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APPROXIMATING SOLUTIONS OF NONLINEAR HYBRID CAPUTO FRACTIONAL INTEGRO–DIFFERENTIAL EQUATIONS VIA DHAGE ITERATION PRINCIPLE

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Abstract: In this article, we prove the existence and approximation of solutions of the initial value problems of nonlinear hybrid Caputo fractional integro-differential equations. The main tool employed here is the Dhage iteration principle in a partially ordered normed linear space. An example is also given to illustrate the main results.

Keywords: Approximating solutions, Initial value problems, Dhage iteration principle, Hybrid fixed point theorem.

1. Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1–4, 9, 11–13, 15–20] and the references therein.

Hybrid differential equations involve the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. This class of equations arises from a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [1, 2, 5, 7, 9, 10, 20].

Dhage and Lakshmikantham [10] discussed the existence of solutions for the following first-order hybrid differential equation

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t)}{g(t,x(t))}\right) = f(t,x(t)) & a.e.\ t \in [t_0, t_0 + T], \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases}$$

where $t_0, T \in \mathbb{R}$ with $T > 0$, $g : [t_0, t_0 + T] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $f : [t_0, t_0 + T] \times \mathbb{R} \to \mathbb{R}$ are continuous

functions. By using the fixed point theorem in Banach algebra, the authors obtained the existence results.

Let $J = [0, a]$ be a closed and bounded interval of the real line $\mathbb{R}$ for some $a \in \mathbb{R}$ with $a > 0$. The hybrid fractional differential equation

$$\begin{cases} D^\alpha \left(\dfrac{x(t)}{g(t, x(t))} \right) = f(t, x(t)) \quad a.e.\ t \in J, \\ x(0) = 0, \end{cases}$$

has been investigated in [20], where D^α is the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$, $g : J \times \mathbb{R} \to \mathbb{R} \backslash \{0\}$ and $f : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions. By employing the fixed point theorem in Banach algebra, the authors obtained the existence of a solution.

Dhage et al. [9] studied the existence and approximation of the solutions of the following nonlinear fractional integro-differential equation

$$\begin{cases} {}^C D^\alpha \left(\dfrac{x(t) - I^\beta h(t, x(t))}{g(t, x(t))} \right) = f\left(t, x(t), \displaystyle\int_0^t k(s, x(s))\, ds \right), \quad t \in J, \\ x(0) = x_0 \in \mathbb{R}_+, \end{cases}$$

where ${}^C D^\alpha$ is the Caputo fractional derivative of order $0 < \alpha < 1$, I^β is the Riemann-Liouville fractional integral of order $0 < \beta < 1$, $g : J \times \mathbb{R} \to \mathbb{R} \backslash \{0\}$, $h, k : J \times \mathbb{R} \to \mathbb{R}$ and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. By using the Dhage iteration principle, the authors obtained the existence and approximation of solutions.

Inspired and motivated by the works mentioned above and some recent studies on hybrid fractional differential equations, we consider the existence and approximation of solutions for the following initial value problem (in short IVP) of the nonlinear hybrid Caputo fractional integro-differential equation

$$\begin{cases} {}^C D^\alpha \left(\dfrac{x(t)}{p(t) + \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1} g(s, x(s))\, ds} \right) = f(t, x(t)),\ t \in J, \\ x(0) = p(0)\, \theta, \end{cases} \tag{1.1}$$

where $0 < \alpha \le 1$, $0 < \beta \le 1$, $\theta \in \mathbb{R}$, $g, f : J \times \mathbb{R} \to \mathbb{R}$ are given functions and $p : J \to \mathbb{R}$ is a given function.

By a solution of the IVP (1.1) we mean a function $x \in C(J, \mathbb{R})$ that satisfies the corresponding integral equation of (1.1), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The purpose of this paper is to use the Dhage iteration principle to show the existence and approximation of solutions of (1.1) under weaker partially continuity and partially compactness type conditions.

The article is organized as follows. In Section 2 we give some preliminaries and key fixed point theorem that will be used in later sections. In Section 3 we prove some sufficient conditions of the existence and approximation of solutions of (1.1) by using the Dhage iteration principle. For details on the Dhage iteration principle we refer the reader to [6]. Finally, an example is given to illustrate our main results.

2. Preliminaries

We introduce some necessary definitions, lemmas and theorems which will be used in this paper. For more details, see [15, 19].

Definition 1 [15, 19]. *The fractional integral of order $\alpha > 0$ of a function $x : J \to \mathbb{R}$ is given by*

$$I^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

provided the right-hand side is pointwise defined on J.

Definition 2 [15, 19]. *The Caputo fractional derivative of order $\alpha > 0$ of a function $x : J \to \mathbb{R}$ is given by*

$$^C D^{\alpha} x(t) = D^{\alpha} \left[x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k \right],$$

where

$$n = [\alpha] + 1 \text{ for } \alpha \notin \mathbb{N}_0, \ n = \alpha \text{ for } \alpha \in \mathbb{N}_0, \tag{2.1}$$

and D^{α} is the Riemann-Liouville fractional derivative of order α defined by

$$D^{\alpha} x(t) = D^n I^{n-\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} x(s) ds.$$

The Caputo fractional derivative $^C D^{\alpha} x$ exists for x belonging to $AC^n (J, \mathbb{R})$ the space of functions which have continuous derivatives up to order $(n-1)$ on J such that $x^{(n-1)} \in AC^1 (J, \mathbb{R})$. $AC^1 (J, \mathbb{R})$ also denoted $AC (J, \mathbb{R})$ is the space of absolutely continuous functions. In this case, the Caputo fractional derivative is defined by

$$^C D^{\alpha} x(t) = I^{n-\alpha} x^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds.$$

Remark that when $\alpha = n$, we have $^C D^{\alpha} x(t) = x^{(n)}(t)$.

Lemma 1. [15, Lemma 2.22] *Let $\alpha > 0$ and let n be given by (2.1). If $x \in AC^n (J, \mathbb{R})$, then*

$$\left(I^{\alpha} \, {}^C D^{\alpha} x \right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k.$$

In particular, when $0 < \alpha \leq 1$, $\left(I^{\alpha} \, {}^C D^{\alpha} x \right)(t) = x(t) - x(0)$.

Let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\| \cdot \|$. It is known that E is called regular if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [14] and the references therein.

Definition 3. *A mapping $\mathcal{A} : E \to E$ is called isotone or monotone nondecreasing if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{A}x \preceq \mathcal{A}y$ for all $x, y \in E$. Similarly, $\mathcal{A}$ is called monotone nonincreasing if $x \preceq y$ implies $\mathcal{A}x \succeq \mathcal{A}y$ for all $x, y \in E$. Finally, $\mathcal{A}$ is called monotonic or simply monotone if it is either monotone nondecreasing or monotone nonincreasing on E.*

Definition 4. *An operator $\mathcal{A}$ on a normed linear space E into itself is called compact if $\mathcal{A}(E)$ is a relatively compact subset of E. $\mathcal{A}$ is called totally bounded if for any bounded subset S of E, $\mathcal{A}(S)$ is a relatively compact subset of E. If $\mathcal{A}$ is continuous and totally bounded, then it is called completely continuous on E.*

Definition 5 [6]. *A mapping $\mathcal{A} : E \to E$ is called partially continuous at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{A}x - \mathcal{A}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{A}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{A}$ is partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 6 [5, 6]. *An operator $\mathcal{A}$ on a partially normed linear space E into itself is called partially bounded if $\mathcal{A}(C)$ is bounded for every chain C in E. $\mathcal{A}$ is called uniformly partially bounded if all chains $\mathcal{A}(C)$ in E are bounded by a unique constant. $\mathcal{A}$ is called partially compact if $\mathcal{A}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{A}$ is called partially totally bounded if for any totally ordered and bounded subset C of E, $\mathcal{A}(C)$ is a relatively compact subset of E. If $\mathcal{A}$ is partially continuous and partially totally bounded, then it is called partially completely continuous on E.*

Definition 7 [5]. *The order relation $\preceq$ and the metric d on a non-empty set E are said to be compatible if $\{x_n\}$ is a monotone, that is, monotone nondecreasing or monotone nondecreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function has this property.

Theorem 1 [6]. *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain of E. Let $\mathcal{A} : E \to E$ be a partially continuous, nondecreasing and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0$ or $x_0 \succeq \mathcal{A}x_0$, then the operator equation $\mathcal{A}x = x$ has a solution x^* in E and the sequence $\{\mathcal{A}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Remark 1. [9] The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E is held if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$.

Remark 2. [5] Note that every compact mapping in a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is continuous and partially totally bounded, but the converse may not be true. Then, the hypothesis concerning the partially continuous and partially compact operator in Theorem 1 may be replaced by the continuous and compact operator.

3. Main results

The equivalent integral formulation of the IVP (1.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)|, \tag{3.1}$$

$$x \leq y \iff x(t) \leq y(t), \tag{3.2}$$

for all $t \in J$. Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm, regular and also partially ordered with respect to the above partially order relation $\le$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ has some nice properties with respect to the above order relation in it.

Lemma 2 [8]. *Let $(C(J, \mathbb{R}), \le, \| \cdot \|)$ be a partially ordered Banach space with the norm $\| \cdot \|$ and the order relation $\le$ defined by (3.1) and (3.2) respectively. Then $\| \cdot \|$ and $\le$ are compatible in every partially compact subset of $C(J, \mathbb{R})$.*

We need the following definition in what follows.

Definition 8. *A function $u \in C(J, \mathbb{R})$ is said to be a lower solution of the IVP (1.1) if it satisfies the corresponding integral inequality of*

$$\begin{cases} {}^C D^\alpha \left(\dfrac{u(t)}{p(t) + \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1} g(s, u(s)) ds} \right) \le f(t, u(t)), \quad t \in J, \\ u(0) \le p(0)\theta. \end{cases}$$

Similarly, an upper solution $v \in C(J, \mathbb{R})$ for the IVP (1.1), by reversing the above inequalities.

We consider the following set of assumptions:

(B1) $g, f : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $p : J \to \mathbb{R}$ is a continuous function such that

$$\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds \ge 0,$$

and

$$p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s)) ds > 0,$$

for all $t \in J$ and $x \in C(J, \mathbb{R})$.

(B2) There exist constants $K_g, K_f > 0$ such that

$$|g(t, x)| \le K_g \text{ and } |f(t, x)| \le K_f \text{ for all } t \in J \text{ and } x \in \mathbb{R}.$$

(B3) There exists constant $K_p > 0$ such that

$$|p(t_2) - p(t_1)| \le K_p |t_2 - t_1| \text{ for all } t_1, t_2 \in J.$$

(B4) $g(t, x)$ and $f(t, x)$ are monotone nondecreasing functions in x for all $t \in J$.

(B5) The IVP (1.1) has a lower solution $u \in C(J, \mathbb{R})$.

Lemma 3. *Let $h \in C(J, \mathbb{R})$ and $q \in C(J, (0, \infty))$. If $x/q \in AC(J, \mathbb{R})$, then the IVP*

$$\begin{cases} {}^C D^\alpha \left(\dfrac{x(t)}{q(t)} \right) = h(t), \quad t \in J, \\ x(0) = q(0)\theta, \end{cases}$$

is equivalent to the integral equation

$$x(t) = q(t) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \right), \quad t \in J.$$

Theorem 2. *Assume that hypotheses* (B1)–(B5) *hold. Then the IVP* (1.1) *has a solution* x^* *defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_{n+1}(t) = \left(p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x_n(s)) ds \right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_n(s)) ds \right), \quad (3.3)$$

for all $t \in J$, where $x_0 = u$ converges monotonically to x^*.

P r o o f. Set $E = C(J, \mathbb{R})$. Then by Lemma 2, every compact chain in E is compatible with respect to the norm $\|\cdot\|$ and order relation $\leq$. Define the operator $\mathcal{A}$ on E by

$$(\mathcal{A}x)(t) = \left(p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s)) ds \right)$$
$$\times \left(\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds \right), \ t \in J.$$

From the continuity of the integral, it follows that $\mathcal{A}$ defines the map $\mathcal{A} : E \to E$. Now, by Lemma 3, the IVP (1.1) is equivalent to the operator equation

$$(\mathcal{A}x)(t) = x(t), \ t \in J.$$

We shall show that the operator $\mathcal{A}$ satisfies all the conditions of Theorem 1. This is achieved in the series of following steps.

Step I: $\mathcal{A}$ is a nondecreasing operator on E. Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (B4), we obtain

$$(\mathcal{A}x)(t) = \left(p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s)) ds \right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds \right)$$
$$\leq \left(p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, y(s)) ds \right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds \right) = (\mathcal{A}y)(t),$$

for all $t \in J$. This shows that $\mathcal{A}$ is nondecreasing operator on E into E.

Step II: $\mathcal{A}$ is a partially continuous operator on E. Let $\{x_n\}$ be a sequence in a chain C in E such that $x_n \to x$ when $n \to \infty$. Then, by dominated convergence theorem, we have

$$\lim_{n\to\infty} (\mathcal{A}x_n)(t) = \lim_{n\to\infty} \left[\left(p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x_n(s)) ds \right) \right.$$
$$\left. \times \left(\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_n(s)) ds \right) \right]$$
$$= \left(p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left[\lim_{n\to\infty} g(s, x_n(s)) \right] ds \right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\lim_{n\to\infty} f(s, x_n(s)) \right] ds \right)$$
$$= \left(p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s)) ds \right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds \right) = (\mathcal{A}x)(t),$$

for all $t \in J$. This shows that $\{\mathcal{A}x_n\}$ converges to $\mathcal{A}x$ pointwise on J.

Next, we show that $\{\mathcal{A}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$
|(\mathcal{A}x_n)(t_2) - (\mathcal{A}x_n)(t_1)| \leq \left(|p(t_1)| + \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} |g(s, x_n(s))| \, ds \right)
$$

$$
\times \left(\frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, x_n(s)) \, ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, x_n(s)) \, ds \right| \right)
$$

$$
+ \left(|p(t_2) - p(t_1)| + \frac{1}{\Gamma(\beta)} \left| \int_0^{t_2} (t_2 - s)^{\beta-1} g(s, x_n(s)) \, ds - \int_0^{t_1} (t_1 - s)^{\beta-1} g(s, x_n(s)) \, ds \right| \right)
$$

$$
\times \left(|\theta| + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} |f(s, x_n(s))| \, ds \right)
$$

$$
\leq \left(|p(t_1)| + \frac{K_g a^\beta}{\Gamma(\beta+1)} \right) \left(\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right) |f(s, x_n(s))| \, ds \right.
$$

$$
\left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |f(s, x_n(s))| \, ds \right)
$$

$$
+ \left(K_p |t_2 - t_1| + \frac{1}{\Gamma(\beta)} \int_0^{t_1} \left((t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1} \right) |g(s, x_n(s))| \, ds \right.
$$

$$
\left. + \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} |g(s, x_n(s))| \, ds \right) \left(|\theta| + \frac{K_f a^\alpha}{\Gamma(\alpha+1)} \right)
$$

$$
\leq \left(|p(t_1)| + \frac{K_g a^\beta}{\Gamma(\beta+1)} \right) \frac{2K_f}{\Gamma(\alpha+1)} (t_2 - t_1)^\alpha
$$

$$
+ \left(K_p |t_2 - t_1| + \frac{2K_g}{\Gamma(\beta+1)} (t_2 - t_1)^\beta \right) \left(|\theta| + \frac{K_f a^\alpha}{\Gamma(\alpha+1)} \right) \to 0 \quad \text{as} \quad t_2 - t_1 \to 0,
$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{A}x_n \to \mathcal{A}x$ is uniformly and hence $\mathcal{A}$ is partially continuous on E.

Step III: $\mathcal{A}$ is a partially compact operator on E. Let C be an arbitrary chain in E. We show that $\mathcal{A}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{A}(C)$ is a uniformly bounded. Let $x \in C$ be arbitrary. Then

$$
|(\mathcal{A}x)(t)| \leq \left(|p(t)| + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} |g(s, x(s))| \, ds \right)
$$

$$
\times \left(|\theta| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} |f(s, x(s))| \, ds \right)
$$

$$
\leq \left(K_p t + |p(0)| + \frac{K_g}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} \, ds \right) \left(|\theta| + \frac{K_f}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \, ds \right)
$$

$$
\leq \left(K_p a + |p(0)| + \frac{K_g a^\beta}{\Gamma(\beta+1)} \right) \left(|\theta| + \frac{K_f a^\alpha}{\Gamma(\alpha+1)} \right) = r,
$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{A}x\| \leq r$ for all $x \in C$. Hence $\mathcal{A}(C)$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{A}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$
|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)| \leq \left(|p(t_1)| + \frac{K_g a^\beta}{\Gamma(\beta+1)} \right) \frac{2K_f}{\Gamma(\alpha+1)} (t_2 - t_1)^\alpha
$$

$$
+ \left(K_p |t_2 - t_1| + \frac{2K_g}{\Gamma(\beta+1)} (t_2 - t_1)^\beta \right) \left(|\theta| + \frac{K_f a^\alpha}{\Gamma(\alpha+1)} \right) \to 0 \quad \text{as} \quad t_2 - t_1 \to 0,
$$

uniformly for all $x \in C$. Hence $\mathcal{A}(C)$ is a relatively compact subset of E and consequently $\mathcal{A}$ is a partially compact operator on E into itself.

Step IV: u satisfies the operator inequality $u \leq \mathcal{A}u$. By hypothesis $(B5)$, the IVP (1.1) has a lower solution u on J. Then we have

$$^C D^\alpha \left(\frac{u(t)}{p(t) + \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1} g(s, u(s)) ds} \right) \leq f(t, u(t)), \quad t \in J, \tag{3.4}$$

satisfying

$$u(0) \leq p(0)\,\theta.$$

Applying I^α to both sides of (3.4) and by using Lemma 1, we obtain

$$u(t) \leq \left(p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, u(s)) ds \right) \left(\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))\, ds \right), \quad t \in J.$$

The definition of the operator $\mathcal{A}$ implies that $u(t) \leq (\mathcal{A}u)(t)$ for all $t \in J$. Consequently, u is a lower solution to the operator equation $x = \mathcal{A}x$.

Thus $\mathcal{A}$ satisfies all the conditions of Theorem 1 with $x_0 = u$ and we apply it to conclude that the operator equation $\mathcal{A}x = x$ has a solution. Consequently the integral equation and the IVP (1.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.3) converges monotonically to x^*. This completes the proof. $\qquad\square$

Remark 3. The conclusion of Theorem 2 also remains true if we replace the hypothesis $(B5)$ with the following one

$(B5')$ The IVP (1.1) has an upper solution $v \in C(J, \mathbb{R})$.

Example 1. Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the IVP,

$$\begin{cases} ^C D^{1/2} \left(\dfrac{x(t)}{\pi + \sin t + \dfrac{1}{\Gamma(1/3)} \displaystyle\int_0^t (t-s)^{-2/3} \arctan x(s)\, ds} \right) = \tanh x(t), \quad t \in J, \\[2em] x(0) = \pi, \end{cases} \tag{3.5}$$

where $\alpha = 1/2$, $\beta = 1/3$, $\theta = 1$, $g(t, x) = \arctan x$, $f(t, x) = \tanh x$ and $p(t) = \pi + \sin t$. Clearly, the functions g and f are continuous on $J \times \mathbb{R}$, p is continuous on J and

$$\pi + \sin t + \frac{1}{\Gamma(1/3)} \int_0^t (t-s)^{-2/3} \arctan x(s)\, ds > 0,$$

and

$$1 + \frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} \tanh x(s) ds \geq 0,$$

for all $t \in J$ and $x \in C(J, \mathbb{R})$. The functions g and f satisfy the hypothesis $(B2)$ with $K_g = \pi/2$ and $K_f = 1$. The function p satisfies the hypothesis $(B3)$ with $K_p = 1$. Moreover, the functions g and f are nondecreasing in x for each $t \in J$ and so the hypothesis $(B4)$ is satisfied. Finally the IVP (3.5) has a lower solution

$$u(t) = \left(\pi + \sin t - \frac{\pi t^{1/3}}{2\Gamma(4/3)} \right) \left(1 - \frac{t^{1/2}}{\Gamma(3/2)} \right),$$

defined on J. Thus all hypotheses of Theorem 2 are satisfied. Hence we apply Theorem 2 and conclude that the IVP (3.5) has a solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = \left(\pi + \sin t + \frac{1}{\Gamma(1/3)} \int_0^t (t-s)^{-2/3} \arctan x_n(s)\, ds \right)$$
$$\times \left(1 + \frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} \tanh x_n(s) ds \right),$$

for all $t \in J$, where $x_0 = u$, converges monotonically to x^*.

Remark 4. In view of Remark 3, the existence of the solutions x^* of the IVP (3.5) may be obtained under the upper solution

$$v(t) = \left(\pi + \sin t + \frac{\pi t^{1/3}}{2\Gamma(4/3)} \right) \left(1 + \frac{t^{1/2}}{\Gamma(3/2)} \right),$$

defined on J and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = \left(\pi + \sin t + \frac{1}{\Gamma(1/3)} \int_0^t (t-s)^{-2/3} \arctan x_n(s)\, ds \right)$$
$$\times \left(1 + \frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} \tanh x_n(s) ds \right),$$

for all $t \in J$, where $x_0 = v$, converges monotonically to x^*.

Acknowledgements

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ON THE CHERNOUS'KO TIME-OPTIMAL PROBLEM FOR THE EQUATION OF HEAT CONDUCTIVITY IN A ROD[1]

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Abstract: The time-optimal problem for the controllable equation of heat conductivity in a rod is considered. By means of the Fourier expansion, the problem reduced to a countable system of one-dimensional control systems with a combined constraint joining control parameters in one relation. In order to improve the time of a suboptimal control constructed by F.L. Chernous'ko, a method of grouping coupled terms of the Fourier expansion of a control function is applied, and a synthesis of the improved suboptimal control is obtained in an explicit form.

Keywords: Heat equation, Time-optimal problem, Pontryagin maximum principle, Suboptimal control, Synthesis of control.

Introduction

It is known that a time-optimal problem occupied a very important place in the foundation and development of optimal control theory. Even for simple non-trivial cases, the problem required working-out new approaches and lead after all to Pontryagin's maximum principle [3, 10, 30]. Despite 70 years of development, the solution of concrete non-trivial examples of time-optimal control still needs considerable effort [2, 4, 19]. The problem becomes even more difficult when a control system is described by a partial differential equation [11, 24, 25, 34], particularly, for the heat conductivity equation [12, 22, 26, 29, 35, 36]. In [1], the correctness of parabolic equations for heat propagation is discussed and for that purpose, a parabolic equation with time delay is considered.

Here, the maximum principle can be formally written out as well, but it loses its effectiveness as compared with a finite-dimensional case or on cases when the time interval is fixed [2, 9, 18, 25, 32, 33]. Therefore, Chernous'ko suggested [13] another approach based on the Fourier expansion that allowed him to reduce the problem to an infinite system of one-dimensional problems whose control parameters are connected by a condition in the min-max form (see below (1.4)) generating a closed convex control set in a Hilbert space. Unfortunately, to deal with such a constraint is quite difficult (about other kinds of constraints see [17]). In order to overcome this complexity, the mentioned constraint was replaced [13] by an infinite system of separated conditions for scalar control parameters that can be interpreted as if one took Hilbert's brick inscribed into the control set. As a result, this approach made it possible to construct a suboptimal control and to give an explicit upper estimation for an optimal time. In [5], a co-Hilbert's brick inscribed into the control set was considered, and an improved suboptimal control function was constructed. In the present

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paper, we suggest another way for constructing a suboptimal control function in the case of the heat conductivity equation in a rod.

1. Preliminaries

As it was noted above, Chernous'ko considered the time-optimal problem for an evolutional equation

$$\frac{\partial u(t,x)}{\partial t} = A[u(\cdot,\cdot)](t,x) + v(t,x) \tag{1.1}$$

with the initial and boundary conditions

$$u(0,x) = u^0(x), \quad Mu(t,s) = u^*(t,s), \tag{1.2}$$

where A is a uniformly elliptic differential operator, $t \geq 0$, $x \in D$, D is a regular domain with Lyapunov boundary Γ, $s \in \Gamma$, and M is a boundary operator [13].

The constraint on the control function in problem (1.1), (1.2) is bounded in the norm of the space L_∞; i.e. $|v(t,x)| \leq v_0$ for almost all t and every $x \in \bar{D}$, where v_0 is a given positive number [31]. It is known that, for every control function $v(t,x)$, problem (1.1), (1.2) has a unique solution $u(t,x)$ [14, 21, 28].

If a solution $u(t,x)$ of problem (1.1), (1.2) satisfies the condition $u(T,x) \equiv 0$ at some T, $T \geq 0$, then the corresponding control function $v(t,x)$ is called admissible, and the number T is called the transition time (from the initial state $u_0(\cdot)$ into the equilibrium state $u(t,x) \equiv 0$). Let V be the class of all admissible controls. Then the quantity $T = T[v(\cdot,\cdot)]$ will be a functional on V at every fixed $u^0(x)$ and $u^*(t,s)$.

If an admissible control $v_*(t,x)$ satisfies the condition $T_* = T[v_*(\,\cdot\,,\,\cdot\,)] \leq T[v(\,\cdot\,,\,\cdot\,)]$ for all $v(\,\cdot\,,\,\cdot\,) \in V$, then $v_*(\,\cdot\,,\,\cdot\,)$ is called a time-optimal control, and the value T_* is called optimal transition time.

