \forall\, \Sigma \in \mathcal{E} \ \ \forall\, L \in \mathcal{L}$

$$(\Sigma \subset L) \Longrightarrow (L \in \mathcal{E}). \tag{2.17}$$

Therefore, we obtain that

$$E \in \mathcal{E} \ \ \forall\, \mathcal{E} \in (\mathcal{L}-\mathrm{link})_0[E]. \tag{2.18}$$

The property (2.14) is complemented by the following equality:

$$\mathbb{F}_0^*(\mathcal{L}) = \{\mathcal{U} \in (\mathcal{L}-\mathrm{link})_0[E] \,|\, A \cap B \in \mathcal{U} \ \ \forall\, A \in \mathcal{U} \ \ \forall\, B \in \mathcal{U}\} \in \mathcal{P}'\big((\mathcal{L}-\mathrm{link})_0[E]\big).$$

## 3. Maximal linked systems; topology of the Wallman type

We recall that (2.14)–(2.18) are fulfilled under $\mathcal{L} = \mathcal{P}(E)$ (the lattice of all subsets of $E$). In addition, $(\mathrm{link})[E] = \big(\mathcal{P}(E) - \mathrm{link}\big)[E]$ and $(\mathrm{link})_0[E] = \big(\mathcal{P}(E) - \mathrm{link}\big)_0[E]$ (see (1.34)). As variant of (1.36) and (2.15), we obtain that

$$\begin{aligned}(\mathrm{link})_0[E] = \{\mathcal{E} \in (\mathrm{link})[E] | \,\forall\, \mathcal{S} \in (\mathrm{link})[E] \;\; (\mathcal{E} \subset \mathcal{S}) \Rightarrow (\mathcal{E} = \mathcal{S})\} = \\ = \{\mathcal{E} \in (\mathrm{link})[E] | \,\forall\, L \in \mathcal{P}(E) \;\; (L \cap \Sigma \neq \emptyset \;\; \forall\, \Sigma \in \mathcal{E}) \Rightarrow (L \in \mathcal{E})\},\end{aligned} \tag{3.1}$$

$(\mathrm{link})_0[E] \neq \emptyset$. From (2.18), we obtain that

$$E \in \mathcal{E} \;\; \forall\, \mathcal{E} \in (\mathrm{link})_0[E]. \tag{3.2}$$

By (2.16) we obtain that

$$\forall\, \mathcal{E}_1 \in (\mathrm{link})[E] \;\exists\, \mathcal{E}_2 \in (\mathrm{link})_0[E] : \, \mathcal{E}_1 \subset \mathcal{E}_2. \tag{3.3}$$

Now, we return to arbitrary fixed lattice (2.6). Using (3.1), we consider one property of MLS for lattice (2.6). But, at first, we note one simple corollary of Proposition 1.

**Proposition 2.** *The following property takes place:*

$$\mathcal{E} \cap \mathcal{L} \in (\mathcal{L} - \mathrm{link})[E] \quad \forall\, \mathcal{E} \in (\mathrm{link})_0[E]. \tag{3.4}$$

P r o o f. Let $\mathcal{S} \in (\mathrm{link})_0[E]$. Using (2.6), consider the family $\mathcal{S} \cap \mathcal{L}$. By (1.4), (1.6), (2.6) and (3.2) $E \in \mathcal{S} \cap \mathcal{L}$. So, $\mathcal{S} \cap \mathcal{L} \neq \emptyset$ and by Proposition 1 $\mathcal{S} \cap \mathcal{L} \in (\mathcal{L} - \mathrm{link})[E]$. □

**Proposition 3.** *If* $\mathcal{E} \in (\mathcal{L} - \mathrm{link})_0[E]$, *then*

$$\exists\, \mathcal{S} \in (\mathrm{link})_0[E] : \, \mathcal{E} = \mathcal{S} \cap \mathcal{L}.$$

P r o o f. Fix $\mathcal{E} \in (\mathcal{L} - \mathrm{link})_0[E]$. Then, in particular, $\mathcal{E} \in (\mathcal{L} - \mathrm{link})[E]$ and $\forall\, \mathcal{C} \in (\mathcal{L} - \mathrm{link})[E]$

$$(\mathcal{E} \subset \mathcal{C}) \Longrightarrow (\mathcal{E} = \mathcal{C}). \tag{3.5}$$

By (1.35) $\mathcal{E} \in (\mathrm{link})[E]$ and $\mathcal{E} \subset \mathcal{L}$. Then (see (3.3)), for some MLS $\mathcal{V} \in (\mathrm{link})_0[E]$

$$\mathcal{E} \subset \mathcal{V}. \tag{3.6}$$

In addition, by Proposition 2

$$\mathcal{V} \cap \mathcal{L} \in (\mathcal{L} - \mathrm{link})[E]. \tag{3.7}$$

From (3.6), the inclusion $\mathcal{E} \subset \mathcal{V} \cap \mathcal{L}$ is realized. By (3.5) and (3.7) we obtain the equality

$$\mathcal{E} = \mathcal{V} \cap \mathcal{L}.$$

So, $\mathcal{V} \in (\mathrm{link})_0[E] : \, \mathcal{E} = \mathcal{V} \cap \mathcal{L}$. □

We suppose by analogy with [4, 4.10] that

$$(\mathcal{L} - \mathrm{link})^0[E|\, L] \overset{\triangle}{=} \{\mathcal{E} \in (\mathcal{L} - \mathrm{link})_0[E] |\, L \in \mathcal{E}\} \quad \forall\, L \in \mathcal{L}. \tag{3.8}$$

Of course, we have the following particular cases:

$$\big((\mathcal{L} - \mathrm{link})^0[E|\, \emptyset] = \emptyset\big) \& \big((\mathcal{L} - \mathrm{link})^0[E|\, E] = (\mathcal{L} - \mathrm{link})_0[E]\big) \tag{3.9}$$

(here and later, we follow to [5]). Using (3.8) and (3.9), we obtain that

$$\mathfrak{C}_0^*[E;\mathcal{L}] \overset{\triangle}{=} \{(\mathcal{L}-\mathrm{link})^0[E|\,L]:\ L\in\mathcal{L}\}\in\mathcal{P}'\Big(\mathcal{P}\big((\mathcal{L}-\mathrm{link})_0[E]\big)\Big); \tag{3.10}$$

in addition, $\emptyset\in\mathfrak{C}_0^*[E;\mathcal{L}]$ and $(\mathcal{L}-\mathrm{link})_0[E]\in\mathfrak{C}_0^*[E;\mathcal{L}]$. The basic properties of the family (3.10) are considered later. Now, we pass to equipment by topology of the Wallman type. For this, we note that

$$\mathbf{C}_E[\mathcal{L}]=\{E\setminus L:\ L\in\mathcal{L}\}\in(\mathrm{LAT})_0[E]. \tag{3.11}$$

In the form of (3.11), we obtain the lattice dual with respect to $\mathcal{L}$.

*Remark 3.1*. We recall (see [5]) that $\mathcal{L}=\mathbf{C}_E[\mathcal{L}]$ under $\mathcal{L}\in(\mathrm{alg})[E]$. So, for the particular case, when $(E,\mathcal{L})$ is a measurable space with algebra of sets, the dual lattice (3.11) coincides with $\mathcal{L}$. □

Under $\Lambda\in\mathbf{C}_E[\mathcal{L}]$, we suppose that

$$(\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda]\overset{\triangle}{=}\{\mathcal{E}\in(\mathcal{L}-\mathrm{link})_0[E]|\ \exists\,\Sigma\in\mathcal{E}:\ \Sigma\subset\Lambda\}; \tag{3.12}$$

of course, we can consider that $\Lambda=E\setminus L$, where $L\in\mathcal{L}$. In this connection, we note that

$$(\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,E\setminus L]=(\mathcal{L}-\mathrm{link})_0[E]\setminus(\mathcal{L}-\mathrm{link})^0[E|\,L]\quad\forall\,L\in\mathcal{L}. \tag{3.13}$$

Of course, by (3.11) $\emptyset\in\mathbf{C}_E[\mathcal{L}]$ and $E\in\mathbf{C}_E[\mathcal{L}]$; in addition,

$$\big((\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\emptyset]=\emptyset\big)\&\big((\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,E]=(\mathcal{L}-\mathrm{link})_0[E]\big).$$

As a corollary, we obtain that by statements of Section 1

$$\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}]\overset{\triangle}{=}\{(\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda]:\ \Lambda\in\mathbf{C}_E[\mathcal{L}]\}\in(\mathrm{p}-\mathrm{BAS})_\emptyset\big[(\mathcal{L}-\mathrm{link})_0[E]\big]. \tag{3.14}$$

As a corollary, in the form of $\{\cap\}_\sharp(\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}])\in(\mathrm{op}-\mathrm{BAS})_\emptyset\big[(\mathcal{L}-\mathrm{link})_0[E]\big]$, we obtain an open base and

$$\mathbb{T}_0(E|\,\mathcal{L})\overset{\triangle}{=}\{\cup\}\big(\{\cap\}_\sharp(\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}])\big)\in(\mathrm{top})\big[(\mathcal{L}-\mathrm{link})_0[E]\big]. \tag{3.15}$$

So, we have the following TS

$$\big((\mathcal{L}-\mathrm{link})_0[E],\ \mathbb{T}_0(E|\,\mathcal{L})\big). \tag{3.16}$$

Of course, $\{\cap\}_\sharp(\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}])\in\big(\mathbb{T}_0(E|\,\mathcal{L})-\mathrm{BAS}\big)_0\big[(\mathcal{L}-\mathrm{link})_0[E]\big]$ and, as a corollary,

$$\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}]\in(\mathrm{p}-\mathrm{BAS})^0_\emptyset\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]. \tag{3.17}$$

In addition, by (3.13) the following equality is realized:

$$\mathfrak{C}_0^*[E;\mathcal{L}]=\mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}\big[\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}]\big]. \tag{3.18}$$

From (3.17) and (3.18), by duality we obtain (see (1.16)) that

$$\mathfrak{C}_0^*[E;\mathcal{L}]\in(\mathrm{p}-\mathrm{BAS})^0_{\mathrm{cl}}\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]. \tag{3.19}$$

From (3.17) and (3.19), we have dual construction for TS (3.16). In addition, (3.17) and (3.19) are open and closed subbases of this TS respectively. By (3.18) self these subbases are situated in a duality. Now, we note the statements of [5] connected with supercompactness of TS (3.16). At first, we recall the notion of closed binary subbases. Namely, by (1.38)

$$\begin{gathered}\big((\mathrm{p},\mathrm{bin})-\mathrm{cl}\big)\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]=\big\{\mathfrak{L}\in(\mathrm{p}-\mathrm{BAS})^0_{\mathrm{cl}}\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]\big|\\ \bigcap_{\mathbb{L}\in\lambda}\mathbb{L}\neq\emptyset\quad\forall\,\lambda\in(\mathfrak{L}-\mathrm{link})\big[(\mathcal{L}-\mathrm{link})_0[E]\big]\big\}\end{gathered} \tag{3.20}$$

is the family of all closed binary subbases. We recall that by (1.40)

$$\Big(\mathbb{T}_0(E|\,\mathcal{L}) \in \big((\mathbb{SC})-\mathrm{top}\big)\big[(\mathcal{L}-\mathrm{link})_0[E]\big]\Big) \Leftrightarrow \Big(\big((\mathrm{p},\mathrm{bin})-\mathrm{cl}\big)\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big] \neq \emptyset\Big). \quad (3.21)$$

In [5], the following statement was established: $\mathfrak{C}_0^*[E;\mathcal{L}] \in \big((\mathrm{p},\mathrm{bin})-\mathrm{cl}\big)\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]$. From (3.21), we obtain that

$$\mathbb{T}_0(E|\,\mathcal{L}) \in \big((\mathbb{SC})-\mathrm{top}\big)\big[(\mathcal{L}-\mathrm{link})_0[E]\big]. \quad (3.22)$$

So, (3.16) is a supercompact TS. With employment of (1.39), (3.18), and the above-mentioned property of $\mathfrak{C}_0^*[E;\mathcal{L}]$, we have the following statement:

$$\forall\, \mathcal{C} \in (\mathrm{COV})\big[(\mathcal{L}-\mathrm{link})_0[E]|\,\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}]\big]\ \exists\, C_1 \in \mathcal{C}\ \exists\, C_2 \in \mathcal{C}:\ (\mathcal{L}-\mathrm{link})_0[E] = C_1 \cup C_2. \quad (3.23)$$

Of course, for (3.23), we use the property

$$\mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}] = \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}\big[\mathfrak{C}_o^*[E;\mathcal{L}]\big] \quad (3.24)$$

(indeed, (3.24) is obvious corollary of (3.18)). We consider (3.15) as a topology of Wallman type. We note two obvious property. Namely, for $\Lambda_1 \in \mathbf{C}_E[\mathcal{L}]$ and $\Lambda_2 \in \mathbf{C}_E[\mathcal{L}]$

$$(\Lambda_1 \cap \Lambda_2 = \emptyset) \Rightarrow \big((\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda_1] \cap (\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda_2] = \emptyset\big).$$

Moreover, we have the following property of isotonicity: under $\Lambda_1 \in \mathbf{C}_E[\mathcal{L}]$ and $\Lambda_2 \in \mathbf{C}_E[\mathcal{L}]$

$$(\Lambda_1 \subset \Lambda_2) \Rightarrow \big((\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda_1] \subset (\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda_2]\big).$$

Now, we consider the corresponding equipment for the set of u/f of the lattice $\mathcal{L}$. For $\Lambda \in \mathbf{C}_E[\mathcal{L}]$, we obtain that

$$\widetilde{\mathbb{F}}_{\mathbf{C}}[\mathcal{L}|\,\Lambda] \overset{\triangle}{=} (\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,\Lambda] \cap \mathbb{F}_0^*(\mathcal{L}) = \{\mathcal{U} \in \mathbb{F}_0^*(\mathcal{L})|\,\exists\, U \in \mathcal{U}:\ U \subset \Lambda\} \in \mathcal{P}\big(\mathbb{F}_0^*(\mathcal{L})\big). \quad (3.25)$$

Of course, $\widetilde{\mathbb{F}}_{\mathbf{C}}[\mathcal{L}|\,E \setminus L]$ is defined under $L \in \mathcal{L}$. It is obvious that

$$\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}] \overset{\triangle}{=} \{\widetilde{\mathbb{F}}_{\mathbf{C}}[\mathcal{L}|\,\Lambda]:\ \Lambda \in \mathbf{C}_E[\mathcal{L}]\} = \mathbf{C}_{\mathbb{F}_0^*(\mathcal{L})}\big[(\mathbb{UF})[E;\mathcal{L}]\big]. \quad (3.26)$$

In (3.26), the following equality is used: namely, under $L \in \mathcal{L}$, $\widetilde{\mathbb{F}}_{\mathbf{C}}[\mathcal{L}|\,E \setminus L] = \mathbb{F}_0^*(\mathcal{L}) \setminus \Phi_{\mathcal{L}}(L)$. Using simple corollary of (2.8) and (3.26), we obtain that $\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}] \in (\mathrm{BAS})[\mathbb{F}_0^*(\mathcal{L})]$ (see (1.12)). In addition, by (1.13), (2.8), and (3.26)

$$\mathbf{T}^0_{\mathcal{L}}[E] = \{\cup\}(\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}]). \quad (3.27)$$

So, we obtain the following property (see [5]): namely,

$$\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}] \in (\mathbf{T}^0_{\mathcal{L}}[E]-\mathrm{BAS})_0[\mathbb{F}_0^*(\mathcal{L})]. \quad (3.28)$$

In (3.27) and (3.28), we have analog of (3.15) and (3.17) respectively; in addition, it is useful to note that $\emptyset = \widetilde{\mathbb{F}}_{\mathbf{C}}[\mathcal{L}|\,\emptyset] \in \widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}]$ (we use (3.11)) and therefore

$$\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}] \in (\mathrm{op}-\mathrm{BAS})_{\emptyset}[\mathbb{F}_0^*(\mathcal{L})].$$

On the other hand, by (3.14), (3.25) and (3.26)

$$\widetilde{\mathfrak{F}}_{\mathbf{C}}[\mathcal{L}] = \mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}]|_{\mathbb{F}_0^*(\mathcal{L})}. \quad (3.29)$$

From (3.29), we obtain the following statement of [5]: (2.10) is a subspace of TS (3.16). Namely

$$\mathbf{T}^0_{\mathcal{L}}[E] = \mathbb{T}_0(E|\,\mathcal{L})|_{\mathbb{F}_0^*(\mathcal{L})}. \quad (3.30)$$

As a corollary, we obtain the useful property: the set $\mathbb{F}_0^*(\mathcal{L})$ is compact in TS (3.16):

$$\mathbb{F}_0^*(\mathcal{L}) \in \big(\mathbb{T}_0(E|\,\mathcal{L})-\mathrm{comp}\big)\big[(\mathcal{L}-\mathrm{link})_0[E]\big]. \quad (3.31)$$

Now, the following statement is obvious.

**Proposition 4.** *If* (3.16) *is $T_2$-space, then $\mathbb{F}_0^*(\mathcal{L})$ is closed in this space:*

$$\mathbb{F}_0^*(\mathcal{L}) \in \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}[\mathbb{T}_0(E|\,\mathcal{L})].$$

In connection with Proposition 4, we note the known property concerning to [4, 4.16] (see too [16, p. 65]).

We note that, for every $\mathcal{E} \in (\mathcal{L} - \mathrm{link})_0[E]$, the following equality is realized:

$$\bigcap_{\Sigma\in\mathcal{E}} (\mathcal{L} - \mathrm{link})^0[E|\,\Sigma] = \{\mathcal{E}\};$$

as a corollary, by (3.19) we obtain that

$$\{\mathcal{E}\} \in \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}[\mathbb{T}_0(E|\,\mathcal{L})].$$

So, we have the following statement of [5].

**Proposition 5.** *By* (3.16) *a supercompact $T_1$-space is realized.*

Of course, if (3.16) is a $T_2$-space, then it is a supercompactum. We note that by the maximality property $\forall\,\mathcal{E}_1 \in (\mathcal{L} - \mathrm{link})_0[E]\ \ \forall\,\mathcal{E}_2 \in (\mathcal{L} - \mathrm{link})_0[E]$

$$(\mathcal{E}_1 \neq \mathcal{E}_2) \Longleftrightarrow \big((\mathcal{E}_1 \setminus \mathcal{E}_2 \neq \emptyset)\&(\mathcal{E}_2 \setminus \mathcal{E}_1 \neq \emptyset)\big).$$

Moreover, it is obvious that $\forall\,\mathcal{E}_1 \in (\mathcal{L} - \mathrm{link})_0[E]\ \ \forall\,\mathcal{E}_2 \in (\mathcal{L} - \mathrm{link})_0[E]$

$$(\mathcal{E}_1 \neq \mathcal{E}_2) \Longleftrightarrow (\exists\,\Sigma_1 \in \mathcal{E}_1\ \ \exists\,\Sigma_2 \in \mathcal{E}_2 :\ \Sigma_1 \cap \Sigma_2 = \emptyset). \tag{3.32}$$

## 4. Maximal linked systems as elements of zero-dimensional $T_2$-space and bitopological structure

In this section, we introduce TS analogous to (2.5). Elements of this new TS are MLS. We recall that by (1.14), (3.8), (3.10), and (3.9)

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p} - \mathrm{BAS})_\emptyset\big[(\mathcal{L} - \mathrm{link})_0[E]\big]. \tag{4.1}$$

From (4.1), the obvious property $\{\cap\}_\sharp(\mathfrak{C}_0^*[E;\mathcal{L}]) \in (\mathrm{op} - \mathrm{BAS})_\emptyset\big[(\mathcal{L} - \mathrm{link})_0[E]\big]$ follows. As a corollary,

$$\mathbb{T}_*(E|\,\mathcal{L}) \overset{\triangle}{=} \{\cup\}\big(\{\cap\}_\sharp(\mathfrak{C}_0^*[E;\mathcal{L}])\big) \in (\mathrm{top})\big[(\mathcal{L} - \mathrm{link})_0[E]\big]. \tag{4.2}$$

So, by (4.2) we obtain the required TS

$$\big((\mathcal{L} - \mathrm{link})_0[E], \mathbb{T}_*(E|\,\mathcal{L})\big). \tag{4.3}$$

For this TS, by (4.2) we have the inclusion

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p} - \mathrm{BAS})_\emptyset^0\big[(\mathcal{L} - \mathrm{link})_0[E]; \mathbb{T}_*(E|\,\mathcal{L})\big]. \tag{4.4}$$

So, we obtain the following statement

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p} - \mathrm{BAS})_\emptyset^0\big[(\mathcal{L} - \mathrm{link})_0[E]; \mathbb{T}_*(E|\,\mathcal{L})\big] \cap \big((\mathrm{p}, \mathrm{bin}) - \mathrm{cl}\big)\big[(\mathcal{L} - \mathrm{link})_0[E]; \mathbb{T}_0(E|\,\mathcal{L})\big]. \tag{4.5}$$

We obtain some analog of (2.13). So, the family $\mathfrak{C}_0^*[E;\mathcal{L}]$ «serves» both topology $\mathbb{T}_*(E|\,\mathcal{L})$ and topology $\mathbb{T}_0(E|\,\mathcal{L})$. But, now we focus on consideration of TS (4.3).

It is easy proved that $\forall L_1 \in \mathcal{L} \ \ \forall L_2 \in \mathcal{L}$

$$(L_1 \cap L_2 = \emptyset) \Longleftrightarrow \big((\mathcal{L}-\mathrm{link})^0[E|\,L_1] \cap (\mathcal{L}-\mathrm{link})^0[E|\,L_2] = \emptyset\big); \tag{4.6}$$

in (4.5), we use the property analogous to (3.3). We use (4.6) for verification of separability of the TS (4.2). For this, we introduce the next notion: if $\mathcal{E}_1$ and $\mathcal{E}_2$ are nonempty families, then

$$(\mathrm{Dis})[\mathcal{E}_1; \mathcal{E}_2] \overset{\triangle}{=} \{z \in \mathcal{E}_1 \times \mathcal{E}_2 |\, \mathrm{pr}_1(z) \cap \mathrm{pr}_2(z) = \emptyset\}. \tag{4.7}$$

Of course, in (4.7), we can use arbitrary MLS from $(\mathcal{L}-\mathrm{link})_0[E]$ as $\mathcal{E}_1$ and $\mathcal{E}_2$. Then, by (4.6) and (4.7)

$$\begin{gathered}(\mathcal{L}-\mathrm{link})^0[E|\,\mathrm{pr}_1(z)] \cap (\mathcal{L}-\mathrm{link})^0[E|\,\mathrm{pr}_2(z)] = \emptyset \\ \forall \mathcal{E}_1 \in (\mathcal{L}-\mathrm{link})_0[E] \ \ \forall \mathcal{E}_2 \in (\mathcal{L}-\mathrm{link})_0[E] \ \ \forall z \in (\mathrm{Dis})]\mathcal{E}_1; \mathcal{E}_2].\end{gathered} \tag{4.8}$$

If $(X, \tau)$ is TS and $x \in X$, then $N_\tau^0(x) \overset{\triangle}{=} \{G \in \tau |\, x \in G\}$. We confine ourselves to employment of open neighborhoods. Of course, by (3.10) and (4.2) we obtain the following obvious property: if $\mathcal{E} \in (\mathcal{L}-\mathrm{link})_0[E]$ and $\Sigma \in \mathcal{E}$, then

$$(\mathcal{L}-\mathrm{link})^0[E|\,\Sigma] \in N^0_{\mathbb{T}_*(E|\mathcal{L})}(\mathcal{E}). \tag{4.9}$$

We note that by (3.32), for $\mathcal{E}_1 \in (\mathcal{L}-\mathrm{link})_0[E]$ and $\mathcal{E}_2 \in (\mathcal{L}-\mathrm{link})_0[E] \setminus \{\mathcal{E}_1\}$, the property

$$(\mathrm{Dis})[\mathcal{E}_1; \mathcal{E}_2] \neq \emptyset$$

is realized (see (4.7)). As a corollary, by (4.8) and (4.9) $\forall \mathcal{E}_1 \in (\mathcal{L}-\mathrm{link})_0[E]$ $\forall \mathcal{E}_2 \in (\mathcal{L}-\mathrm{link})_0[E] \setminus \{\mathcal{E}_1\}$ $\exists \mathbb{G}_1 \in N^0_{\mathbb{T}_*(E|\mathcal{L})}(\mathcal{E}_1)$ $\exists \mathbb{G}_2 \in N^0_{\mathbb{T}_*(E|\mathcal{L})}(\mathcal{E}_2):$

$$\mathbb{G}_1 \cap \mathbb{G}_2 = \emptyset. \tag{4.10}$$

So, (4.3) is a $T_2$-space. Moreover, we note that by (2.15)

$$(\mathcal{L}-\mathrm{link})^0[E|\,L] = \{\mathcal{E} \in (\mathcal{L}-\mathrm{link})_0[E]|\, L \cap \Sigma \neq \emptyset \ \ \forall \Sigma \in \mathcal{E}\} \quad \forall L \in \mathcal{L}. \tag{4.11}$$

On the other hand, from (4.11) the following property (see [5]) is extracted:

$$(\mathcal{L}-\mathrm{link})^0[E|\,L] \in \mathbb{T}_*(E|\,\mathcal{L}) \cap \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}[\mathbb{T}_*(E|\mathcal{L})] \quad \forall L \in \mathcal{L}. \tag{4.12}$$

From (3.10) and (4.12), we obtain that

$$\mathfrak{C}_0^*[E; \mathcal{L}] \subset \mathbb{T}_*(E|\,\mathcal{L}) \cap \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}[\mathbb{T}_*(E|\,\mathcal{L})]. \tag{4.13}$$

Using axioms of TS, from (4.13), we obtain that

$$\{\cap\}_\sharp(\mathfrak{C}_0^*[E; \mathcal{L}]) \subset \mathbb{T}_*(E|\,\mathcal{L}) \cap \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}[\mathbb{T}_*(E|\mathcal{L})], \tag{4.14}$$

where $\{\cap\}_\sharp(\mathfrak{C}_0^*[E; \mathcal{L}]) \in (\mathrm{BAS})\big[(\mathcal{L}-\mathrm{link})_0[E]\big]$ and by (4.2)

$$\{\cap\}_\sharp(\mathfrak{C}_0^*[E; \mathcal{L}]) \in \big(\mathbb{T}_*(E|\,\mathcal{L}) - \mathrm{BAS}\big)_0\big[(\mathcal{L}-\mathrm{link})_0[E]\big]. \tag{4.15}$$

**Proposition 6.** *In the form of* (4.3) *a zero-dimensional $T_2$-space is realized.*

The corresponding proof (see [5]) is immediate combination of (4.10), (4.14), and (4.15) (see [13, 6.2]).

We note the following obvious property (see (3.8) and definitions of Section 2)

$$\Phi_{\mathcal{L}}(L) = (\mathcal{L} - \mathrm{link})^0[E|\, L] \cap \mathbb{F}_0^*(\mathcal{L}) \quad \forall\, L \in \mathcal{L}. \tag{4.16}$$

Therefore, by (2.4), (3.10), and (4.16) we obtain the equality

$$(\mathbb{UF})[E; \mathcal{L}] = \mathfrak{C}_0^*[E; \mathcal{L}]|_{\mathbb{F}_0^*(\mathcal{L})}.$$

As a corollary, the following important property (see [5]) is realized:

$$\mathbf{T}_{\mathcal{L}}^*[E] = \mathbb{T}_*(E|\, \mathcal{L})|_{\mathbb{F}_0^*(\mathcal{L})}. \tag{4.17}$$

From (4.17), we obtain the next statement: (2.5) is a subspace of the TS (4.3). So, by (3.30) and (4.17)

$$\big(\mathbf{T}_{\mathcal{L}}^0[E] = \mathbb{T}_0(E|\, \mathcal{L})|_{\mathbb{F}_0^*(\mathcal{L})}\big) \& \big(\mathbf{T}_{\mathcal{L}}^*[E] = \mathbb{T}_*(E|\, \mathcal{L})|_{\mathbb{F}_0^*(\mathcal{L})}\big). \tag{4.18}$$

In (4.18), we have the natural connection for topological equipments of the spaces of MLS and u/f. In addition, by [5, Proposition 6.5]

$$\mathbb{T}_0(E|\, \mathcal{L}) \subset \mathbb{T}_*(E|\, \mathcal{L}). \tag{4.19}$$

So, by (4.19) we obtain the following BTS

$$\big((\mathcal{L} - \mathrm{link})_0[E], \mathbb{T}_0(E|\, \mathcal{L}), \mathbb{T}_*(E|\, \mathcal{L})\big). \tag{4.20}$$

Of course, by (4.18) we can consider BTS (2.12) as a subspace of BTS (4.20).

## 5. Ultrafilters of separable lattice of sets

In present section, we suppose that

$$\mathcal{L} \in (\mathrm{LAT})_0[E] \cap \widetilde{\pi}^0[E]. \tag{5.1}$$

By (5.1) we obtain the case of separable lattice. Using (2.3) and (5.1), we obtain that

$$(\mathcal{L} - \mathrm{triv})[x] \in \mathbb{F}_0^*(\mathcal{L}) \quad \forall\, x \in E. \tag{5.2}$$

By (5.2) we can introduce operator

$$x \longmapsto (\mathcal{L} - \mathrm{triv})[x] : \, E \longrightarrow \mathbb{F}_0^*(\mathcal{L}) \tag{5.3}$$

denoted by $(\mathcal{L} - \mathrm{triv})[\cdot]$. Of course, (5.3) is an immersion of $E$ into $\mathbb{F}_0^*(\mathcal{L})$. Therefore, we can consider sets-images

$$(\mathcal{L} - \mathrm{triv})[\cdot]^1(A) \stackrel{\triangle}{=} \{(\mathcal{L} - \mathrm{triv})[x] : \, x \in A\} \in \mathcal{P}\big(\mathbb{F}_0^*(\mathcal{L})\big) \quad \forall\, A \in \mathcal{P}(E). \tag{5.4}$$

We note that by (2.11) and (5.4) the inclusions

$$\mathrm{cl}\big((\mathcal{L} - \mathrm{triv})[\cdot]^1(A), \mathbf{T}_{\mathcal{L}}^*[E]\big) \subset \mathrm{cl}\big((\mathcal{L} - \mathrm{triv})[\cdot]^1(A), \mathbf{T}_{\mathcal{L}}^0[E]\big) \quad \forall\, A \in \mathcal{P}(E) \tag{5.5}$$

are realized. By [5, Proposition 6.6] we have the system of equalities

$$\mathrm{cl}\big((\mathcal{L} - \mathrm{triv})[\cdot]^1(L), \mathbf{T}_{\mathcal{L}}^*[E]\big) = \mathrm{cl}\big((\mathcal{L} - \mathrm{triv})[\cdot]^1(L), \mathbf{T}_{\mathcal{L}}^0[E]\big) = \Phi_{\mathcal{L}}(L) \quad \forall\, L \in \mathcal{L}. \tag{5.6}$$

We recall [17, (4.7)] that $\forall L_1 \in \mathcal{L} \ \ \forall L_2 \in \mathcal{L}$

$$(L_1 \subset L_2) \Longleftrightarrow \big(\Phi_{\mathcal{L}}(L_1) \subset \Phi_{\mathcal{L}}(L_2)\big). \tag{5.7}$$

As an obvious corollary, for $L_1 \in \mathcal{L}$ and $L_2 \in \mathcal{L}$

$$(L_1 = L_2) \Longleftrightarrow \big(\Phi_{\mathcal{L}}(L_1) = \Phi_{\mathcal{L}}(L_2)\big).$$

**Proposition 7.** *If $L_1 \in \mathcal{L}$ and $L_2 \in \mathcal{L}$, then*

$$(L_1 \subset L_2) \Longleftrightarrow \big((\mathcal{L}-\mathrm{link})^0[E|\,L_1] \subset (\mathcal{L}-\mathrm{link})^0[E|\,L_2]\big). \tag{5.8}$$

P r o o f. By [5, (2.15)] we have implication

$$(L_1 \subset L_2) \Longrightarrow \big((\mathcal{L}-\mathrm{link})^0[E|\,L_1] \subset (\mathcal{L}-\mathrm{link})^0[E|\,L_2]\big). \tag{5.9}$$

Let $(\mathcal{L}-\mathrm{link})^0[E|\,L_1] \subset (\mathcal{L}-\mathrm{link})^0[E|\,L_2]$. We prove that $L_1 \subset L_2$. Indeed, suppose the contrary: let

$$L_1 \setminus L_2 \neq \emptyset. \tag{5.10}$$

With employment of (5.10), we choose $x_* \in L_1 \setminus L_2$. Then, $(\mathcal{L}-\mathrm{triv})[x_*] \in \mathbb{F}_0^*(\mathcal{L})$ and, in particular (see (2.14)),

$$(\mathcal{L}-\mathrm{triv})[x_*] \in (\mathcal{L}-\mathrm{link})_0[E]. \tag{5.11}$$

In addition, by the choice of $x_*$ we obtain (see Section 2) that $L_1 \in (\mathcal{L}-\mathrm{triv})[x_*]$. Then, by (3.8) and (5.11)

$$(\mathcal{L}-\mathrm{triv})[x_*] \in (\mathcal{L}-\mathrm{link})^0[E|\,L_1].$$

Therefore, $(\mathcal{L}-\mathrm{triv})[x_*] \in (\mathcal{L}-\mathrm{link})^0[E|\,L_2]$ (we use our supposition). Using (3.8), we obtain that $L_2 \in (\mathcal{L}-\mathrm{triv})[x_*]$ and, as a corollary, $x_* \in L_2$. But, this inclusion contradicts to the choice of $x_*$ (recall that $x_* \notin L_2$). The obtained contradiction proves the required inclusion $L_1 \subset L_2$. So, implication

$$\big((\mathcal{L}-\mathrm{link})^0[E|\,L_1] \subset (\mathcal{L}-\mathrm{link})^0[E|\,L_2]\big) \Longrightarrow (L_1 \subset L_2) \tag{5.12}$$

is established. From (5.9) and (5.12), we obtain (5.8). □

**Corollary 1.** *If $L_1 \in \mathcal{L}$ and $L_2 \in \mathcal{L}$, then*

$$(L_1 = L_2) \Longleftrightarrow \big((\mathcal{L}-\mathrm{link})^0[E|\,L_1] = (\mathcal{L}-\mathrm{link})^0[E|\,L_2]\big).$$

The corresponding proof is obvious (see Proposition 7). So, mapping

$$L \longmapsto (\mathcal{L}-\mathrm{link})^0[E|\,L] : \mathcal{L} \longrightarrow \mathfrak{C}_0^*[E;\mathcal{L}]$$

is a bijection from $\mathcal{L}$ onto $\mathfrak{C}_0^*[E;\mathcal{L}]$ (see (3.10) and Corollary 1). We note that from (5.6) the next density property follows:

$$\mathrm{cl}\big((\mathcal{L}-\mathrm{triv})[\cdot]^1(E), \mathbf{T}^*_{\mathcal{L}}[E]\big) = \mathrm{cl}\big((\mathcal{L}-\mathrm{triv})[\cdot]^1(E),\, \mathbf{T}^0_{\mathcal{L}}[E]\big) = \mathbb{F}_0^*(\mathcal{L}); \tag{5.13}$$

in (5.13), we use the obvious equality $\Phi_{\mathcal{L}}(E) = \mathbb{F}_0^*(\mathcal{L})$.

## 6. Some additions

In this sections, at first, we consider questions meaningful of a duality for families $\mathfrak{C}_0^*[E;\mathcal{L}]$ and $\mathfrak{C}_{\mathrm{op}}^0[E;\mathcal{L}]$. For this, we recall that (see Section 3)

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}^0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]. \tag{6.1}$$

As a corollary, by (4.4) and (6.1) we have the property

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p}-\mathrm{BAS})_0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_*(E|\,\mathcal{L})\big]\cap(\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}^0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]. \tag{6.2}$$

**Proposition 8.** *The family $\mathfrak{C}_{\mathrm{op}}^0[E;\mathcal{L}]$ is a closed subbase of the TS* (4.3)*:*

$$\mathfrak{C}_{\mathrm{op}}^0[E;\mathcal{L}] \in (\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}^0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_*(E|\,\mathcal{L})\big]. \tag{6.3}$$

P r o o f. We recall (3.24). So, by (4.5) we have the following statement

$$\mathfrak{C}_0^*[E;\mathcal{L}] \in (\mathrm{p}-\mathrm{BAS})_{\emptyset}^0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_*(E|\,\mathcal{L})\big]:\ \mathfrak{C}_{\mathrm{op}}^0[E;\mathcal{L}] = \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}\big[\mathfrak{C}_0^*[E;\mathcal{L}]\big].$$

Then, by (1.16) we obtain (6.3). □

From (3.17) and Proposition 8 we have the following property

$$\mathfrak{C}_{\mathrm{op}}^0[E;\mathcal{L}] \in (\mathrm{p}-\mathrm{BAS})_0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_0(E|\,\mathcal{L})\big]\cap(\mathrm{p}-\mathrm{BAS})_{\mathrm{cl}}^0\big[(\mathcal{L}-\mathrm{link})_0[E];\mathbb{T}_*(E|\,\mathcal{L})\big]. \tag{6.4}$$

In (6.2) and (6.4), we obtain a duality of subbases.