The direct application of the Pontryagin maximum principle to problem (1.1), (1.2) is a very hard task, unlike optimization problems on a finite interval of time (see [4, 8, 15]). For example, in [25], only theorems on the existence of optimal control and the bang-bang principle are given, but no specific example of a solution was considered. In monograph [11], the time-optimal problem when a control parameter participates in boundary conditions was considered [11, Ch. 5, Sect. 1] and, instead of the necessary conditions, the method of the L-momentum of N.N. Krasovskii [19] was applied [11, Sect. 2]. In the recently published article [20], Butkovsky's approach was applied to the case of a fractional-order diffusion equation. It should be noted that the L-momentum method only allows one to simplify to some degree the time-optimal problem and rarely gives an explicit solution. Therefore, the approach suggested by Chernous'ko [13], where the method of expansions on the system of eigenfunctions of the operator A was used, seems to be more effective. That helped to reduce considering problem to the infinite system of one-dimensional control problems:

$$\dot{y}_k = -\lambda_k y_k + v_k, \quad y_k(0) = y_{k0}, \quad k = 0, 1, 2, \ldots . \tag{1.3}$$

(About solution of systems of this kind, see [16]).

In terms of system (1.3), the condition $|v(t,s)| \leq v_0$ means that a counting system of the control parameters v_k, $k = 0, 1, 2, \ldots$, should satisfy the combined constraint

$$\max_{x \in \bar{D}} \left| \sum_{k=0}^{\infty} \varphi_k(x) v_k \right| \leq v_0. \tag{1.4}$$

where φ_k are eigenfunctions of the problem.

Condition (1.4) defines some closed convex set L in the Hilbert space l_2, which is difficult to deal with. In this connection, it is natural to try to solve the problem of finding a suboptimal control. (It is essential to note that, if a time interval is fixed, then the method of penalty functions is enough effective for the construction of a suboptimal control. It would be interesting to apply this method for the time-interval problem as well [6].) For this purpose, in [13], constraint (1.4) was replaced by a more rigid system of constraints in the form

$$|v_k| \leq U_k, \quad k = 0, 1, 2, \ldots, \tag{1.5}$$

where $\alpha_k = \max\limits_{x \in \bar{D}} |\varphi_k(x)|$. Wherein, nonnegative numbers U_k should be chosen satisfying the condition $\sum_{k=0}^{\alpha} \alpha_k U_k = v_0$.

Let T_{*k} be an optimal transition time in the problem

$$\dot{y}_k = -\lambda_k y_k + v_k, \quad y_k(0) = y_k^0, \tag{1.6}$$

such that $y_k(T_{*k}) = 0$, $k = 0, 1, 2 \ldots$. In [13], it is shown that the numbers U_k can be chosen so that all T_{*k} coincide: $T_{*k} = \hat{T}$ for some $\hat{T}$. Let $\hat{v}_k(t)$ be the sequence of the corresponding optimal controls. Then $T_* \leq \hat{T}$ and $\hat{v}_*(t, x) = \sum_{k=0}^{\infty} \varphi_k(x) v_{*k}(t)$ may serve as the sought suboptimal control.

A new problem arises here: is it possible to use a more exact reduction of the constraint than (1.5)? As mentioned above in [5] it was used Hilbert's co-cube instead of (1.5). Here, we are going to follow another approach based on a special grouping of terms of (1.5). Effectiveness of this approach is tightly related to specific properties of eigenfunctions $\varphi_k(\cdot)$, so here it will be demonstrated for the operator $A = \partial^2/\partial x^2$ connected with the process of the heat conductivity in a rod.

2. A method of grouping terms of the Fourier expansion

Consider the following concretization of problem (1.1), (1.2):

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + v(t, x), & |v(t, x)| \leq v_0, \quad t \geq 0, \quad 0 \leq x \leq \pi, \\ u(0, x) = u^0(x), \quad u(t, 0) = 0, \quad u(t, \pi) = 0. \end{cases} \tag{2.1}$$

The system of eigenfunctions $\varphi_k(t) = \sin kx$, $k = 1, 2, \ldots$, of the operator $\partial^2/\partial x^2$ forms a complete orthogonal basis of the space $L_2[0, \pi]$ [21, 28].

Let $u(t, x) = \sum_{k=1}^{\infty} y_k \sin kx$ and $v(t, x) = \sum_{k=1}^{\infty} v_k \sin kx$ be the Fourier expansions on the basis $\{\sin kx\}$. Then the restriction (1.4) takes the form

$$\max_{0 \leq x \leq \pi} \left| \sum_{k=1}^{\infty} v_k \sin kx \right| \leq v_0. \tag{2.2}$$

Let us consider a more rigid restriction

$$\max_{0 \leq x \leq \pi} \sum_{k \in Q} \left| v_k \sin kx + v_{3k} \sin 3kx \right| \leq v_0 \tag{2.3}$$

instead of (2.2), thereby replacing the optimal control problem with a suboptimal control problem.

System (1.6) takes the form

$$\dot{y}_k = -k^2 y_k + v_k, \quad k \in Z^+. \tag{2.4}$$

Let Q be the set of all positive integers having the form $3^{2p}q$, where $p = 0, 1, 2, \ldots$, and q is relatively prime with 3. It is obvious that the set of all positive integers Z^+ is the union of the two disjoint sets Q and $3Q$. Then (2.4) can be rewritten in the form

$$\dot{y}_k = -k^2 y_k + v_k, \quad \dot{y}_{3k} = -9k^2 y_{3k} + v_{3k}, \quad k \in Q. \tag{2.5}$$

After the substitutions

$$y_k = \frac{\mu_k}{k^2} x^1, \quad y_{3k} = \frac{\mu_k}{k^2} x^2, \quad t = \frac{1}{k^2} \tau, \quad v_k = \mu_k w^1, \quad v_{3k} = \mu_k w^2,$$

all systems (2.5) will be reformulated to the following two-dimensional control system:

$$\dot{x}^1 = -x^1 + w^1, \quad \dot{x}^2 = -9x^2 + w^2. \tag{2.6}$$

Now, following the Chernous'ko way, we replace (2.3) by the even more rigid restriction

$$\max_{0 \leq x \leq \pi} \left| w_k^1 \sin kx + w_k^2 \sin 3kx \right| \leq 1, \quad k \in Q, \tag{2.7}$$

that implies (2.3) if $\sum_{k \in Q} \mu_k = v_0$. Thus, we have reduced the infinite dimensional control problem to the two-dimensional problem.

3. Solution of the auxiliary time-optimal problem on the plane

Let P_k denote the set of all pairs (w_k^1, w_k^2) for which (2.7) holds. Setting

$$P = \left\{ w = (w^1, w^2) \in R^2 : \max_{0 \leq t \leq \pi} \left| w^1 \sin t + w^2 \sin 3t \right| \leq 1 \right\},$$

we have $P_k = \mu_k P$. As a result, the considered problem of constructing a suboptimal control reduces to the concrete problem of time-optimal control for the following two-dimensional system:

$$\dot{x}^1 = -x^1 + w^1, \quad \dot{x}^2 = -9x^2 + w^2, \quad (w^1, w^2) \in P. \tag{3.1}$$

Obviously, P is a convex and compact set with non-empty interior (i.e., a convex body). Since P is symmetric with respect to the origin, we may restrict ourselves to considering only the case $w^1 \geq 0$. It is more convenient to set $\sin t = y$. Then, by the formula

$$\sin 3t = 3 \sin t - 4 \sin^3 t,$$

we get

$$P = \left\{ w = (w^1, w^2) \in R^2 : \max_{0 \leq y \leq 1} \left| (w^1 + 3w^2) y - 4w^2 y^3 \right| \leq 1 \right\}.$$

Just this transformation lay on the base of the separation $Z^+ = Q \cup 3Q$.

After elementary calculations, we find that the part of the boundary of the set P lying in the half-plane $w^1 \geq 0$ is given by the formula

$$w^1 = \begin{cases} w^2 + 1 & \text{if} \quad -1 \leq w^2 < 0.125, \\ 3(\sqrt[3]{w^2} - w^2) & \text{if} \quad 0.125 \leq w^2 \leq 1, \end{cases}$$

while the other part is found by central symmetry (see Fig. 1).

Let us recall that, in the auxiliary problem (3.1), a unique optimal time-control function exists at each initial point (x_0^1, x_0^2) [7, 23, 27]. The existence follows from the property $O \in \text{Int} \, P$. The
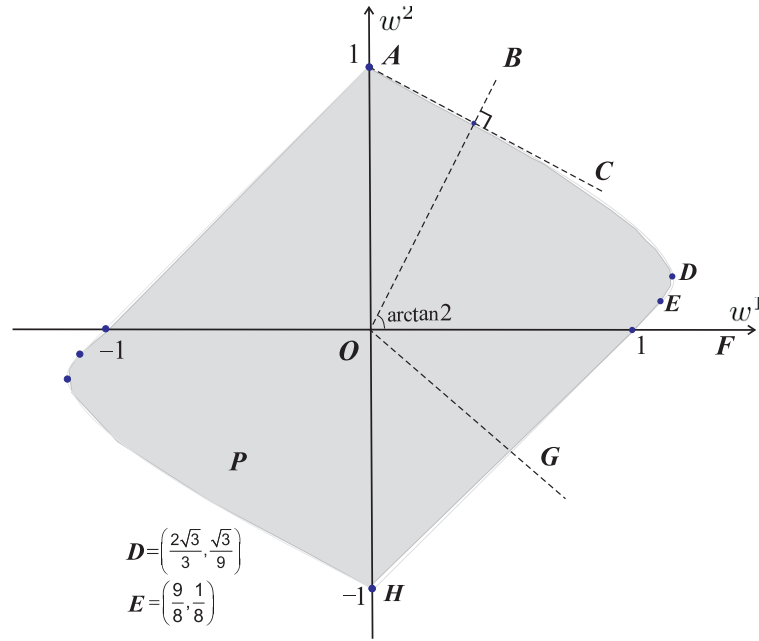
Figure 1. The straight ray AC is tangent at the point A to the curve AD, which is a part of the boundary of P.

uniqueness is a consequence of the following feature of P: the vector $(1,1)$, which is orthogonal to the segment AC, is not an eigenvector of the matrix of system (3.1). Therefore, the optimal control problem (2.5) coincides with the extremal controls of Pontryagin's maximum principle [23, 30].

To calculate the latter, we prefer to use the "backward motion" principle. Let $T(x_0^1, x_0^2)$ be a transition time for the initial point (x_0^1, x_0^2) in the system (3.1). If we set $\tau = T(x_0^1, x_0^2) - t$, then extremals of Pontryagin's maximum principle are defined by the system

$$\begin{cases} \dfrac{dx^1}{d\tau} = x^1 - \bar{w}^1, & \dfrac{dx^2}{d\tau} = 9x^2 - \bar{w}^2, & \dfrac{d\psi_1}{d\tau} = -\psi_1, & \dfrac{d\psi_2}{d\tau} = -9\psi_2. \\ x(0) = y(0) = 0, & \psi_1(0) = \cos s, & \psi_2(0) = \sin s, & -\pi \leq s \leq \pi. \end{cases} \quad (3.2)$$

Since $\psi_1(\tau, s) = e^{-\tau} \cos s$ and $\psi_2(\tau, s) = e^{-9\tau} \sin s$, an extremal control $\bar{w}(\tau, s)$ should be found by the Pontryagin's maximum principle, i.e., from the equation

$$\bar{w}^1(\tau, s) e^{-\tau} \cos s + \bar{w}^2(\tau, s) e^{-9\tau} \sin s = \max_{w \in P} [w^1 e^{-\tau} \cos s + w^2 e^{-9\tau} \sin s]. \quad (3.3)$$

Equation (3.3) leads to the following construction of the extremal controls.

If $\psi(\tau, s)$ lies in the open angle AOB, then obviously $\bar{w}(\tau, s) = (0, 1)$. Note that, if $s = \pi/2$, then $\psi_1(\tau, s) \equiv 0$. Therefore, $\bar{w}(\tau, \pi/2) = (0, 1)$. Similarly, if $s = 0$, then $\psi_2(\tau, s) \equiv 0$; thus, $\bar{w}(\tau, 0) = (2\sqrt{3}/3, \sqrt{3}/9)$.

Consider now the dynamics of $\psi(\tau, s)$. In the case $0 < s < \pi/2$, the vector $\psi(\tau, s)$ lies in the quarter $\psi_1 > 0$, $\psi_2 > 0$ and turns clockwise. Moreover, its direction tends to the axis of abscissas OE as $\tau \to +\infty$. (Similarly, if $-\pi/2 < s < 0$, then $\psi(\tau, s)$ lies in the quarter $\psi_1 > 0$, $\psi_2 < 0$ and turns counterclockwise with the same limit direction.)

Thus, the extremal control has the following structure: if $0 < s \leq \arctan 2$ (see Fig. 1), then $\psi(\tau, s)$ lies in the angle BOD for all τ ($\tau \geq 0$), and, hence, $\bar{w}(\tau, s)$ is a point of the arc AD such that its projection to the direction $\psi(\tau, s)$ is maximal (the analytical expression for $\bar{w}(\tau, s)$ is given in Table 1).

Table 1. The analytical expression for $\bar{w}^1(\tau, s)$ and $\bar{w}^2(\tau, s)$.

	$\bar{w}^1(\tau, s)$	$\bar{w}^2(\tau, s)$
$-\dfrac{\pi}{2} < s \le -\dfrac{\pi}{4}$	$\begin{array}{ll} 0 & \text{if } 0 \le \tau \le \tau_{**}, \\ 3(2 - e^{-8\tau}\tan s)M & \text{if } \tau_{**} \le \tau \end{array}$	$\begin{array}{ll} -1 & \text{if } 0 \le \tau \le \tau_{**}, \\ M & \text{if } \tau_{**} \le \tau \end{array}$
$-\pi/4 < s < 0$	$3(2 - e^{-8\tau}\tan s)M$	M
$s = 0$	$2\sqrt{3}/3$	$\sqrt{3}/9$
$0 < s \le \arctan 2$	$3(2 - e^{-8\tau}\tan s)M$	M
$\arctan 2 < s < \dfrac{\pi}{2}$	$\begin{array}{ll} 0 & \text{if } 0 \le \tau \le \tau_*, \\ 3(2 - e^{-8\tau}\tan s)M & \text{if } \tau_* \le \tau \end{array}$	$\begin{array}{ll} 1 & \text{if } 0 \le \tau \le \tau_*, \\ M & \text{if } \tau_* \le \tau \end{array}$
$s = \pi/2$	0	1

Table 2. The analytical expression for $x^1(\tau, s)$ and $x^2(\tau, s)$.

	$x^1(\tau, s)$	$x^2(\tau, s)$
$-\dfrac{\pi}{2} < s \le -\dfrac{\pi}{4}$	$\begin{array}{ll} 0 & \text{if } 0 \le \tau \le \tau_{**}, \\ \dfrac{3e^\tau}{4\tan s}\displaystyle\int_m^n \dfrac{1-p^2}{p^2 q^7}dp & \text{if } \tau_{**} \le \tau \end{array}$	$\begin{array}{ll} \left(e^{9\tau}-1\right)/9 & \text{if } 0 \le \tau \le \tau_{**}, \\ \dfrac{e^{9\tau}}{4\tan s}\displaystyle\int_m^n q\,dp & \text{if } \tau_{**} \le \tau \end{array}$
$-\pi/4 < s < 0$	$\dfrac{3e^\tau}{4\tan s}\displaystyle\int_m^n \dfrac{1-p^2}{p^2 q^7}dp$	$\dfrac{e^{9\tau}}{4\tan s}\displaystyle\int_m^n q\,dp$
$s = 0$	$2\sqrt{3}(1-e^\tau)/3$	$\sqrt{3}(1-e^{9\tau})/81$
$0 < s \le \arctan 2$	$\dfrac{3e^\tau}{4\tan s}\displaystyle\int_m^n \dfrac{1-p^2}{p^2 q^7}dp$	$\dfrac{e^{9\tau}}{4\tan s}\displaystyle\int_m^n q\,dp$
$\arctan 2 < s < \dfrac{\pi}{2}$	$\begin{array}{ll} 0 & \text{if } 0 \le \tau \le \tau_*, \\ \dfrac{3e^\tau}{4\tan s}\displaystyle\int_m^n \dfrac{1-p^2}{p^2 q^7}dp & \text{if } \tau_* \le \tau \end{array}$	$\begin{array}{ll} \left(1-e^{9\tau}\right)/9 & \text{if } 0 \le \tau \le \tau_*, \\ \dfrac{e^{9\tau}}{4\tan s}\displaystyle\int_m^n q\,dp & \text{if } \tau_* \le \tau \end{array}$
$s = \pi/2$	0	$\left(1-e^{9\tau}\right)/9$

Further, in the case $\arctan 2 < s < \pi/2$, we have $\bar{w}(\tau, s) = (0, 1)$ on the interval $[0, \tau_*)$, where $\tau_* = -1/8 \cdot \log(2\cot s)$. At the time $\tau = \tau_*$, the vector $\psi(\tau, s)$ becomes orthogonal to the right side tangent to the curve ∂P at the point $(0, 1)$ and it occurs "switching" of the extremal control from the value $(0, 1)$ to a continuous mode. Namely, $\bar{w}(\tau, s)$ begins sliding along the arc AC (see Table 1) and tends to the point C as $\tau \to +\infty$.

Similarly, if $(-\pi/2 < s < -\pi/4)$, then $\bar{w}(\tau, s) = (0, -1)$ at $0 \le \tau < \tau_{**}$, where $\tau_{**} = -1/8 \cdot \log(-\cot s)$ and $w(\tau, s)$ is a switching time. On the interval $(\tau_{**}, +\infty)$, $\bar{w}(\tau, s)$ slides along the arc ED tending to the point D.

The entire synthesis of the extremal control is given in Table 1. Due to the central symmetry, the values of s are considered only on the range $-\pi/2 < s \le \pi/2$ and the following notation is used:

$$M = (3 - e^{-8\tau}\tan s)^{-3/2}, \quad m = (3e^{8\tau}\cot s - 1)^{-1/2}, \quad n = (3\cot s - 1)^{-1/2},$$

$$p = (3 - e^{-8\tau}\tan s)^{-1/2}, \quad q = ((3 - p^{-2})\cot s)^{1/8}.$$

Now, extremal trajectories can be easily calculated by (3.2). The corresponding formulas are gathered in Table 2. They are illustrated in Fig. 2.
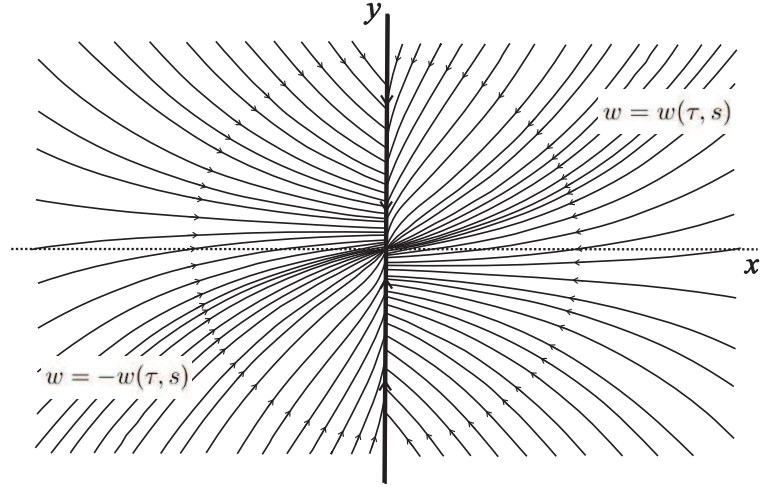
Figure 2. The extremal trajectories.
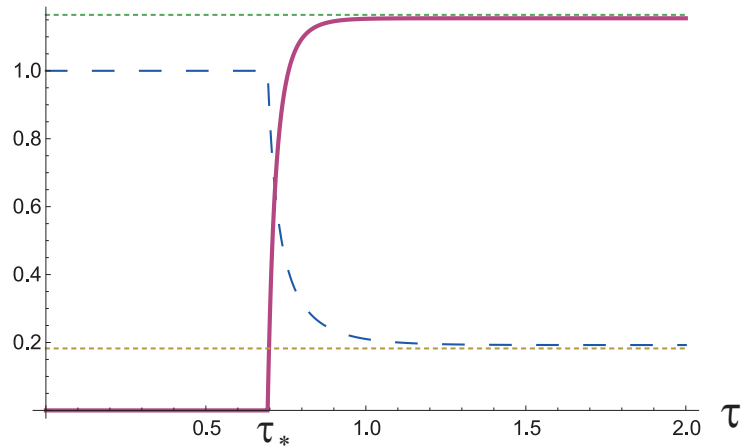
Figure 3. The graphs of the functions $\bar{w}^1(\tau, s)$ (the continuous line) and $\bar{w}^2(\tau, s)$ (the dashed line) for $\arctan 2 < s < \pi/2$.

4. Construction of a suboptimal control in the initial problem

Let us now derive the solution of problem (2.1), (2.3), (2.5) basing one of reduced problems.

Extremals $x(t, s), y(t, s)$ cover R^2. Therefore, for every $(x_0^1, x_0^2) \in R^2 \setminus (0, 0)$, there exists a pair (τ_0, s_0) such that $x^1(\tau_0, s_0) = x_0^1$, $x^2(\tau_0, s_0) = x_0^2$. Further, in the system (3.1), for every fixed $(x_0^1, x_0^2) \neq 0$, an optimal control is unique, which implies the uniqueness of the value τ_0 (while corresponding values of s_0 may be not unique, but one can choose any of them).

Then $T(x_0^1, x_0^2) = \tau_0$ is the transition time and

$$\bar{v}^1(t) = v^1(\tau_0 - t, s_0), \quad \bar{v}^2(t) = v^2(\tau_0 - t, s_0)$$
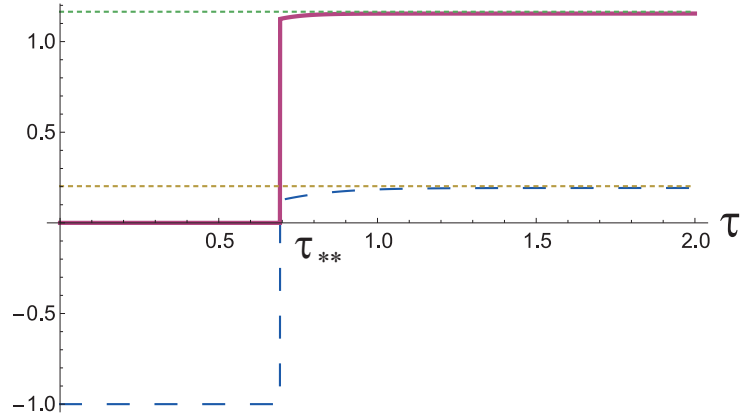
Figure 4. The graphs of the functions $\bar{w}^1(\tau, s)$ (the continuous line) and $\bar{w}^2(\tau, s)$ (the dashed line) for $-\pi/2 < s < \arctan 2$.

is the suboptimal control for (2.1).

Let us now consider system (2.5). For an initial point (y_k^0, y_{3k}^0), the corresponding trajectory $(y_k(t), y_{3k}(t))$ satisfies the condition

$$y_k(T_k) = y_{3k}(T_k) = 0,$$

where

$$T_k(\mu_k) = \frac{1}{k^2}\left(\frac{k^2}{\mu_k}y_k^0, \ \frac{k^2}{\mu_k}y_{3k}^0\right).$$

The constructed synthesis implies that T_k is monotonically decreasing in μ_k, and it is easy to see that $T_k \to 0$ as $\mu_k \to +\infty$ and $T_k \to \infty$ as $\mu_k \to 0$. Therefore, for every k, there exists a unique value μ_k^* such that $T_k(\mu_k^*)$ is the same for all k. Moreover, μ_k^* can be chosen satisfying the condition $\sum \mu_k^* = v_0$. One can easily see that

$$\frac{\alpha}{k^2} \leq \mu_k^* \leq \frac{\beta}{k^2}$$

for some positive α and β.

Finally, we consider the initial problem (2.1), (2.3), (2.5). Let $u_0(x) = \sum_{k=1}^{\infty} u_k^0 \sin x$ be the Fourier expansion of the initial function $u_0(x)$. Taking (y_k^0, y_{3k}^0), $k \in Q$, as an initial point for system (2.6), we find

$$\bar{w}_k^0(t) = \frac{1}{\mu_k}\bar{v}_k^1(t), \quad \bar{w}_{3k}^0(t) = \frac{1}{\mu_k}\bar{v}_k^2(t), \quad k \in Q.$$

Thus, the following statement holds.

Theorem 1. *The function*

$$\bar{v}(t, x) = \sum_{k=1}^{\infty} \bar{w}_k^0(t) \sin kx$$

is a suboptimal control in problem (2.1) for the initial state $u_0(x)$.

5. Conclusion

The paper is devoted to the time-optimal problem for the process of heat conductivity in a rod when the control parameter is the intensity of external heat sources. A suboptimal control is

constructed by the combination of the Chernous'ko approach with the method of grouping terms of the Fourier expansion.

This method may be applied to the time-optimal control problem for other systems given in an evolutionary form.