## 7. Bitopological space of closed ultrafilters and maximal linked systems

We recall (5.6). Then, by this property the topologies $\mathbf{T}_{\mathcal{L}}^0[E]$ and $\mathbf{T}_{\mathcal{L}}^*[E]$ are similar (later, we show that in many cases the above-mentioned topologies are equal). But, now we consider the variant of the set lattice for which the above-mentioned topologies differ typically. Namely, we fix $\tau \in (\mathcal{D}-\mathrm{top})[E]$; so, $\tau \in (\mathrm{top})[E]$ for which $(E,\tau)$ is a $T_1$-space and (in this section) we suppose that

$$\mathcal{L} = \mathbf{C}_E[\tau]. \tag{7.1}$$

Under (7.1), we call u/f of the set $\mathbb{F}_0^*(\mathcal{L})$ as closed u/f. Analogously, for MLS of $(\mathcal{L}-\mathrm{link})_0[E]$, under (7.1), we use the term closed MLS. In addition, in our case by (5.1) and (7.1)

$$\mathbf{C}_E[\tau] \in (\mathrm{LAT})_0[E] \cap \widetilde{\pi}^0[E]. \tag{7.2}$$

Of course, $\mathbf{C}_E[\tau] \in (\mathcal{D}-\mathrm{clos})[E]$. By (7.2) we have the separable lattice (7.1). Indeed, $\{x\} \in \mathbf{C}_E[\tau]$ under $x \in E$ (really, by (1.19) $(E,\tau)$ is a $T_1$-space). So, in our case, by (2.3) and (5.2)

$$(\mathcal{L}-\mathrm{triv})[x] \in \mathbb{F}_0^*(\mathbf{C}_E[\tau]) \quad \forall\, x \in E. \tag{7.3}$$

Of course, by (7.1) and (7.2) we can use statements of Section 5. In particular, by (5.6), (7.1), and (7.2)

$$\mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(F),\mathbf{T}_{\mathbf{C}_E[\tau]}^*[E]\big)= \mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(F),\mathbf{T}_{\mathbf{C}_E[\tau]}^0[E]\big)= \Phi_{\mathbf{C}_E[\tau]}(F)\ \ \forall\, F \in \mathbf{C}_E[\tau]. \tag{7.4}$$

At the same time, we have (see [5, § 7]) the property

$$\mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(A),\mathbf{T}^0_{\mathbf{C}_E[\tau]}[E]\big)=\Phi_{\mathbf{C}_E[\tau]}\big(\mathrm{cl}(A,\tau)\big)\quad\forall\,A\in\mathcal{P}(E). \tag{7.5}$$

So, by (7.5) the following statement is realized: TS (2.10) «feels» subsets of $E$ accurate to closure. We recall [5, (7.3)]: for $A\in\mathcal{P}(E)$ and $x_*\in\mathrm{cl}(A,\tau)\setminus A$

$$(\mathbf{C}_E[\tau]-\mathrm{triv})[x_*]\in\mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(A),\mathbf{T}^0_{\mathbf{C}_E[\tau]}[E]\big)\setminus\mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(A),\mathbf{T}^*_{\mathbf{C}_E[\tau]}[E]\big). \tag{7.6}$$

With employment of (7.6), we obtain (see [5, (7.4)]) in our case

$$\mathbf{C}_E[\tau]=\big\{A\in\mathcal{P}(E)|\,\mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(A),\mathbf{T}^0_{\mathbf{C}_E[\tau]}[E]\big)=\mathrm{cl}\big((\mathbf{C}_E[\tau]-\mathrm{triv})[\cdot]^1(A),\mathbf{T}^*_{\mathbf{C}_E[\tau]}[E]\big)\big\}. \tag{7.7}$$

Finally, by [5, Theorem 7.1] we obtain the following implication:

$$\big(\tau\neq\mathcal{P}(E)\big)\Longrightarrow(\mathbf{T}^0_{\mathbf{C}_E[\tau]}[E]\neq\mathbf{T}^*_{\mathbf{C}_E[\tau]}[E]). \tag{7.8}$$

So, for (7.1) and nondiscrete $T_1$-space $(E,\tau)$, BTS (2.12) is nondegenerate. From (4.18) and (7.8), we obtain that

$$\big(\tau\neq\mathcal{P}(E)\big)\Longrightarrow\big(\mathbb{T}_0(E|\,\mathbf{C}_E[\tau])\neq\mathbb{T}_*(E|\,\mathbf{C}_E[\tau])\big). \tag{7.9}$$

We use (7.8) and (7.9) in connection with lattices of the family (1.17).

## 8. Some particular cases

In this section, we fix a lattice

$$\widetilde{\mathcal{L}}\in(\downarrow-\mathrm{LAT})^0[E]. \tag{8.1}$$

Then, by (1.18) we obtain that $\widetilde{\mathcal{L}}\cup\{E\}\in(\mathrm{clos})[E]$ and (in particular) $\widetilde{\mathcal{L}}\cup\{E\}\in(\mathrm{LAT})_0[E]$. In addition,

$$\tau^0_{\widetilde{\mathcal{L}}}[E]=\mathbf{C}_E[\widetilde{\mathcal{L}}\cup\{E\}]=\mathbf{C}_E[\widetilde{\mathcal{L}}]\cup\{\emptyset\}\in(\mathcal{D}-\mathrm{top})[E] \tag{8.2}$$

realizes the following $T_1$-space:

$$(E,\tau^0_{\widetilde{\mathcal{L}}}[E]). \tag{8.3}$$

We recall that (see Section 1), for (8.2) and (8.3), the following property takes place: (8.3) is not $T_2$-space. From (8.2), we have the equality

$$\widetilde{\mathcal{L}}\cup\{E\}=\mathbf{C}_E\big[\tau^0_{\widetilde{\mathcal{L}}}[E]\big] \tag{8.4}$$

(see (1.21)). In addition, by (1.22) we obtain that

$$\tau^0_{\widetilde{\mathcal{L}}}[E]\neq\mathcal{P}(E). \tag{8.5}$$

We recall that by (1.20) $\widetilde{\mathcal{L}}\cup\{E\}\in(\mathcal{D}-\mathrm{clos})[E]$. In addition,

$$\Big(\mathbb{T}_0(E|\,\widetilde{\mathcal{L}}\cup\{E\})\in(\mathrm{top})\big[\big((\widetilde{\mathcal{L}}\cup\{E\})-\mathrm{link}\big)_0[E]\big]\Big)\,\&\,\Big(\mathbb{T}_*(E|\,\widetilde{\mathcal{L}}\cup\{E\})\in(\mathrm{top})\big[\big((\widetilde{\mathcal{L}}\cup\{E\})-\mathrm{link}\big)_0[E]\big]\Big). \tag{8.6}$$

In the form of the triplet

$$\Big(\big((\widetilde{\mathcal{L}}\cup\{E\})-\mathrm{link}\big)_0[E],\mathbb{T}_0(E|\,\widetilde{\mathcal{L}}\cup\{E\}),\mathbb{T}_*(E|\,\widetilde{\mathcal{L}}\cup\{E\})\Big), \tag{8.7}$$

we obtain a BTS. Of course, (8.7) is a variant of BTS (4.20). By (7.9), (8.4), and (8.5) we obtain that

$$\mathbb{T}_0(E|\,\widetilde{\mathcal{L}}\cup\{E\})\neq\mathbb{T}_*(E|\,\widetilde{\mathcal{L}}\cup\{E\}). \tag{8.8}$$

So, by (8.8) the BTS (8.7) is non-degenerate. Moreover, we have topologies

$$\big(\mathbf{T}^0_{\widetilde{\mathcal{L}}\cup\{E\}}[E] \in (\mathrm{top})[\mathbb{F}^*_0(\widetilde{\mathcal{L}} \cup \{E\})]\big)\&\big(\mathbf{T}^*_{\widetilde{\mathcal{L}}\cup\{E\}}[E] \in (\mathrm{top})[\mathbb{F}^*_0(\widetilde{\mathcal{L}} \cup \{E\})]\big).$$

In addition, in the form of the triplet

$$\big(\mathbb{F}^*_0(\widetilde{\mathcal{L}} \cup \{E\}), \mathbf{T}^0_{\widetilde{\mathcal{L}}\cup\{E\}}[E], \mathbf{T}^*_{\widetilde{\mathcal{L}}\cup\{E\}}[E]\big) \tag{8.9}$$

we have BTS. Of course, (8.9) is a variant of BTS (2.12). By (7.8), (8.4), and (8.5) we obtain that

$$\mathbf{T}^0_{\widetilde{\mathcal{L}}\cup\{E\}}[E] \neq \mathbf{T}^*_{\widetilde{\mathcal{L}}\cup\{E\}}[E]. \tag{8.10}$$

So, by (8.10) the BTS (8.9) is non-degenerate. We recall that, in Section 1, the concrete examples of the realization of (8.9) and (8.10) were identified (see Examples 1.1–1.4). Now, we consider yet one example of such type.

*Example 8.1.* Let $\sqsubseteq$ be a direction on the (nonempty) set $E$. So, we consider the case of nonempty directed set $(E, \sqsubseteq)$. Suppose that

$$(\sqsubseteq -\mathrm{Ma})_E[Y] \overset{\triangle}{=} \{z \in E |\, y \sqsubseteq z \ \ \forall\, y \in Y\} \ \ \forall\, Y \in \mathcal{P}(E).$$

Then $\mathfrak{M}[E; \sqsubseteq] \overset{\triangle}{=} \{Y \in \mathcal{P}(E) |\, (\sqsubseteq -\mathrm{Ma})_E[Y] \neq \emptyset\}$ is the family of all majorized subsets of $E$. Since $E \neq \emptyset$, we have the obvious property $\emptyset \in \mathfrak{M}[E; \sqsubseteq]$ (moreover, by the choice of $\sqsubseteq$ we obtain that $\{x; y\} \in \mathfrak{M}[E; \sqsubseteq] \ \ \forall\, x \in E \ \ \forall\, y \in E$). From properties of directed sets, the statement $\mathfrak{M}[E; \sqsubseteq\,] \in (\mathrm{LAT})[E]$ is realized. It is obvious that $\{x\} \in \mathfrak{M}[E; \sqsubseteq] \ \ \forall\, x \in E$. Finally,

$$\bigcap_{\mathcal{H}\in\mathfrak{H}} \mathcal{H} \in \mathfrak{M}[E; \sqsubseteq] \ \ \forall\, \mathfrak{H} \in \mathcal{P}'(\mathfrak{M}[E; \sqsubseteq]).$$

As a corollary, by (1.17) we obtain the implication

$$(E \notin \mathfrak{M}[E; \sqsubseteq]) \Longrightarrow (\mathfrak{M}[E; \sqsubseteq] \in (\downarrow -\mathrm{LAT})^0[E]).$$

So, under $E \notin \mathfrak{M}[E; \sqsubseteq]$, in the form of $\mathfrak{M}[E; \sqsubseteq]$, we obtain yet one variant of the family of $(\downarrow -\mathrm{LAT})^0[E]$ : $\mathfrak{M}[E; \sqsubseteq] \in (\downarrow -\mathrm{LAT})^0[E]$. □

## 9. Measurable space with algebra of sets

Recall that by (1.6) and (1.7) $(\mathrm{alg})[E] \subset (\mathrm{LAT})_0[E]$. Using this property, in the present section, we consider the case

$$\mathcal{L} \in (\mathrm{alg})[E]. \tag{9.1}$$

By (9.1) we have that (in the present section) $(E, \mathcal{L})$ is a measurable space with algebra of sets. We recall Remark 2.1: in the form of

$$\big(\mathbb{F}^*_0(\mathcal{L}), (\mathbb{UF})[E; \mathcal{L}]\big),$$

a measurable space with algebra of sets is realized also. Moreover, we have BTS (2.12). But, by [6, Proposition 9.2] this BTS is degenerate:

$$\mathbf{T}^0_{\mathcal{L}}[E] = \mathbf{T}^*_{\mathcal{L}}[E]. \tag{9.2}$$

By (9.2) we obtain the following equality of TS:

$$\big(\mathbb{F}^*_0(\mathcal{L}), \mathbf{T}^0_{\mathcal{L}}[E]\big) = \big(\mathbb{F}^*_0(\mathcal{L}), \mathbf{T}^*_{\mathcal{L}}[E]\big); \tag{9.3}$$

of course, (9.3) is a nonempty zero-dimensional compactum. Moreover, by (9.1)

$$\mathcal{L} = \mathbf{C}_E[\mathcal{L}]. \tag{9.4}$$

Therefore, the sets $(\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,L],\ L \in \mathcal{L}$, are defined. In addition,

$$(\mathcal{L}-\mathrm{link})^0_{\mathrm{op}}[E|\,L] = (\mathcal{L}-\mathrm{link})^0[E|\,L]$$

under $L \in \mathcal{L}$. We recall that, under $\Lambda \in \mathcal{L}$, the inclusion $E \setminus \Lambda \in \mathcal{L}$ is realized; and what is more, by (3.13)

$$(\mathcal{L}-\mathrm{link})^0[E|\,E\setminus\Lambda] = (\mathcal{L}-\mathrm{link})_0[E] \setminus (\mathcal{L}-\mathrm{link})^0[E|\,\Lambda].$$

As a simple corollary, in our case, the equality

$$\mathfrak{C}^*_0[E;\mathcal{L}] = \mathfrak{C}^0_{\mathrm{op}}[E;\mathcal{L}]$$

is realized. Therefore, by (3.15) and (4.2)

$$\mathbb{T}_0(E|\,\mathcal{L}) = \mathbb{T}_*(E|\,\mathcal{L}). \tag{9.5}$$

So, by (9.5) we have the following important property: TS

$$\big((\mathcal{L}-\mathrm{link})_0[E], \mathbb{T}_0(E|\,\mathcal{L})\big) = \big((\mathcal{L}-\mathrm{link})_0[E], \mathbb{T}_*(E|\,\mathcal{L})\big) \tag{9.6}$$

is a nonempty supercompactum. In particular, (9.6) is a nonempty compactum.

**Proposition 9.** *The set $\mathbb{F}^*_0(\mathcal{L})$ is closed in TS (9.6):*

$$\mathbb{F}^*_0(\mathcal{L}) \in \mathbf{C}_{(\mathcal{L}-\mathrm{link})_0[E]}[\mathbb{T}_0(E|\,\mathcal{L})]. \tag{9.7}$$

The corresponding proof follows from Proposition 4 (indeed, for (9.6) we have the separability property). In connection with Proposition 9, we recall (3.31).

## 10. Open maximal linked systems

In this section, we suppose that

$$\mathcal{L} = \tau, \tag{10.1}$$

where $\tau \in (\mathrm{top})[E]$. So, $(E,\mathcal{L}) = (E,\tau)$ is a TS. We consider the lattice of open sets. In this connection, we recall (see [15, Section 8]) that

$$\mathbf{T}^0_\tau[E] = \mathbf{T}^*_\tau[E]. \tag{10.2}$$

Of course, in the form of

$$\big(\mathbb{F}^*_0(\tau), \mathbf{T}^0_\tau[E]\big) = \big(\mathbb{F}^*_0(\tau), \mathbf{T}^*_\tau[E]\big), \tag{10.3}$$

we obtain a nonempty zero-dimensional compactum of open u/f (using (10.1), we consider u/f consisting of open sets as open u/f). On the other hand, by (10.1) we can consider MLS consisting of open sets. We call such MLS open also (recall that $(\mathrm{top})[E] \subset (\mathrm{LAT})_0[E]$). By [5, Proposition 9.1]

$$(\tau-\mathrm{link})_0[E] \setminus (\tau-\mathrm{link})^0[E|\,G] = (\tau-\mathrm{link})^0[E|\,E\setminus \mathrm{cl}(G,\tau)] \quad \forall\, G \in \tau.$$

With employment of this property, in [5, Proposition 9.2], the equality

$$\mathbb{T}_0(E|\,\tau) = \mathbb{T}_*(E|\,\tau) \tag{10.4}$$

was established. By (10.4) we obtain that

$$\big((\tau - \mathrm{link})_0[E], \mathbb{T}_0(E|\,\tau)\big) = \big((\tau - \mathrm{link})_0[E], \mathbb{T}_*(E|\,\tau)\big) \tag{10.5}$$

is a zero-dimensional supercompactum. In addition, by Proposition 4

$$\mathbb{F}_0^*(\tau) \in \mathbf{C}_{(\tau-\mathrm{link})_0[E]}[\mathbb{T}_0(E|\,\tau)];$$

so, $\mathbb{F}_0^*(\tau)$ is the closed in the supercompactum (10.5). We obtain that compactum (10.3) is a closed subspace of the supercompactum (10.5).

## 11. Conclusion

We reviewed two BTS. In the first case point of BTS are MLS and, in the second case, similar points are u/f of a set lattice. It is established that the second BTS can be considered as a subspace of the first BTS. We indicated the natural variants of our lattice for which the above-mentioned BTS are degenerate and, opposite, the variants with degeneracy of the corresponding BTS is absent. Our consideration is connected with ideas of supercompactness and superextension of a TS. For degenerate BTS the corresponding space of MLS is a supercompactum. Under consideration of the lattice of closed MLS, we obtain a non-degenerate BTS typically.

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# ON THE OSCILLATION OF A THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH NEUTRAL TYPE

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**Abstract:** In this article, we investigate the oscillation behavior of the solutions of the third-order nonlinear differential equation with neural type of the form

$$\Big(a_1(t)\big(a_2(t)Z'(t)\big)'\Big)' + q(t)f\big(x(\sigma(t))\big) = 0, \quad t \geq t_0 > 0,$$

where $Z(t) := x(t) + p(t)x^{\alpha}(\tau(t))$. Some new oscillation results are presented that extend those results given in the literature.
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## 1. Introduction

Consider the third order non-linear neutral delay differential equation

$$\Big(a_1(t)\big(a_2(t)Z'(t)\big)'\Big)' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0 > 0, \tag{$E$}$$

where $Z(t) := x(t) + p(t)x^{\alpha}(\tau(t))$ and $0 < \alpha \leq 1$ is a ratio of odd positive integers. Throughout this paper, without further mention, let

$(A_1)$ $a_i(t) \in C([t_0, +\infty))$, $a_i(t) > 0$ for $i = 1, 2$ and $p(t), q(t) \in C([t_0, +\infty))$, $q(t) > 0$;

$(A_2)$ $\tau(t) \in C([t_0, +\infty))$, $\tau(t) \leq t$, $\sigma(t) \in C([t_0, +\infty))$, $\sigma(t) \leq t$;

$(A_3)$ $f$ is nondecreasing and $uf(u) \geq k > 0$ for $u \neq 0$ and $\lim\limits_{t\to+\infty} \tau(t) = \lim\limits_{t\to+\infty} \sigma(t) = \infty$.

By a solution of equation $(E)$ we mean a nontrivial real valued function $x(t) \in C([T_x, \infty)), T_x \geq t_0$, which has the property $Z'(t) \in C^1([T_x, \infty))$, $a_2(t)Z'(t) \in C^1([T_x, \infty))$, $a_1(t)(a_2(t)Z'(t))' \in C^1([T_x, \infty))$ and satisfies $(E)$ on $[T_x, \infty)$. We consider only those solutions $x(t)$ of $(E)$ which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of $(E)$ is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$ and otherwise, it is said to be non-oscillatory. Equation $(E)$ is called almost oscillatory if all its solutions are oscillatory or convergent to zero asymptotically.

In the last years, a great deal of interest in oscillatory properties of neutral functional differential equations has been shown, we refer the reader to [1–8] and the references cited therein. A number of authors including B. Baculíková and J. Džurina [10], T. Candan and Dahiya [11, 12], Graef et al. [7], and E. Thandapani and Li [8] have studied the oscillatory behavior of solutions of third order neutral differential equations in the form of equation $(E)$ when $\alpha = 1$.