The following question naturally arises: how effective is the method of grouping? First of all, let us bring general considerations. The set of all admissible controls in the initial problem (1.1)–(1.2) can be identified with the subset

$$U_{Initial} = \left\{ u \in l_2 \mid \sup_{0 \leq x \leq \pi} |\sum_{k=0}^{\infty} u_k \sin kx| \leq v_0 \right\}.$$

As noted in Section 1, Chernous'ko restricted the set of controls using

$$U_{Ch} = \left\{ u \in l_2 \mid |u_k| \leq U_k, \quad k = 1, 2, 3, \ldots \right\},$$

where U_k is a sequence chosen from the condition $\sum U_k \leq v_0$ and guaranteeing the equality $u(t, x) \equiv 0$ for some $T = T_{Ch} > 0$.

The considerations in this paper are based on the set

$$U_{gr} = \left\{ u \in l_2 \mid \max_{0 \leq x \leq \pi} |u_k \sin kx + u_{3kx} \sin 3kx| \leq U_k \right\}$$

taken as a region of admissible controls.

One can easily see that

$$U_{gr} \subset U_{Ch} \subset U_{Initial}.$$

These relations imply $T_2 \leq T_1 \leq T_0$ for optimal and suboptimal times of transition respectively.

If one takes an initial point of the form $(0, 0, \ldots, x_m^0, 0, \ldots, 0)$, i.e., in terms of the initial problem (1.1)–(1.2), $\varphi(x) = (0, 0, \ldots, x_m^0 \sin mx, 0, \ldots, 0)$, then, obviously, $U_{Initial} = U_{Ch} = U_{gr}$ and, thus, $T_2 = T_1 = T_0$. But if an initial point is taken in the form $(0, 0, \ldots, 0, x_k^0, 0, \ldots, 0, x_{3k}^0, 0, \ldots)$, then $U_{In} = U_{gr}$ while $U_{In} \supset U_{Ch}$ and, thus, $T_2 = T_0 < T_1$. Table 3 contains values for specific cases.

Table 3

Initial point	T_1	$T_0 = T_2$
$(1, 0, 1, 0, 0, 0, \ldots)$	0.7	0.64
$(1, 0, 2, 0, 0, 0, \ldots)$	0.67	0.61
$(1, 0, -1, 0, 0, 0, \ldots)$	0.72	0.65
$(1, 0, -2, 0, 0, 0, \ldots)$	0.69	0.62

Obviously, $x_k^0 \neq 0$ at least for three values of the index k when $T_2 < T_0 < T_1$.

The final note is that the method of grouping can be applied only if there some algebraic relations between the eigenfunctions of the operator A.

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ON NECESSARY OPTIMALITY CONDITIONS
FOR RAMSEY-TYPE PROBLEMS[1]

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Abstract: We study an optimal control problem in infinite time, where the integrand does not depend explicitly on the state variable. A special case of such problem is the Ramsey optimal capital accumulation in centralized economy. To complete the optimality conditions of Pontryagin's maximum principle, so called transversality conditions of different types are used in the literature. Here, instead of a transversality condition, an additional maximum condition is considered.

Keywords: Pontryagin maximum principle, Transversality condition, Optimal control, Ramsey problem.

1. Introduction

Economic theory has been involving optimal control theory since invention of Pontryagin's maximum principle. Based on the maximum principle [1], David Cass in [2] elaborated the problem of optimal capital accumulation first discussed by Frank Ramsey [3]. Since then, this model, sometimes referred to as Ramsey–Cass–Koopmans model, is a component of advanced books on macroeconomics and growth theory, see, e.g. [4]. This model has the infinite planning horizon as well as many other models in economics.

Optimal control with infinite horizon still has theoretical issues to investigate, see [5]. The proof of the maximum principle for the infinite time horizon one can find, e.g., in [6]. The proved theorem does not include *transversality conditions*. Moreover, it is known [6–8] the that usually used forms of transversality conditions

$$\lim_{t \to \infty} \psi(t) = 0, \tag{1.1}$$

$$\lim_{t \to \infty} \langle \hat{x}(t), \psi(t) \rangle = 0, \tag{1.2}$$

might be not necessary, where $\hat{x}$ is the optimal *state variable*, ψ is the corresponding *adjoint variable*, and the angle brackets $\langle \cdot, \cdot \rangle$ denote the scalar product of two vectors.

The condition considered by Philippe Michel in [9], under assumptions including that the objective functional takes only finite values, has the form of the *Hamiltonian* $\mathcal{H}$ converging to zero

$$\lim_{t \to \infty} \mathcal{H}(\hat{x}(t), \hat{u}(t), t, \psi(t)) = 0, \tag{1.3}$$

where $\hat{u}$ is the optimal control.

Karl Shell [7, Lecture III] proposes, without proof, the condition

$$\limsup_{t \to \infty} \langle \psi(t), x(t) \rangle \geq \liminf_{t \to \infty} \langle \psi(t), \hat{x}(t) \rangle, \tag{1.4}$$

where x is any admissible path of the state variable. In [7, footnote 4 for Lecture III], he says that

> "this conjecture is related to a conjecture made by Kenneth J. Arrow in private correspondence."

Note that Arrow's sufficiency theorem contains a condition that follows from (1.4), which one can expect for problems, where the maximum principle provides both necessary and sufficient conditions of optimality.

In [10–14], the authors determine the adjoint variable uniquely by a Cauchy-type formula, that solves the adjoint equation with transversality conditions in the following form:

$$\lim_{t \to \infty} Y(t)\,\psi(t) = 0, \tag{1.5}$$

where $Y(t)$ is the *fundamental matrix* of the state equation linearized about the optimal solution, see eg. [15, 16].

Conditions (1.2)–(1.5) do hold for an optimal solution of the Ramsey problem with exponential discounting, when $\psi(t) \to 0$ as $t \to \infty$. But all (1.2)–(1.5) fail for the zero discount rate, when the objective functional diverges and $\psi(t) \to \text{const} > 0$ as $t \to \infty$. This case was considered by Ramsay in [3, p. 543] with moral motivation:

> "One point should perhaps be emphasised more particularly; we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination; we shall, however, in Section II include such a rate of discount in some of our investigations."

Even-though condition (1.3) is proved only for converging functionals, it can hold if we modify the objective improper integral subtracting from its integrand a constant, called "bliss" in [3], such that, for the optimal solution, the integral converges, see e.g., [2, Section 7].

In this paper, a new necessary condition is obtained for Ramsey-type problems both with and without discounting, where the integrand of an objective functional does not explicitly depend on the state variable. Since, without discounting, the objective functional diverges, we involve the concept of overtaking optimality. The new condition is the maximum of the integrand that completes the Pontryagin's maximum principal conditions. In contrast to the maximum condition for the Hamiltonian, the maximum of the integrand is taken w.r.t. control within admissible trajectories that satisfy the maximum principle (the maximum of the Hamiltonian, state and adjoint equations). The example of the Ramsey problem without discounting is considered.

2. Statement of the problem

Let X be a nonempty open convex subset of $\mathbb{R}$, and let U be an arbitrary nonempty set in $\mathbb{R}$. Let us consider the following optimal control problem:

$$\int_{t_0}^{\infty} e^{-\rho t} g(u(t))\,\mathrm{d}t \to \max_u, \tag{2.1}$$

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0, \tag{2.2}$$

where the control $u(t) \in U$ and the state variable $x(t) \in X$ for all $t \in (t_0, +\infty)$. We call such control $u(\cdot)$ and state variable $x(\cdot)$ trajectories *admissible*. The functions f and g are differentiable w.r.t. all their arguments, and f together with its partial derivatives is continuous in (x, u). Moreover, the function g is strictly concave and $\rho \geq 0$.

2.1. The concept of optimality

The improper integral in (2.1) might not converge for any candidate for optimal control $\hat{u}(\cdot)$; i.e., the limit

$$\lim_{T \to \infty} J(\hat{u}(\cdot), t_0, T), \tag{2.3}$$

might fail to exist, or might be infinite, where we introduce the finite time horizon functional:

$$J(u(\cdot), t_0, T) = \int_{t_0}^{T} e^{-\rho t} g(u(t)) \, \mathrm{d}t,$$

subject to state equation (2.2). Thus, the functional J may be unbounded as $T \to \infty$. So, we use the following more general concept of optimality.

Definition 1. *An admissible control $\hat{u}(\cdot)$ is **overtaking optimal (OO)** if for every admissible control $u(\cdot)$:*

$$\limsup_{T \to \infty} \left(J(u(\cdot), t_0, T) - J(\hat{u}(\cdot), t_0, T) \right) \leq 0.$$

When the ordinary optimality holds, i.e. a finite limit exists in (2.3) and, for all admissible controls $u(\cdot)$,

$$\limsup_{T \to \infty} J(u(\cdot), t_0, T) \leq \lim_{T \to \infty} J(\hat{u}(\cdot), t_0, T),$$

then $\hat{u}(\cdot)$ is also OO.

3. Optimality conditions

3.1. Pontryagin's maximum principle

With the use of the adjoint variable ψ, we introduce current value *Hamiltonian*

$$\mathcal{H}(x, u, \psi, \lambda) = \lambda \, g(u) + \psi \, f(x, u).$$

Theorem 1 [1, 6, 7]. *There exist $\lambda \geq 0$ and ψ_0 such that $(\lambda, \psi_0) \neq 0$ and the **maximum condition***

$$\mathcal{H}(\hat{x}(t), \hat{u}(t), \psi(t), \lambda) = \max_{u \in U} \mathcal{H}(\hat{x}(t), u, \psi(t), \lambda) \tag{3.1}$$

*holds along with the **adjoint equation***

$$-\dot{\psi}(t) = -\rho \psi(t) + \psi(t) \frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t)), \quad \psi(t_0) = \psi_0. \tag{3.2}$$

In this theorem, ψ_0 remains undetermined. Note that, for $\psi \equiv \psi_0 = 0$, the maximum condition (3.1) might have no solution with $\lambda > 0$, while $\lambda = \psi_0 = 0$ contradicts the theorem. Additional arguments are used to refine solutions of (3.1)–(3.2) and single out a nonzero value of ψ_0.

But it turns out that the maximum condition (3.1) with $\psi \equiv 0$ and $\lambda = 1$ can yield an additional necessary optimality condition if we replace the set U by the set $\hat{U}(\hat{x}(t))$ defined as follows.

Definition 2.

$$\hat{U}(x) = \{u : (u, x) \in G\},$$

where $G \subset U \times X$ consists of graphs of all admissible trajectories $(u(\cdot), x(\cdot))$ satisfying maximum principle (3.1)–(3.2) and the state equation (2.2). We denote the set of such trajectories by D, so that

$$G = \{(u(t), x(t)) : t \in (t_0, \infty), \ (u(\cdot), x(\cdot)) \in D\}.$$

3.2. An additional maximum condition

In order to use the condition in the main Proposition 1, we first need to calculate the sets $\hat{U}(x)$, thus making the synthesis of control. Due to the continuous dependence of solutions of the differential equation (3.2) on the initial value ψ_0 and due to the convexity of the set X, we state the following.

Assertion 1. If an admissible pair $(\hat{u}(\cdot), \hat{x}(\cdot))$ belongs to D, i.e., satisfies the maximum principle (3.1)–(3.2) and the state equation (2.2), then, for almost all $\tau > t_0$, there exists $\alpha_0 > 0$ such that, for all $\alpha \in (0, \alpha_0)$, the control

$$\begin{cases} \hat{u}(t), & t \le \tau - \alpha, \\ u(t), & t \in (\tau - \alpha, \tau], \\ \hat{u}(t), & t > \tau, \end{cases}$$

is admissible, where $u(t)$ for $t \in (\tau - \alpha, \tau]$ satisfies the maximum principle (3.1)–(3.2) and the state equation

$$\dot{x}(t) = f(x(t), u(t)), \quad x(\tau - \alpha) = \hat{x}(\tau - \alpha),$$

with the current state $\hat{x}(t)$ at time $t = \tau - \alpha$ as the initial condition.

The following main proposition is proved with the needle variation technique similar to the one in [11], provided that Assertion 1 holds.

Proposition 1 (Necessary optimality condition). *Let there exists an admissible pair $(\hat{u}(\cdot), \hat{x}(\cdot)) \in D$. If the control $\hat{u}$ is optimal, then*

$$g(u) \le g(\hat{u}(\tau)) \tag{3.3}$$

for almost all $\tau \in [t_0, \infty)$ and all $u \in \hat{U}(\hat{x}(\tau))$.

P r o o f. According to Assertion 1, we can define an admissible *needle variation* of optimal pair $(\hat{u}(\cdot), \hat{x}(\cdot)) \in D$ at almost any time τ, i.e., excluding discontinuity time instances of control, as

$$u_\alpha(t) := \begin{cases} \hat{u}(t), & t \le \tau - \alpha, \\ u(t), & t \in (\tau - \alpha, \tau], \\ \hat{u}(t), & t > \tau, \end{cases}$$

where $u(t)$ satisfies the maximum principle (3.1)–(3.2) and the state equation $\dot{x}(t) = f(x(t), u(t))$ with the condition $x(\tau - \alpha) = \hat{x}(\tau - \alpha)$. Thus,

$$J(u_\alpha(\cdot), t_0, T) = \int_{t_0}^{\tau - \alpha} e^{-\rho t} g(\hat{u}(t)) \, \mathrm{d}t + \int_{\tau - \alpha}^{\tau} e^{-\rho t} g(u(t)) \, \mathrm{d}t + \int_{\tau}^{T} e^{-\rho t} g(\hat{u}(t)) \, \mathrm{d}t.$$

The corresponding increment in the value of the functional can be written as follows:

$$\Delta J_\alpha(T) := J(u_\alpha(\cdot), t_0, T) - J(\hat{u}(\cdot), t_0, T) = J(u_\alpha(\cdot), \tau - \alpha, \tau) - J(\hat{u}(\cdot), \tau - \alpha, \tau) =$$

$$= \int_{\tau-\alpha}^{\tau} e^{-\rho t}\left(g(u(t)) - g(\hat{u}(t))\right) \mathrm{d}t,$$

Hence,

$$\frac{\Delta J_\alpha(T)}{\alpha} \geq \frac{1}{\alpha}\int_{\tau-\alpha}^{\tau} e^{-\rho t}\left(g(u(t)) - g(\hat{u}(t))\right)\mathrm{d}t. \tag{3.4}$$

We have the following approximation of the latter term in (3.4), due to the continuity of u and $\hat{u}$ at τ and the continuity of g w.r.t. u:

$$\frac{1}{\alpha}\int\limits_{\tau-\alpha}^{\tau} e^{-\rho t}\left(g(u(t)) - g(\hat{u}(t))\right)\mathrm{d}t = e^{-\rho\tau}\left(g(u(\tau)) - g(\hat{u}(\tau))\right) + O(\alpha),$$

where $\lim\limits_{\alpha\to 0} O(\alpha) = 0$. Hence, inequality (3.4) takes the form

$$\frac{\Delta J_\alpha(T)}{\alpha} \geq e^{-\rho\tau}\left(g(u(\tau)) - g(\hat{u}(\tau))\right) + O(\alpha).$$

Definition 1 of OO for the initial problem means that, for all $\varepsilon_2 > 0$, there exists $T_2(\varepsilon_2) > t_0$ such that, for all $T' \geq T_2(\varepsilon_2)$, we have $\Delta J_\alpha(T') \leq \varepsilon_2$. Let us take $\varepsilon_2 = \alpha\,\varepsilon$. Then the inequality $\Delta J_\alpha(T') \leq \alpha\,\varepsilon$ results in

$$\varepsilon \geq e^{-\rho\tau}\left(g(u(\tau)) - g(\hat{u}(\tau))\right) + O(\alpha). \tag{3.5}$$

Suppose that (3.3) is violated, i.e., there exist $\varepsilon > 0$ and $u \in \hat{U}(\hat{x}(\tau))$ such that the following inequality holds:

$$e^{-\rho\tau}\left(g(u) - g(\hat{u}(\tau))\right) \geq 2\varepsilon,$$

this contradicts to (3.5) for $u = u(\tau)$, when α is small enough. Hence, (3.3) should hold. $\qquad\square$

Corollary 1. *If g is a strictly monotone function and there exists an optimal control, then the optimal trajectory $(\hat{u}(\cdot), \hat{x}(\cdot))$ is unique and its graph belongs to the boundary of the set G.*

This is the case in the Ramsey problem of optimal capital accumulation, where the optimal trajectory is on the saddle path separating the domain of admissible pairs $(c(\cdot), k(\cdot))$ of consumption c and capital k that does not deplete capital, $k(t) > 0$ for all $t > 0$, see the domain of thin solid lines in Fig. 1.

Example 1. [The Ramsey problem with $\rho = 0$.] We maximize the aggregated constant relative risk aversion utility

$$\int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta}\,\mathrm{d}t \to \max_{c>0}$$

subject to the dynamics of the capital

$$\dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t), \quad k(t) > 0,$$

where $k(0) = k_0 > 0$, $\theta \neq 1$, $\theta > 0$, and $\alpha \in (0,1)$.

The Hamiltonian:

$$\mathcal{H}(k, c, \psi, \lambda) = \lambda\,\frac{c^{1-\theta}}{1-\theta} + \psi\left(k^\alpha - \delta k - c\right), \quad \lambda \geq 0, \quad (\lambda, \psi) \neq 0.$$
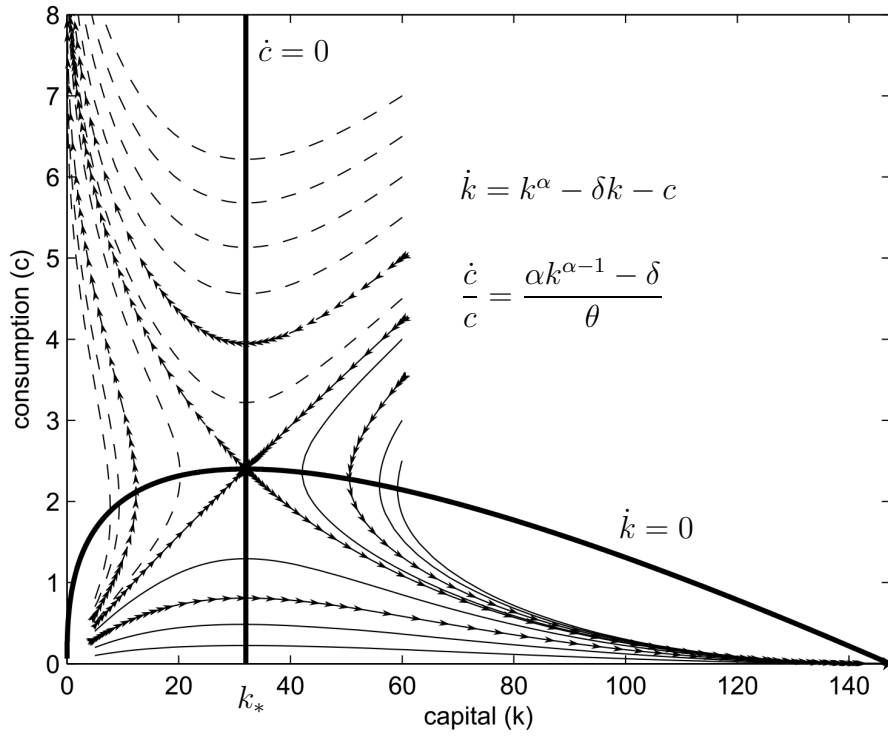
Figure 1. The bold lines are the stationary curves, $\dot{k} = 0$ and $\dot{c} = 0$. The solid lines are the trajectories governed by the state and Euler equations, for which $k(t) > 0$ for all $t > t_0$.

The abnormal case $(\lambda = 0)$ would lead to $\psi = 0$ and thus impossible.
The stationarity condition

$$\lambda\, c(t)^{-\theta} = \psi(t)$$

and the adjoint equation

$$-\dot{\psi}(t) = \left(\alpha k(t)^{\alpha-1} - \delta\right)\psi(t)$$

result in the Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha k(t)^{\alpha-1} - \delta}{\theta}.$$

Due to the Euler and state equations, any admissible pair $(k(\cdot), c(\cdot))$ not violating the constraints $c(t) \geq 0$ and $k(t) > 0$ converges either to the steady state (k_*, c_*), where $k_* = (\delta/\alpha)^{1/(\alpha-1)}$ and $c_* = (1-\alpha)\, k_* > 0$, or to $(\delta^{1/(\alpha-1)}, 0)$, where $k_* < \delta^{1/(\alpha-1)}$. The solid lines in Fig. 1 belong to D and their graphs constitute the set G, which is whole space below the saddle path and the horizontal line $c = 0$.[2] Thus, condition (3.3)

$$\frac{\hat{c}(t)^{1-\theta}}{1-\theta} \geq \frac{c^{1-\theta}}{1-\theta} \quad \text{for all } c \in \{c : (c, \hat{k}(t)) \in G\}$$

selects the saddle path as unique possible optimal.

_____

[2]It is easy to see in Fig. 1, that Assertion 1 holds, since such a variation of the saddle path trajectory diverts the remaining part of the trajectory below the saddle path so that its graph remains in G, even though the trajectory itself might not belong to D anymore, because its remaining part does not have to satisfy the maximum principle.

4. Conclusion

A quite simple additional necessary optimality condition is derived in the form of maximization of the integrand of the objective functional in each state x on the subset of control values $\hat{U}(x) \subset U$. Even though one should first determine the set D of admissible trajectories that solve state and adjoint equations subject to the maximum condition, it is anyway done in practice when possible. The substantial assumption is that the objective functional does not explicitly depend on the state. The functional is subject to the state equation indirectly via the form of an open admissible set X. In a more general case, the additional maximum condition is more involved, see [17].

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TO A QUESTION ON THE SUPERCOMPACTNESS OF ULTRAFILTER SPACES[1]

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Abstract: The space of ultrafilters of a π-system endowed with the topology of Wallman type is considered. The question on the supercompactness of this space is investigated. For this, the enveloping space of maximal linked systems with the corresponding topology of Wallman type is used. Necessary and sufficient conditions for the coincidence of the set of all ultrafilters of the initial π-system and the set of all maximal linked systems for this π-system are obtained. Specific variants of wide sense measurable spaces with this coincidence property are given.

Keywords: Maximal linked system, Topology, Supercompactness, Ultrafilter.

Introduction

In the present paper, we study ultrafilters of wide sense measurable spaces. Every such measurable space is defined by introducing a π-system of subsets of a nonempty set. More precisely, a π-system is a family of sets closed with respect to finite intersections; see [1, p. 14]. We consider only π-systems with "zero" and "unit," i.e., the empty set and the enveloping set are elements of our π-systems. This approach enables the precise description of the ultrafilter set for a given measurable space (in this connection, see [2, 3]). The ultrafilter set can be equipped with a Wallman type topology. We consider the equipping of this type as a basic thing. In addition, we use a Stone type topology. As a result, a bitopological space is realized. This bitopological space can be considered as a kind of a subspace of the bitopological space of maximal linked systems with Wallman and Stone equippings.

Note that Wallman and Stone type topologies are used in the theory of extension of topological spaces (see [4; 5, Sect. 3.6] and others). Ultrafilters play an important role in general topology. Constructions involving maximal linked systems are used in connection with the notions of topological space; in addition, usually, maximal linked systems of the family of closed sets in a T_1-space are considered (see [4, Sect. 5.11], [6–8], and [9, Ch. VII, Sect. 4]). We emphasize the important result of [8] on the supercompactness of metrizable compact sets.

In [10], for superextension constructions, a natural bitopological equipping was realized. In the author's consequent papers [11–14], this approach of [10] was extended to the case when a π-system is used as an anticipating measurable structure (the approach of [6–8] corresponds to the case when this π-system is the lattice of closed sets in a T_1-space). Thus, in the above-mentioned general case of an arbitrary π-system, bitopological spaces of ultrafilters and maximal linked systems were constructed (see [11–14]). In addition, in the case of maximal linked systems, a topology of Wallman type generates a supercompact space.

Recall that ultrafilters are maximal linked systems. But often there are maximal linked systems that are not ultrafilters. The simplest example of such maximal linked system was given in [9, 4.18]. This example plays an important role in questions of representation of maximal linked systems.

At the same time, for some (widely understood) measurable space, all maximal linked systems are ultrafilters; in this connection, see [15, 16]. In the present paper, we investigate this case in more details. It can be interesting because the question on the supercompactness of the ultrafilter space with a Wallman type topology arises. In addition, the supercompactness property is not, generally speaking, hereditary (see [4, Sect. 5.11]). However, in the case when all maximal linked systems are ultrafilters, the required supercompactness property of the ultrafilter space can be obtained from the similar property of the space of maximal linked systems. This approach reduces to an interesting representation, but requires finding conditions for the above-mentioned coincidence of the ultrafilter set and the set of maximal linked systems. The present paper is devoted to this issue.

1. General notions and notation

We use the standard set-theoretic symbolism (quantifiers and connectives); $\stackrel{\triangle}{=}$ stands for the equality by definition and $\varnothing$ for the empty set. In what follows, a family is a set whose elements are also sets. We accept the axiom of choice. For any objects x and y, we denote by $\{x; y\}$ the set containing x and y (as proper elements) and not containing other elements; thus, $\{x; y\}$ is an unordered pair of the objects x and y. If z is an object, then $\{z\} \stackrel{\triangle}{=} \{z; z\}$ is a singleton containing z as a proper element. For any three objects u, v, and w, we set $\{u; v; w\} \stackrel{\triangle}{=} \{u; v\} \cup \{w\}$. For any objects p and q, $(p, q) \stackrel{\triangle}{=} \{\{p\}; \{p; q\}\}$ is the ordered pair [17, Ch. II, Sect. 3] with the first element p and the second element q. If z is an ordered pair, then $\mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$ are the first and the second elements of z, respectively.