Recently, Lin and Tang [13] explored the oscillation of first-order neutral differential equation with a super-linear neutral term

$$[x(t) - p(x^{\alpha}(t-\tau)]' + q(t)\prod_{j=1}^{m}|x(t-\sigma_j)|^{\beta_j}\operatorname{sgn}[x(t-\sigma_j)] = 0,$$

where $\alpha > 1$. Ravi P. Agarwal et al. [14] concerned with oscillation of a certain class of second-order differential equations with a sub-linear neutral term

$$\big(a(t)\,[x(t) + p(t)x^{\alpha}(\tau(t))]'\,\big)' + q(t)x(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

where $0 < \alpha \leq 1$ is a ratio of odd positive integers and E. Thandapani et al. [9] established sufficient conditions for the oscillation of all solutions of a nonlinear differential equation

$$\big(a(t)\,[x(t) + p(t)x^{\alpha}(\tau(t))]'\,\big)' + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

where $\alpha$ and $\beta$ are ratio of odd positive integers. The above observation shows that this paper extend the results in third order.

This article presents the further investigation of the oscillations of $(E)$. The following two cases:

$$\int_{t_0}^{\infty}\frac{1}{a_1(t)}dt = \infty, \quad \int_{t_0}^{\infty}\frac{1}{a_2(t)}dt = \infty, \tag{1.1}$$

$$\int_{t_0}^{\infty}\frac{1}{a_1(t)}dt < \infty, \quad \int_{t_0}^{\infty}\frac{1}{a_2(t)}dt = \infty, \tag{1.2}$$

are studied.

The paper is organized as follows. In Section 2, we present sufficient conditions for the oscillation of all solutions of $(E)$ and in Section 3, we provide some examples to illustrate the main results.

In the following, all functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all $t$ large enough. Without loss of generality, we can deal only with the positive solutions of $(E)$.

## 2. Main result

In this section, we state and prove our main results for the equation $(E)$. For convenience, we use the notations

$$p_*(t) = \left(1 - \frac{p(\sigma(t))}{M^{1-\alpha}}\right), \quad \Theta(t) = \frac{\displaystyle\int_{t_2}^{\sigma(t)}\left(\frac{1}{a_2(s)}\int_{t_1}^{s}\frac{du}{a_1(u)}\right)ds}{\displaystyle\int_{t_1}^{t}\frac{du}{a_1(u)}} \tag{2.1}$$

**Theorem 1.** *Let $0 \leq p(t) \leq p_1 \leq 1$. If (1.1) holds and it there exists a positive function $\phi \in C^1([t_0,\infty),\mathbb{R})$, such that for all sufficiently large $t_3 > t_2 > t_1 \geq t_0$ we have*

$$\limsup_{t\to\infty}\int_{t_3}^{t}\left(\phi(s)kq(s)p_*(s)\Theta(s) - \frac{a_1(s)(\phi'(s))^2}{4\phi(s)}\right)ds = \infty \tag{2.2}$$

*and*

$$\int_{t_0}^{\infty} \frac{1}{a_2(v)} \int_{v}^{\infty} \frac{1}{a_1(u)} \left[ \int_{u}^{\infty} q(s)ds \right] du\, dv = \infty \tag{2.3}$$

*holds for all constants $M > 0$, then $(E)$ is almost oscillatory.*

P r o o f. Suppose that $x(t)$ is a positive solution of $(E)$. By condition (1.1), there exist two possible cases:

(1) $Z(t) > 0$, $Z'(t) > 0$, $(a_2(t)Z'(t))' > 0$, $(a_1(t)(a_2(t)Z'(t))')' < 0$,

(2) $Z(t) > 0$, $Z'(t) < 0$, $(a_2(t)Z'(t))' > 0$, $(a_1(t)(a_2(t)Z'(t))')' < 0$, for $t \geq t_1$, $t_1$ is large enough.

Assume $Z(t)$ satisfying property (1), then

$$\Big(a_1(t)\big(a_2(t)Z'(t)\big)'\Big)' = -q(t)f(x(\sigma(t))) \leq -kq(t)x(\sigma(t)) < 0.$$

If there exists $t \geq t_1$ such that $Z(t) > 0$, $Z(\sigma(t)) > 0$, $Z'(t) > 0$, then $Z(t)$ is monotonically increasing, there exists a constant $M > 0$ such that $Z(t) \geq M$ and by the definition of $Z$ we have

$$x(t) = Z(t) - p(t)x^{\alpha}(\sigma(t)) \geq Z(t) - p(t)Z^{\alpha}(\sigma(t)) \geq \left(1 - \frac{p(\sigma(t))}{M^{1-\alpha}}\right) Z(t) = p_*(t)Z(t), \tag{2.4}$$

where $p_*(t)$ is defined in (2.1). Let

$$\omega(t) = \phi(t)\frac{a_1(t)(a_2(t)Z'(t))'}{a_2(t)Z'(t)}, \tag{2.5}$$

$\omega(t) > 0$ for $t \geq t_1$. Differentiating (2.5), we obtain

$$\begin{aligned} \omega'(t) = {} & \phi'(t)\frac{a_1(t)(a_2(t)Z'(t))'}{a_2(t)Z'(t)} + \phi(t)\frac{(a_1(t)(a_2(t)Z'(t))')'}{a_2(t)Z'(t)} \\ & -\phi(t)\frac{a_1(t)(a_1(t)(a_2(t)Z'(t))')(a_2(t)Z'(t))'}{(a_2(t)Z'(t))^2}. \end{aligned}$$

Since $(a_1(t)(a_2(t)Z'(t))')' < 0$, then $a_1(t)(a_2(t)Z'(t))'$ is decreasing, so

$$a_2(t)Z'(t) \geq \int_{t_1}^{t} \frac{a_1(s)(a_2(s)Z'(s))'}{a_1(s)} ds \geq a_1(t)(a_2(t)Z'(t))' \int_{t_1}^{t} \frac{ds}{a_1(s)},$$

which implies that

$$\left(\frac{a_2(t)Z'(t)}{\int_{t_1}^{t} ds/a_1(s)}\right)' \leq 0. \tag{2.6}$$

Thus,

$$Z(t) = Z(t_2) + \int_{t_2}^{t} \frac{a_2(s)Z'(s)}{\displaystyle\int_{t_1}^{s}\frac{du}{a_1(u)}} \frac{\displaystyle\int_{t_1}^{s}\frac{du}{a_1(u)}}{a_2(s)} ds \geq \frac{a_2(t)Z'(t)}{\displaystyle\int_{t_1}^{t}\frac{du}{a_1(u)}} \int_{t_2}^{t} \frac{\displaystyle\int_{t_1}^{s}\frac{du}{a_1(u)}}{a_2(s)} ds, \tag{2.7}$$

for $t \geq t_2 \geq t_1$.
It follows from $(E)$, (2.4), and (2.5) that

$$\omega'(t) \leq \frac{\phi'(t)}{\phi(t)}\omega(t) - \frac{\omega^2(t)}{\phi(t)a_1(t)} - \phi(t)kq(t)p_*(t)\frac{Z(\sigma(t))}{a_2(t)Z'(t)},$$

that is,

$$\omega'(t) \leq \frac{\phi'(t)}{\phi(t)}\omega(t) - \frac{\omega^2(t)}{\phi(t)a_1(t)} - \phi(t)kq(t)p_*(t)\frac{Z(\sigma(t))}{a_2(\sigma(t))Z'(\sigma(t))}\frac{a_2(\sigma(t))Z'(\sigma(t))}{a_2(t)Z'(t)}.$$

From (2.6) and (2.7) follows

$$\begin{aligned}
\omega'(t) &\leq \frac{\phi'(t)}{\phi(t)}\omega(t) - \frac{\omega^2(t)}{\phi(t)a_1(t)} - \phi(t)kq(t)p_*(t)\frac{\displaystyle\int_{t_2}^{\sigma(t)}\left(\frac{1}{a_2(s)}\int_{t_1}^{s}\frac{du}{a_1(u)}\right)ds}{\displaystyle\int_{t_1}^{\sigma(t)}\frac{du}{a_1(u)}}\frac{\displaystyle\int_{t_1}^{\sigma(t)}\frac{du}{a_1(u)}}{\displaystyle\int_{t_1}^{t}\frac{du}{a_1(u)}} \\
&= \frac{\phi'(t)}{\phi(t)}\omega(t) - \frac{\omega^2(t)}{\phi(t)a_1(t)} - \phi(t)kq(t)p_*(t)\frac{\displaystyle\int_{t_2}^{\sigma(t)}\left(\frac{1}{a_2(s)}\int_{t_1}^{s}\frac{du}{a_1(u)}\right)ds}{\displaystyle\int_{t_1}^{t}\frac{du}{a_1(u)}} \\
&\leq -\left[\frac{\omega(t)}{\sqrt{\phi(t)a_1(t)}} - \frac{1}{2}\sqrt{\frac{a_1(t)}{\phi(t)}}\phi'(t)\right]^2 - \phi(t)q(t)kp_*(t)\Theta(t) + \frac{a_1(t)(\phi'(t))^2}{4\phi(t)},
\end{aligned}$$

which implies

$$\omega'(t) \leq -\phi(t)q(t)kp_*(t)\Theta(t) + \frac{a_1(t)(\phi'(t))^2}{4\phi(t)}.$$

Integrating the last inequality from $t_3$ $(> t_2)$ to $t$ we obtain

$$\int_{t_3}^{t}\left(\phi(s)q(s)kp_*(s)\Theta(s) - \frac{a_1(s)(\phi'(s))^2}{4\phi(s)}\right)ds \leq \omega(t_3).$$

Letting $t \to \infty$, it contradicts to (2.2).

Assume the case (2) holds. Using the similar proof of [10, Lemma 2], we can get $\lim_{t\to\infty} x(t) = 0$ due to condition (2.3). □

**Theorem 2.** *Let $0 \leq p(t) \leq p_1 \leq 1$. If (1.2) holds and there exists a positive function $\varphi \in C^1([t_0, \infty), \mathbb{R})$, such that for all sufficiently large $t_3 > t_2 > t_1 \geq t_0$, one has (2.2) and (2.3). If*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\delta(s)q(s)kp_*(s)\left(\int_{t_1}^{\sigma(s)}\frac{dv}{a_2(v)}\right) - \frac{1}{4\delta(s)a_1(s)}\right)ds = \infty, \tag{2.8}$$

*where*

$$\delta(t) := \int_{t}^{\infty}\frac{1}{a_1(s)}ds,$$

*holds for all constants $M > 0$, then $(E)$ is almost oscillatory.*

P r o o f. Suppose that $x(t)$ is a positive solution of $(E)$. By condition (1.2), there exist three possible cases:

(1) $Z(t) > 0$, $Z'(t) > 0$, $(a_2(t)Z'(t))' > 0$, $(a_1(t)(a_2(t)Z'(t))')' < 0$,

(2) $Z(t) > 0$, $Z'(t) < 0$, $(a_2(t)Z'(t))' > 0$, $(a_1(t)(a_2(t)Z'(t))')' < 0$, and

(3) $Z(t) > 0$, $Z'(t) > 0$, $(a_2(t)Z'(t))' < 0$, $(a_1(t)(a_2(t)Z'(t))')' < 0$, for $t \geq t_1$, $t_1$ is large enough.

In cases (1) and (2) we can obtain the conclusion of Theorem 2 by applying the proof of Theorem 1. Assume that case (3) holds, $(a_1(t)(a_2(t)Z'(t))')' < 0$ and $a_1(t)(a_2(t)Z'(t))'$ is nonincreasing. Thus, we get

$$a_1(s)(a_2(s)Z'(s))' \leq a_1(t)(a_2(t)Z'(t))', \quad s \geq t \geq t_1.$$

Dividing the above inequality by $a_1(s)$ and integrating from $t$ to $l$, we obtain

$$a_2(l)Z'(l) \leq a_2(t)Z'(t) + a_1(t)(a_2(t)Z'(t))' \int_t^l \frac{ds}{a_1(s)}.$$

Letting $l \to \infty$, we have

$$0 \leq a_2(t)Z'(t) + a_1(t)(a_2(t)Z'(t))' \int_t^\infty \frac{ds}{a_1(s)},$$

that is,

$$-\frac{a_1(t)(a_2(t)Z'(t))'}{a_2(t)Z'(t)} \int_t^\infty \frac{ds}{a_1(s)} \leq 1. \tag{2.9}$$

Now define $\varphi$ as

$$\varphi(t) := \frac{a_1(t)(a_2(t)Z'(t))'}{a_2(t)Z'(t)}, \quad t \geq t_1. \tag{2.10}$$

Then $\varphi(t) < 0$ for $t \geq t_1$. Therefore, by (2.9) and (2.10), we obtain

$$-\delta(t)\varphi(t) \leq 1. \tag{2.11}$$

Differentiating (2.10) gives

$$\varphi'(t) = \frac{(a_1(t)(a_2(t)Z'(t))')'}{a_2(t)Z'(t)} - \frac{a_1(t)a_1(t)(a_2(t)Z'(t))'(a_2(t)Z'(t))'}{(a_2(t)Z'(t))^2}.$$

Now $Z'(t) > 0$, so from $(E)$ and (2.4) we have

$$\varphi'(t) \leq -q(t)kp_*(t)\frac{Z(\sigma(t))}{a_2(t)Z'(t)} - \frac{a_1(t)a_1(t)(a_2(t)Z')'(a_2(t)Z'(t))'}{(a_2(t)Z'(t))^2}. \tag{2.12}$$

In view of case (3), we see that

$$Z(t) \geq a_2(t) \int_{t_1}^t \frac{ds}{a_2(s)} Z'(t). \tag{2.13}$$

Thus, $\left(\frac{Z(t)}{\int_{t_1}^{t} ds/a_2(s)}\right)' \leq 0$ implies

$$\frac{Z(\sigma(t))}{Z(t)} \geq \frac{\displaystyle\int_{t_1}^{\sigma(t)} \frac{ds}{a_2(s)}}{\displaystyle\int_{t_1}^{t} \frac{ds}{a_2(s)}}. \tag{2.14}$$

From (2.10) and (2.12)–(2.14) we have

$$\varphi'(t) \leq -q(t)kp_*(t) \int_{t_1}^{\sigma(t)} \frac{ds}{a_2(s)} - \frac{\varphi^2(t)}{a_1(t)}.$$

Multiplying the above inequality by $\delta(t)$ and integrating it from $t_2$ ($> t_1$) to $t$, we have

$$\begin{aligned}\varphi(t)\delta(t) - \varphi(t_2)\delta(t_2) + \int_{t_2}^{t} \delta(s)kq(s)p_*(s)\Big(\int_{t_1}^{\sigma(s)} \frac{dv}{a_2(v)}\Big)ds \\ + \int_{t_2}^{t} \frac{\varphi^2(s)\delta(s)}{a_1(s)}ds + \int_{t_2}^{t} \frac{\varphi(s)}{a_1(s)}ds \leq 0,\end{aligned}$$

$$\begin{aligned}\varphi(t)\delta(t) - \varphi(t_2)\delta(t_2) + \int_{t_2}^{t} \delta(s)kq(s)p_*(s)\Big(\int_{t_1}^{\sigma(s)} \frac{dv}{a_2(v)}\Big)ds - \int_{t_2}^{t} \frac{ds}{4\delta(s)a_1(s)} \\ + \int_{t_2}^{t} \left[\sqrt{\frac{\delta(s)}{a_1(s)}}\varphi(s) + \frac{1}{2}\frac{1}{\sqrt{a_1(s)\delta(s)}}\right]^2 ds \leq 0,\end{aligned}$$

from which it follows that

$$\int_{t_2}^{t} \left(\delta(s)q(s)kp_*(s)\left(\int_{t_1}^{\sigma(s)} \frac{dv}{a_2(v)}\right) - \frac{1}{4\delta(s)a_1(s)}\right) ds \leq 1 + \varphi(t_2)\delta(t_2)$$

due to (2.11). Letting $t \to \infty$, we come to the contradiction (2.8). Then the result of the Theorem follows. □

## 3. Examples

In this section we will present some examples to illustrate the main results.

*Example 1.* Consider a third-order neutral differential equation

$$\left(t^{-1/2}\left(t^{1/2}\left[x(t) + \frac{1}{4}x^{3/5}(t-1)\right]'\right)'\right)' + \frac{\lambda}{t^{1/2}}x(t-2) = 0, \quad t \geq 1, \tag{3.1}$$

where $\lambda > 0$ is a constant. Let $\alpha = 3/5$, $a_1(t) = t^{-1/2}$, $a_2(t) = t^{1/2}$, $p(t) = 1/4$, $q(t) = \frac{\lambda}{t^{1/2}}$, $\tau(t) = t-1$, and $\sigma(t) = t-2$. We obtain $p_*(t) = 1 - \frac{1/4}{M^{2/5}}$,

$$\int_{t_0}^{\infty} \frac{1}{a_2(v)} \int_{v}^{\infty} \frac{1}{a_1(u)} \left[\int_{u}^{\infty} \frac{\lambda}{s^{1/2}}ds\right] du\, dv = \infty.$$

and

$$\Theta(t)=\frac{\int_{t_2}^{\sigma(t)}\left(\frac{1}{a_2(s)}\int_{t_1}^{s}\frac{du}{a_1(u)}\right)ds}{\int_{t_1}^{t}\frac{du}{a_1(u)}}=\frac{(t-2)^3+6(t-2)^{1/2}t_1^{1/2}-3c}{t^{3/2}-t_1^{3/2}},$$

where $c=t_2^3/3+2t_1^{3/2}t_2^{1/2}$. Pick $\phi(t)=1$, then

$$\int_{t_3}^{\infty}q(s)p_*(s)\Theta(s)ds=\frac{\lambda}{3}\Big(1-\frac{1/4}{M^{2/5}}\Big)\int_{t_3}^{\infty}\frac{(s-2)^3+6(s-2)^{1/2}t_1^{1/2}-3c}{s^2-s^{1/2}t_1^{3/2}}=\infty,$$

if $1/4<M^{2/5}$. Hence, by Theorem 1, every solution of equation (3.1) is either oscillatory or converges to zero as $t\to\infty$ when $1/4<M^{2/5}$.

*Example 2.* Consider a third-order neutral differential equation

$$\left(t^2\left[x(t)+\frac{1}{2}x^{1/3}(t/8)\right]''\right)'+\frac{1}{t}\Big(1+\frac{2}{27}t^{2/3}\Big)x(t/2)=0,\quad t\geq 1. \tag{3.2}$$

Let $\alpha=1/3$, $a_1(t)=t^2$, $a_2(t)=1$, $p(t)=1/2$, $q(t)=1/t\Big(1+2/27\,t^{2/3}\Big)$, $\tau(t)=t/8$ and $\sigma(t)=t/2$. We obtain

$$p_*(t)=1-\frac{1/2}{M^{2/3}},\quad \delta(t)=\int_t^{\infty}\frac{ds}{s^2}=\frac{1}{t}$$

and

$$\int_{t_1}^{t/2}\frac{ds}{a_2(s)}=\frac{1}{2}(t-2t_1),$$

then

$$\int_{t_2}^{\infty}\left(\delta(s)q(s)p_*(s)\left(\int_{t_1}^{\sigma(s)}\frac{dv}{a_2(v)}\right)-\frac{1}{4\delta(s)a_1(s)}\right)ds$$
$$=\int_{t_2}^{\infty}\left(\Big(1-0.5M^{-2/3}\Big)\left[\frac{1}{2s}+\frac{1}{27}s^{-1/3}-\frac{t_1}{s^2}-\frac{2t_1}{27}s^{-4/3}\right]-\frac{1}{4s}\right)ds=\infty,$$

if $0.5<M^{2/3}$. Hence, by Theorem 2, every solution of equation (3.2) is either oscillatory or converges to zero as $t\to\infty$ when $0.5<M^{2/3}$ and $x(t)=t^{-1}$ is such a solution of (3.2).