If X is a set, then $\mathcal{P}(X)$ is, by definition, the family of all subsets of X, $\mathcal{P}'(X) \stackrel{\triangle}{=} \mathcal{P}(X) \setminus \{\varnothing\}$, and $\mathrm{Fin}(X)$ is the family of all finite sets of $\mathcal{P}'(X)$; thus, $\mathrm{Fin}(X)$ is the family of all nonempty finite subsets of X. Certainly, a family can be used as X. In this connection, we introduce the following notation [12, Sect. 2] for a fixed nonempty family $\mathfrak{X}$:

$$\{\cup\}(\mathfrak{X}) \stackrel{\triangle}{=} \{ \bigcup_{X \in \mathcal{X}} X : \mathcal{X} \in \mathcal{P}(\mathfrak{X})\}, \quad \{\cap\}(\mathfrak{X}) \stackrel{\triangle}{=} \{ \bigcap_{X \in \mathcal{X}} X : \mathcal{X} \in \mathcal{P}'(\mathfrak{X})\},$$

$$\{\cup\}_\sharp(\mathfrak{X}) \stackrel{\triangle}{=} \{ \bigcup_{X \in \mathcal{K}} X : \mathcal{K} \in \mathrm{Fin}(\mathfrak{X})\}, \quad \{\cap\}_\sharp(\mathfrak{X}) \stackrel{\triangle}{=} \{ \bigcap_{X \in \mathcal{K}} X : \mathcal{K} \in \mathrm{Fin}(\mathfrak{X})\}.$$

If $\mathbb{M}$ is a set and $\mathcal{M} \in \mathcal{P}'(\mathcal{P}(\mathbb{M}))$, then

$$\mathbf{C}_{\mathbb{M}}[\mathcal{M}] \stackrel{\triangle}{=} \{\mathbb{M} \setminus M : M \in \mathcal{M}\} \in \mathcal{P}'(\mathcal{P}(\mathbb{M}))$$

is the family dual to $\mathcal{M}$. For any set H and a family $\mathcal{H} \in \mathcal{P}'(\mathcal{P}(H))$, we have

$$\mathcal{H} = \mathbf{C}_H[\mathbf{C}_H[\mathcal{H}]].$$

For any nonempty family $\mathcal{S}$ and a set L, define

$$(\mathcal{S}|_L \stackrel{\triangle}{=} \{S \cap L : S \in \mathcal{S}\} \in \mathcal{P}'(\mathcal{P}(L)))\&([\mathcal{S}](L) \stackrel{\triangle}{=} \{S \in \mathcal{S}|L \subset S\} \in \mathcal{P}(\mathcal{S})). \tag{1.1}$$

If $\mathbb{X}$ is a nonempty set and $\mathcal{X} \in \mathcal{P}'(\mathcal{P}(\mathbb{X}))$, then

$$(\mathrm{COV})[\mathbb{X}|\mathcal{X}] \stackrel{\triangle}{=} \{\chi \in \mathcal{P}'(\mathcal{X})|\mathbb{X} = \bigcup_{X \in \chi} X\} \in \mathcal{P}(\mathcal{P}'(\mathcal{X}))$$

is the family of all coverings of the set $\mathbb{X}$ by sets of $\mathcal{X}$; a covering $\kappa \in (\mathrm{COV})[\mathbb{X}|\mathcal{X}]$ is called binary if $\kappa = \{X_1; X_2\}$, where $X_1 \in \mathcal{X}$ and $X_2 \in \mathcal{X}$.

Let $\mathbb{R}$ be the real line and $\mathbb{N} \stackrel{\triangle}{=} \{1; 2; \ldots\}$; $\mathbb{N} \in \mathcal{P}'(\mathbb{R})$. For $n \in \mathbb{N}$, we use the notation $\overline{1, n} \stackrel{\triangle}{=} \{k \in \mathbb{N} | k \le n\}$. For a set H, denote by H^n the set of all mappings from $\overline{1, n}$ into H; thus, elements of H^n are n-tuples $(h_i)_{i \in \overline{1, n}}$, where $h_j \in H$ for $j \in \overline{1, n}$.

Some special families. In this subsection, we fix a nonempty set $\mathbf{I}$ and consider families of $\mathcal{P}'(\mathcal{P}(\mathbf{I}))$, i.e., nonempty families of subsets of $\mathbf{I}$, especially, π-systems. More precisely, let

$$\pi[\mathbf{I}] \stackrel{\triangle}{=} \big\{\mathcal{I} \in \mathcal{P}'(\mathcal{P}(\mathbf{I}))| \ (\varnothing \in \mathcal{I}) \& (\mathbf{I} \in \mathcal{I}) \& (A \cap B \in \mathcal{I} \ \ \forall A \in \mathcal{I} \ \ \forall B \in \mathcal{I})\big\} \tag{1.2}$$

be the family of all π-systems of subsets of $\mathbf{I}$ with "zero" and "unit." If $\mathcal{I} \in \pi[\mathbf{I}]$, then the pair $(\mathbf{I}, \mathcal{I})$ is considered as a measurable space in the wide sense. Denote by

$$\tilde{\pi}^0[\mathbf{I}] \stackrel{\triangle}{=} \big\{\mathcal{I} \in \pi[\mathbf{I}]| \ \forall L \in \mathcal{I} \ \ \forall x \in \mathbf{I} \setminus L \ \ \exists \Lambda \in \mathcal{I} : \ (x \in \Lambda) \& (\Lambda \cap L = \varnothing)\big\}$$

the family of all separable π-systems of (1.2). Finally, we introduce the family

$$\pi_*^\sharp[\mathbf{I}] \stackrel{\triangle}{=} \big\{\mathcal{I} \in \pi[\mathbf{I}]| \ \ \forall \Sigma_1 \in \mathcal{I} \ \ \forall \Sigma_2 \in \mathcal{I} \ \ \forall \Sigma_3 \in \mathcal{I}$$
$$\big((\Sigma_1 \cap \Sigma_2 \ne \varnothing) \& (\Sigma_2 \cap \Sigma_3 \ne \varnothing) \& (\Sigma_1 \cap \Sigma_3 \ne \varnothing)\big) \Longrightarrow (\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \ne \varnothing)\big\}. \tag{1.3}$$

In (1.3), we have a special variant of π-systems connected with the linkedness property that plays a basic role in this paper. We introduce the family of lattices by using only subsets of $\mathbf{I}$. More precisely, let

$$(\mathrm{LAT})_0[\mathbf{I}] \stackrel{\triangle}{=} \big\{\mathfrak{L} \in \pi[\mathbf{I}]| \ A \cup B \in \mathfrak{L} \ \ \forall A \in \mathfrak{L} \ \ \forall B \in \mathfrak{L}\big\}$$

be the family of all lattices of subsets of $\mathbf{I}$. In addition, the following two particular cases are realized:

$$(\mathrm{alg})[\mathbf{I}] \stackrel{\triangle}{=} \{\mathcal{A} \in \pi[\mathbf{I}]| \ \mathbf{I} \setminus A \in \mathcal{A} \ \ \forall A \in \mathcal{A}\} = \{\mathcal{A} \in (\mathrm{LAT})_0[\mathbf{I}]| \ \mathbf{I} \setminus A \in \mathcal{A} \ \ \forall A \in \mathcal{A}\},$$

$$(\mathrm{top})[\mathbf{I}] \stackrel{\triangle}{=} \Big\{\tau \in \pi[\mathbf{I}]| \ \bigcup_{G \in \mathcal{G}} G \in \tau \ \ \forall \mathcal{G} \in \mathcal{P}'(\tau)\Big\} = \Big\{\tau \in (\mathrm{LAT})_0[\mathbf{I}]| \ \bigcup_{G \in \mathcal{G}} G \in \tau \ \ \forall \mathcal{G} \in \mathcal{P}'(\tau)\Big\}.$$

Certainly, $(\mathbf{I}, \mathcal{A})$ with $\mathcal{A} \in (\mathrm{alg})[\mathbf{I}]$ is a usual measurable space with algebra of sets. Moreover, $(\mathbf{I}, \tau)$ with $\tau \in (\mathrm{top})[\mathbf{I}]$ is a topological space. Finally,

$$(\mathrm{clos})[\mathbf{I}] \stackrel{\triangle}{=} \Big\{\mathcal{F} \in (\mathrm{LAT})_0[\mathbf{I}]| \ \bigcap_{F \in \mathcal{F}'} F \in \mathcal{F} \ \ \forall \mathcal{F}' \in \mathcal{P}'(\mathcal{F})\Big\} = \{\mathbf{C}_{\mathbf{I}}[\tau] : \ \tau \in (\mathrm{top})[\mathbf{I}]\}. \tag{1.4}$$

Elements of (1.4) are closed topologies in the sense of P.S. Aleksandrov [18, Ch. 4, Sect. 1]. We introduce the family of all nonempty centered subfamilies of a π-system:

$$(\mathrm{Cen})[\mathcal{I}] \stackrel{\triangle}{=} \Big\{\mathcal{Z} \in \mathcal{P}'(\mathcal{I})| \ \bigcap_{Z \in \mathcal{K}} Z \ne \varnothing \ \ \forall \mathcal{K} \in \mathrm{Fin}(\mathcal{Z})\Big\} \ \ \forall \mathcal{I} \in \pi[\mathbf{I}].$$

Now, we recall notions connected with bases and subbases of topological spaces. In addition,

$$(\mathrm{BAS})[\mathbf{I}] \stackrel{\triangle}{=} \Big\{\beta \in \mathcal{P}'(\mathcal{P}(\mathbf{I})) \ | \ (\mathbf{I} = \bigcup_{B \in \beta} B) \ \& \ (\forall B_1 \in \beta \ \ \forall B_2 \in \beta \ \ \forall x \in B_1 \cap B_2$$
$$\exists B_3 \in \beta : (x \in B_3) \ \& \ (B_3 \subset B_1 \cap B_2))\Big\}, \tag{1.5}$$

$$(\mathrm{cl} - \mathrm{BAS})[\mathbf{I}] \stackrel{\triangle}{=} \Big\{\beta \in \mathcal{P}'(\mathcal{P}(\mathbf{I})) \ | \ (\mathbf{I} \in \beta) \ \& \ (\bigcap_{B \in \beta} B = \varnothing)$$
$$\& \ (\forall B_1 \in \beta \ \ \forall B_2 \in \beta \ \ \forall x \in \mathbf{I} \setminus (B_1 \cup B_2) \ \ \exists B_3 \in \beta : (B_1 \cup B_2 \subset B_3) \ \& \ (x \notin B_3))\Big\}. \tag{1.6}$$

In (1.5), bases of sets open in some topology of $\mathbf{I}$ are introduced. Similarly, in (1.6), bases of sets closed in some topology of $\mathbf{I}$ are introduced. For brevity, we call bases from (1.5) and (1.6) open and closed, respectively. Certainely, the property $\{\cup\}(\beta) \in (\text{top})[\mathbf{I}]$ (the property $\{\cap\}(\beta) \in (\text{clos})[\mathbf{I}]$) is realized for $\beta \in (\text{BAS})[\mathbf{I}]$ (for $\beta \in (\text{cl} - \text{BAS})[\mathbf{I}]$). Moreover, we consider open and closed bases of a specific topological space. More precisely, for any topology $\tau \in (\text{top})[\mathbf{I}]$, let

$$\big((\tau - \text{BAS})_0[\mathbf{I}] \triangleq \{\beta \in (\text{BAS})[\mathbf{I}] \mid \tau = \{\cup\}(\beta)\}\big)$$
$$\&\big((\text{cl} - \text{BAS})_0[\mathbf{I}; \tau] \triangleq \{\beta \in (\text{cl} - \text{BAS})[\mathbf{I}] \mid \mathbf{C}_\mathbf{I}[\tau] = \{\cap\}(\beta)\}\big)$$

(thus, open and closed bases of the topological space $(\mathbf{I}, \tau)$ are introduced). We introduce subbases of open and closed sets in some topology on the set $\mathbf{I}$:

$$\big((\text{p} - \text{BAS})[\mathbf{I}] \triangleq \{\mathbf{x} \in \mathcal{P}'(\mathcal{P}(\mathbf{I})) \mid \{\cap\}_\sharp(\mathbf{x}) \in (\text{BAS})[\mathbf{I}]\} = \{\mathbf{x} \in \mathcal{P}'(\mathcal{P}(\mathbf{I})) \mid \mathbf{I} = \bigcup_{\mathbb{X} \in \mathbf{x}} \mathbb{X}\}\big)$$
$$\&\big((\text{p} - \text{BAS})_{\text{cl}}[\mathbf{I}] \triangleq \{\chi \in \mathcal{P}'(\mathcal{P}(\mathbf{I})) \mid \{\cup\}_\sharp(\chi) \in (\text{cl} - \text{BAS})[\mathbf{I}]\}\big).$$

Now, we introduce open and closed subbases of a specific topological space. More precisely, for $\tau \in (\text{top})[\mathbf{I}]$, let

$$\begin{aligned}&\big((\text{p} - \text{BAS})_0[\mathbf{I}; \tau] \triangleq \{\chi \in (\text{p} - \text{BAS})[\mathbf{I}] \mid \{\cap\}_\sharp(\chi) \in (\tau - \text{BAS})_0[\mathbf{I}]\}\big)\\&\&\big((\text{p} - \text{BAS})^0_{\text{cl}}[\mathbf{I}; \tau] \triangleq \{\chi \in (\text{p} - \text{BAS})_{\text{cl}}[\mathbf{I}] \mid \{\cup\}_\sharp(\chi) \in (\text{cl} - \text{BAS})_0[\mathbf{I}; \tau]\}\big).\end{aligned} \tag{1.7}$$

In (1.7), the families of open and closed subbases of the topological space $(\mathbf{I}, \tau)$ are introduced. For $\tau \in (\text{top})[\mathbf{I}]$ and $x \in \mathbf{I}$, we set $N^0_\tau(x) \triangleq \{G \in \tau \mid x \in G\}$ and

$$N_\tau(x) \triangleq \{H \in \mathcal{P}(\mathbf{I}) \mid \exists G \in N^0_\tau(x) : G \subset H\}$$

obtaining the family of all neighborhoods of x in the topological space $(\mathbf{I}, \tau)$. In addition, for $\tau \in (\text{top})[\mathbf{I}]$ and $A \in \mathcal{P}(\mathbf{I})$, we define the closure of A in the topological space $(\mathbf{I}, \tau)$:

$$\text{cl}(A, \tau) \triangleq \{x \in \mathbf{I} \mid A \cap H \neq \varnothing \ \ \forall H \in N_\tau(x)\}.$$

In what follows, the supercompactness property plays an important role. In this connection, we set

$$\begin{aligned}((\mathbb{SC}) - \text{top})[\mathbf{I}] \triangleq \{\tau \in (\text{top})[\mathbf{I}] \mid \exists \mathcal{S} \in (\mathbf{p} - \text{BAS})_0[\mathbf{I}; \tau] \ \ \forall \mathcal{G} \in (\text{COV})[\mathbf{I} \mid \mathcal{S}]\\\exists G_1 \in \mathcal{G} \ \ \exists G_2 \in \mathcal{G} : \mathbf{I} = G_1 \cup G_2\}.\end{aligned} \tag{1.8}$$

Then, (1.8) is the family of all topologies converting $\mathbf{I}$ to a supercompact topological space. A supercompact T_2-space is called [4] a supercompact set. Certainly, (1.8) can be defined in terms of closed subbases (see [4–9]).

2. The family of ultrafilters and its Stone equipping

In what follows, we fix a nonempty set E and consider its π-systems. Moreover, in this section, we also fix $\mathcal{I} \in \pi[E]$. Then

$$\mathbb{F}^\star(\mathcal{I}) \triangleq \big\{\mathcal{F} \in \mathcal{P}'(\mathcal{I} \setminus \{\varnothing\}) \mid \big(A \cap B \in \mathcal{F} \ \forall A \in \mathcal{F} \ \forall B \in \mathcal{F}\big) \& \big(\forall F \in \mathcal{F} \ \forall I \in \mathcal{I} \ (F \subset I) \Longrightarrow (I \in \mathcal{F})\big)\big\}$$

is the set of all filters of the (widely understood) measurable space $(E, \mathcal{I})$. The simplest variant of a filter is given by the following obvious definition: for $x \in E$,

$$(\mathcal{I} - \text{triv})[x] \triangleq \{I \in \mathcal{I} \mid x \in I\} \in \mathbb{F}^\star(\mathcal{I}). \tag{2.1}$$

In (2.1), we obtain the trivial filter corresponding to the point x. Maximal filters are called ultra-filters. Then

$$
\begin{aligned}
\mathbb{F}_0^\star(\mathcal{I}) &\stackrel{\triangle}{=} \{\mathcal{U} \in \mathbb{F}^\star(\mathcal{I}) \mid \forall \mathcal{F} \in \mathbb{F}^\star(\mathcal{I}) \ (\mathcal{U} \subset \mathcal{F}) \Longrightarrow (\mathcal{U} = \mathcal{F})\} \\
&= \{\mathcal{U} \in \mathbb{F}^*(\mathcal{I}) \mid \forall I \in \mathcal{I} \ (I \cap U \neq \varnothing \ \forall U \in \mathcal{U}) \Longrightarrow (I \in \mathcal{U})\} \\
&= \{\mathcal{U} \in (\mathrm{Cen})[\mathcal{I}] \mid \forall \mathcal{V} \in (\mathrm{Cen})[\mathcal{I}] \ (\mathcal{U} \subset \mathcal{V}) \Longrightarrow (\mathcal{U} = \mathcal{V})\}
\end{aligned}
\tag{2.2}
$$

is the set of all ultrafilters of the measurable space $(E, \mathcal{I})$. It is known (see [19, (5.9)]) that

$$
\big((\mathcal{I} - \mathrm{triv})[x] \in \mathbb{F}_0^\star(\mathcal{I}) \ \forall x \in E\big) \Longleftrightarrow (\mathcal{I} \in \tilde{\pi}^0[E]).
$$

In the general case of $\mathcal{I}$, $\forall \mathcal{F} \in \mathbb{F}^*(\mathcal{I}) \ \exists \mathcal{U} \in \mathbb{F}_0^*(\mathcal{I}) \colon \mathcal{F} \subset \mathcal{U}$ (this property is established by the Zorn lemma). Thus, by (2.1), we obtain $\mathbb{F}_0^*(\mathcal{I}) \neq \varnothing$. Let

$$
\Phi_{\mathcal{I}}(L) \stackrel{\triangle}{=} \{\mathcal{U} \in \mathbb{F}_0^\star(\mathcal{I}) \mid L \in \mathcal{U}\} \quad \forall L \in \mathcal{I}.
\tag{2.3}
$$

In addition, (2.2) and (2.3) imply the following property:

$$
\Phi_{\mathcal{I}}(L) \stackrel{\triangle}{=} \{\mathcal{U} \in \mathbb{F}_0^\star(\mathcal{I}) \mid U \cap L \neq \varnothing \ \forall U \in \mathcal{U}\} \quad \forall L \in \mathcal{I}.
\tag{2.4}
$$

Representation (2.4) plays an important role in the theory of ultrafilters. Note that

$$
(\mathbb{UF})[E; \mathcal{I}] \stackrel{\triangle}{=} \{\Phi_{\mathcal{I}}(I) : \ I \in \mathcal{I}\} \in \pi[\mathbb{F}_0^\star(\mathcal{I})].
$$

In addition, $(\mathbb{UF})[E; \mathcal{I}] \in \pi[\mathbb{F}_0^\star(\mathcal{I})]$ and, in particular, $(\mathbb{UF})[E; \mathcal{I}] \in (\mathrm{BAS})[\mathbb{F}_0^\star(\mathcal{I})]$. Thus, we obtain the topology of Stone type:

$$
\mathbf{T}_{\mathcal{I}}^\star[E] \stackrel{\triangle}{=} \{\cup\}\big((\mathbb{UF})[E; \mathcal{I}]\big) \in (\mathrm{top})[\mathbb{F}_0^\star(\mathcal{I})].
$$

Thus, $(\mathbb{UF})[E; \mathcal{I}] \in (\mathbf{T}_{\mathcal{I}}^\star[E] - \mathrm{BAS})_0[\mathbb{F}_0^\star(\mathcal{I})]$. In addition, by (2.4), we have the following inclusion:

$$
(\mathbb{UF})[E; \mathcal{I}]) \subset \mathbf{T}_{\mathcal{I}}^\star[E] \cap \mathbf{C}_{\mathbb{F}_0^\star(\mathcal{I})}[\mathbf{T}_{\mathcal{I}}^\star[E]]
\tag{2.5}
$$

(for $\mathcal{I} \in (\mathrm{alg})[E]$, (2.5) becomes an equality, see [20, Remark 3.3]). Note that

$$
(\mathbb{F}_0^\star(\mathcal{I}), \mathbf{T}_{\mathcal{I}}^\star[E])
\tag{2.6}
$$

is a zero-dimensional T_2-space (if $\mathcal{I} \in (\mathrm{alg})[E]$, then (2.6) is a zero-dimensional compact set, see [20, Sect. 3]).

For $H \in \mathcal{P}(E)$, we set

$$
\mathbb{F}_{\mathbf{C}}^\flat[\mathcal{I}|H] \stackrel{\triangle}{=} \{\mathcal{U} \in \mathbb{F}_0^\star(\mathcal{I}) \mid \exists U \in \mathcal{U} : \ U \subset H\}.
$$

By (2.4), we have

$$
\mathbb{F}_{\mathbf{C}}^\flat[\mathcal{I}|E \setminus I] = \mathbb{F}_0^\star(\mathcal{I}) \setminus \Phi_{\mathcal{I}}(I) \quad \forall I \in \mathcal{I}.
\tag{2.7}
$$

Using (2.7), we obtain the following representation:

$$
\mathfrak{F}_{\mathbf{C}}^\flat[\mathcal{I}] \stackrel{\triangle}{=} \{\mathbb{F}_{\mathbf{C}}^\flat[\mathcal{I}|\Lambda] : \ \Lambda \in \mathbf{C}_E[\mathcal{I}]\} = \mathbf{C}_{\mathbb{F}_0^\star(\mathcal{I})}[(\mathbb{UF})[E; \mathcal{I}]] \in (\mathrm{cl} - \mathrm{BAS})_0[\mathbb{F}_0^\star(\mathcal{I}); \mathbf{T}_{\mathcal{I}}^\star[E]].
\tag{2.8}
$$

Thus, by (2.8), we obtain a closed base of the topological space (2.6). In this connection, it is useful to note one representation of ultrafilters connected with (2.7) and (2.8). More precisely, from (2.6)

and (2.8), we conclude that, for every $L \in \mathcal{I}$, the set $\Phi_{\mathcal{I}}(L)$ is the intersection of all sets of the family $[\mathfrak{F}_{\mathbf{C}}^{\natural}[\mathcal{I}]](\Phi_{\mathcal{I}}(L))$. In addition, by [12, Proposition 3.1], for $L \in \mathcal{I}$ and $\Lambda \in \mathbf{C}_E[\mathcal{I}]$, we have

$$(L \subset \Lambda) \Longleftrightarrow (\Phi_{\mathcal{I}}(L) \subset \mathbb{F}_{\mathbf{C}}^{\natural}[\mathcal{I}|\Lambda]).$$

As a corollary, we obtain the following property:

$$[\mathfrak{F}_{\mathbf{C}}^{\natural}[\mathcal{I}]](\Phi_{\mathcal{I}}(L)) = \{\mathbb{F}_{\mathbf{C}}^{\natural}[\mathcal{I}|\Lambda] : \ \Lambda \in [\mathbf{C}_E[\mathcal{I}]](L)\} \quad \forall L \in \mathcal{I}. \tag{2.9}$$

From (2.9), we extract the useful equality

$$\mathbb{F}_0^{\star}(\mathcal{I}) = \{\mathcal{U} \in \mathbb{F}^{\star}(\mathcal{I}) | \ \forall I \in \mathcal{I} \ (I \in \mathcal{U}) \vee (\exists \Lambda \in [\mathbf{C}_E[\mathcal{I}]](I) : \ E \setminus \Lambda \in \mathcal{U})\}. \tag{2.10}$$

In connection with (2.10), we emphasize the constructions from [16, Sect. 2].