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# EVALUATION OF THE NON-ELEMENTARY INTEGRAL $\int e^{\lambda x^\alpha} dx$, $\alpha \geq 2$, AND OTHER RELATED INTEGRALS

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**Abstract:** A formula for the non-elementary integral $\int e^{\lambda x^\alpha} dx$ where $\alpha$ is real and greater or equal two, is obtained in terms of the confluent hypergeometric function ${}_1F_1$ by expanding the integrand as a Taylor series. This result is verified by directly evaluating the area under the Gaussian Bell curve, corresponding to $\alpha = 2$, using the asymptotic expression for the confluent hypergeometric function and the Fundamental Theorem of Calculus (FTC). Two different but equivalent expressions, one in terms of the confluent hypergeometric function ${}_1F_1$ and another one in terms of the hypergeometric function ${}_1F_2$, are obtained for each of these integrals, $\int \cosh(\lambda x^\alpha)dx$, $\int \sinh(\lambda x^\alpha)dx$, $\int \cos(\lambda x^\alpha)dx$ and $\int \sin(\lambda x^\alpha)dx$, $\lambda \in \mathbb{C}, \alpha \geq 2$. And the hypergeometric function ${}_1F_2$ is expressed in terms of the confluent hypergeometric function ${}_1F_1$. Some of the applications of the non-elementary integral $\int e^{\lambda x^\alpha} dx, \alpha \geq 2$ such as the Gaussian distribution and the Maxwell-Bortsman distribution are given.

**Key words:** Non-elementary integral, Hypergeometric function, Confluent hypergeometric function, Asymptotic evaluation, Fundamental theorem of calculus, Gaussian, Maxwell-Bortsman distribution.

## 1. Introduction

**Definition 1.** *An elementary function is a function of one variable built up using that variable and constants, together with a finite number of repeated algebraic operations and the taking of exponentials and logarithms* [6].

In 1835, Joseph Liouville established conditions in his theorem, known as Liouville 1835's Theorem [4, 6], which can be used to determine whether an indefinite integral is elementary or non-elementary. Using Liouville 1835's Theorem, one can show that the indefinite integral $\int e^{\lambda x^\alpha} dx$, $\alpha \geq 2$, is non-elementary [4], and to my knowledge, no one has evaluated this non-elementary integral before.

For instance, if $\alpha = 2$, $\lambda = -\beta^2 < 0$, where $\beta$ is a real constant, the area under the Gaussian Bell curve can be calculated using double integration and then polar coordinates to obtain

$$\int\limits_{-\infty}^{+\infty} e^{-\beta^2 x^2} dx = \frac{\sqrt{\pi}}{\beta}. \tag{1.1}$$

Is that possible to evaluate (1.1) by directly using the Fundamental Theorem of Calculus (FTC) as in equation (1.2)?

$$\int\limits_{-\infty}^{+\infty} e^{-\beta^2 x^2} dx = \lim_{t \to -\infty} \int\limits_{t}^{0} e^{-\beta^2 x^2} dx + \lim_{t \to +\infty} \int\limits_{0}^{t} e^{-\beta^2 x^2} dx. \tag{1.2}$$

The Central limit Theorem (CLT) in Probability theory [2] states that the probability that a random variable $x$ does not exceed some observed value $z$ is

$$P(X < z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx. \tag{1.3}$$

So if we know the antiderivative of the function $g(x) = e^{\lambda x^2}$, we may choose to use the FTC to calculate the cumulative probability $P(X < z)$ in (1.3) when the value of $z$ is given or is known, rather than using numerical integration.

The Maxwell-Boltsman distribution in gas dynamics,

$$F(v) = \theta \int_{0}^{v} x^2 e^{-\gamma x^2} dx, \tag{1.4}$$

where $\theta$ and $\gamma$ are some positive constants that depend on the properties of the gas and $v$ is the gas speed, is another application.

There are many other examples where the antiderivative of $g(x) = e^{\lambda x^\alpha}$, $\alpha \geq 2$ can be useful. For example, using the FTC, formulas for integrals such as

$$\int_{x}^{\infty} e^{t^{2n+1}} dt, x < \infty; \quad \int_{x}^{\infty} e^{-t^{2n+1}} dt, x > -\infty; \quad \int_{x}^{\infty} t^{2n} e^{-t^2} dt, x \leq \infty, \tag{1.5}$$

where $n$ is a positive integer, can be obtained if the antiderivative of $g(x) = e^{\lambda x^\alpha}$, $\alpha \geq 2$ is known.

In this paper, the antiderivative of $g(x) = e^{\lambda x^\alpha}$, $\alpha \geq 2$, is expressed in terms of a special function, the confluent hypergeometric ${}_1F_1$ [1]. And the confluent hypergeometric ${}_1F_1$ is an entire function [3], and its properties are well known [1, 5]. The main goal here is to consider the most general case with $\lambda$ complex ($\lambda \in \mathbb{C}$), evaluate the non-elementary integral $\int e^{\lambda x^\alpha}$, $\alpha \geq 2$ and thus make possible the use of the FTC to compute the definite integral

$$\int_{A}^{B} e^{\lambda x^\alpha} dx, \tag{1.6}$$

for any A and B. And once (1.6) is evaluated, then integrals such as (1.1), (1.2), (1.3), (1.4) and (1.5) can also be evaluated using the FTC.

Using the hyperbolic and Euler identities,

$$\cosh(\lambda x^\alpha) = (e^{\lambda x^\alpha} + e^{-\lambda x^\alpha})/2, \quad \sinh(\lambda x^\alpha) = (e^{\lambda x^\alpha} - e^{-\lambda x^\alpha})/2,$$
$$\cos(\lambda x^\alpha) = (e^{i\lambda x^\alpha} + e^{-i\lambda x^\alpha})/2, \quad \sin(\lambda x^\alpha) = (e^{i\lambda x^\alpha} - e^{-i\lambda x^\alpha})/(2i),$$

the integrals

$$\int \cosh(\lambda x^\alpha) dx, \quad \int \sinh(\lambda x^\alpha) dx, \quad \int \cos(\lambda x^\alpha) dx \quad \text{and} \quad \int \sin(\lambda x^\alpha) dx, \alpha \geq 2, \tag{1.7}$$

are evaluated in terms of ${}_1F_1$ for any constant $\lambda$. They are also expressed in terms of the hypergeometric ${}_1F_2$. And some expressions of the hypergeometric function ${}_1F_2$ in terms of the confluent hypergeometric function ${}_1F_1$ are therefore obtained.

For reference, we shall first define the confluent confluent hypergeometric function ${}_1F_1$ and the hypergeometric function ${}_1F_2$ before we proceed to the main aims of this paper (see sections 2 and 3).

**Definition 2.** *The confluent hypergeometric function, denoted as* ${}_1F_1$*, is a special function given by the series* [1, 5]

$$ {}_1F_1(a;b;x)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{x^n}{n!}, \tag{1.8}$$

*where* $a$ *and* $b$ *are arbitrary constants,* $(\vartheta)_n=\Gamma(\vartheta+n)/\Gamma(\vartheta)$ *(Pochhammer's notation* [1]*) for any complex* $\vartheta$*, with* $(\vartheta)_0=1$*, and* $\Gamma$ *is the standard gamma function* [1].

**Definition 3.** *The hypergeometric function* ${}_1F_2$ *is a special function given by the series* [1, 5]

$$ {}_1F_2(a;b,c;x)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n(c)_n}\frac{x^n}{n!}, \tag{1.9}$$

*where* $a, b$ *and* $c$ *are arbitrary constants, and* $(\vartheta)_n=\Gamma(\vartheta+n)/\Gamma(\vartheta)$ *(Pochhammer's notation* [1]*) as in Definition 2.*

## 2. Evaluation of $\int_A^B e^{\lambda x^\alpha} dx$

**Proposition 1.** *The function* $G(x)=x\ {}_1F_1\left(\frac{1}{\alpha};\frac{1}{\alpha}+1;\lambda x^\alpha\right)$*, where* ${}_1F_1$ *is a confluent hypergeometric function* [1], $\lambda$ *is an arbitrarily constant and* $\alpha\geq 2$*, is the antiderivative of the function* $g(x)=e^{\lambda x^\alpha}$*. Thus,*

$$\int e^{\lambda x^\alpha}dx = x\ {}_1F_1\left(\frac{1}{\alpha};\frac{1}{\alpha}+1;\lambda x^\alpha\right)+C. \tag{2.1}$$

P r o o f. We expand $g(x)=e^{\lambda x^\alpha}$ as a Taylor series and integrate the series term by term. We also use the Pochhammer's notation [1] for the gamma function, $\Gamma(a+n)=\Gamma(a)(a)_n$, where $(a)_n=a(a+1)\cdots(a+n-1)$, and the property of the gamma function $\Gamma(a+1)=a\Gamma(a)$ [1]. For example, $\Gamma(n+a+1)=(n+a)\Gamma(n+a)$. We then obtain

$$\begin{aligned}\int g(x)dx &= \int e^{\lambda x^\alpha}dx=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int x^{\alpha n}dx\\ &=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\frac{x^{\alpha n+1}}{\alpha n+1}+C=\frac{x}{\alpha}\sum_{n=0}^{\infty}\frac{(\lambda x^\alpha)^n}{\left(n+\frac{1}{\alpha}\right)n!}+C\\ &=\frac{x}{\alpha}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{\alpha}\right)}{\Gamma\left(n+\frac{1}{\alpha}+1\right)}\frac{(\lambda x^\alpha)^n}{n!}+C\\ &=x\sum_{n=0}^{\infty}\frac{\left(\frac{1}{\alpha}\right)_n}{\left(\frac{1}{\alpha}+1\right)_n}\frac{(\lambda x^\alpha)^n}{n!}+C\\ &=x\ {}_1F_1\left(\frac{1}{\alpha};\frac{1}{\alpha}+1;\lambda x^\alpha\right)+C=G(x)+C. \quad\square\end{aligned} \tag{2.2}$$

*Example 1.* We can now evaluate $\int x^{2n}e^{\lambda x^2}dx$ in terms of the confluent hypergeometric function. Using integration by parts,

$$\int x^{2n}e^{\lambda x^2}dx=\frac{x^{2n-1}}{2\lambda}e^{\lambda x^2}-\frac{2n-1}{2\lambda}\int x^{2n-2}e^{\lambda x^2}dx. \tag{2.3}$$

1. For instance, for $n = 1$,

$$\int x^2 e^{\lambda x^2} dx = \frac{x}{2\lambda}e^{\lambda x^2} - \frac{1}{2\lambda}\int e^{\lambda x^2} dx = \frac{x}{2\lambda}e^{\lambda x^2} - \frac{x}{2\lambda}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};\lambda x^2\right) + C. \tag{2.4}$$

2. For $n = 2$,

$$\int x^4 e^{\lambda x^2} dx = \frac{x^3}{2\lambda}e^{\lambda x^2} - \frac{3}{2\lambda}\int x^2 e^{\lambda x^2} dx = \frac{x^3}{2\lambda}e^{\lambda x^2} - \frac{3x}{4\lambda^2}e^{\lambda x^2} + \frac{3x}{4\lambda^2}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};\lambda x^2\right) + C. \tag{2.5}$$

*Example 2.* Using the method of integrating factor, the first-order ordinary differential equation

$$y' + 2xy = 1 \tag{2.6}$$

has solution

$$y(x) = e^{-x^2}\left(\int e^{x^2} dx + C\right) = xe^{-x^2}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};x^2\right) + Ce^{-x^2}. \tag{2.7}$$

Assuming that the function $G(x)$ (see Proposition 1) is unknown, in the following lemma, we use the properties of function $g(x)$ to establish the properties of $G(x)$ such as the inflection points and the behavior as $x \to \pm\infty$.

**Lemma 1.** *Let the function $G(x)$ be an antiderivative of $g(x) = e^{\lambda x^\alpha}, \lambda \in \mathbb{C}$ with $\alpha \geq 2$.*

1. *If the real part of $\lambda$ is negative ($< 0$) and $\alpha$ is even, then the limits $lim_{x\to-\infty}G(x)$ and $lim_{x\to+\infty}G(x)$ are finite (constants). And thus the Lebesgue integral $\int_{-\infty}^{\infty}|e^{\lambda x^\alpha}|dx < \infty$.*
2. *If $\lambda$ is real ($\lambda \in \mathbb{R}$), then the point $(0, G(0)) = (0, 0)$ is an inflection point of the curve $Y = G(x)$, $x \in \mathbb{R}$.*
3. *And if $\lambda \in \mathbb{R}$ and $\lambda < 0$, and $\alpha$ is even, then the limits $\lim_{x\to-\infty} G(x)$ and $\lim_{x\to+\infty} G(x)$ are finite. And there exists real constant $\theta > 0$ such that limits $\lim_{x\to-\infty} G(x) = -\theta$ and $\lim_{x\to+\infty} G(x) = \theta$.*

P r o o f.

1. For complex $\lambda = \lambda_r + i\lambda_i$, where the subscript $r$ and $i$ stand for real and imaginary parts respectively, the function $g(x) = g(z) = e^{z^\alpha}$ where $z = (\lambda_r + i\lambda_i)^{1/\alpha}x$, $\alpha \geq 2$, is an entire function on $\mathbb{C}$. And if $\lambda_r < 0$ and $\alpha$ is even implies $\mathrm{Re}(z^\alpha)$ is always negative regardless of the values of $x$. And so, if $|z| \to \infty$ (or $x \to \pm\infty$), then $g(z) = 0$ ($g(z) \to 0$) (or $g(x) = 0$ as $x \to \pm\infty$). Therefore by Liouville theorem, $G(z)$ has to be constant as $|z| \to \infty$, and so is $G(x)$ as $x \to \pm\infty$. Hence, the Lebesgue integral

$$\int_{-\infty}^{\infty} |e^{\lambda x^\alpha}|dx = \int_{-\infty}^{\infty} e^{\lambda_r x^\alpha}|e^{\lambda_i x^\alpha}|dx = \int_{-\infty}^{\infty} e^{\lambda_r x^\alpha}dx < \infty$$

since $G(x)$ is constant as $x \to \pm\infty$. For $\lambda_r < 0$ and $\alpha$ odd, the limit $\lim_{x\to-\infty} e^{\lambda_r x^\alpha}$ diverges and so does the integral $\int_{-\infty}^{\infty} e^{\lambda_r x^\alpha}dx$. Therefore, the Lebesgue integral $\int_{-\infty}^{\infty}|e^{\lambda x^\alpha}|dx$ has to diverge too. On the other hand, for $\lambda_r > 0$, the limit $\lim_{x\to+\infty} e^{\lambda_r x^\alpha}$ diverges, and so does the integral $\int_{-\infty}^{\infty} e^{\lambda_r x^\alpha}dx$ regardless of the value of $\alpha$. Therefore, the Lebesgue integral $\int_{-\infty}^{\infty}|e^{\lambda x^\alpha}|dx$ has to diverge too.

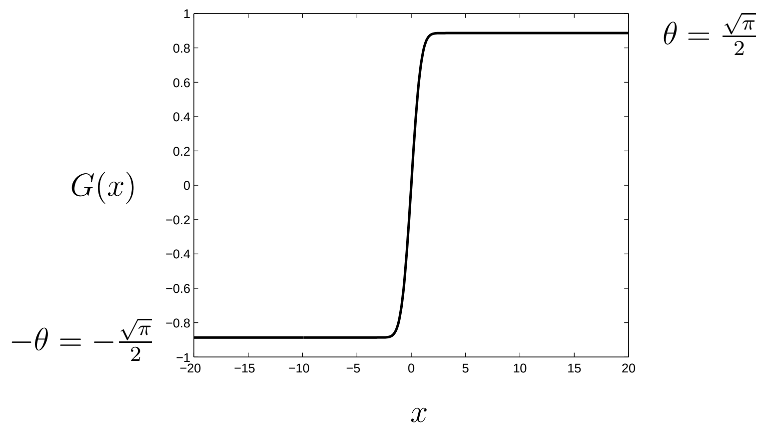


Figure 1. $G(x)$ is the antiderivative of $e^{-x^2}$ given by (2.8).

2. At $x = 0$, $g(0) = 1$. And so, around $x = 0$, the antiderivative $G(x) \sim x$ because $G'(0) = g(0) = 1$. And so $(0, G(0)) = (0, 0)$. Moreover, $G''(x) = g'(x) = \lambda\alpha x^{\alpha-1}e^{\lambda x^\alpha}, \alpha \geq 2$, gives $G''(0) = 0$. Hence, by the second derivative test, if $\lambda$ is real ($\lambda = \lambda_r$), the point $(0, G(0)) = (0, 0)$ is an inflection point of the curve $Y = G(x)$, $x \in \mathbb{R}$.

3. For $\lambda = \lambda_r$ ($\lambda \in \mathbb{R}$), both $g(x)$ and $G(x)$ are analytic on $\mathbb{R}$. Using this fact and the fact that for even $\alpha$ and $\lambda_r < 0$, $\int_{-\infty}^{\infty} |e^{\lambda x^\alpha}|dx < \infty$ implies that for even $\alpha$ and $\lambda_r < 0$, $G(x)$ has to be constant as $x \to \pm\infty$. In addition, the fact that $G''(x) < 0$ if $x < 0$ and $G''(x) > 0$ if $x > 0$ implies that, $G(x)$ is concave upward on the interval $(\infty, 0)$ while is concave downward on the interval $(0, +\infty)$. Moreover, the fact that $g(x) = G'(x)$ is symmetric about the $y$-axis (even) implies that $G(x)$ has to be antisymmetric about the $y$-axis (odd). Hence there exists a real positive constant $\theta > 0$ such that limits $\lim_{x\to-\infty} G(x) = -\theta$ and $\lim_{x\to+\infty} G(x) = \theta$.$\square$

*Example 3.* If $\lambda = -1$ and $\alpha = 2$, then

$$\int e^{-x^2} dx = x \; {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x^2\right) + C. \tag{2.8}$$

According to (2.8), the antiderivative of $g(x) = e^{-x^2}$ is $G(x) = x \; {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x^2\right)$. Its graph as a function of $x$, sketched using MATLAB, is shown in Figure 1. It is in agreement with Lemma 1. It is actually seen in Figure 1 that $(0, 0)$ is an inflection point and that $G(x)$ reaches some constants as $x \to \pm\infty$ as predicted by Lemma 1.