3.   Ultrafilters and maximal linked systems; topologies of Wallman type

In what follows, unless otherwise stated, we fix a π-system $\mathcal{E} \in \pi[E]$. Since $\mathfrak{F}_{\mathbf{C}}^{\natural}[\mathcal{E}] \in (\mathrm{p} - \mathrm{BAS})[\mathbb{F}_0^{\star}(\mathcal{E})]$, we introduce the topology of Wallman type

$$\mathbf{T}_{\mathcal{E}}^0 \langle E \rangle \stackrel{\triangle}{=} \{\cup\}\big(\{\cap\}_{\sharp}(\mathfrak{F}_{\mathbf{C}}^{\natural}[\mathcal{E}])\big) \in (\mathrm{top})[\mathbb{F}_0^{\star}(\mathcal{E})]. \tag{3.1}$$

In addition, we obtain the topological space

$$(\mathbb{F}_0^{\star}(\mathcal{E}), \mathbf{T}_{\mathcal{E}}^0 \langle E \rangle). \tag{3.2}$$

Using (3.1), we also call (3.2) a topological space of Wallman type. To investigate the properties of this topological space, we consider maximal linked systems of the π-system $\mathcal{E}$. More precisely, for $\mathfrak{X} \in \mathcal{P}'(\mathcal{P}(E))$, we set

$$\langle \mathfrak{X} - \mathrm{link} \rangle[E] \stackrel{\triangle}{=} \{\mathcal{X} \in \mathcal{P}'(\mathfrak{X}) | \ X_1 \cap X_2 \neq \varnothing \ \ \forall X_1 \in \mathcal{X} \ \ \forall X_2 \in \mathcal{X}\}. \tag{3.3}$$

Thus, (3.3) is the family of all linked subfamilies of $\mathfrak{X}$. Moreover, for $\mathfrak{Y} \in \mathcal{P}'(\mathcal{P}(E))$, we consider the family of all maximal linked systems of the family $\mathfrak{Y}$:

$$\langle \mathfrak{Y} - \mathrm{link} \rangle_0[E] \stackrel{\triangle}{=} \{\mathcal{Y} \in \langle \mathfrak{Y} - \mathrm{link} \rangle[E] | \ \forall \mathcal{Z} \in \langle \mathfrak{Y} - \mathrm{link} \rangle[E] \ (\mathcal{Y} \subset \mathcal{Z}) \Longrightarrow (\mathcal{Y} = \mathcal{Z})\}. \tag{3.4}$$

By (3.4), the family

$$\begin{aligned}
\langle \mathcal{E} - \mathrm{link} \rangle_0[E] &= \{\mathcal{S} \in \langle \mathcal{E} - \mathrm{link} \rangle[E] | \ \forall \tilde{\mathcal{S}} \in \langle \mathcal{E} - \mathrm{link} \rangle[E] \ (\mathcal{S} \subset \tilde{\mathcal{S}}) \Longrightarrow (\mathcal{S} = \tilde{\mathcal{S}})\} \\
&= \{\mathcal{S} \in \langle \mathcal{E} - \mathrm{link} \rangle[E] | \ \forall L \in \mathcal{E} \ (L \cap S \neq \varnothing \ \ \forall S \in \mathcal{S}) \Longrightarrow (L \in \mathcal{S})\}
\end{aligned}$$

is defined (see [12, Sect. 4]). Note that $\forall \mathcal{S} \in \langle \mathcal{E} - \mathrm{link} \rangle[E] \ \exists \tilde{\mathcal{S}} \in \langle \mathcal{E} - \mathrm{link} \rangle_0[E] : \mathcal{S} \subset \tilde{\mathcal{S}}$; see [12, (4.5)]. Moreover (see [12, (4.5)]),

$$\forall \mathcal{S} \in \langle \mathcal{E} - \mathrm{link} \rangle_0[E] \quad \forall S \in \mathcal{S} \quad \forall \Sigma \in \mathcal{E} \ (S \subset \Sigma) \Longrightarrow (\Sigma \in \mathcal{S}).$$

Note the following property [12, Sect. 4]:

$$\mathbb{F}_0^{\star}(\mathcal{E}) = \{\mathcal{U} \in \langle \mathcal{E} - \mathrm{link} \rangle_0[E] | \ A \cap B \in \mathcal{U} \ \ \forall A \in \mathcal{U} \ \ \forall B \in \mathcal{U}\} \in \mathcal{P}'(\langle \mathcal{E} - \mathrm{link} \rangle_0[E]). \tag{3.5}$$

For $L \in \mathcal{E}$, we set $\langle \mathcal{E} - \mathrm{link} \rangle^0[E|L] \stackrel{\triangle}{=} \{\mathcal{S} \in \langle \mathcal{E} - \mathrm{link} \rangle_0[E] | \ L \in \mathcal{S}\}$. Moreover, for $H \in \mathcal{P}(E)$, let

$$\langle \mathcal{E} - \mathrm{link} \rangle_{\mathrm{op}}^0[E|H] \stackrel{\triangle}{=} \{\mathcal{S} \in \langle \mathcal{E} - \mathrm{link} \rangle_0[E] | \ \exists \Sigma \in \mathcal{S} : \ \Sigma \subset H\}.$$

Now, we introduce the following families:

$$\hat{\mathfrak{C}}^0_{\mathrm{op}}[E;\mathcal{E}] \triangleq \big\{ \langle \mathcal{E} - \mathrm{link} \rangle^0_{\mathrm{op}}[E|\Lambda] : \ \Lambda \in \mathbf{C}_E[\mathcal{E}] \big\} \in \mathcal{P}'\big(\mathcal{P}(\langle \mathcal{E} - \mathrm{link} \rangle_0[E])\big), \tag{3.6}$$

$$\hat{\mathfrak{C}}^\star_0[E;\mathcal{E}] \triangleq \big\{ \langle \mathcal{E} - \mathrm{link} \rangle^0[E|\Sigma] : \ \Sigma \in \mathcal{E} \big\} = \mathbf{C}_{\langle \mathcal{E} - \mathrm{link} \rangle_0[E]}[\hat{\mathfrak{C}}^0_{\mathrm{op}}[E;\mathcal{E}]].$$

In connection with (3.6), we note that $\hat{\mathfrak{C}}^0_{\mathrm{op}}[E;\mathcal{E}] \in (\mathrm{p} - \mathrm{BAS})[\langle \mathcal{E} - \mathrm{link} \rangle_0[E]]$. As a corollary, the following topology of Wallman type is defined (see [12, Sect. 5]):

$$\mathbb{T}_0\langle E|\mathcal{E} \rangle \triangleq \{\cup\}(\{\cap\}_\sharp(\hat{\mathfrak{C}}^0_{\mathrm{op}}[E;\mathcal{E}])) \in ((\mathbb{SC}) - \mathrm{top})\big[\langle \mathcal{E} - \mathrm{link} \rangle_0[E]\big].$$

In addition,

$$\big(\langle \mathcal{E} - \mathrm{link} \rangle_0[E], \mathbb{T}_0\langle E|\mathcal{E} \rangle\big) \tag{3.7}$$

is a supercompact T_1-space for which (see (3.5))

$$\mathbf{T}^0_\mathcal{E}\langle E \rangle = \mathbb{T}_0\langle E|\mathcal{E} \rangle|_{\mathbb{F}^\star_0(\mathcal{E})}. \tag{3.8}$$

Thus, (3.2) is a subspace of the topological space (3.7) and

$$\hat{\mathfrak{C}}^0_{\mathrm{op}}[E;\mathcal{E}] \in (\mathrm{p} - \mathrm{BAS})_0\big[\langle \mathcal{E} - \mathrm{link} \rangle_0[E]; \mathbb{T}_0\langle E|\mathcal{E} \rangle\big]. \tag{3.9}$$

Now, we recall the useful corollary of the supercompactness of the topological space (3.7) connected with (3.9); more precisely, $\forall \mathcal{G} \in \mathcal{P}'(\mathbf{C}_E[\mathcal{E}])$

$$\big(\langle \mathcal{E} - \mathrm{link} \rangle_0[E] = \bigcup_{G \in \mathcal{G}} \langle \mathcal{E} - \mathrm{link} \rangle^0_{\mathrm{op}}[E|G]\big) \Longrightarrow \big(\exists G_1 \in \mathcal{G} \ \ \exists G_2 \in \mathcal{G} :$$
$$\langle \mathcal{E} - \mathrm{link} \rangle_0[E] = \langle \mathcal{E} - \mathrm{link} \rangle^0_{\mathrm{op}}[E|G_1] \cup \langle \mathcal{E} - \mathrm{link} \rangle^0_{\mathrm{op}}[E|G_2]\big).$$

More detailed information on the topological spaces (3.2) and (3.7) can be found in [12, 15, 16].

4. Ultrafilters and maximal linked systems; topologies of Stone type

We recall that, as a variant of (2.6), the zero-dimensional T_2-space

$$(\mathbb{F}^\star_0(\mathcal{E}), \mathbf{T}^\star_\mathcal{E}[E]) \tag{4.1}$$

is defined. We consider (4.1) as a topological space of Stone type. Now, let us recall an analog of (4.1) for the space of maximal linked systems. To this end, note that

$$\hat{\mathfrak{C}}^\star_0[E;\mathcal{E}] \in (\mathrm{p} - \mathrm{BAS})\big[\langle \mathcal{E} - \mathrm{link} \rangle_0[E]\big].$$

Therefore, the following topology is defined:

$$\mathbb{T}_\star\langle E|\mathcal{E} \rangle \triangleq \{\cup\}(\{\cap\}_\sharp(\hat{\mathfrak{C}}^\star_0[E;\mathcal{E}])) \in (\mathrm{top})\big[\langle \mathcal{E} - \mathrm{link} \rangle_0[E]\big]. \tag{4.2}$$

In addition, topology (4.2) converts $\langle \mathcal{E} - \mathrm{link} \rangle_0[E]$ to the zero-dimensional T_2-space

$$\big(\langle \mathcal{E} - \mathrm{link} \rangle_0[E], \mathbb{T}_\star\langle E|\mathcal{E} \rangle\big). \tag{4.3}$$

Note that, by [12, Proposition 6.5], we have

$$\mathbf{T}^\star_\mathcal{E}[E] = \mathbb{T}_\star\langle E|\mathcal{E} \rangle|_{\mathbb{F}^\star_0(\mathcal{E})}. \tag{4.4}$$

Thus, in view of (4.4), we conclude that the topological space

$$(\mathbb{F}_0^\star(\mathcal{E}), \mathbf{T}_\mathcal{E}^\star[E])$$

is a subspace of the topological space (4.3). Moreover, by [12, Proposition 7.1], we have the inclusion

$$\mathbb{T}_0\langle E|\mathcal{E}\rangle \subset \mathbb{T}_\star\langle E|\mathcal{E}\rangle.$$

If X is a nonempty set, $\tau_1 \in (\text{top})[X]$, $\tau_2 \in (\text{top})[X]$, and $\tau_1 \subset \tau_2$, then we call the triplet (X, τ_1, τ_2) a bitopological space. Then $(\langle \mathcal{E} - \text{link}\rangle_0[E], \mathbb{T}_0\langle E|\mathcal{E}\rangle, \mathbb{T}_\star\langle E|\mathcal{E}\rangle)$ and $(\mathbb{F}_0^\star(\mathcal{E}), \mathbf{T}_\mathcal{E}^0\langle E\rangle, \mathbf{T}_\mathcal{E}^\star[E])$ are two bitopological spaces. General properties of these two bitopological spaces are given in [12, 13, 16].

   Now, let us recall some statements from [16, Sect. 7]. More precisely, to the end of this section, we suppose that $\mathcal{E} \in \tilde{\pi}^0[E]$. Therefore (see Sect. 2), $(\mathcal{E} - \text{triv})[x] \in \mathbb{F}_0^\star(\mathcal{E})$ $\forall x \in E$. As a result, we obtain the mapping

$$(\mathcal{E} - \text{triv})[\cdot] \stackrel{\triangle}{=} \big((\mathcal{E} - \text{triv})[x]\big)_{x \in E}$$

acting from E to $\mathbb{F}_0^\star(\mathcal{E})$ and the corresponding sets-images

$$(\mathcal{E} - \text{triv})[\cdot]^1(A) = \{(\mathcal{E} - \text{triv})[x] \colon x \in A\} \in \mathcal{P}(\mathbb{F}_0^\star(\mathcal{E}))$$

for $A \in \mathcal{P}(E)$. We can consider the closure of every such set both in the topology $\mathbf{T}_\mathcal{E}^0\langle E\rangle$ and in the topology $\mathbf{T}_\mathcal{E}^\star[E]$. In addition (see [16, (7.2)]),

$$\text{cl}\big((\mathcal{E} - \text{triv})[\cdot]^1(A), \mathbf{T}_\mathcal{E}^\star[E]\big) = \{\mathcal{U} \in \mathbb{F}_0^\star(\mathcal{E})|\ A \cap U \neq \varnothing\ \forall U \in \mathcal{U}\} \quad \forall A \in \mathcal{P}(E).$$

Moreover, by [16, Proposition 7.1], we obtain

$$\Phi_\mathcal{E}(\Sigma) = \text{cl}\big((\mathcal{E} - \text{triv})[\cdot]^1(\Sigma), \mathbf{T}_\mathcal{E}^0\langle E\rangle\big) = \text{cl}\big((\mathcal{E} - \text{triv})[\cdot]^1(\Sigma), \mathbf{T}_\mathcal{E}^\star[E]\big) \quad \forall \Sigma \in \mathcal{E}. \tag{4.5}$$

From (4.5), we obtain the corresponding density property:

$$\mathbb{F}_0^\star(\mathcal{E}) = \text{cl}\big((\mathcal{E} - \text{triv})[\cdot]^1(E), \mathbf{T}_\mathcal{E}^0\langle E\rangle\big) = \text{cl}\big((\mathcal{E} - \text{triv})[\cdot]^1(E), \mathbf{T}_\mathcal{E}^\star[E]\big). \tag{4.6}$$

Relations (4.5) and (4.6) imply the following fact: in the general case $\mathcal{E} \in \tilde{\pi}^0[E]$, the properties of the topologies $\mathbf{T}_\mathcal{E}^0\langle E\rangle$ and $\mathbf{T}_\mathcal{E}^\star[E]$ are similar. In [16, Sects. 5, 6], sufficient conditions for the coincidence of these topologies and conditions under which $\mathbf{T}_\mathcal{E}^0\langle E\rangle \neq \mathbf{T}_\mathcal{E}^\star[E]$ are given as well as similar conditions for the topologies $\mathbb{T}_0\langle E|\mathcal{E}\rangle$ and $\mathbb{T}_\star\langle E|\mathcal{E}\rangle$.

5. Proper maximal linked systems

   Let us emphasize some properties of the set $\langle \mathcal{E} - \text{link}\rangle_0[E] \setminus \mathbb{F}_0^\star(\mathcal{E})$. We call elements of this set proper maximal linked systems. Let us observe some properties of such maximal linked systems (more details can be found in [15, 16]). We first note the following statement from [16]:

$$\langle \mathcal{E} - \text{link}\rangle_0[E] \setminus \mathbb{F}_0^\star(\mathcal{E}) = \big\{\mathcal{S} \in \langle \mathcal{E} - \text{link}\rangle_0[E]|\ \exists S_1 \in \mathcal{S}\ \exists S_2 \in \mathcal{S}\ \exists S_3 \in \mathcal{S}:\ S_1 \cap S_2 \cap S_3 = \varnothing\big\}. \tag{5.1}$$

Hence, $\forall \mathcal{M} \in \langle \mathcal{E} - \text{link}\rangle_0[E]$,

$$\big(A \cap B \cap C \neq \varnothing\ \ \forall A \in \mathcal{M}\ \ \forall B \in \mathcal{M}\ \ \forall C \in \mathcal{M}\big) \Longrightarrow (\mathcal{M} \in \mathbb{F}_0^\star(\mathcal{E})). \tag{5.2}$$

Moreover, (4.2) and (5.1) imply the following property:

$$\langle \mathcal{E} - \text{link}\rangle_0[E] \setminus \mathbb{F}_0^\star(\mathcal{E}) \in \mathbb{T}_\star\langle E|\mathcal{E}\rangle.$$

Finally, using (5.1), we obtain the following equivalence:

$$\begin{aligned} \big(\langle \mathcal{E} - \text{link}\rangle_0[E] \setminus \mathbb{F}_0^\star(\mathcal{E}) \neq \varnothing\big) &\Longleftrightarrow \big(\exists \Sigma_1 \in \mathcal{E}\ \ \exists \Sigma_2 \in \mathcal{E}\ \ \exists \Sigma_3 \in \mathcal{E}: \\ (\Sigma_1 \cap \Sigma_2 \neq \varnothing)\&(\Sigma_2 \cap \Sigma_3 \neq \varnothing)&\&(\Sigma_1 \cap \Sigma_3 \neq \varnothing)\&(\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 = \varnothing)\big). \end{aligned} \tag{5.3}$$

Theorem 1. *The following equivalence holds*:

$$(\mathcal{E} \in \pi_\star^\sharp[E]) \Longleftrightarrow \big(\langle\mathcal{E}-\text{link}\rangle_0[E] = \mathbb{F}_0^\star(\mathcal{E})\big). \tag{5.4}$$

P r o o f. Let $\mathcal{E} \in \pi_\star^\sharp[E]$. Consider an arbitrary $\mathcal{M} \in \langle\mathcal{E}-\text{link}\rangle_0[E]$. Let

$$(M_1 \in \mathcal{M})\&(M_2 \in \mathcal{M})\&(M_3 \in \mathcal{M}). \tag{5.5}$$

Then, by (1.3), we obtain the following implication:

$$\big((M_1 \cap M_2 \neq \varnothing)\&(M_2 \cap M_3 \neq \varnothing)\&(M_1 \cap M_3 \neq \varnothing)\big) \Longrightarrow (M_1 \cap M_2 \cap M_3 \neq \varnothing). \tag{5.6}$$

Now, we use (5.2). More precisely, by the choice of $\mathcal{M}$, we have the following property (see (3.3)):

$$(M_1 \cap M_2 \neq \varnothing)\&(M_2 \cap M_3 \neq \varnothing)\&(M_1 \cap M_3 \neq \varnothing).$$

From (5.6), we conclude that $M_1 \cap M_2 \cap M_3 \neq \varnothing$. Since the choice of (5.5) was arbitrary, we have the property

$$A \cap B \cap C \neq \varnothing \quad \forall A \in \mathcal{M} \quad \forall B \in \mathcal{M} \quad \forall C \in \mathcal{M}.$$

From (5.2), the inclusion $\mathcal{M} \in \mathbb{F}_0^\star(\mathcal{E})$ follows. Thus,

$$\langle\mathcal{E}-\text{link}\rangle_0[E] \subset \mathbb{F}_0^\star(\mathcal{E}). \tag{5.7}$$

Using (3.5) and (5.7), we obtain the equality $\langle\mathcal{E}-\text{link}\rangle_0[E] = \mathbb{F}_0^\star(\mathcal{E})$. Thus, we have the implication

$$(\mathcal{E} \in \pi_\star^\sharp[E]) \Longrightarrow \big(\langle\mathcal{E}-\text{link}\rangle_0[E] = \mathbb{F}_0^\star(\mathcal{E})\big). \tag{5.8}$$

Let $\langle\mathcal{E}-\text{link}\rangle_0[E] = \mathbb{F}_0^\star(\mathcal{E})$. Let us prove that $\mathcal{E} \in \pi_\star^\sharp[E]$. On the contrary, let $\mathcal{E} \notin \pi_\star^\sharp[E]$. From (1.3), we obtain for some $\Sigma_1 \in \mathcal{E}$, $\Sigma_2 \in \mathcal{E}$, and $\Sigma_3 \in \mathcal{E}$

$$(\Sigma_1 \cap \Sigma_2 \neq \varnothing)\&(\Sigma_2 \cap \Sigma_3 \neq \varnothing)\&(\Sigma_1 \cap \Sigma_3 \neq \varnothing)\&(\Sigma_1 \cap \Sigma_2 \cap \Sigma_2 = \varnothing).$$

Then, by (5.3), $\langle\mathcal{E}-\text{link}\rangle_0[E] \setminus \mathbb{F}_0^\star(\mathcal{E}) \neq \varnothing$, a contradiction. Thus, the required inclusion $\mathcal{E} \in \pi_\star^\sharp[E]$ holds. Therefore, we have the implication

$$\big(\langle\mathcal{E}-\text{link}\rangle_0[E] = \mathbb{F}_0^\star(\mathcal{E})\big) \Longrightarrow (\mathcal{E} \in \pi_\star^\sharp[E]).$$

Using (5.8), we obtain (5.4). $\qquad\square$

Remark 1. Theorem 1 implies the following equality:

$$\pi_\star^\sharp[E] = \big\{\mathcal{L} \in \pi[E] \,|\, \langle\mathcal{L}-\text{link}\rangle_0[E] = \mathbb{F}_0^\star(\mathcal{L})\big\}.$$

6. Supercompact spaces of ultrafilters

Consider some corollaries of Theorem 1. The π-systems from (1.3) play an important role in these constructions. The following statement is an obvious corollary of (3.8) and Theorem 1.

Proposition 1. *If $\mathcal{E} \in \pi_\star^\sharp[E]$, then* $\mathbf{T}_\mathcal{E}^0\langle E\rangle \in ((\mathbb{SC})-\text{top})[\mathbb{F}_0^\star(\mathcal{E})]$.

Proposition 1 motivates special investigation of specific variants of π-systems from $\pi_\star^\sharp[E]$. Note first the important property connected with the Cartesian product. Let us start with the simplest case. We need new notation. For nonempty families $\mathcal{X}$ and $\mathcal{Y}$, we set

$$\mathcal{X}\{\times\}\mathcal{Y} \stackrel{\triangle}{=} \{\mathrm{pr}_1(z) \times \mathrm{pr}_2(z) : z \in \mathcal{X} \times \mathcal{Y}\}. \tag{6.1}$$

Family (6.1) is nonempty. We can use π-systems of nonempty sets as $\mathcal{X}$ and $\mathcal{Y}$. In addition, for all nonempty sets X and Y such that $\mathcal{X} \in \pi[X]$ and $\mathcal{Y} \in \pi[Y]$, we have

$$\mathcal{X}\{\times\}\mathcal{Y} \in \pi[X \times Y]. \tag{6.2}$$

Proposition 2. *If X and Y are nonempty sets, $\mathcal{X} \in \pi_\star^\sharp[X]$ and $\mathcal{Y} \in \pi_\star^\sharp[Y]$, then*

$$\mathcal{X}\{\times\}\mathcal{Y} \in \pi_\star^\sharp[X \times Y]. \tag{6.3}$$

P r o o f. Fix $X, Y, \mathcal{X}$, and $\mathcal{Y}$ satisfying our conditions. We have (6.2). Consider arbitrary sets

$$(\Lambda_1 \in \mathcal{X}\{\times\}\mathcal{Y}) \& (\Lambda_2 \in \mathcal{X}\{\times\}\mathcal{Y}) \& (\Lambda_3 \in \mathcal{X}\{\times\}\mathcal{Y}). \tag{6.4}$$

Then, for some sets $U_1 \in \mathcal{X}$, $V_1 \in \mathcal{Y}$, $U_2 \in \mathcal{X}$, $V_2 \in \mathcal{Y}$, $U_3 \in \mathcal{X}$, and $V_3 \in \mathcal{Y}$, the following representation is valid:

$$(\Lambda_1 = U_1 \times V_1) \& (\Lambda_2 = U_2 \times V_2) \& (\Lambda_3 = U_3 \times V_3). \tag{6.5}$$

Let relations (6.4) satisfy the conditions

$$(\Lambda_1 \cap \Lambda_2 \neq \varnothing) \& (\Lambda_2 \cap \Lambda_3 \neq \varnothing) \& (\Lambda_1 \cap \Lambda_3 \neq \varnothing). \tag{6.6}$$

From (6.5) and (6.6), we obtain the following obvious properties:

$$(U_1 \cap U_2 \neq \varnothing) \& (V_1 \cap V_2 \neq \varnothing) \& (U_2 \cap U_3 \neq \varnothing) \& (V_2 \cap V_3 \neq \varnothing) \& (U_1 \cap U_3 \neq \varnothing) \& (V_1 \cap V_3 \neq \varnothing). \tag{6.7}$$

From (1.3) and (6.7), we obtain

$$(U_1 \cap U_2 \cap U_3 \neq \varnothing) \& (V_1 \cap V_2 \cap V_3 \neq \varnothing). \tag{6.8}$$

From (6.5) and (6.8), we obtain $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \neq \varnothing$. Thus (see (6.6)), the following implication hols:

$$\big((\Lambda_1 \cap \Lambda_2 \neq \varnothing) \& (\Lambda_2 \cap \Lambda_3 \neq \varnothing) \& (\Lambda_1 \cap \Lambda_3 \neq \varnothing)\big) \Longrightarrow (\Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \neq \varnothing).$$

Since the choice of (6.4) was arbitrary, we have proved (6.3) (see (1.3) and (6.2)). $\square$

We now return to the fixed nonempty set E and suppose, unless otherwise stated, that $\mathcal{E} \in \pi_\star^\sharp[E]$ (see Proposition 1). In view of Proposition 1, we conclude that

$$(\mathbb{F}_0^\star(\mathcal{E}), \mathbf{T}_\mathcal{E}^0\langle E\rangle) \tag{6.9}$$

is a supercompact T_1-space (see [12, Sect. 5]). Recall Remark 1: in our case,

$$\langle \mathcal{E} - \mathrm{link}\rangle_0[E] = \mathbb{F}_0^\star(\mathcal{E}). \tag{6.10}$$

From (3.8) and (6.10), we obtain the following equality for the topologies:

$$\mathbf{T}_\mathcal{E}^0\langle E\rangle = \mathbb{T}_0\langle E|\mathcal{E}\rangle \tag{6.11}$$

(we recall that, in our constructions, $\mathcal{E} \in \pi_\star^\sharp[E]$). Moreover, from (6.10), we obtain

$$\langle \mathcal{E} - \mathrm{link}\rangle_{\mathrm{op}}^0[E|H] = \mathbb{F}_\mathbf{C}^\natural[\mathcal{E}|H] \quad \forall H \in \mathcal{P}(E).$$

As a corollary, we obtain the equality

$$\hat{\mathfrak{C}}_{\mathrm{op}}^0[E; \mathcal{E}] = \mathfrak{F}_\mathbf{C}^\natural[\mathcal{E}]. \tag{6.12}$$

Proposition 3. *If* $\mathcal{G} \in \mathcal{P}'(\mathbf{C}_E[\mathcal{E}])$, *then*

$$\left(\mathbb{F}_0^\star(\mathcal{E}) = \bigcup_{\mathbb{G} \in \mathcal{G}} \mathbb{F}_\mathbf{C}^\natural[\mathcal{E}|\mathbb{G}]\right) \Longrightarrow \left(\exists G_1 \in \mathcal{G} \; \exists G_2 \in \mathcal{G} : \; \mathbb{F}_0^\star(\mathcal{L}) = \mathbb{F}_\mathbf{C}^\natural[\mathcal{E}|G_1] \cup \mathbb{F}_\mathbf{C}^\natural[\mathcal{E}|G_2]\right). \qquad (6.13)$$

The corresponding proof follows from (6.10), (6.12), and the properties of the family $\hat{\mathfrak{C}}_{\mathrm{op}}^0[E; \mathcal{E}]$ mentioned in [12, Sect. 5]. In (6.13), we obtained a natural sharpening of Proposition 1.