In the following lemma, we obtain the values of $G(x)$, the antiderivative of the function $g(x) = e^{\lambda x^\alpha}$, as $x \to \pm\infty$ using the asymptotic expansion of the confluent hypergeometric function ${}_1F_1$.

**Lemma 2.** *Consider $G(x)$ in Proposition 1.*

1. *Then for $|x| \gg 1$,*

$$G(x) = x {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha} + 1; \lambda x^\alpha\right) \sim \begin{cases} \Gamma\left(\frac{1}{\alpha} + 1\right) \dfrac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} \dfrac{x}{|x|} + \dfrac{e^{\lambda x^\alpha}}{\alpha\lambda x^{\alpha-1}}, \textit{if } \alpha \textit{ is even}, \\ \Gamma\left(\frac{1}{\alpha} + 1\right) \dfrac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} + \dfrac{e^{\lambda x^\alpha}}{\alpha\lambda x^{\alpha-1}}, \textit{if } \alpha \textit{ is odd}. \end{cases} \tag{2.9}$$

2. *Let $\alpha \geq 2$ and be even, and let $\lambda = -\beta^2$, where $\beta$ is a real number, preferably positive. Then*

$$G(-\infty) = \lim_{x\to-\infty} G(x) = \lim_{x\to-\infty} x_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; -\beta^2 x^{\alpha}\right) = -\frac{1}{\beta^{\frac{2}{\alpha}}}\Gamma\left(\frac{1}{\alpha}+1\right) \tag{2.10}$$

*and*

$$G(+\infty) = \lim_{x\to+\infty} G(x) = \lim_{x\to+\infty} x_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; -\beta^2 x^{\alpha}\right) = \frac{1}{\beta^{\frac{2}{\alpha}}}\Gamma\left(\frac{1}{\alpha}+1\right). \tag{2.11}$$

3. *And by the FTC,*

$$\int_{-\infty}^{\infty} e^{-\beta^2 x^{\alpha}} dx = G(+\infty) - G(-\infty)$$

$$= \frac{1}{\beta^{\frac{2}{\alpha}}}\Gamma\left(\frac{1}{\alpha}+1\right) - \left(-\frac{1}{\beta^{\frac{2}{\alpha}}}\Gamma\left(\frac{1}{\alpha}+1\right)\right) = \frac{2}{\beta^{\frac{2}{\alpha}}}\Gamma\left(\frac{1}{\alpha}+1\right). \tag{2.12}$$

P r o o f.

1. To prove (2.9), we use the asymptotic series for the confluent hypergeometric function that is valid for $|z| \gg 1$ ([1], formula 13.5.1),

$$\frac{{}_1F_1(a;b;z)}{\Gamma(b)} = \frac{e^{\pm i\pi a} z^{-a}}{\Gamma(b-a)}\left\{\sum_{n=0}^{R-1} \frac{(a)_n(1+a-b)_n}{n!}(-z)^{-n} + O(|z|^{-R})\right\}$$
$$+ \frac{e^z z^{a-b}}{\Gamma(a)}\left\{\sum_{n=0}^{S-1} \frac{(b-a)_n(1-a)_n}{n!}(z)^{-n} + O(|z|^{-S})\right\}, \tag{2.13}$$

where $a$ and $b$ are constants, and the upper sign being taken if $-\pi/2 < \arg(z) < 3\pi/2$ and the lower sign if $-3\pi/2 < \arg(z) \leq -\pi/2$. We set $z = \lambda x^{\alpha}, a = \frac{1}{\alpha}$ and $b = \frac{1}{\alpha}+1$, and obtain

$$\frac{{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; \lambda x^{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}+1\right)} = \frac{e^{i\frac{\pi}{\alpha}}}{(\lambda x^{\alpha})^{\frac{1}{\alpha}}}\left\{\sum_{n=0}^{R-1} \frac{\left(\frac{1}{\alpha}\right)_n}{n!}(\lambda x^{\alpha})^{-n} + O\left\{\lambda x^{\alpha}\right)^{-R}\right\}$$
$$+ \frac{e^{\lambda x^{\alpha}}(\lambda x^{\alpha})^{-1}}{\Gamma\left(\frac{1}{\alpha}\right)}\left\{\sum_{n=0}^{S-1}\left(1-\frac{1}{\alpha}\right)_n (\lambda x^{\alpha})^{-n} + O\left(\lambda x^{\alpha}\right)^{-S}\right\}. \tag{2.14}$$

Then, for $|x| \gg 1$,

$$\frac{e^{i\frac{\pi}{\alpha}}}{(\lambda x^{\alpha})^{\frac{1}{\alpha}}}\left\{\sum_{n=0}^{R-1} \frac{\left(\frac{1}{\alpha}\right)_n}{n!}(\lambda x^{\alpha})^{-n} + O\left\{\lambda x^{\alpha}\right)^{-R}\right\} \sim \begin{cases} \dfrac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\dfrac{1}{|x|}, \text{if } \alpha \text{ is even,} \\ \dfrac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\dfrac{1}{x}, \text{if } \alpha \text{ is odd,} \end{cases} \tag{2.15}$$

while

$$\frac{e^{\lambda x^{\alpha}}(\lambda x^{\alpha})^{-1}}{\Gamma\left(\frac{1}{\alpha}\right)}\left\{\sum_{n=0}^{S-1}\left(1-\frac{1}{\alpha}\right)_n (\lambda x^{\alpha})^{-n} + O\left(\lambda x^{\alpha}\right)^{-S}\right\} \sim \frac{e^{\lambda x^{\alpha}}}{\Gamma\left(\frac{1}{\alpha}\right)\lambda x^{\alpha}}. \tag{2.16}$$

And so, for $|x| \gg 1$,

$$\frac{{}_1F_1\left(\frac{1}{\alpha};\frac{1}{\alpha}+1;\lambda x^\alpha\right)}{\Gamma\left(\frac{1}{\alpha}+1\right)} \sim \begin{cases} \dfrac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\dfrac{1}{|x|} + \dfrac{e^{\lambda x^\alpha}}{\Gamma\left(\frac{1}{\alpha}\right)\lambda x^\alpha}, \text{if } \alpha \text{ is even,} \\ \dfrac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\dfrac{1}{x} + \dfrac{e^{\lambda x^\alpha}}{\Gamma\left(\frac{1}{\alpha}\right)\lambda x^\alpha}, \text{if } \alpha \text{ is odd.} \end{cases} \tag{2.17}$$

Hence,

$$G(x) = x_1F_1\left(\frac{1}{\alpha};\frac{1}{\alpha}+1;\lambda x^\alpha\right) \sim \begin{cases} \Gamma\left(\frac{1}{\alpha}+1\right)\dfrac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\dfrac{x}{|x|} + \dfrac{e^{\lambda x^\alpha}}{\alpha\lambda x^{\alpha-1}}, \text{if } \alpha \text{ is even,} \\ \Gamma\left(\frac{1}{\alpha}+1\right)\dfrac{e^{i\frac{\pi}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} + \dfrac{e^{\lambda x^\alpha}}{\alpha\lambda x^{\alpha-1}}, \text{if } \alpha \text{ is odd.} \end{cases} \tag{2.18}$$

2. Setting $\lambda = -\beta^2$, where $\beta$ is real and positive and using (2.9), then for $\alpha$ even,

$$G(x) = x_1F_1\left(\frac{1}{\alpha};\frac{1}{\alpha}+1;-\beta^2 x^\alpha\right) \sim \frac{1}{\beta^{\frac{2}{\alpha}}}\Gamma\left(\frac{1}{\alpha}+1\right)\frac{x}{|x|} - \frac{e^{-\beta^2 x^\alpha}}{\alpha\beta^2 x^{\alpha-1}}. \tag{2.19}$$

Therefore,

$$G(-\infty) = \lim_{x\to-\infty} G(x) = \lim_{x\to-\infty} x_1F_1\left(\frac{1}{\alpha};\frac{1}{\alpha}+1;-\beta^2 x^\alpha\right) = -\frac{1}{\beta^{\frac{2}{\alpha}}}\Gamma\left(\frac{1}{\alpha}+1\right) \tag{2.20}$$

and

$$G(+\infty) = \lim_{x\to+\infty} G(x) = \lim_{x\to+\infty} x_1F_1\left(\frac{1}{\alpha};\frac{1}{\alpha}+1;-\beta^2 x^\alpha\right) = \frac{1}{\beta^{\frac{2}{\alpha}}}\Gamma\left(\frac{1}{\alpha}+1\right). \tag{2.21}$$

3. By the Fundamental Theorem of Calculus, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-\beta^2 x^\alpha}dx &= \lim_{y\to-\infty}\int_y^0 e^{-\beta^2 x^\alpha}dx + \lim_{y\to+\infty}\int_0^y e^{-\beta^2 x^\alpha}dx \\ &= \lim_{y\to+\infty} y\ {}_1F_1\left(\frac{1}{\alpha};\frac{1}{\alpha}+1;-\beta^2 y^\alpha\right) - \lim_{y\to-\infty} y\ {}_1F_1\left(\frac{1}{\alpha};\frac{1}{\alpha}+1;-\beta^2 y^\alpha\right) \\ &= G(+\infty) - G(-\infty) \\ &= \frac{1}{\beta^{\frac{2}{\alpha}}}\Gamma\left(\frac{1}{\alpha}+1\right) - \left(-\frac{1}{\beta^{\frac{2}{\alpha}}}\Gamma\left(\frac{1}{\alpha}+1\right)\right) = \frac{2}{\beta^{\frac{2}{\alpha}}}\Gamma\left(\frac{1}{\alpha}+1\right). \end{aligned} \tag{2.22}$$

We now verify whether (2.22) is correct or not by double integration. We first observe that (2.22) is valid for all even $\alpha \geq 2$. And so, if (2.22) is verified for $\alpha = 2$, we are done since (2.22) is valid for all even $\alpha \geq 2$. For $\alpha = 2$, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-\beta^2 x^2}dx &= \lim_{y\to-\infty}\int_y^0 e^{-\beta^2 x^2}dx + \lim_{y\to+\infty}\int_0^y e^{-\beta^2 x^2}dx \\ &= \lim_{y\to+\infty} y\ {}_1F_1\left(\frac{1}{2};\frac{3}{2};-\beta^2 y^2\right) - \lim_{y\to-\infty} y\ {}_1F_1\left(\frac{1}{2};\frac{3}{2};-\beta^2 y^2\right) \\ &= G(+\infty) - G(-\infty) = \frac{2}{\beta}\Gamma\left(\frac{3}{2}\right) = \frac{2}{\beta}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{\beta}. \end{aligned} \tag{2.23}$$

On the other hand,

$$\left(\int_{-\infty}^{\infty} e^{-\beta^2 x^2} dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-\beta^2 x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-\beta^2 y^2} dy\right) \tag{2.24}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta^2 (x^2+y^2)} dydx. \tag{2.25}$$

In polar coordinate,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta^2 (x^2+y^2)} dydx = \int_{0}^{2\pi}\int_{0}^{\infty} e^{-\beta^2 r^2} rdrd\theta = \frac{1}{2\beta^2}\int_{0}^{2\pi} d\theta = \frac{\pi}{\beta^2}. \tag{2.26}$$

This gives

$$\int_{-\infty}^{\infty} e^{-\beta^2 x^2} dx = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dydx} = \frac{\sqrt{\pi}}{\beta} \tag{2.27}$$

as before. $\square$

*Example 4.* Setting $\lambda = -\beta^2 = -1$, $\beta = 1$ and $\alpha = 2$ in Lemma 2 gives

$$G(-\infty) = \lim_{x\to-\infty} G(x) = \lim_{x\to-\infty} x\ {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x^2\right) = -\frac{\sqrt{\pi}}{2} \tag{2.28}$$

and

$$G(+\infty) = \lim_{x\to+\infty} G(x) = \lim_{x\to+\infty} x\ {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x^2\right) = \frac{\sqrt{\pi}}{2}. \tag{2.29}$$

This implies $\theta = \sqrt{\pi}/2$ in Lemma 1. And this is exactly the value of $G(x)$ as $x \to \infty$ in Figure 1. We also have $\lim_{x\to-\infty} G(x) = -\theta = -\sqrt{\pi}/2$ as in Figure 1. Using the FTC, we readily obtain

$$\int_{-\infty}^{0} e^{-x^2} dx = G(0) - G(-\infty) = 0 - \left(-\frac{\sqrt{\pi}}{2}\right) = \frac{\sqrt{\pi}}{2}, \tag{2.30}$$

$$\int_{0}^{+\infty} e^{-x^2} dx = G(+\infty) - G(0) = \frac{\sqrt{\pi}}{2} - 0 = \frac{\sqrt{\pi}}{2} \tag{2.31}$$

and

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = G(+\infty) - G(-\infty) = \frac{\sqrt{\pi}}{2} - \left(-\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}. \tag{2.32}$$

*Example 5.* In this example, the integral

$$\int_{-\infty}^{x} e^{t^{2n+1}} dt, \quad x < \infty, \tag{2.33}$$

where $n$ is a positive integer, is evaluated using Proposition 1 and the asymptotic expression (2.9). Setting $\lambda = 1$ and $\alpha = 2n+1$ in Proposition 1 , and using (2.9) gives

$$\begin{aligned}
\int\limits_{-\infty}^{x} e^{t^{2n+1}} dt &= \lim_{y\to-\infty} \int\limits_{y}^{x} e^{t^{2n+1}} dt \\
&= x\ {}_1F_1\left(\frac{1}{2n+1}; \frac{2n+2}{2n+1}; x^{2n+1}\right) - \lim_{y\to-\infty} y\ {}_1F_1\left(\frac{1}{2n+1}; \frac{2n+2}{2n+1}; y^{2n+1}\right) \\
&= x\ {}_1F_1\left(\frac{1}{2n+1}; \frac{2n+2}{2n+1}; x^{2n+1}\right) - \Gamma\left(\frac{2n+2}{2n+1}\right), \quad x < \infty.
\end{aligned} \tag{2.34}$$

One can also obtain

$$\begin{aligned}
\int\limits_{x}^{+\infty} e^{-t^{2n+1}} dt &= \lim_{y\to+\infty} \int\limits_{x}^{y} e^{-t^{2n+1}} dt \\
&= \lim_{y\to-\infty} y\ {}_1F_1\left(\frac{1}{2n+1}; \frac{2n+2}{2n+1}; -y^{2n+1}\right) - x\ {}_1F_1\left(\frac{1}{2n+1}; \frac{2n+2}{2n+1}; -x^{2n+1}\right) \\
&= \Gamma\left(\frac{2n+2}{2n+1}\right) - x\ {}_1F_1\left(\frac{1}{2n+1}; \frac{2n+2}{2n+1}; -x^{2n+1}\right), \quad x > -\infty.
\end{aligned} \tag{2.35}$$

**Theorem 1.** *For any $A$ and $B$, the FTC gives*

$$\int\limits_{A}^{B} e^{\lambda x^{\alpha}} dx = G(B) - G(A), \tag{2.36}$$

*where $G$ is the antiderivative of the function $g(x) = e^{\lambda x^{\alpha}}$ and is given in Proposition 1. And $\lambda$ is any complex or real constant, and $\alpha \geq 2$.*

P r o o f. $G(x) = x\ {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; \lambda x^{\alpha}\right)$, where $\lambda$ is any constant, is the antiderivative of $g(x) = e^{\lambda x^{\alpha}}, \alpha \geq 2$ by Proposition 1, Lemma 1 and Lemma 2. And since the FTC works for $A = -\infty$ and $B = 0$ in (2.30), $A = 0$ and $B = +\infty$ in (2.31) and $A = -\infty$ and $B = +\infty$ in (2.32) by Lemma 2 if $\lambda = 1$ and $\alpha = 2$, and for all $\lambda < 0$ and all even $\alpha \geq 2$, then it has to work for other values of $A, B \in \mathbb{R}$ and for any $\lambda \in \mathbb{C}$ and $\alpha \geq 2$. This completes the proof. □

*Example 6.* In this example, we apply Theorem 1 to the Central Limit Theorem in Probability theory [2]. The normal zero-one distribution of a random variable X is the measure $\mu(dx) = g_X(x)dx$, where $dx$ is the Lebesgue measure and the function $g_X(x)$ is the probability density function (p.d.f) of the normal zero-one distribution [2], and is

$$g_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < +\infty. \tag{2.37}$$

A comparison with the function g(x) in Proposition 1 and Lemma 1 gives $\lambda = \beta^2 = -1/2$ and $\alpha = 2$. By Theorem 1, the cumulative probability, $P(X < z)$, is then given by

$$P(X < z) = \mu\{(-\infty, z)\} = \int_{-\infty}^{z} g_X(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx = \frac{1}{2} + \frac{z}{\sqrt{2\pi}}\ {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -\frac{z^2}{2}\right). \quad (2.38)$$

For example, we can also use Theorem 1 to obtain $P(-2 < X < 2) = \mu(-2, 2) = 0.4772 - (-0.4772) = 0.9544$, $P(-1 < X < 2) = \mu(-1, 2) = 0.4772 - (-0.3413) = 0.8185$ and so on.