Remark 2. Note that, by duality, the family $(\mathbb{UF})[E; \mathcal{E}]$ in the considered case is a closed binary subbase (see [12, (5.1)]) of the topological space (6.9). This property follows from [12, Proposition 5.1] and (6.12).

In the next section, we consider specific examples, for which (6.10) and (6.11) are realized.

7. Examples of supercompact ultrafilter spaces

We first consider the cases when $E \subset \mathbb{R}$, where $\mathbb{R}$ is the real line (Proposition 2 enables extending our constructions to a vector case). Consider the case when this set E is bounded. More precisely, suppose that E is an interval of $\mathbb{R}$. In addition, we will consider specific sets E.

Let us agree to use only square brackets in the notation of all intervals in $\mathbb{R}$: $]x, y[$, $[x, y[$, $]x, y]$, and $[x, y]$ for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. In addition, relations between x and y can be arbitrary: $x < y$, $x = y$, and $y < x$. Thus, for example, $[1, 0] \triangleq \{\xi \in \mathbb{R}|(1 \le \xi)\&(\xi \le 0)\} = \varnothing$. In other words, we consider the empty set as an interval. Moreover, for $c \in \mathbb{R}$, we set (as usually) $] -\infty, c[\triangleq \{\xi \in \mathbb{R}|\xi < c\}$ and $]c, \infty[\triangleq \{\xi \in \mathbb{R}|c < \xi\}$.

For $a \in \mathbb{R}$ and $b \in]a, \infty[$, define

$$\mathcal{J}_1(a, b) \triangleq \{]\mathrm{pr}_1(z), \mathrm{pr}_2(z)[: \; z \in [a, b] \times [a, b]\},$$

$$\mathcal{J}_2(a, b) \triangleq \{[\mathrm{pr}_1(z), \mathrm{pr}_2(z)] : \; z \in [a, b] \times [a, b]\},$$

$$\mathcal{J}_3(a, b) \triangleq \{[\mathrm{pr}_1(z), \mathrm{pr}_2(z)[: \; z \in [a, b] \times [a, b]\},$$

$$\mathcal{J}_4(a, b) \triangleq \{]\mathrm{pr}_1(z), \mathrm{pr}_2(z)] : \; z \in [a, b] \times [a, b]\}.$$

We obtain four nonempty families of subsets of $\mathbb{R}$.

Proposition 4. *If* $a \in \mathbb{R}$ *and* $b \in]a, \infty[$, *then*

$$\left(\mathcal{J}_1(a, b) \in \pi_\star^\sharp[]a, b[]\right) \& \left(\mathcal{J}_2(a, b) \in \pi_\star^\sharp[[a, b]]\right) \& \left(\mathcal{J}_3(a, b) \in \pi_\star^\sharp[[a, b[]\right) \& \left(\mathcal{J}_4(a, b) \in \pi_\star^\sharp[]a, b]]\right). \quad (7.1)$$

P r o o f. The proofs of all four statements in (7.1) are similar. Therefore, let us consider only one of them and establish the property $\mathcal{J}_1(a, b) \in \pi_\star^\sharp[]a, b[]$, where $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a < b$. Note the obvious property:

$$\mathcal{J}_1(a, b) \in \pi[]a, b[]. \qquad (7.2)$$

Let $\Lambda_1 \in \mathcal{J}_1(a, b)$, $\Lambda_2 \in \mathcal{J}_1(a, b)$, and $\Lambda_3 \in \mathcal{J}_1(a, b)$. Then, for some numbers $u_1 \in [a, b]$, $u_2 \in [a, b]$, $v_1 \in [a, b]$, $v_2 \in [a, b]$, $w_1 \in [a, b]$, and $w_2 \in [a, b]$, we have

$$\left(\Lambda_1 =]u_1, u_2[\right) \& \left(\Lambda_2 =]v_1, v_2[\right) \& \left(\Lambda_3 =]w_1, w_2[\right). \qquad (7.3)$$

Let $(\Lambda_1 \cap \Lambda_2 \ne \varnothing) \& (\Lambda_2 \cap \Lambda_3 \ne \varnothing) \& (\Lambda_1 \cap \Lambda_3 \ne \varnothing)$. We introduce $\mu \triangleq \sup(\{u_1; v_1; w_1\}) \in [a, b]$ and $\nu \triangleq \inf(\{u_2; v_2; w_2\}) \in [a, b]$. Then, by (7.3), $]\mu, \nu[\subset \Lambda_1 \cap \Lambda_2 \cap \Lambda_3$ and $\mu < \nu$ (the latter inequality

follows from the linkedness of the family $\{\Lambda_1; \Lambda_2; \Lambda_3\}$). As a result, $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \neq \varnothing$. Thus, we obtain the implication

$$\big((\Lambda_1 \cap \Lambda_2 \neq \varnothing)\&(\Lambda_2 \cap \Lambda_3 \neq \varnothing)\&(\Lambda_1 \cap \Lambda_3 \neq \varnothing)\big) \Longrightarrow (\Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \neq \varnothing). \tag{7.4}$$

Since Λ_1, Λ_2, and Λ_3 were chosen arbitrary, the required property of the family $\mathcal{J}_1(a,b)$ follows from (1.3), (7.2), and (7.4). $\qquad\square$

Thus, by Propositions 2 and 4, we obtain many examples of realization of (6.10) and (6.11). In these examples, (6.9) is a supercompact T_1-space (see Proposition 1). Later, we will provide analogs of (7.1) for the case $E = \mathbb{R}$. Now we first observe a very simple general construction.

Unless otherwise stated, we suppose that E is an arbitrary nonempty set. Suppose that

$$\pi_E^0 \triangleq \{\mathcal{L} \in \mathcal{P}'(\mathcal{P}(E))|\ (\varnothing \in \mathcal{L})\&(A \cap B \in \mathcal{L}\ \ \forall A \in \mathcal{L}\ \ \forall B \in \mathcal{L})\}. \tag{7.5}$$

Certainly, $\pi[E] = \{\mathcal{L} \in \pi_E^0 | E \in \mathcal{L}\}$. It is obvious that $\mathcal{L} \cup \{E\} \in \pi[E]\ \ \forall \mathcal{L} \in \pi_E^0$. Moreover, let

$$\pi_E^\sharp \triangleq \big\{\mathcal{L} \in \pi_E^0|\ \forall L_1 \in \mathcal{L}\ \ \forall L_2 \in \mathcal{L}\ \ \forall L_3 \in \mathcal{L}$$
$$\big((L_1 \cap L_2 \neq \varnothing)\&(L_2 \cap L_3 \neq \varnothing)\&(L_1 \cap L_3 \neq \varnothing)\big) \Longrightarrow (L_1 \cap L_2 \cap L_3 \neq \varnothing)\big\}. \tag{7.6}$$

Then, we obtain the following obvious property:

$$\mathcal{L} \cup \{E\} \in \pi_\star^\sharp[E]\quad \forall \mathcal{L} \in \pi_E^\sharp. \tag{7.7}$$

In (7.5)–(7.7), we have simple modifications of definitions of Section 1 (see (1.2) and (1.3)). Consider now the case when $E = \mathbb{R}$. In addition,

$$\big(\mathcal{J}_{\mathbb{R}}^{(1)} \triangleq \{]\mathrm{pr}_1(z), \mathrm{pr}_2(z)[:\ z \in \mathbb{R} \times \mathbb{R}\} \in \pi_{\mathbb{R}}^\sharp\big)$$
$$\&\big(\mathcal{J}_{\mathbb{R}}^{(2)} \triangleq \{[\mathrm{pr}_1(z), \mathrm{pr}_2(z)]:\ z \in \mathbb{R} \times \mathbb{R}\} \in \pi_{\mathbb{R}}^\sharp\big)$$
$$\&\big(\mathcal{J}_{\mathbb{R}}^{(3)} \triangleq \{[\mathrm{pr}_1(z), \mathrm{pr}_2(z)[:\ z \in \mathbb{R} \times \mathbb{R}\} \in \pi_{\mathbb{R}}^\sharp\big)$$
$$\&\big(\mathcal{J}_{\mathbb{R}}^{(4)} \triangleq \{]\mathrm{pr}_1(z), \mathrm{pr}_2(z)]:\ z \in \mathbb{R} \times \mathbb{R}\} \in \pi_{\mathbb{R}}^\sharp\big). \tag{7.8}$$

The corresponding proof is an obvious analog of constructions used in the proof of Proposition 4. By (7.8), we obtain new variants of (6.10), (6.11), and Proposition 1 for the case when $E = \mathbb{R}$ and

$$(\mathcal{E} = \mathcal{J}_{\mathbb{R}}^{(1)} \cup \{\mathbb{R}\}) \vee (\mathcal{E} = \mathcal{J}_{\mathbb{R}}^{(2)} \cup \{\mathbb{R}\}) \vee (\mathcal{E} = \mathcal{J}_{\mathbb{R}}^{(3)} \cup \{\mathbb{R}\}) \vee (\mathcal{E} = \mathcal{J}_{\mathbb{R}}^{(4)} \cup \{\mathbb{R}\}).$$

The direct combination of Propositions 2 and 4 and similar combination of Proposition 2 and (7.8) enable obtaining variants of (6.10), (6.11), and Proposition 1 for subsets of $\mathbb{R} \times \mathbb{R}$.

8. The case of generalized Cartesian product

In this section, we consider a natural generalization of Proposition 2. More precisely, we fix nonempty sets X and E. In what follows, for every nonempty set S, we denote by S^X (as in [17, Ch. II, Sect. 6]) the set of all mappings from X to S; we can use a nonempty family as S. In addition, we fix a (set-valued) mapping $(E_x)_{x \in X} \in \mathcal{P}'(E)^X$; thus, we have a mapping

$$x \longmapsto E_x:\ X \longrightarrow \mathcal{P}'(E). \tag{8.1}$$

Using (8.1) (and the axiom of choice), we obtain the (nonempty) set

$$\mathbb{E} \overset{\triangle}{=} \prod_{x \in X} E_x = \{ f \in E^X | f(x) \in E_x \ \ \forall x \in X \} \in \mathcal{P}'(E^X). \tag{8.2}$$

In connection with (8.1) and (8.2), note that

$$\prod_{x \in X} \pi[E_x] = \{ (\mathcal{L}_x)_{x \in X} \in \mathcal{P}'(\mathcal{P}(E))^X | \mathcal{L}_y \in \pi[E_y] \ \ \forall y \in X \}. \tag{8.3}$$

Similarly, we define the set

$$\prod_{x \in X} \pi_\star^\sharp[E_x] = \{ (\mathcal{L}_x)_{x \in X} \in \mathcal{P}'(\mathcal{P}(E))^X | \mathcal{L}_y \in \pi_\star^\sharp[E_y] \ \ \forall y \in X \} \in \mathcal{P}(\prod_{x \in X} \pi[E_x]).$$

From (8.3), we obtain the following obvious representation:

$$\prod_{x \in X} \mathcal{L}_x = \{ (L_x)_{x \in X} \in \mathcal{P}(E)^X | \ L_y \in \mathcal{L}_y \ \forall y \in X \} \in \mathcal{P}'(\mathcal{P}(E)^X) \ \ \ \forall (\mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \pi[E_x]. \tag{8.4}$$

As a corollary, by (8.4), the following product of sets is defined:

$$\prod_{x \in X} L_x = \{ f \in E^X | f(x) \in L_x \ \forall x \in X \} \ \ \ \forall (\mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \pi[E_x] \ \ \ \forall (L_x)_{x \in X} \in \prod_{x \in X} \mathcal{L}_x. \tag{8.5}$$

Now, we introduce two variants of the π-system product: usual (similar to the variant applied in the general topology in the Tychonoff product construction) and "box" variants. The latter variant corresponds to the way used in the construction of the box topology.

More precisely (see (8.4) and (8.5)), we introduce the usual π-system product for $(\mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \pi[E_x]$:

$$\bigotimes_{x \in X} \mathcal{L}_x \overset{\triangle}{=} \{ H \in \mathcal{P}(E^X) | \exists (L_x)_{x \in X} \in \prod_{x \in X} \mathcal{L}_x : \ (H = \prod_{x \in X} L_x) \& (\exists K \in \mathrm{Fin}(X) :$$
$$L_s = E_s \ \ \forall s \in X \setminus K) \} = \{ H \in \mathcal{P}(\mathbb{E}) | \exists (L_x)_{x \in X} \in \prod_{x \in X} \mathcal{L}_x : \tag{8.6}$$
$$(H = \prod_{x \in X} L_x) \& (\exists K \in \mathrm{Fin}(X) : \ L_s = E_s \ \ \forall s \in X \setminus K) \}.$$

The "box" variant is defined easier. For $(\mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \pi[E_x]$, we set

$$\bigodot_{x \in X} \mathcal{L}_x \overset{\triangle}{=} \{ \prod_{x \in X} L_x : \ (L_x)_{x \in X} \in \prod_{x \in X} \mathcal{L}_x \}. \tag{8.7}$$

We note two obvious (but useful) properties. More precisely, for $(H_x^{(1)})_{x \in X} \in \mathcal{P}(E)^X$ and $(H_x^{(2)})_{x \in X} \in \mathcal{P}(E)^X$, we have

$$(\prod_{x \in X} H_x^{(1)}) \cap (\prod_{x \in X} H_x^{(2)}) = \prod_{x \in X} (H_x^{(1)} \cap H_x^{(2)}). \tag{8.8}$$

Similarly, for $(H_x^{(1)})_{x \in X} \in \mathcal{P}(E)^X$, $(H_x^{(2)})_{x \in X} \in \mathcal{P}(E)^X$, and $(H_x^{(3)})_{x \in X} \in \mathcal{P}(E)^X$, we have

$$(\prod_{x \in X} H_x^{(1)}) \cap (\prod_{x \in X} H_x^{(2)}) \cap (\prod_{x \in X} H_x^{(3)}) = \prod_{x \in X} (H_x^{(1)} \cap H_x^{(2)} \cap H_x^{(3)}). \tag{8.9}$$

From (8.6) and (8.8), we obtain

$$\bigotimes_{x \in X} \mathcal{L}_x \in \pi[\mathbb{E}] \quad \forall (\mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \pi[E_x].$$

Moreover, (1.2), (8.7), and (8.8) imply

$$\bigodot_{x \in X} \mathcal{L}_x \in \pi[\mathbb{E}] \quad \forall (\mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \pi[E_x]. \tag{8.10}$$

In addition, we obtain the following obvious inclusion for $(\mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \pi[E_x]$:

$$\bigotimes_{x \in X} \mathcal{L}_x \subset \bigodot_{x \in X} \mathcal{L}_x. \tag{8.11}$$

Theorem 2. *If* $(\mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \pi_\star^\sharp[E_x]$, *then*

$$\Big(\bigotimes_{x \in X} \mathcal{L}_x \in \pi_\star^\sharp[\mathbb{E}] \Big) \& \Big(\bigodot_{x \in X} \mathcal{L}_x \in \pi_\star^\sharp[\mathbb{E}] \Big). \tag{8.12}$$

P r o o f. We restrict ourselves to the verification of the latter inclusion in (8.12) (the corresponding verification of the former inclusion is an obvious corollary; see (8.11)). In this proof, we fix

$$(\mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \pi_\star^\sharp[E_x] \tag{8.13}$$

and use (8.10). From (8.10) and (8.13), we obtain

$$(\mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \pi[E_x] : \ \mathfrak{L} \overset{\triangle}{=} \bigodot_{x \in X} \mathcal{L}_x \in \pi[\mathbb{E}]. \tag{8.14}$$

Thus, by (8.7) and (8.14), we have the following equality:

$$\mathfrak{L} = \Big\{ \prod_{x \in X} L_x : \ (L_x)_{x \in X} \in \prod_{x \in X} \mathcal{L}_x \Big\}. \tag{8.15}$$

Consider arbitrary sets $\Lambda_1 \in \mathfrak{L}$, $\Lambda_2 \in \mathfrak{L}$, and $\Lambda_3 \in \mathfrak{L}$. Taking into account (8.15), we choose

$$\Big((U_x)_{x \in X} \in \prod_{x \in X} \mathcal{L}_x \Big) \& \Big((V_x)_{x \in X} \in \prod_{x \in X} \mathcal{L}_x \Big) \& \Big((W_x)_{x \in X} \in \prod_{x \in X} \mathcal{L}_x \Big)$$

with the following properties:

$$\Big(\Lambda_1 = \prod_{x \in X} U_x \Big) \& \Big(\Lambda_2 = \prod_{x \in X} V_x \Big) \& \Big(\Lambda_3 = \prod_{x \in X} W_x \Big). \tag{8.16}$$

Then, by (8.8) and (8.16), we conclude that $\Lambda_1 \cap \Lambda_2$, $\Lambda_2 \cap \Lambda_3$, and $\Lambda_1 \cap \Lambda_3$ are realized as follows:

$$\Big(\Lambda_1 \cap \Lambda_2 = \prod_{x \in X} (U_x \cap V_x) \Big) \& \Big(\Lambda_2 \cap \Lambda_3 = \prod_{x \in X} (V_x \cap W_x) \Big) \& \Big(\Lambda_1 \cap \Lambda_3 = \prod_{x \in X} (U_x \cap W_x) \Big). \tag{8.17}$$

Similarly, from (8.9) and (8.16), we obtain

$$\Lambda_1 \cap \Lambda_2 \cap \Lambda_3 = \prod_{x \in X} (U_x \cap V_x \cap W_x). \tag{8.18}$$

Suppose now that

$$(\Lambda_1 \cap \Lambda_2 \neq \varnothing) \& (\Lambda_2 \cap \Lambda_3 \neq \varnothing) \& (\Lambda_1 \cap \Lambda_3 \neq \varnothing). \tag{8.19}$$

Then, from (8.17) and (8.19), we obtain, for $x \in X$,

$$(U_x \cap V_x \neq \varnothing) \& (V_x \cap W_x \neq \varnothing) \& (U_x \cap W_x \neq \varnothing), \tag{8.20}$$

where $U_x \in \mathcal{L}_x$, $V_x \in \mathcal{L}_x$, and $W_x \in \mathcal{L}_x$. However, for $x \in X$, the inclusion $\mathcal{L}_x \in \pi_\star^\sharp[E_x]$ holds; therefore, by (1.3) and (8.20), we have $U_x \cap V_x \cap W_x \neq \varnothing$. Using (8.18) (and the axiom of choice), we obtain $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \neq \varnothing$. Thus, we have proved the implication

$$\big((\Lambda_1 \cap \Lambda_2 \neq \varnothing) \& (\Lambda_2 \cap \Lambda_3 \neq \varnothing) \& (\Lambda_1 \cap \Lambda_3 \neq \varnothing)\big) \implies (\Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \neq \varnothing). \tag{8.21}$$

Since Λ_1, Λ_2, and Λ_3 were chosen arbitrary, by (8.21), we obtain (see (1.3)) $\mathfrak{L} \in \pi_\star^\sharp[\mathbb{E}]$. By (8.14), we have

$$\bigodot_{x \in X} \mathcal{L}_x \in \pi_\star^\sharp[\mathbb{E}].$$

Thus, the theorem is proved (see (1.3) and (8.11)). $\square$

Now, we observe some cases connected with the realization of our measurable space equipped with the semi-algebra of sets. These measurable spaces are widely used in the probability theory (see [1]). Let us first give a general definition in terms of π-systems. For this, we fix a nonempty set $\mathbb{X}$. If $\mathcal{L} \in \pi[\mathbb{X}]$, $A \in \mathcal{P}(\mathbb{X})$, and $n \in \mathbb{N}$, then

$$\Delta_n(A, \mathcal{L}) \triangleq \Big\{ (L_i)_{i \in \overline{1,n}} \in \mathcal{L}^n \,\big|\, (A = \bigcup_{i=1}^n L_i) \& (L_p \cap L_q = \varnothing \;\; \forall p \in \overline{1,n} \;\; \forall q \in \overline{1,n} \setminus \{p\}) \Big\}.$$

We set

$$\Pi[\mathbb{X}] \triangleq \big\{ \mathcal{L} \in \pi[\mathbb{X}] \,\big|\, \forall L \in \mathcal{L} \;\; \exists n \in \mathbb{N}: \; \Delta_n(\mathbb{X} \setminus L, \mathcal{L}) \neq \varnothing \big\};$$

thus, the family of all semi-algebras of subsets of $\mathbb{X}$ is introduced. Certainly, $(\mathrm{alg})[\mathbb{X}] \subset \Pi[\mathbb{X}]$. Moreover,

$$\Pi_\star^\sharp[\mathbb{X}] \triangleq \big\{ \mathcal{L} \in \Pi[\mathbb{X}] \,\big|\, \forall L_1 \in \mathcal{L} \;\forall L_2 \in \mathcal{L} \;\forall L_3 \in \mathcal{L}$$
$$\big((L_1 \cap L_2 \neq \varnothing) \& (L_2 \cap L_3 \neq \varnothing) \& (L_1 \cap L_3 \neq \varnothing)\big) \implies (L_1 \cap L_2 \cap L_3 \neq \varnothing) \big\} \tag{8.22}$$
$$= \Pi[\mathbb{X}] \cap \pi_\star^\sharp[\mathbb{X}].$$

Elements of family (8.22) are special semi-algebras of subsets of $\mathbb{X}$. Note that, for $a \in \mathbb{R}$ and $b \in]a, \infty[$, by Proposition 4, $\mathcal{J}_3(a, b) \in \Pi_*^\sharp[[a, b[]$ and

$$([a, b[, \mathcal{J}_3(a, b)) \tag{8.23}$$

is the usual space-pointer. Of course, (8.23) is a specific measurable space with a semi-algebra of sets. In addition, by Proposition 4, $\mathcal{J}_3(a, b) \in \pi_\star^\sharp[[a, b[]$. Thus, for $E = [a, b[$ and $\mathcal{E} = \mathcal{J}_3(a, b)$, we obtain (6.10) and (6.11).

Consider a more complicated case. We return to (8.1) and (8.2). For a nonempty set X and $(E_x)_{x \in X} \in \mathcal{P}'(E)^X$, we introduce

$$\prod_{x \in X} \Pi[E_x] \triangleq \big\{ (\mathcal{L}_x)_{x \in X} \in \mathcal{P}'(\mathcal{P}(E))^X \,\big|\, \mathcal{L}_y \in \Pi[E_y] \;\forall y \in X \big\} \in \mathcal{P}\big(\prod_{x \in X} \pi[E_x]\big). \tag{8.24}$$

By (8.14) and (8.24), we define

$$\bigotimes_{x \in X} \mathcal{L}_x \in \pi[\mathbb{E}] \quad \forall \mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \Pi[E_x].$$

Moreover, the following property holds:

$$\bigotimes_{x \in X} \mathcal{L}_x \in \Pi[\mathbb{E}] \quad \forall \mathcal{L}_x)_{x \in X} \in \prod_{x \in X} \Pi[E_x]. \tag{8.25}$$

From Theorem 2 and (8.25), we obtain for $(\mathcal{L}_x)_{x \in X} \in \prod\limits_{x \in X} \Pi_\star^\sharp[E_x]$

$$\bigotimes_{x \in X} \mathcal{L}_x \in \Pi_\star^\sharp[\mathbb{E}]. \tag{8.26}$$

Note an obvious corollary concerning the case $E = \mathbb{R}$.

Proposition 5. *Let* $(a_x)_{x \in X} \in \mathbb{R}^X$ *and* $(b_x)_{x \in X} \in \prod\limits_{x \in X}]a_x, \infty[$. *Then*

$$\bigotimes_{x \in X} \mathcal{J}_3(a_x, b_x) \in \Pi_\star^\sharp\big[\{f \in \mathbb{R}^X | \ \forall x \in X \ (a_x \leq f(x)) \& (f(x) < b_x)\}\big].$$

The corresponding proof is obtained by the direct combination of (8.26) and the above-mentioned properties of the measurable space (8.23). Thus, by (8.22) and (8.25), we obtain one more variant of realization of (6.10) and (6.11) in the class of measurable spaces with a semi-algebra of sets (see Proposition 1).

9. Conclusion

In this paper, relations between ultrafilters and maximal linked systems on a π-system were considered. The emphasis was on the conditions under which all maximal linked systems are ultrafilters on the π-system. The investigation of the ultrafilter space is important, since in the class of ultrafilters the density properties (see (4.5) and (4.6)) are realized. Therefore, this space can be considered as a kind of extension of the initial set E whose elements play the role of usual solutions. Such interpretation is very natural for extension procedures of extremal problems and attainability problems (see [2, 3, 19, 20]). More precisely, the ultrafilters can be used as generalized elements. Therefore, it is useful to know as much as possible properties of topological nature. The supercompactness is a very interesting property of such type. We investigate this property identifying ultrafilters and maximal linked systems for a π-system. Of course, this approach is not universal. However, we obtain a positive result for many interesting variants of π-systems. In addition, we use the supercompactness property of the space of maximal linked systems with Wallman equipping.

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COMMUTATIVE WEAKLY INVO–CLEAN GROUP RINGS

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Abstract: A ring R is called *weakly invo–clean* if any its element is the sum or the difference of an involution and an idempotent. For each commutative unital ring R and each abelian group G, we find only in terms of R, G and their sections a necessary and sufficient condition when the group ring $R[G]$ is weakly invo-clean. Our established result parallels to that due to Danchev-McGovern published in J. Algebra (2015) and proved for weakly nil-clean rings.

Keywords: Invo-clean rings, Weakly invo-clean rings, Group rings.

Introduction and conventions

Throughout the current paper, we shall assume that all rings R are associative, containing the identity element 1 which differs from the zero element 0. Our standard terminology and notation are in agreement with [9] and [10], while the specific notions and notations will be stated explicitly below. As usual, $J(R)$ denotes the Jacobson radical of a ring R and G is a multiplicative group. Both objects R and G forming the symbol $R[G]$ will stand for the group ring of G over R.

The next concept appeared in [1], [2] and [3], respectively.

Definition 1. *A ring R is said to be invo-clean if, for every $r \in R$, there exist an involution v and an idempotent e such that $r = v + e$. If $r = v + e$ or $r = v - e$, the ring is called weakly invo-clean.*

The next necessary and sufficient condition for a commutative ring R to be invo-clean was established in [1, 2], namely: *A ring R is invo-clean if, and only if, $R \cong R_1 \times R_2$, where R_1 is a nil-clean ring with $z^2 = 2z$ for all $z \in J(R_1)$, and R_2 is a ring of characteristic 3 whose elements satisfy the equation $x^3 = x$.*

Let us recall that a ring is *nil-clean* if every its element is a sum of a nilpotent and an idempotent, and it is *weakly nil-clean* if every its element is a sum or a difference of a nilpotent and an idempotent (see, for more details, [6]).

A criterion for an arbitrary commutative group ring to be nil-clean was recently obtained in [8]. Specifically, the following holds: *A commutative ring $R[G]$ is nil-clean if, and only if, the ring R is nil-clean and the group G is a 2-group.* This was generalized in [6, Theorem 2.1] by finding a suitable criterion when $R[G]$ is weakly nil-clean.

Some other related results in this subject can be found by the interested reader in [4] too.

So, the aim of this brief article is to obtain a paralleling result for the class of weakly invo-clean rings. This is successfully done below in our main Theorem 1.

1. The characterization result and a problem

We begin here with the following key formula from [7]: Suppose that R is a commutative ring and G is an abelian group. Then

$$J(R[G]) = J(R)[G] + \langle r(g-1) \mid g \in G_p,\ pr \in J(R) \rangle,$$

where G_p designates the p-primary component of G.

The next technicality already was mentioned above, but for the sake of completeness and reader's convenience, we will state it once again.

Lemma 1. [1, 2] *Let R be a commutative ring. Then the following two points hold:*
(i) *If $2 \in J(R)$, then R is invo-clean $\iff$ R is nil-clean and $z^2 = 2z$ for each $z \in J(R)$.*

(ii) *If $\operatorname{char}(R) = 3$, then R is invo-clean $\iff$ $x^3 = x$ for all $x \in R$.*

We also need the following two technical claims.

Lemma 2. *The direct product $K \times L$ of two rings K, L is invo-clean $\iff$ both K and L are invo-clean rings.*

P r o o f. It is straightforward by using of results from [1] and [2]. $\qquad\square$

Lemma 3. *A ring R is weakly invo-clean $\iff$ either R is invo-clean or R can be decomposed as $R = K \times \mathbb{Z}_5$, where $K = \{0\}$ or K is invo-clean.*

P r o o f. It is straightforward by the utilization of results from [2] and [3]. $\qquad\square$

We are now ready to proceed by proving the following preliminary statement (see [5] as well).

Proposition 1. *Suppose R is a non-zero commutative ring and G is an abelian group. Then $R[G]$ is invo-clean if, and only if, R is invo-clean having the decomposition $R = R_1 \times R_2$ such that precisely one of the next three items holds:*

(0) $G = \{1\}$

or

(1) $|G| > 2$, $G^2 = \{1\}$, $R_1 = \{0\}$ or R_1 is a ring of $\operatorname{char}(R_1) = 2$, and $R_2 = \{0\}$ or R_2 is a ring of $\operatorname{char}(R_2) = 3$

or

(2) $|G| = 2$, $2r_1^2 = 2r_1$ for all $r_1 \in R_1$ *(in addition $4 = 0$ in R_1)*, and $R_2 = \{0\}$ or R_2 is a ring of $\operatorname{char}(R_2) = 3$.

P r o o f. If G is the trivial identity group, there is nothing to do, so we shall assume hereafter that G is non-identity.

"**Necessity.**" Since there is an epimorphism $R[G] \to R$, and an epimorphic image of an invo-clean ring is obviously an invo-clean ring (see, e.g., [1]), it follows at once that R is again an invo-clean ring. According to the criterion for invo-cleanness alluded to above, one writes that $R = R_1 \times R_2$, where R_1 is a nil-clean ring with $a^2 = 2a$ for all $a \in J(R_1)$ and R_2 is a ring whose

elements satisfy the equation $x^3 = x$. Therefore, it must be that $R[G] \cong R_1[G] \times R_2[G]$, where it is not too hard to verify by Lemma 2 that both $R_1[G]$ and $R_2[G]$ are invo-clean rings.

First, we shall deal with the second direct factor $R_2[G]$ being invo-clean. Since $\text{char}(R_2) = 3$, it follows immediately that $\text{char}(R_2[G]) = 3$ too. Thus an application of Lemma 1 (ii) (which is an assemble of facts from [1, 2]) allows us to deduce that all elements in $R_2[G]$ also satisfy the equation $y^3 = y$. So, given $g \in G \subseteq R[G]$, it follows that $g^3 = g$, that is, $g^2 = 1$.

Next, we shall treat the invo-cleanness of the group ring $R_1[G]$. Since $\text{char}(R_1)$ is a power of 2 (see [1]), it follows the same for $R_1[G]$. Consequently, utilizing once again Lemma 1 (i) (being an assortment of results from [1, 2]), we infer that $R_1[G]$ should be nil-clean, so that $z^2 = 2z$ for all $z \in J(R_1[G])$. That is why, invoking the criterion from [8], listed above, we have that G is a 2-group. We claim that even $G^2 = 1$. In fact, for an arbitrary $g \in G$, we derive with the aid of the aforementioned formula from [7] that $1 - g \in J(R_1[G])$, because $2 \in J(R_1)$. Hence $(1 - g)^2 = 2(1 - g)$ which forces that $1 - 2g + g^2 = 2 - 2g$ and that $g^2 = 1$, as desired. We now assert that $\text{char}(R_1) = 2$ whenever $|G| > 2$. To that purpose, there are two nonidentity elements $g \neq h$ in G with $g^2 = h^2 = 1$. Furthermore, again appealing to the formula from [7], the element $1 - g + 1 - h = 2 - g - h$ lies in $J(R_1[G])$, because $2 \in J(R_1)$. Thus $(2 - g - h)^2 = 2(2 - g - h)$ which yields that $2 - 2g - 2h + 2gh = 0$. Since $gh \neq 1$ as for otherwise $g = h^{-1} = h$, a contradiction, this record is in canonical form. This assures that $2 = 0$, as wanted.

However, in the case when $|G| = 2$, i.e. when $G = \{1, g \mid g^2 = 1\} = \langle g \rangle$, we can conclude that $2r^2 = 2r$ for any $r \in R_1$. Indeed, in view of the already cited formula from [7], the element $r(1 - g)$ will always lie in $J(R_1[G])$, because $2 \in J(R_1)$. We therefore may write $[r(1 - g)]^2 = 2r(1 - g)$ which ensures that $2r^2 - 2r^2 g = 2r - 2rg$ is canonically written on both sides. But this means that $2r^2 = 2r$, as pursued. Substituting $r = 2$, one obtains that $4 = 0$. Notice also that $2r^2 = 2r$ for all $r \in R_1$ and $a^2 = 2a$ for all $a \in J(R_1)$ will imply that $a^2 = 0$.

"**Sufficiency.**" Foremost, assume that (1) is true. Since R_1 has characteristic 2, whence it is nil-clean, and G is a 2-group, an appeal to [8] allows us to get that $R_1[G]$ is nil-clean as well. Since $z^2 = 2z = 0$ for every $z \in J(R_1)$, it is routinely checked that $\delta^2 = 2\delta = 0$ for each $\delta \in J(R_1[G])$, exploiting the formula from [7] for $J(R_1[G])$ and the fact that $R_1[G]$ is a modular group algebra of characteristic 2. That is why, by a consultation with Lemma 1 (i), one concludes that $R_1[G]$ is invo-clean, as expected. Further, by the usage of Lemma 1 (ii) above, we derive that $R_2[G]$ is an invo-clean ring of characteristic 3. To see that, given $x \in R_2[G]$, we write $x = \sum_{g \in G} r_g g$ with $r_g \in R_2$ satisfying $r_g^3 = r_g$. Since $G^2 = 1$ will easily imply that $g^3 = g$, one obtains that

$$x^3 = (\sum_{g \in G} r_g g)^3 = \sum_{g \in G} r_g^3 g^3 = \sum_{g \in G} r_g g = x,$$

as needed. We finally conclude with the help of Lemma 2 that $R[G] \cong R_1[G] \times R_2[G]$ is invo-clean, as expected.

Let us now point (2) be fulfilled. Since $G^2 = 1$, similarly to (1), R_2 being invo-clean of characteristic 3 implies that $R_2[G]$ is invo-clean, too. In order to prove that $R_1[G]$ is invo-clean, we observe that R_1 is nil-clean with $2 \in J(R_1)$. According to [8], the group ring $R_1[G]$ is also nil-clean. What remains to show is that for any element δ of $J(R_1[G])$ the equality $\delta^2 = 2\delta$ is valid. Since in conjunction with the explicit formula quoted above for the Jacobson radical, an arbitrary element in $J(R_1[G])$ has the form $j + j'g + r(1 - g)$, where $j, j' \in J(R_1)$ and $r \in R_1$, we have that $[j + j'g + r(1 - g)]^2 \in (J(R_1)^2 + 2J(R_1))[G] + r^2(1 - g)^2$. However, using the given conditions, $z^2 = 2z = 2z^2$ and thus $z^2 = 2z = 0$ for any $z \in J(R_1)$. Consequently, one checks that $[j + j'g + r(1 - g)]^2 = r^2(1 - g)^2 = 2r^2(1 - g) = 2r(1 - g) = 2[j + j'g + r(1 - g)]$, because $2r^2 = 2r$, as required. Therefore, $R_1[G]$ is invo-clean with Lemma 1 (i) at hand. Finally, Lemma 2 gives that $R[G] \cong R_1[G] \times R_2[G]$ is invo-clean, as promised. $\square$

It is worthwhile noticing that concrete examples of an invo-clean ring of characteristic 4, such that its elements are solutions of the equation $2r^2 = 2r$, are the rings $\mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$.

We will prove now the following reduction of weak invo-cleanness.

Proposition 2. *Suppose that R is a commutative non-zero ring and G is an abelian group. Then $R[G]$ is weakly invo-clean which is not invo-clean if, and only if, R is a weakly invo-clean ring which is not invo-clean and $G = \{1\}$.*

P r o o f. "**Necessity.**" As it is well known and easy to establish that there is a surjection $R[G] \to R$, we may apply [2] to get that R is weakly invo-clean as well. According now to Lemma 3 we obtain that R is either invo-clean, or isomorphic to $\mathbb{Z}_5$, or decomposed as $K \times \mathbb{Z}_5$, where K is non-zero invo-clean. We will consider these three possibilities separately:

Case 1: *R is invo-clean.* Since both $R[G]$ and R have equal characteristics, it follows once again with the aid of Lemma 3 that $R[G]$ must be invo-clean too, a contrary to our assumption.

Case 2: *$R \cong \mathbb{Z}_5$.* It follows that $R[G] \cong \mathbb{Z}_5[G]$ has to be weakly invo-clean of characteristic 5. Employing [2], one infers that $\mathbb{Z}_5[G] \cong \mathbb{Z}_5$ whence these two rings have equal cardinalities. This, however, implies by a simple comparison of elements that $G = \{1\}$.

Case 3: *$R \cong K \times \mathbb{Z}_5$ with $K \neq \{0\}$ invo-clean.* Hence $R[G] \cong K[G] \times \mathbb{Z}_5[G]$. It follows as is Case 1 that $K[G]$ is necessarily invo-clean, whereas $\mathbb{Z}_5[G]$ is weakly invo-clean. Similarly to Case 2, we detect once again that $G = \{1\}$.

"**Sufficiency.**" It is immediate, because of the fulfillment of the isomorphism $R[G] \cong R$. $\square$

So, combining both Propositions 1 and 2, we come to our chief result. Specifically, the following assertion is true:

Theorem 1. *Let G be an abelian group and let R be a commutative non-zero ring. Then the group ring $R[G]$ is weakly invo-clean if, and only if, at most one of the following points is true*:

(1) $G = \{1\}$ *and R is weakly invo-clean.*

(2) $G \neq \{1\}$ *and $R \cong R_1 \times R_2$ is invo-clean such that either*

(2.1) $|G| > 2$, $G^2 = \{1\}$, $R_1 = \{0\}$ *or R_1 is a ring of $\mathrm{char}(R_1) = 2$, and $R_2 = \{0\}$ or R_2 is a ring of $\mathrm{char}(R_2) = 3$*
or

(2.2) $|G| = 2$, $2r_1^2 = 2r_1$ *for all $r_1 \in R_1$ (in addition $4 = 0$ in R_1), and $R_2 = \{0\}$ or R_2 is a ring of $\mathrm{char}(R_2) = 3$.*

P r o o f. If G is trivial, there is nothing to prove because of the isomorphism $R[G] \cong R$, so let us assume henceforth that G is non-trivial.

"**Necessity.**" As already observed in Proposition 2 alluded to above, if $G \neq \{1\}$, then the ring R must be invo-clean but *not* properly weakly invo-clean, i.e., it does not contain $\mathbb{Z}_5$ as a (proper) direct factor. Thus $R[G]$ has to be invo-clean too, as $\mathrm{char}(R[G]) = \mathrm{char}(R)$. We, therefore, appeal to Proposition 1 getting the listed above two items, as desired.

"**Sufficiency.**" As in the previous direction, Proposition 1 is in use to infer that $R[G]$ is invo-clean and hence weakly invo-clean, as wanted. $\square$

In closing, we state one more intriguing problem.

Problem 1. *Find a suitable criterion only in terms of the commutative unital ring R and the abelian group G when the group ring $R[G]$ is feebly invo-clean as defined in* [3].

In that direction, similarly to Lemma 3, the question of whether or not any (commutative) feebly invo-clean ring R which is possibly *not* weakly invo-clean possesses the decomposition $R = K \times P$, where K is a weakly invo-clean ring and P is a ring whose elements satisfy the equation $x^5 = x$ such that $P \not\cong \mathbb{Z}_5$, is of some interest.

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JACOBI TRANSFORM OF (ν, γ, p)-JACOBI–LIPSCHITZ FUNCTIONS IN THE SPACE $\mathrm{L}^p(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)$[1]

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Abstract: Using a generalized translation operator, we obtain an analog of Younis' theorem [Theorem 5.2, Younis M. S. *Fourier transforms of Dini–Lipschitz functions*, Int. J. Math. Math. Sci., 1986] for the Jacobi transform for functions from the (ν, γ, p)-Jacobi–Lipschitz class in the space $\mathrm{L}^p(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)$.

Keywords: Jacobi operator, Jacobi transform, Generalized translation operator.

1. Introduction and preliminaries

Younis [8, Theorem 5.2] characterized the set of functions in $\mathrm{L}^2(\mathbb{R})$ satisfying the Dini–Lipschitz condition by means of an asymptotic estimate of the growth of the norm of their Fourier transforms.

Theorem 1. [8, Theorem 5.2] *Let $f \in \mathrm{L}^2(\mathbb{R})$. Then the following conditions are equivalent*:

(1) $\|f(.+h) - f(.)\|_{\mathrm{L}^2(\mathbb{R})} = O\left(\dfrac{h^\alpha}{(\log 1/h)^\beta}\right)$ *as* $h \to 0, \quad 0 < \alpha < 1, \quad \beta > 0,$

(2) $\int_{|\lambda| \geq r} |\mathcal{F}(f)(\lambda)|^2 d\lambda = O(r^{-2\alpha}(\log r)^{-2\beta})$ *as* $r \to +\infty,$

where $\mathcal{F}$ stands for the Fourier transform of f.

The main aim of this paper is to establish an analog of Theorem 1 for the Jacobi transform in the space $\mathrm{L}^p(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)$. For this purpose, we use a generalized translation operator which was defined by Flensted–Jensen and Koornwinder [5].

In order to confirm the basic and standard notation, we briefly overview the theory of Jacobi operators and related harmonic analysis. The main references are [1, 4, 6].

Let $\lambda \in \mathbb{C}$, $\alpha \geq \beta \geq -1/2$, and $\alpha \neq 0$. The Jacobi function ϕ_λ of order (α, β) is the unique even C^∞-solution of the differential equation

$$(\mathrm{D}_{\alpha,\beta} + \lambda^2 + \rho^2)u = 0, \quad u(0) = 1, \quad u'(0) = 0,$$

where $\rho = \alpha + \beta + 1$, $\mathrm{D}_{\alpha,\beta}$ is the Jacobi differential operator defined as

$$\mathrm{D}_{\alpha,\beta} = \frac{d^2}{dx^2} + \left(\frac{\Delta'_{(\alpha,\beta)}(x)}{\Delta_{(\alpha,\beta)}(x)}\right)\frac{d}{dx}$$

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[1]Dedicated to Professor Radouan Daher for his 61's birthday.

with
$$\Delta_{(\alpha,\beta)}(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1},$$
and $\Delta'_{(\alpha,\beta)}(x)$ is the derivative of $\Delta_{(\alpha,\beta)}(x)$.

The Jacobi functions ϕ_λ can be expressed in terms of Gaussian hypergeometric functions as
$$\phi_\lambda(x) = \phi_\lambda^{(\alpha,\beta)}(x) = \mathrm{F}\left(\frac{1}{2}(\rho - i\lambda),\ \frac{1}{2}(\rho + i\lambda),\ \alpha + 1,\ -\sinh^2 x\right),$$
where the Gaussian hypergeometric function is defined as
$$F(a,b,c,z) = \sum_{m=0}^{\infty} \frac{a_m b_m}{c_m m!} z^m, \quad |z| < 1,$$
with $a, b, z \in \mathbb{C}$, $c \notin -\mathbb{N}$, $a_0 = 1$, and $a_m = a(a+1)\cdots(a+m-1)$.

The function $z \to F(a,b,c,z)$ is the unique solution of the differential equation
$$z(1-z)u''(z) + (c - (a+b+1)z)u'(z) - abu(z) = 0,$$
which is regular at 0 and equals 1 there.

From [7, Lemmas 3.1–3.3], we obtain the following statement.

Lemma 1. *The following inequalities are valid for a Jacobi function $\phi_\lambda(t)$ $(\lambda, t \in \mathbb{R}^+)$:*

(1) $|\phi_\lambda(t)| \le 1$;

(2) $|1 - \phi_\lambda(t)| \le t^2(\lambda^2 + \rho^2)$;

(3) *there is a constant $d > 0$ such that*
$$1 - \phi_\lambda(t) \ge d \quad for \quad \lambda t \ge 1.$$

Let $\mathrm{L}_{\alpha,\beta}^p(\mathbb{R}^+) = \mathrm{L}^p(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)$, $1 \le p \le 2$, be the space of p-power integrable functions on $\mathbb{R}^+$ endowed with the norm
$$\|f\|_p = \left(\int_0^\infty |f(x)|^p \Delta_{(\alpha,\beta)}(x)dx\right)^{1/p} < \infty.$$

Let $\mathrm{L}_\mu^p(\mathbb{R}^+) = \mathrm{L}^p(\mathbb{R}^+, d\mu(\lambda)/2\pi)$, $1 \le p \le 2$, be the space of measurable functions f on $\mathbb{R}^+$ such that
$$\|f\|_{p,\mu} = \left(\frac{1}{2\pi}\int_0^\infty |f(x)|^p d\mu(\lambda)\right)^{1/p},$$
where $d\mu(\lambda) = |c(\lambda)|^{-2}d\lambda$ and the c-function $c(\lambda)$ is defined as
$$c(\lambda) = \frac{2^{\rho - i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(1/2 \cdot (i\lambda + \alpha + \beta + 1))\Gamma(1/2 \cdot (i\lambda + \alpha - \beta + 1))}.$$

Now, we define the Jacobi transform
$$\widehat{f}(\lambda) = \int_0^\infty f(x)\phi_\lambda(x)\Delta_{(\alpha,\beta)}(x)dx,$$
for all functions f on $\mathbb{R}^+$ and complex numbers λ for which the right-hand side is well defined.

The Jacobi transform reduces to the Fourier transform when $\alpha = \beta = -1/2$.

We have the following inversion formula [6].

Theorem 2. *If* $f \in \mathrm{L}_{\alpha,\beta}^p(\mathbb{R}^+)$, *then*

$$f(x) = \frac{1}{2\pi} \int_0^\infty \widehat{f}(\lambda)\phi_\lambda(x)d\mu(\lambda).$$

From [3], we have the Hausdorff–Young inequality

$$\|\widehat{f}\|_{q,\mu} \le C_2\|f\|_p \quad \text{for all} \quad f \in \mathrm{L}_{\alpha,\beta}^p(\mathbb{R}^+),$$

where $1/p + 1/q = 1$ and C_2 is a positive constant.

The generalized translation operator T_h of a function $f \in \mathrm{L}_{\alpha,\beta}^p(\mathbb{R}^+)$ is defined as

$$\mathrm{T}_h f(x) = \int_0^\infty f(z)K(x,h,z)\Delta_{(\alpha,\beta)}(z)dz,$$

where K is an explicity known kernel function such that

$$K(x,y,z) = \frac{2^{-2\rho}\Gamma(\alpha+1)(\cosh x \cosh y \cosh z)^{\alpha-\beta-1}}{\Gamma(1/2)\Gamma(\alpha+1/2)(\sinh x \sinh y \sinh z)^{2\alpha}}(1-B^2)^{\alpha-1/2}$$

$$\times F\left(\alpha+\beta,\ \alpha-\beta,\ \alpha+\frac{1}{2},\ \frac{1}{2}(1-B)\right) \quad \text{for} \quad |x-y| < z < x+y,$$

and $K(x,y,z) = 0$ elsewhere and

$$B = \frac{\cosh^2 x + \cosh^2 y + \cosh^2 z - 1}{2\cosh x \cosh y \cosh z}.$$

From [2], we have

$$\widehat{(\mathrm{T}_h f)}(\lambda) = \phi_\lambda(h)\widehat{f}(\lambda).$$

2. Main results

In this section, we give the main result of this paper. We need first to define the (ν, γ, p)-Jacobi–Lipschitz class.