*Example 7.* Using integration by parts and applying Theorem 1, the Maxwell-Bortsman distribution is written in terms of the confluent hypergeometric ${}_1F_1$ as

$$F(v) = \theta \int_0^v x^2 e^{-\gamma x^2} dx = -\frac{\theta v}{2\gamma} e^{-\gamma v^2} + \frac{\theta v}{2\gamma}\ {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -\gamma v^2\right) = \frac{\theta v}{2\gamma}\left[{}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -\gamma v^2\right) - e^{-\gamma v^2}\right]. \quad (2.39)$$

## 3. Other related non-elementary integrals

**Proposition 2.** *The function* $G(x) = x\ {}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2}, \frac{1}{2\alpha} + 1; \frac{\lambda^2 x^{2\alpha}}{4}\right)$, *where* ${}_1F_2$ *is a hypergeometric function* [1], $\lambda$ *is an arbitrarily constant and* $\alpha \geq 2$, *is the antiderivative of the function* $g(x) = \cosh(\lambda x^\alpha)$. *Thus,*

$$\int \cosh(\lambda x^\alpha) dx = x\ {}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2}, \frac{1}{2\alpha} + 1; \frac{\lambda^2 x^{2\alpha}}{4}\right) + C. \quad (3.1)$$

P r o o f. We proceed as before. We expand $g(x) = \cosh(\lambda x^\alpha)$ as a Taylor series and integrate the series term by term, use the Pochhammers notation [1] for the gamma function, $\Gamma(a + n) = \Gamma(a)(a)_n$, where $(a)_n = a(a+1)\cdots(a+n-1)$, and the property of the gamma function $\Gamma(a+1) = a\Gamma(a)$ [1]. We also use the Gamma duplication formula [1]. We then obtain

$$\begin{aligned}
\int g(x)dx &= \int \cosh(\lambda x^\alpha)dx = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} \int x^{2\alpha n} dx \\
&= \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} \frac{x^{2\alpha n+1}}{2\alpha n + 1} + C \\
&= \frac{x}{2\alpha} \sum_{n=0}^{\infty} \frac{(\lambda^2 x^{2\alpha})^n}{(2n)!\left(n + \frac{1}{2\alpha}\right)} + C \\
&= \frac{x}{2\alpha} \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1}{2\alpha}\right)}{\Gamma(2n+1)\Gamma\left(n + \frac{1}{2\alpha} + 1\right)} (\lambda^2 x^{2\alpha})^n + C \\
&= x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2\alpha}\right)_n}{\left(\frac{1}{2}\right)_n \left(\frac{1}{2\alpha} + 1\right)_n} \frac{(\lambda^2 x^{2\alpha})^n}{n!} + C \\
&= x\ {}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2}, \frac{1}{2\alpha} + 1; \frac{\lambda^2 x^{2\alpha}}{4}\right) + C = G(x) + C. \quad \square
\end{aligned} \quad (3.2)$$

**Proposition 3.** *The function*

$$G(x) = \frac{\lambda x^{\alpha+1}}{\alpha + 1}\ {}_1F_2\left(\frac{1}{2\alpha} + \frac{1}{2}; \frac{3}{2}, \frac{1}{2\alpha} + \frac{3}{2}; \frac{\lambda^2 x^{2\alpha}}{4}\right),$$

*where* ${}_1F_2$ *is a hypergeometric function* [1], $\lambda$ *is an arbitrarily constant and* $\alpha \geq 2$, *is the antiderivative of the function* $g(x) = \sinh(\lambda x^{\alpha})$. *Thus,*

$$\int \sinh(\lambda x^{\alpha})dx = \frac{\lambda x^{\alpha+1}}{\alpha+1}\ {}_1F_2\left(\frac{1}{2\alpha}+\frac{1}{2}; \frac{3}{2}, \frac{1}{2\alpha}+\frac{3}{2}; \frac{\lambda^2 x^{2\alpha}}{4}\right) + C. \tag{3.3}$$

P r o o f. As above, we expand $g(x) = \sinh(\lambda x^{\alpha})$ as a Taylor series and integrate the series term by term, use the Pochhammers notation [1] for the gamma function, $\Gamma(a+n) = \Gamma(a)(a)_n$, where $(a)_n = a(a+1)\cdots(a+n-1)$, and the property of the gamma function $\Gamma(a+1) = a\Gamma(a)$ [1]. We also use the Gamma duplication formula [1]. We then obtain

$$\begin{aligned}
\int g(x)dx = \int \sinh(\lambda x^{\alpha})dx &= \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{(2n+1)!} \int x^{2\alpha n+\alpha} dx \\
&= \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{(2n+1)!} \frac{x^{2\alpha n+\alpha+1}}{2\alpha n+\alpha+1} + C \\
&= \frac{\lambda x^{\alpha+1}}{2\alpha} \sum_{n=0}^{\infty} \frac{(\lambda^2 x^{2\alpha})^n}{(2n+1)!\left(n+\frac{1}{2\alpha}+\frac{1}{2}\right)} + C \\
&= \frac{\lambda x^{\alpha+1}}{2\alpha} \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2\alpha}+\frac{1}{2}\right)}{\Gamma(2n+2)\Gamma\left(n+\frac{1}{2\alpha}+\frac{3}{2}\right)} (\lambda^2 x^{2\alpha})^n + Cr \\
&= \frac{\lambda x^{\alpha+1}}{\alpha+1} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2\alpha}+\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n \left(\frac{1}{2\alpha}+\frac{3}{2}\right)_n} \frac{(\lambda^2 x^{2\alpha})^n}{n!} + C \\
&= \frac{\lambda x^{\alpha+1}}{\alpha+1}\ {}_1F_2\left(\frac{1}{2\alpha}+\frac{1}{2}; \frac{3}{2}, \frac{1}{2\alpha}+\frac{3}{2}; \frac{\lambda^2 x^{2\alpha}}{4}\right) + C = G(x) + C. \quad \square
\end{aligned} \tag{3.4}$$

We also can show as above that

$$\int \cos(\lambda x^{\alpha})dx = x\ {}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2}, \frac{1}{2\alpha}+1; -\frac{\lambda^2 x^{2\alpha}}{4}\right) + C \tag{3.5}$$

and

$$\int \sin(\lambda x^{\alpha})dx = \frac{\lambda x^{\alpha+1}}{\alpha+1}\ {}_1F_2\left(\frac{1}{2\alpha}+\frac{1}{2}; \frac{3}{2}, \frac{1}{2\alpha}+\frac{3}{2}; -\frac{\lambda^2 x^{2\alpha}}{4}\right) + C. \tag{3.6}$$

**Theorem 2.** *For any constants* $\alpha$ *and* $\lambda$,

$${}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2}, \frac{1}{2\alpha}+1; \frac{\lambda^2 x^{2\alpha}}{4}\right) = \frac{1}{2}\left[{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; \lambda x^{\alpha}\right) + {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; -\lambda x^{\alpha}\right)\right] \tag{3.7}$$

*and*

$${}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2}, \frac{1}{2\alpha}+1; -\frac{\lambda^2 x^{2\alpha}}{4}\right) = \frac{1}{2}\left[{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; i\lambda x^{\alpha}\right) + {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; -i\lambda x^{\alpha}\right)\right]. \tag{3.8}$$

P r o o f. Using Proposition 1, we obtain

$$\begin{aligned}
\int \cosh(\lambda x^{\alpha})dx = \int \frac{e^{\lambda x^{\alpha}} + e^{-\lambda x^{\alpha}}}{2} dx \\
= \frac{x}{2}\left[{}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; \lambda x^{\alpha}\right) + {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; -\lambda x^{\alpha}\right)\right] + C.
\end{aligned} \tag{3.9}$$

Hence, comparing (3.1) with (3.9) gives (3.7). Using Proposition 1, on the other hand, we obtain

$$\int \cos(\lambda x^\alpha) dx = \int \frac{e^{i\lambda x^\alpha} + e^{-i\lambda x^\alpha}}{2} dx$$
$$= \frac{x}{2} \left[ {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; i\lambda x^\alpha\right) + {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; -i\lambda x^\alpha\right)\right] + C. \quad (3.10)$$

Hence, comparing (3.5) with (3.10) gives (3.8). □

**Theorem 3.** *For any constants $\alpha$ and $\lambda$,*

$$\frac{\lambda x^\alpha}{\alpha+1} {}_1F_2\left(\frac{1}{2\alpha}+\frac{1}{2}; \frac{3}{2}, \frac{1}{2\alpha}+\frac{3}{2}; -\frac{\lambda^2 x^{2\alpha}}{4}\right)$$
$$= \frac{1}{2}\left[ {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; \lambda x^\alpha\right) - {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; -\lambda x^\alpha\right)\right] \quad (3.11)$$

*and*

$$\frac{\lambda x^\alpha}{\alpha+1} {}_1F_2\left(\frac{1}{2\alpha}+\frac{1}{2}; \frac{3}{2}, \frac{1}{2\alpha}+\frac{3}{2}; -\frac{\lambda^2 x^{2\alpha}}{4}\right)$$
$$= \frac{1}{2i}\left[ {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; i\lambda x^\alpha\right) - {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; -i\lambda x^\alpha\right)\right]. \quad (3.12)$$

P r o o f. Using Proposition 1, we obtain

$$\int \sinh(\lambda x^\alpha) dx = \int \frac{e^{\lambda x^\alpha} + e^{-\lambda x^\alpha}}{2} dx$$
$$= \frac{x}{2}\left[ {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; \lambda x^\alpha\right) - {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; -\lambda x^\alpha\right)\right] + C. \quad (3.13)$$

Hence, comparing (3.3) with (3.13) gives (3.11). Using Proposition 1, on the other hand, we obtain

$$\int \sin(\lambda x^\alpha) dx = \int \frac{e^{i\lambda x^\alpha} + e^{-i\lambda x^\alpha}}{2i} dx$$
$$= \frac{x}{2i}\left[ {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; i\lambda x^\alpha\right) - {}_1F_1\left(\frac{1}{\alpha}; \frac{1}{\alpha}+1; -i\lambda x^\alpha\right)\right] + C. \quad (3.14)$$

Hence, comparing (3.6) with (3.14) gives (3.12). □

## 4. Conclusion

The non-elementary integral $\int e^{\lambda x^\alpha} dx$, where $\lambda$ is an arbitrary constant and $\alpha \geq 2$, was expressed in term of the confluent hypergeometric function ${}_1F_1$. And using the properties of the confluent hypergeometric function ${}_1F_1$, the asymptotic expression for $|x| \gg 1$ of this integral was derived too. As established in Theorem 1, the definite integral (1.6) can now be computed using the FTC. For example, one can evaluate the area under the Gaussian Bell curve using the FTC rather than using double integration and then polar coordinates. One can also choose to use Theorem 1 to compute the cumulative probability for the normal distribution or that for the Maxwell-Bortsman distribution as shown in examples 6 and 7.

On one hand, the integrals $\int \cosh(\lambda x^{\alpha})dx$, $\int \sinh(\lambda x^{\alpha})dx$, $\int \cos(\lambda x^{\alpha})dx$ and $\int \sin(\lambda x^{\alpha})dx$, $\alpha \geq 2$, were evaluated in terms of the confluent hypergeometric function ${}_1F_1$, while on another hand, they were expressed in terms of the hypergeometric ${}_1F_2$. This allowed to express the hypergeometric function${}_1F_2$ in terms of the confluent hypergeometric function ${}_1F_1$ (Theorems 2 and 3).

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# NEW METHOD OF REFLECTOR SURFACE SHAPING TO PRODUCE A PRESCRIBED CONTOUR BEAM[1]

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**Abstract:** In this paper a simple iterative synthesis method is presented for the formation of the shape of the reflector surface with a single feed element to produce the desired contour beam. This is the method of the optimal phase synthesis of the appropriate field in the reflector aperture similar to other works. But unlike them, we solve the problem in a very simple way using the properties of complex-valued functions and Fourier transforms and not applying complicated methods of numerical minimization theory.

**Key words:** Antenna, Shaped reflector, Radiation pattern, Contour beam, Synthesis.

## Introduction

We consider the problem of synthesis of antenna reflector surface with a single feed. Such surfaces are constructed to generate a desirable far-field pattern available for the reflector aperture.

There are antennas with a single feed that form contour beams by means of appropriate profiled reflector surfaces for serving separated districts from spacecraft. They have a very simple construction, are reliable in exploitation, and optimally solve problems of electromagnetic compatibility.

Several synthesizing methods are known for the problem. They proved themselves to be efficient but are related to the minimization problems of multi-parameter goal functions. Some of these methods are direct methods for optimal modification of the basis reflector surface represented by polynomials, splines, and wavelets [1–4]. Other methods are related to preliminary synthesizing the electromagnetic fields in the reflector aperture which generate the assembly of narrow partial beams with subsequent optimization of the disposition of their maximums and selection of their superposition parameters. After that, the computation of the reflector surface form is carried out to generate the synthesized optimal aperture field [5–10]. The developed methods are very efficient for the synthesis of the reflector surface with the diameter of several tenth of the wavelength.

In this paper, another method is presented for contour beam synthesizing by antenna with a single feed element and a special reflector shape. It is also related to solving optimization problems but without the application of multi-parameter nonlinear optimization theory. The method is entirely based on the specific setting of the reflector surface synthesizing problem and on the properties of complex-valued functions and Fourier transforms. It is possible to adapt the method to the problem of phase control of large radiating arrays in real time.

The computation of the electric field in opening of the initial reflector (for example, parabolic revolution with focus at the phase center of the feeding element) is carried out in the knots of the uniform lattice at very close aperture. It is implemented by means of a vector radiation model of

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the array, which has a sufficiently dense set of points on the initial reflector surface. Dimensions of the lattice cells must guarantee the exact calculation of the Kirchhoff integral for the far zone field.

Our method, like many others, is iterative. The phase synthesis of a given pattern for the minimization of the amplitude distribution deviation from the desired far-field pattern is performed in $L^2(\mathbb{R}^2)$ space. We apply (here, perhaps, for the first time) the iterative methods alternately using direct and inverse Fourier transforms. It does not require any classic ways of numerical differentiation and solutions of large systems of linear equations. The mentioned iteration procedure is implemented until a completely suitable phase distribution at the reflector aperture is found. Moreover, the procedure may be accompanied by a proper shaping of the reflector surface at every step of the iteration. To represent intermediate variants of the constructed surface, we make it "scaly". Every "scale" is a fragment of a parabolic surface having the same focus as the initial one. Of course, we do not consider the diffraction at the edges of these (virtual) "scales", because, at the final stage of optimization, the weakly discontinuous surface will be changed by a continuous and smooth one.

## 1. Physical and phenomenological basis of the method

Bellow, the size and contour of the aperture, the pattern of the feed element, and model $D(\theta,\varphi)$ of the required far-field radiation pattern are supposed to be given. The latter can be a model of different beams, including contour ones, whose parameters conform to the reflector opening.

Getting to the problem, we assume that the initial reflector surface is cut off from a paraboloid of revolution by a plane orthogonal to its axis. The corresponding part of the plane is assumed to be its aperture $A$ containing the origin $O$ and the axes $OX$ and $OY$ of the Cartesian coordinate system, whose axis $OZ$ coincides with the axis of the parabolic surface and contains its focus $F$ coincident with the phase center of the feed element.

At first, it is necessary to the calculate electric field at the aperture $A$ (more exactly, its component $\dot{E}(x,y)$ on the chosen polarization). It should be done very precisely, because it is possible (although not necessary) do not change the amplitude distribution $E(x,y) = |\dot{E}(x,y)|$ in the aperture $A$ in future (neglecting the weak influence of local shifts of primary reflector surface cells). The phase $S_0(x,y) = \arg\dot{E}(x,y)/(2\pi)$ will be the initial functional "parameter" which we are going to change in the course of the reflector surface synthesis. The electrodynamic problem of searching the field $\dot{E}(x,y)$ is not considered in this article, since it can be solved by other known methods. We found that the computation according to the vector model formulas (see, for example, [13]) is very efficient.

## 2. Model of the algorithm

First we expound a continuous version of the algorithm for the synthesis of the reflector surface, which will be necessary to carry out numerically. Knowing $\dot{E}(x,y)$ in the reflector aperture $A$, we have a representation of the electric field expected in the far zone as the Kirchhoff integral

$$\widehat{E}(u,v) = \iint\limits_A \dot{E}(x,y)\mathrm{e}^{-2\pi\mathrm{i}(ux+vy)}\,dxdy, \tag{2.1}$$

where $u = k\sin\theta\cos\varphi$, $v = k\sin\theta\sin\varphi$, $(\theta,\varphi)$ are the angles of the spherical coordinate system, $k = 1/\lambda$, and $u,v$ are dimensionless variables, because $x$, $y$ are measured in wavelengths. The function $S_0(x,y)$ is determined from the condition $\dot{E}(x,y) = E(x,y)\exp(2\pi i S_0(x,y))$. Although it involves the phase component of the feed, it can be interpreted as the length of the optical path from a point $(x,y)\in A$ through the corresponding point on the reflector surface and then up to the phase centre of the feed (also measured in wavelength). Disregarding diffraction at the edge of the reflector, we suppose that the antenna radiation extends only to the half-space $z > 0$ and $E(x, y) \equiv 0$ under the condition $(x, y) \in R^2 \backslash A$. Thus, from the previous reasoning and (2.1), we have

$$\widehat{E}(u, v) = \widehat{E}(u, v, S) = \iint\limits_{\mathbb{R}^2} E(x, y) e^{2\pi i S(x,y)} e^{-2\pi i(ux+vy)}\, dxdy, \quad S = S_0, \tag{2.2}$$

that is the Fourier transform of the predetermined function $\dot{E}(x, y)$. If needed, we smoothly extend the given contour model $D(u, v)$ of the radiation pattern a little outside the domain of interest and, denoting the new domain by $\Omega$, we set $D(u, v) \equiv 0$ for $(u, v) \in \mathbb{R}^2 \backslash \Omega$. We can define the mean square deviation by the formula

$$\Delta(D, \widehat{E}) =: \left\| D(u, v) - |\widehat{E}(u, v; S)| \right\| = \left( \iint\limits_{\mathbb{R}^2} (D(u, v) - |\widehat{E}(u, v; S)|)^2\, dudv \right)^{1/2}, \tag{2.3}$$

and formulate the following problem of antenna phase synthesis: to find a function $S(x, y) = S(x, y; D)$ for which the value

$$\delta =: \inf_S \Delta(D(u, v), \widehat{E}(u, v; S)) \tag{2.4}$$

is attained. Here, $S$ belongs to the class of all measurable real-valued functions. It should be noted that only a part of $R^2$ (the ball $u^2 + v^2 \leq 1$) really lies in the physical space. But, in problem (2.4), we also minimize the energy flow

$$\iint\limits_{u^2+v^2>1} \left|\widehat{E}(u, v; S)\right|^2 dudv = \iint\limits_{u^2+v^2>1} \left|D(u, v) - \widehat{E}(u, v; S)\right|^2 dudv$$

in the antenna reactive zone. Further, we apply the usual iterative procedures for phase synthesis worked out for hybrid reflector antennas (HRA) (see, for example, [11, 12]), but here they are especially clear because do not use a finite-dimensional approximation of the antenna radiation $\widehat{E}(u, v; S)$.

Obviously, we have

$$\begin{aligned} \delta^2 &= \inf_S \inf_\psi \iint\limits_{\mathbb{R}^2} \left| D(u, v) e^{i\psi(u,v)} - F^-(E(x, y) e^{2\pi i S(x,y)})(u, v) \right|^2 dudv = \\ &= \inf_\psi \inf_S \iint\limits_{\mathbb{R}^2} \left| F^+(D(u, v) e^{i\psi(u,v)})(x, y) - E(x, y) e^{2\pi i S(x,y)} \right|^2 dxdy, \end{aligned} \tag{2.5}$$

where $\psi(u, v)$ is a real-valued measurable function like $S(x, y)$ and functions $(F^\pm g)(s, t)$ are inverse (with $+$) and direct (with $-$) Fourier transforms defined for a function $g \in L(\mathbb{R}^2)$ by

$$(F^\pm g)(s, t) = \iint\limits_{\mathbb{R}^2} g(\xi, \zeta) e^{\pm 2\pi i(s\xi + t\zeta)}\, d\xi d\zeta,$$

and then extended to $L^2(\mathbb{R}^2)$ in a reasonable well-known way. To represent $\delta$ defined by (2.4) as like (2.5), we have used the following considerations:

1) from the properties of complex-valued functions, it follows that the first integrand in (2.5) is minimal for $\psi$ equal to $\psi_{\mathrm{s}}$ defined

$$\psi_S(u, v) =: \arg \widehat{E}(u, v; S) = \arg(F^-(E \mathrm{e}^{2\pi \mathrm{i} S}))(u, v) = \arg F^-(\dot{E})), \tag{2.6}$$

so the inner infimum over $\psi$ coincides with $\Delta^2(D, \dot{E})$ (see (2.3)) in view of formulas (2.2)–(2.4) and the definition of $F^{-}g$;

2) it is easy to see that, for any positive functional $G(S,\psi)$ on the spaces of real-valued measurable functions $S(x,y)$ and $\psi(u,v)$, the following formula $\inf\limits_{S}\inf\limits_{\psi} G(S,\psi) = \inf\limits_{\psi}\inf\limits_{S} G(S,\psi)$ holds;

3) the equality of integrals in (2.5) follows from the Parseval equality.

Reasoning as in 1), we see that the inner infimum in the last part of (2.5) is attained for $2\pi S$ equal to

$$2\pi S(x,y) = 2\pi S_\psi(u,v) =: \arg(F^{+}(De^{i\psi}))(x,y). \tag{2.7}$$

From these considerations, it follows that the solution $S(x,y)$ ($(x,y)\in A$) of problem (2.4) together with the solution $\psi$ of the problem

$$\inf_{\psi} \iint\limits_{\mathbb{R}^2} \Big(E(x,y) - \Big|F^{+}(De^{i\psi})(x,y)\Big|\Big)^2\, dxdy$$

must be connected by the nonlinear equations (2.6) and (2.7). Except for special cases, this system can be solved only approximately by a numerical method.

To construct such a method, we use the obvious fact that, for every above mentioned functions $S(x,y)$ and $\psi(u,v)$, we have the following inequalities for the norms in the space $L^2(\mathbb{R}^2)$ hold:

$$\begin{aligned}\Big\|De^{i\psi_s} - F^{-}(Ee^{2\pi iS})\Big\| &\le \Big\|De^{i\psi} - F^{-}(Ee^{2\pi iS})\Big\| \\ &= \Big\|F^{+}(De^{i\psi}) - Ee^{2\pi iS}\Big\|,\end{aligned} \tag{2.8}$$

$$\Big\|F^{+}(De^{i\psi}) - Ee^{2\pi iS_\psi}\Big\| \le \Big\|F^{+}(De^{i\psi}) - Ee^{2\pi iS}\Big\|_{L^2(R)}, \tag{2.9}$$

where $\psi_S$ and $S_\psi$ are defined in (2.6) and (2.7). Further, beginning with the function $S_0(x,y)$ and alternately using formulas (2.6) and (2.7), we construct the following chain of functions:

$$\psi_{S_0}(u,v),\ S_{\psi_{S_0}}(x,y) =: S_1(x,y),\ \psi_{S_1}(u,v),$$

$$S_2(x,y) =: S_{\psi_{S_1}}(x,y), \ldots, \psi_{S_{n-1}}(u,v),\ S_n =: S_{\psi_{S_{n-1}}}, \ldots.$$

Denoting

$$\begin{cases} S_n(x,y) - S_{n-1}(x,y) =: \Delta_{n-1}(x,y) \quad (n\in\mathbb{N},\ (x,y)\in A), \\ \delta^n = \big\|De^{i\psi_{S_n}} - F^{-}(Ee^{2\pi iS_n})\big\| = \big\|F^{+}(De^{i\psi_{S_n}}) - Ee^{2\pi iS_n})\big\|, \\ \delta_n = \big\|F^{+}(De^{i\psi_{S_n}}) - Ee^{2\pi iS_{n+1}}\big\| = \big\|De^{i\psi_{S_n}} - F^{-}(Ee^{2\pi iS_{n+1}})\big\| \end{cases} \tag{2.10}$$

and assuming that $S = S_n(x,y)$ and $\psi = \psi_{S_{n-1}}(u,v)$ in (2.8) and $S = S_n(x,y)$ and $\psi = \psi_{S_n}(u,v)$ in (2.9), we deduce the inequalities

$$\cdots \ge \delta_{n-1} \ge \delta^n \ge \delta_n \ge \cdots \quad (n = 1,2,\ldots).$$

Since the formulas $\psi_{S_n} = \arg F^{-}(Ee^{2\pi iS_n})$ and $F^{-}(Ee^{2\pi iS_n}) = \widehat{E}(u,v;S_n)$ hold, we have that $\delta^n = \Big\|D - |\widehat{E}(u,v;S_n)|\Big\|$ is the distance between the desired radiation pattern and the realizable pattern $|\widehat{E}(u,v;S_n)|$. This distance decreases to some value $\delta_\infty(S_0)$ as $n\to\infty$, which cannot be zero, because the finitely supported function $D(u,v)$ is not an entire function. For different $S_0(x,y)$, the sequences $\{S_n(x,y)\}$ may differ too, since problem (2.4) is a set-valued extremal one (for example, $|\widehat{E}(u,v;S)| \equiv |\widehat{E}(u,v;S+\text{const})|$). Nevertheless, for large $n$, the function $|\widehat{E}(u,v;S_n)|$ inherits the main features of $D(u,v)$. We verified this fact in many computing experiments.

## 3. Computation procedures

It is natural to begin the construction of the sequence $\{S_n(x,y)\}$ with calculating $S_0(x,y) = \arg \dot{E}(x,y) = \dot{E}(x,y)/|\dot{E}(x,y)|$ over the nodes of a chosen lattice and stop it when the replacement of $\psi_{S_{n-1}}$ by $\psi_{S_n}$ negligibly changes the value $\max\limits_A |\Delta_{n-1}(x,y)|$ or the value

$$\max_\Omega |\widehat{E}(u,v;S_{n-1}) - \widehat{E}(u,v;S_n)|.$$

In practice, the range of values $(ux+vy)$ is not large, so oscillations of the function $\exp(-2\pi i(ux+vy))$ are also not large. Therefore, an approximate computation of the integral in the Fourier transforms over any dense lattice does not cause any difficulties and the construction of the sequence $\{S_n\}$ almost does not need any additional calculations.

## 4. Synthesis of the reflector surface

The initial parabolic reflector surface $z = \dfrac{x^2+y^2}{4f} - z_0$ $(z_0 > 0,\ x^2+y^2 < 4fz_0)$ can be corrected after each step of replacement of $S_{n-1}(x,y)$ by $S_n(x,y)$. It is possible to do this more rarely or at the end, i.e., immediately after the computation of $S_N$ based on the difference $S_n(x,y) - S_0(x,y)$. Below, we use the first strategy.

Let cells of the chosen lattice over the aperture $A$ be the squares with size $h \times h$ and midpoints $M_{ij}$ with coordinates $x_i = ih$, $y_j = jh$ $(i,j = 0, \pm 1, \dots, \pm k$, where $(k+1/2)h =: r$, and $r$ is the radius of $A)$ and with the conditions $(x_i, y_j) \in A$. Here, we have $r = \sqrt{4fz_0}$, $F(0,0,f,-z_0)$ is the focus of the paraboloid, the cells $m_{ij}$ of the lattice are framed by the lines $(x = x_i \pm h/2,\ z = 0)$ and $(y = y_j \pm h/2,\ z = 0)$. Replacing the condition $z = 0$ by $z < 0$, we obtain, instead of the cells, the set of tubes $t_{ij}$ in the half-space $z < 0$ of the space $R^3$ whose orthogonal cross-sections are the squares

$$\left(-\frac{h}{2} + x_i < x < x_i + \frac{h}{2},\ -\frac{h}{2} + y_j < y < y_j + \frac{h}{2},\ z = \text{const} < 0\right).$$

Every tube $t_{ij}$ cuts off an initial scale $\sigma^0_{ij}$ from the initial paraboloid which is part of the paraboloid $z = \dfrac{x^2+y^2}{4f} - z_0$, $(x,y) \in m_{ij}$, intersecting the axis at $t_{ij}$ at the point $M^0_{ij}\left(x_i,\ y_j,\ \dfrac{x_i^2+y_j^2}{4f} - z_0\right)$. Further, we use analogous “scales” which are cuttings of any other paraboloid $P_{ij}$ with the same axis of symmetry $OZ$ and focus $F$. The cuttings are embedded in the tubes $t_{ij}$ and contain any given point $M(x_i, y_j, z_{ij})$ on their axes. It is easy to verify that the equation of the paraboloid $P_{ij}$ for the given $z_{ij}$ is

$$z = \frac{x^2+y^2}{4f_{ij}} - \widetilde{z}_{ij}, \quad (x,y) \in m_{ij}, \tag{4.11}$$

where the focal distance $f_{ij}$ and the $z$th vertex coordinate $\widetilde{z}_{ij}$ are defined by the relations

$$\left\{4f_{ij}(z_{ij} + \widetilde{z}_{ij}) = x_i^2 + y_j^2,\ f_{ij} - \widetilde{z}_{ij} = f - z_0,\ f_{ij} > 0,\ z_{ij} > 0\right\}. \tag{4.12}$$

Since the function $\exp(-2\pi i S(x,y))$ of $S$ is $\lambda$-periodic, it is quite possible to locally change the function $S(x,y)$ in the Kirchhoff integral (2.2) by $(S(x,y) - n)$ with an arbitrary integer $n$ up to $[S(x,y)]$. Hence, taking into account the first notation in (2.10) and determining $S_1(x,y)$, we can compensate the difference $\Delta_1(x,y)$ by small shifts and simultaneously change all initial scales $\sigma^0_{ij}$ by $\sigma^1_{ij}$ whose coordinates $z^0_{ij}$ are determined from the conditions

$$|FM^0_{ij}| + |M^0_{ij}M_{ij}| = S_0(M_{ij}) + \Delta_0(M_{ij}) - \Delta_0(0,0) + (n^0_{ij})\lambda. \tag{4.13}$$

Here, we assume that $n_{00}^0 = 0$, so, the location of $\sigma_{00}^0$ is not changed. The other numbers $n_{ij}^0$ determining the points $M_{ij}^0$ must be chosen layerwise around $M_{00}(0, 0, -z_0)$ to minimize the distance $|z_{ij}^0 - z_{i_1 j_1}^0|$ between the neighbouring points $M_{ij}$ and $M_{i_1 j_1}$.

The further steps of the iteration procedures are implemented according to the same principles: using (2.6) and (2.7), we find the recurrent phase distribution $S_0(M_{ij})$ ($M_{ij} \in A$) and compensate the differences $\Delta_{n-1}(M_{ij})$ by numerical construction of the corresponding new scaled surface of the reflector as explained above in the case $n = 1$. Just in this construction, we must replace the set of points $M_{ij}^0$ by $M_{ij}^n$ and perform all calculations for $n = 1, 2, \dots$ according formulas (4.11)–(4.13).

In this section, we described how to choose a number $N$ to stop the iterations. In the final stage, it will be necessary to smooth the constructed weakly discontinuous surface $z = f_N(x, y)$ built of little scales $n_{ij}^N$. A fairly good result could be obtained as a close solution of the approximation problem

$$\|f_N(x, y) - S_2(x, y)\| + \alpha \left\|S_2'(x, y)\right\| \to \inf_S$$

for the function $f_N(x, y)$ by using quadratic or cubic splines $S(x, y)$. This problem was completely investigated in [14]. Perhaps it would be useful to recompute the amplitude distribution on the aperture $A$ after each next version of the modified reflector surface and to change the function $E(x, y)$ in the corresponding Fourier transforms (2.2).

Fig. 1 shows an example of a far-field pattern of the cosecant type for the antenna with a single feed element and the reflector surface shaped by the described method. The results of the synthesized contour beam for Europe and the corresponding reflector surface form are presented in Fig. 2. Other examples can be found in [15].

## 5. Conclusion

In this work, we describe a new iterative method for numerical shaping of the locally curved shape of the reflector surface for the antenna with a single feed element. The antenna must generate a beam with a prescribed contour of its cross-section. The method has some common features with other known methods, in particular, stated in the papers from the list of references. Our method mainly differs by the technique of minimization in the problem of aperture phase synthesis, which makes it possible to shape reflectors of large diameter. The method was tested in many computing experiments and showed itself as efficient.

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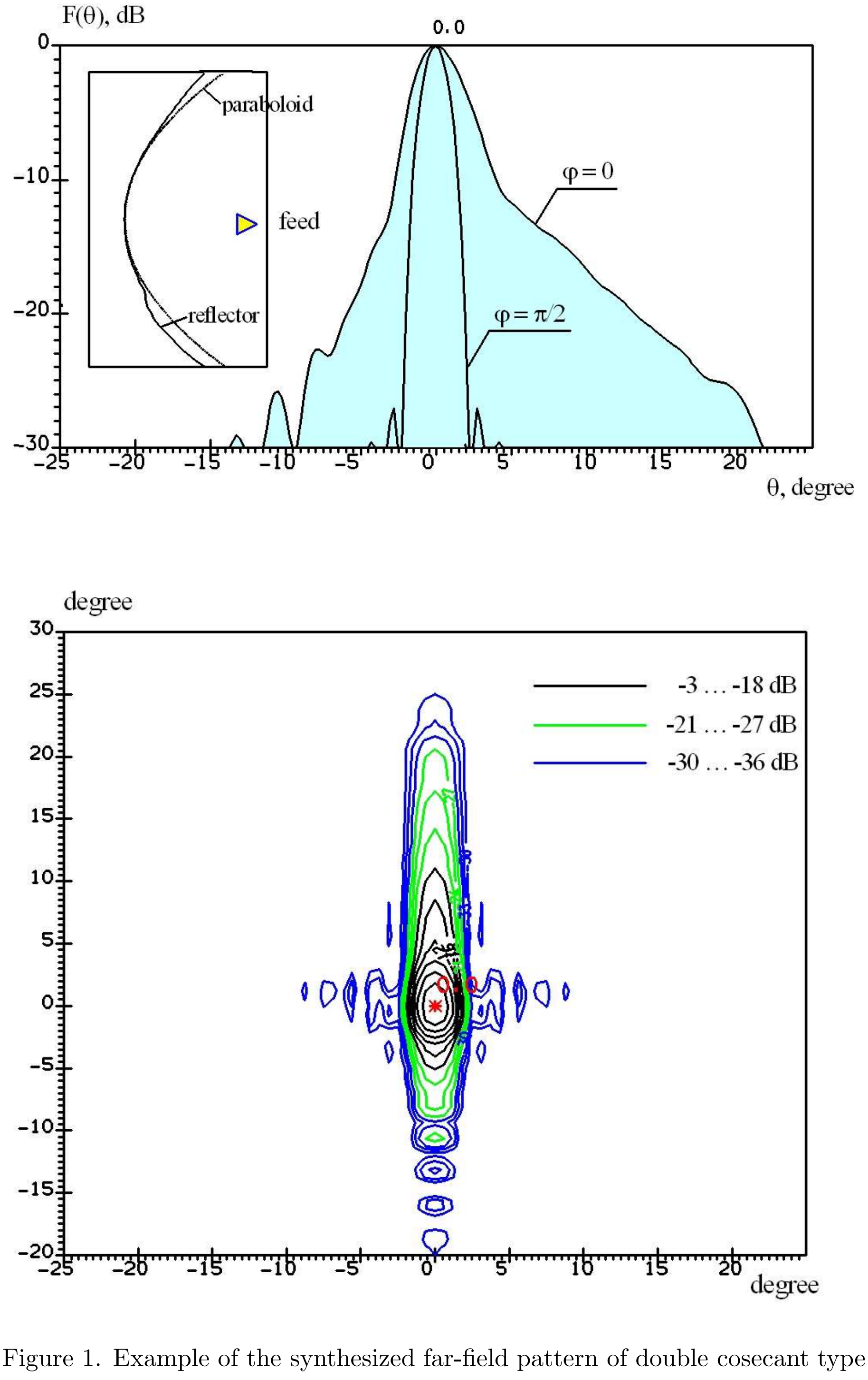


Figure 1. Example of the synthesized far-field pattern of double cosecant type

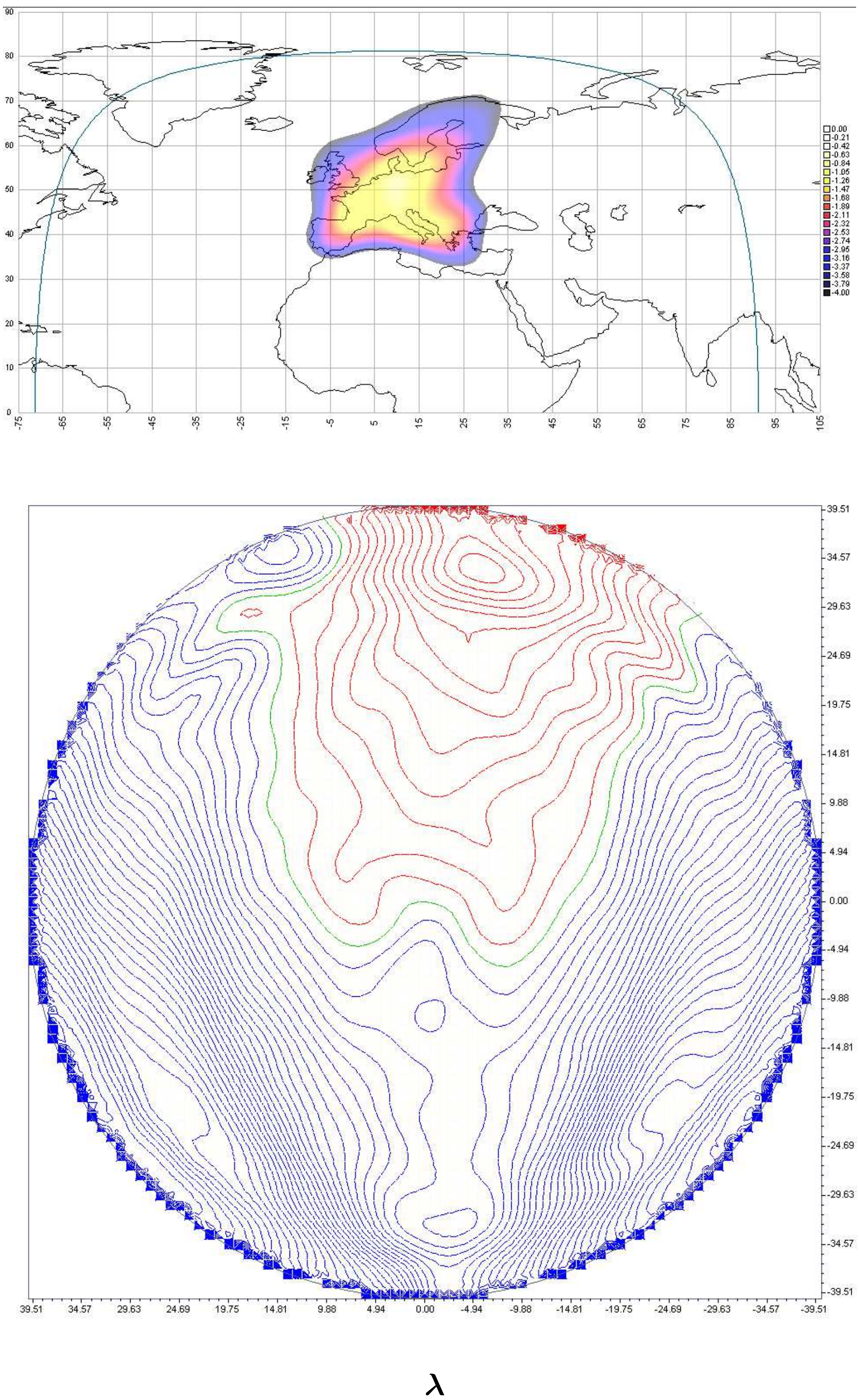


Figure 2. Example of the synthesized contour beam for Europe and the corresponding reflector surface form (contour plot of a deflection of a of the synthesized reflector surface from an initial paraboloid are shown with a step 0,01 wavelength $\lambda$)

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