Definition 1. *Let* $\nu, \gamma > 0$. *A function* $f \in \mathrm{L}_{\alpha,\beta}^p(\mathbb{R}^+)$ *is said to be in the* (ν, γ, p)-*Jacobi–Lipschitz class, denoted by* $\mathrm{Lip}(\nu, \gamma, p)$, *if*

$$\|\mathrm{T}_h f(x) - f(x)\|_p = O\left(\frac{h^\nu}{(\log 1/h)^\gamma}\right) \quad as \quad h \to 0.$$

Theorem 3. *Let* f *belong to* $\mathrm{Lip}(\nu, \gamma, p)$. *Then*

$$\int_N^{+\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = O(N^{-q\nu}(\log N)^{-q\gamma}) \quad as \quad N \to +\infty.$$

P r o o f. Let $f \in \mathrm{Lip}(\nu, \gamma, p)$. Then we have

$$\|\mathrm{T}_h f(x) - f(x)\|_p = O\left(\frac{h^\nu}{(\log 1/h)^\gamma}\right) \quad \text{as} \quad h \to 0.$$

Therefore,

$$\int_0^{+\infty} |1 - \phi_\lambda(h)|^q|\widehat{f}(\lambda)|^q d\mu(\lambda) \le C_2^q\|\mathrm{T}_h f(x) - f(x)\|_p^q.$$

If $\lambda \in [1/h, 2/h]$, then $\lambda h \geq 1$ and inequality (3) of Lemma 1 implies that

$$1 \leq \frac{1}{d^{qk}}|1 - \phi_\lambda(h)|^{qk}.$$

Then

$$\int_{1/h}^{2/h} |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq \frac{1}{d^{qk}} \int_{1/h}^{2/h} |1 - \phi_\lambda(h)|^{qk}|\widehat{f}(\lambda)|^q d\mu(\lambda)$$

$$\leq \frac{1}{d^{qk}} \int_0^{+\infty} |1 - \phi_\lambda(h)|^{qk}|\widehat{f}(\lambda)|^q d\mu(\lambda) \leq \frac{1}{d^{qk}} C_2^q \|T_h f(x) - f(x)\|_p^q = O\left(\frac{h^{q\nu}}{(\log 1/h)^{q\gamma}}\right).$$

Then

$$\int_N^{2N} |\widehat{f}(\lambda)|^q d\mu(\lambda) = O\left(N^{-q\nu}(\log N)^{-q\gamma}\right) \quad \text{as} \quad N \to +\infty.$$

Thus, there exists C_4 such that

$$\int_N^{2N} |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq C_4 N^{-q\nu}(\log N)^{-q\gamma}.$$

Furthermore, we have

$$\int_N^{+\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = \left[\int_N^{2N} + \int_{2N}^{4N} + \int_{4N}^{8N} + ...\right] |\widehat{f}(\lambda)|^q d\mu(\lambda)$$

$$\leq C_4 N^{-q\nu}(\log N)^{-q\gamma} + C_4(2N)^{-q\nu}(\log 2N)^{-q\gamma} + C_4(4N)^{-q\nu}(\log 4N)^{-q\gamma} + ...$$

$$\leq C_4 N^{-q\nu}(\log N)^{-q\gamma}(1 + 2^{-q\nu} + (2^{-q\nu})^2 + (2^{-q\nu})^3 + ...$$

$$\leq C_4 C_k N^{-q\nu}(\log N)^{-q\gamma}),$$

where $C_k = (1 - 2^{-q\nu})^{-1}$ since $2^{-q\nu} < 1$.

This proves that

$$\int_N^{+\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = O\left(N^{-q\nu}(\log N)^{-q\gamma}\right) \quad \text{as} \quad N \to +\infty,$$

and this completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 2. *A function $f \in L_{\alpha,\beta}^p(\mathbb{R}^+)$ is said to be in the (ψ, p)-Jacobi–Lipschitz class, denoted by $\mathrm{Lip}(\psi, p)$, if*

$$\|T_h f(x) - f(x)\|_p = O\left(\frac{\psi(h)}{(\log 1/h)^\gamma}\right), \quad \gamma > 0, \quad as \quad h \to 0,$$

where

(1) $\psi(t)$ *is a continuous increasing function on $[0, \infty)$;*

(2) $\psi(0) = 0$;

(3) $\psi(ts) \leq \psi(t)\psi(s)$ *for all $s, t \in [0, \infty)$.*

Theorem 4. *Let $f \in L_{\alpha,\beta}^p(\mathbb{R}^+)$, ψ be a fixed function satisfying the conditions of Definition 2, and let $f(x)$ belong to $\mathrm{Lip}(\psi, p)$. Then*

$$\int_N^{+\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = O(\psi(N^{-q})(\log N)^{-q\gamma}) \quad as \quad r \to +\infty.$$

P r o o f. Let $f \in \mathrm{Lip}(\psi, p)$. Then we have

$$\|\mathrm{T}_h f(x) - f(x)\|_p = O\left(\frac{\psi(h)}{(\log 1/h)^\gamma}\right) \quad \text{as} \quad h \to 0$$

and

$$\int_0^{+\infty} |1 - \phi_\lambda(h)|^q |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq C_2^q \|\mathrm{T}_h f(x) - f(x)\|_p^q.$$

If $\lambda \in [1/h, 2/h]$, then $\lambda h \geq 1$ and, similarly to the proof of Theorem 3, by inequality (3) of Lemma 1, we obtain

$$1 \leq \frac{1}{d^{qk}} |1 - \phi_\lambda(h)|^{qk}.$$

Then

$$\int_{1/h}^{2/h} |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq \frac{1}{d^{qk}} \int_{1/h}^{2/h} |1 - \phi_\lambda(h)|^{qk} |\widehat{f}(\lambda)|^q d\mu(\lambda)$$

$$\leq \frac{1}{d^{qk}} C_2^q \|\mathrm{T}_h f(x) - f(x)\|_p^q = O\left(\frac{\psi(h^q)}{(\log 1/h)^{q\gamma}}\right).$$

There exists a positive constant C_5 such that

$$\int_N^{2N} |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq C_5 \frac{\psi(N^{-q})}{(\log N)^{q\gamma}}.$$

Thus,

$$\int_N^{+\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = \left[\int_N^{2N} + \int_{2N}^{4N} + \int_{4N}^{8N} + ...\right] |\widehat{f}(\lambda)|^q d\mu(\lambda)$$

$$\leq C_5 \frac{\psi(N^{-q})}{(\log N)^{q\gamma}} + C_5 \frac{\psi((2N)^{-q})}{(\log 2N)^{q\gamma}} + C_5 \frac{\psi((4N)^{-q})}{(\log 4N)^{q\gamma}} + ...$$

$$\leq C_5 \frac{\psi(N^{-q})}{(\log N)^{q\gamma}} + C_5 \frac{\psi((2N)^{-q})}{(\log N)^{q\gamma}} + C_5 \frac{\psi((4N)^{-q})}{(\log N)^{q\gamma}} + ...$$

$$\leq C_5 \frac{\psi(N^{-q})}{(\log N)^{q\gamma}} (1 + \psi(2^{-q}) + (\psi(2^{-q}))^2 + (\psi(2^{-q}))^3 + ...$$

$$\leq C_5 K_1 \frac{\psi(N^{-q})}{(\log N)^{q\gamma}},$$

where $K_1 = (1 - \psi(2^{-q}))^{-1}$ since (1) and (3) from Definition 2 imply that $\psi(2^{-q}) < 1$.

This proves that

$$\int_N^{+\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = O\left(\psi(N^{-q})(\log N)^{-q\gamma}\right) \quad \text{as} \quad N \longrightarrow +\infty,$$

and this completes the proof. $\square$

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REGULAR GLOBAL ATTRACTORS FOR WAVE EQUATIONS WITH DEGENERATE MEMORY

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Abstract: We consider the wave equation with degenerate viscoelastic dissipation recently examined in Cavalcanti, Fatori, and Ma, *Attractors for wave equations with degenerate memory*, J. Differential Equations (2016). Under certain extra assumptions (namely on the nonlinear term), we show the existence of a compact attracting set which provides further regularity for the global attractor and show that it consists of regular solutions.

Keywords: Degenerate viscoelasticity, Relative displacement history, Nonlinear wave equation, Critical exponent, Regular global attractor.

1. Introduction

An elastic body perturbed from equilibrium may undergo a restoring force subject to both frictional and viscoelastic dissipation mechanisms. The problem under consideration is the wave equation with degenerate viscoelastic dissipation in the unknown $u = u(x, t)$

$$u_{tt} - \Delta u + \int_0^\infty g(s)\mathrm{div}[a(x)\nabla u(t-s)]ds + b(x)u_t + f(u) = h(x) \quad \text{in } \Omega \times \mathbb{R}^+, \qquad (1.1)$$

defined on a bounded domain Ω in $\mathbb{R}^3$ with smooth (at least class $\mathcal{C}^2$) boundary Γ. Here, $g(s)$ is an temporal interaction kernel which transmits memory effects to produce the viscoelastic dissipation mechanisms, the function $b(x)$ is the spatially dependent frictional damping coefficient, the nonlinear term $f(u)$ communicates displacement dependent density in the material, and the function $h(x)$ represents a spatially dependent external forcing mechanism. The equation is subject to Dirichlet boundary conditions

$$u(x, t) = 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \qquad (1.2)$$

and the initial conditions

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{at } \Omega \times \{0\}. \qquad (1.3)$$

This problem was recently treated, to the extent of global well-posedness and global attractors, in [4]. The novelty here being the degenerate nature of the viscoelasticity. Similar problems have yielded several important results as well. We mention some other works concerning semilinear wave equations with memory. On the asymptotic behavior of solutions (in the sense of global attractors) see [9, 10, 13, 27–29], and on rates of decay of solutions one can also see [24, 30, 31].

To the problem under consideration here, the well-posedness was carried out under the guise of semigroup methods. Here, local mild solutions and regular (sometimes referred to as "strong") solutions are obtained using the fact that the underlying operator is the infinitesimal generator of a strongly continuous semigroup of contractions on the Hilbertian phase space $\mathcal{H}$, and the

other condition naturally being that the nonlinear term defines a locally Lipschitz continuous functional also on $\mathcal{H}$. The notions of mild solution and regular solution are described below after equation (2.15).

The main result concerning the asymptotic behavior of (1.1)–(1.3) in [4] consists in demonstrating the existence of a finite dimensional global attractor for the associated semidynamical system $(\mathcal{H}, S(t))$. (Throughout, $S(t)$ denotes the semigroup of solution operators generated by problem (1.1)–(1.3).) For this, the authors of [4] rely on [7, Proposition 7.9.4 and Theorem 7.9.6]. That is, the problem is of the asymptotically smooth gradient system class where the set of stationary points is bounded. The so-called quasi-stability of the dynamical system $(\mathcal{H}, S(t))$ involves finding a suitable (relatively) compact seminorm on $\mathcal{H}$ (i.e., the approach is similar to finding a global attractor via an α-contraction method). Instead of characterizing the global attractor as the omega-limit set of some bounded absorbing set $\mathcal{B}$ in $\mathcal{H}$, i.e. $\mathcal{A} = \omega(\mathcal{B})$, the global attractor in this work is characterized with properties from the gradient system so that the global attractor is described by the union of unstable manifolds connecting the set of stationary points $\mathcal{N}$, i.e. $\mathcal{A} = \mathbb{M}^u(\mathcal{N})$. Unlike the methods used to prove the existence of a global attractor by virtue of the former characterization, in the latter no (explicit) bounded absorbing set $\mathcal{B}$ nor any (explicit) uniform bound on solutions is used to prove the existence of the global attractor. Finally, it seems that an explicit bound in terms of some of the parameters of the problem (Lipschitz constant, etc.) can be given to the fractal dimension of the global attractor (indeed, see [6, Theorem 3.4.5]). These results are obtained without assuming the two damping terms satisfy a geometric control condition (cf. e.g. [23]).

To treat the memory term, we define a past history variable using the *relative displacement history*, for all $x \in \Omega \subset \mathbb{R}^3$ and $s, t \in \mathbb{R}^+$,

$$\eta^t(x, s) := u(x, t) - u(x, t - s). \tag{1.4}$$

In order for this formulation to make sense, we also need to prescribe the past history of $u(x, t)$, $t < 0$. Observe, from (1.4) we readily find the useful identity

$$\int_0^\infty g(s)\mathrm{div}[a(x)\nabla u(t - s)]ds = -\int_0^\infty g(s)\mathrm{div}[a(x)\nabla\eta^t(s)]ds + k_0\mathrm{div}[a(x)\nabla u(t)],$$

where $k_0 := \int_0^\infty g(s)ds$ assumed to be sufficiently small below (see (2.3)). Thus, equations (1.1)–(1.3) have an equivalent form in the unknowns $u = u(x, t)$ and $\eta^t = \eta^t(x, s)$, for all $x \in \Omega$ and $s, t \in \mathbb{R}^+$,

$$u_{tt} - \mathrm{div}[(1 - k_0 a(x))\nabla u] - \int_0^\infty g(s)\mathrm{div}[a(x)\nabla\eta^t(s)]ds + b(x)u_t + f(u) = h(x), \tag{1.5}$$

$$\eta_t = -\eta_s + u_t,$$

with boundary conditions, for all $(x, t) \in \Gamma \times \mathbb{R}^+$,

$$u(x, t) = 0 \quad \text{and} \quad \eta^t(x, s) = 0, \tag{1.6}$$

and the following initial conditions at $t = 0$,

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{and} \quad \eta^t(x, 0) = 0, \quad \eta^0(x, s) = \eta_0(x, s). \tag{1.7}$$

In this article, we aim to provide a regularity result to the global attractors found in [4] for the problem (1.1)–(1.3).

2. Preliminaries

This section contains a summary of the assumptions and main results of [4].

A word about notation: we will often drop the dependence on x and even t or s from the unknowns $u(x,t)$ and $\eta^t(x,s)$ writing only u and η^t instead. The norm in the space $L^p(\Omega)$ is denoted $\|\cdot\|_p$ except in the common occurrence when $p = 2$ where we simply write the $L^2(\Omega)$ norm as $\|\cdot\|$. The $L^2(\Omega)$ product is simply denoted $(\cdot, \cdot)$. Other Sobolev norms are denoted by occurrence; in particular, since we are working with the homogeneous Dirichlet boundary conditions (1.6), in $H_0^1(\Omega)$, we will use the equivalent norm

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|,$$

and in particular,

$$\|u\| \leq \frac{1}{\sqrt{\lambda_1}}\|\nabla u\|, \tag{2.1}$$

where $\lambda_1 > 0$ denotes the first eigenvalue of the Dirichlet–Laplacian. With $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, we are able to define, for any $s \geq 0$,

$$H^s := D\big((-\Delta)^{s/2}\big).$$

Given a subset B of a Banach space X, denote by $\|B\|_X$ the quantity $\sup_{x \in B} \|x\|_X$. Finally, in many calculations C denotes a *generic* positive constant which may or may not depend on several of the parameters involved in the formulation of the problem, and $Q(\cdot)$ will denote a *generic* positive nondecreasing function.

Concerning the model problem, we make the following assumptions.

(H1) Let $a \in C^1(\overline{\Omega})$ be such that the space

$$\text{meas}\{x \in \Gamma : a(x) > 0\} > 0,$$

and

$$\mathcal{V}_a^1 := \Big\{\psi \in L^2(\Omega) : \int_\Omega a(x)|\nabla\psi(x)|^2 dx < \infty, \ \psi_{|\Gamma} = 0\Big\},$$

is a Hilbert space endowed with the product

$$(\chi, \psi)_{\mathcal{V}_a^1} := \int_\Omega a(x)\nabla\chi(x) \cdot \nabla\psi(x)dx.$$

(Two examples are given in [4].) Above $\psi_{|\Gamma} = 0$ is meant in the sense of trace which is well-defined when $\mathcal{V}_a^1 \hookrightarrow W^{1,1}(\Omega)$. In addition, we also assume the continuous embeddings hold

$$H_0^1(\Omega) \hookrightarrow \mathcal{V}_a^1 \hookrightarrow L^2(\Omega),$$

and also that $Au := \text{div}(a(x)\nabla u)$ is a self-adjoint non-positive operator.

(H2) Assume $b \in L^\infty(\Omega)$ is a non-negative function and c_0 is a constant satisfying, for all $x \in \Omega$,

$$\inf_{x \in \Omega}\{a(x) + b(x)\} \geq c_0 > 0.$$

(H3) Assume $g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ satisfies, for all $s \geq 0$,

$$g(s) \geq 0 \quad \text{and} \quad g'(s) \leq -\delta g(s). \tag{2.2}$$

We also impose on g the smallness condition

$$k_0 := \int_0^\infty g(s)ds < \|a\|_\infty^{-1}. \tag{2.3}$$

Remark 1. Assumption (H1) allows us to set the space for the past history function η^t. Indeed, define

$$\mathcal{M}^0 := L_g^2(\mathbb{R}^+; \mathcal{V}_a^1) = \left\{ \eta(x,s) : \int_0^\infty g(s)\|\eta(x,s)\|_{\mathcal{V}_a^1}^2 ds < \infty \right\}$$

which is Hilbert with the product

$$(\eta, \zeta)_{\mathcal{M}^0} := \int_0^\infty g(s) \left(\int_\Omega a(x)\nabla\eta(x,s) \cdot \nabla\zeta(x,s)dx \right) ds.$$

It should be noted that in [4], the assumption (H2) allows one to view the role of the frictional damping coefficient b as an arbitrarily small complementary damping in the following sense: if $\omega_0 := \{x \in \mathbb{R}^3 : a(x) = 0\}$, then what is only required is $b(x) > 0$ on any neighborhood of ω_0.

Equation (2.2) of assumption (H3) implies g decays to zero exponentially. Moreover, by (2.3), we have that, for all $x \in \overline{\Omega}$,

$$0 < \ell_0 \leq 1 - k_0 a(x) \tag{2.4}$$

where

$$\ell_0 := 1 - k_0\|a\|_\infty.$$

Now we make our final assumptions.

(H4) Let $f \in C^2(\Omega)$ and assume there exists $C_f > 0$ such that, for all $s \in \mathbb{R}$,

$$|f''(s)| \leq C_f(1 + |s|). \tag{2.5}$$

(Hence, the nonlinear term is allowed to attain critical growth.) We also assume that

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\ell_0\lambda_1 \tag{2.6}$$

cf. (2.1).

Remark 2. The two conditions (2.5) and (2.6) are used in [17] which treats the asymptotic behavior of a phase-field equation with memory. The assumption (2.5) implies there is a constant $C > 0$ such that for all $r, s \in \mathbb{R}$

$$|f(r) - f(s)| \leq C|r - s|(1 + |r|^2 + |s|^2). \tag{2.7}$$

The condition (2.7) appears in many recent works on semilinear wave equations with memory (e.g. [13]) and the strongly damped wave equation (this condition refers to the *sub*critical setting of those problems), see for example [2, 3, 12, 21, 25, 28, 29]. By (2.6) we find that for some $\alpha \in (0, \lambda_1)$, there exists $\rho_f > 0$ so that, for all $s \in \mathbb{R}$, there hold

$$f(s)s \geq -\ell_0\alpha s^2 - \rho_f \tag{2.8}$$

and, for $F(s) := \int_0^s f(\sigma)d\sigma$,

$$F(s) \geq -\frac{\ell_0\alpha}{2}s^2 - \rho_f. \tag{2.9}$$

Observe though both (2.8) and (2.9) follow when (2.6) is replaced by the less general assumption,

$$\liminf_{|s|\to\infty} f'(s) \geq -\ell_0\lambda_1. \tag{2.10}$$

Assumption (2.5) and condition (2.10) appear in equations with memory terms [5, 8, 11, 29].

Concerning the new regularity results described in section 3, we additionally assume the following assumptions hold along with (H1)-(H4).

(H1r) Suppose $a \in C^1(\overline{\Omega})$ is such that the space

$$\mathcal{V}_a^2 := \Big\{\psi \in L^2(\Omega) : \int_\Omega a(x)\left(|\Delta\psi(x)|^2 + |\psi(x)|^2\right)dx < \infty, \ \psi_{|\Gamma} = 0\Big\},$$

is a Hilbert space endowed with the product

$$(\chi,\psi)_{\mathcal{V}_a^2} := \int_\Omega a(x)\left(\Delta\chi(x)\Delta\psi(x) + \chi(x)\psi(x)\right)dx.$$

Also, assume the continuous embedding holds

$$\mathcal{V}_a^2 \hookrightarrow H_0^1(\Omega).$$

Remark 3. It should be noted that the embedding $D(-\Delta) \hookrightarrow \mathcal{V}_a^2$, where $D(-\Delta) := H^2(\Omega) \cap H_0^1(\Omega)$, does not hold. The interested reader should see [1, Section 3] where it is shown $H^2(\Omega) \not\subseteq \mathcal{V}_a^2$.

(H4r) Assume that there exists $\vartheta > 0$ such that, for all $s \in \mathbb{R}$,

$$f'(s) \geq -\vartheta. \tag{2.11}$$

Remark 4. The last assumption (2.11) appears in [5, 14–16, 26]. Such a bound is commonly utilized to obtain the precompactness property for the semigroup of solution operators associated with evolution equations where the use of fractional powers of the Laplace operator present a difficulty, if they are even well-defined.

Throughout the remainder of this article, we simply denote (1.5)–(1.7) under assumptions (H1)–(H4) and (H1r) and (H4r) as problem P.

The finite energy phase-spaces we study problem P in involve the following Hilbert spaces. First,

$$\mathcal{H}^0 := H^1(\Omega) \times L^2(\Omega) \times \mathcal{M}^0,$$

endowed with the norm whose square is given by, for $U = (u,v,\eta) \in \mathcal{H}^0$,

$$\|U\|_{\mathcal{H}^0}^2 := \|\nabla u\|^2 + \|v\|^2 + \|\eta\|_{\mathcal{M}^0}^2.$$

Later we also require

$$\mathcal{M}^1 := L_g^2(\mathbb{R}^+; \mathcal{V}_a^2) = \Big\{\eta : \int_0^\infty g(s)\|\eta(s)\|_{\mathcal{V}_a^2}^2 ds < \infty\Big\}$$

and

$$\mathcal{H}^1 := H^2(\Omega) \times H^1(\Omega) \times \mathcal{M}^1,$$

with the norm whose square is given by, for $U = (u, v, \eta) \in \mathcal{H}^1$,

$$\|U\|_{\mathcal{H}^1}^2 := \|u\|_{H^2(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2 + \|\eta\|_{\mathcal{M}^1}^2.$$

Here $H^1(\Omega)$ is normed with

$$\|\psi\|_{H^1(\Omega)} = \left(\|\nabla\psi\| + \|\psi\|\right)^{1/2},$$

and concerning the $H^2(\Omega)$ norm above, we know by H^2-elliptic regularity theory (cf. e.g. [19, section 8.4]),

$$\|\psi\|_{H^2(\Omega)} \le C\left(\|\Delta\psi\| + \|\psi\|\right), \tag{2.12}$$

for some constant $C > 0$.

So that we may write problem P in an operator formulation, we also define the following spaces,

$$D(T_r) := \{\eta \in \mathcal{M}^0 : \eta_s \in \mathcal{M}^0, \ \eta(0) = 0\},$$

where η_s denotes the distributional derivative of η and the equality $\eta(0) = 0$ is meant as

$$\lim_{s \to 0} \|\eta(s)\| = 0,$$

and

$$D(\mathcal{L}) := \left\{ U = (u, v, \eta) \in \mathcal{H}^0 \ \middle| \ \begin{array}{l} v \in H_0^1(\Omega), \ \eta \in D(T_r), \\ \mathrm{div}[(1 - k_0 a(x))\nabla u] + \int_0^\infty g(s)\mathrm{div}[a(x)\nabla\eta(s)]ds \in L^2(\Omega) \end{array} \right\},$$

to which we observe that there holds $D(\mathcal{L}) \subset \mathcal{H}^1$. On these spaces we defined the associated operators

$$T_r\eta := -\eta_s, \quad \text{for } \eta \in D(T_r),$$

and

$$\mathcal{L}U := \left(\begin{array}{c} v \\ \mathrm{div}[(1 - k_0 a(x))\nabla u] + \int_0^\infty g(s)\mathrm{div}[a(x)\nabla\eta(s)]ds - b(x)v \\ v + T_r\eta \end{array} \right), \quad \text{for } U \in D(\mathcal{L}).$$

For each $t \in [0, T]$, the equation

$$\eta_t^t = T_r\eta^t + v(t) \tag{2.13}$$

holds as an ODE in $\mathcal{M}^0$ subject to the initial condition

$$\eta^0 = \eta_0 \in \mathcal{M}^0. \tag{2.14}$$

Concerning the initial value problem (IVP) (2.13)–(2.14), we have the following proposition (cf. [27]).

Proposition 1. *The operator T_r with domain $D(T_r)$ the generator of the right-translation semigroup. Moreover, η^t can be explicitly represented by*

$$\eta^t(s) = \begin{cases} u(t) - u(t - s) & \text{if } 0 \le s \le t, \\ \eta_0(s - t) + u(t) - u(0) & \text{if } s > t. \end{cases}$$

Next we define the nonlinear functional by

$$\mathcal{F}(U) := (0, -f(u) + h, 0).$$

Problem P can now be written as the abstract Cauchy problem on $\mathcal{H}^0$,

$$\begin{cases} \dfrac{d}{dt} U = \mathcal{L}U + \mathcal{F}(U), & t > 0, \\ U(0) = U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}^0. \end{cases} \tag{2.15}$$

Later, when we are concerned with the regularity properties of problem P, we will also be interested in a more regular subspace of $\mathcal{H}^0$ (this is discussed further below).

Definition 1. *Let $T > 0$ and $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}^0 = H^1(\Omega) \times L^2(\Omega) \times \mathcal{M}^0$ be given. A function $U \in C([0, T]; \mathcal{H}^0)$ is called a **mild solution** to (2.15) on $[0, T]$ if and only if $\mathcal{F}(U(\cdot)) \in L^1(0, T; \mathcal{H}^0)$ and U satisfies the variation of constants formula for all $t \in [0, T]$,*

$$U(t) = e^{\mathcal{L}t} U_0 + \int_0^t e^{\mathcal{L}(t-s)} \mathcal{F}(U(s)) ds.$$

*The map $U = (u, u_t, \eta)$ is a mild solution on $[0, \infty)$ (i.e., is a **global mild solution**) if it is a mild solution on $[0, T]$, for every $T > 0$.*

The notion of *regular solution* used in this article is given precisely in equation (3.1). A regular solution requires better data, e.g. $U_0 \in \mathcal{H}^1 = H^2(\Omega) \times H^1(\Omega) \times \mathcal{M}^1$, and a trajectory that remains in the same space, e.g. $U(t) \in \mathcal{H}^1$. Indeed, our notion will also include the tail spaces defined above. Here, regular solutions are mild solutions that persist in the space

$$H^2(\Omega) \times H^1(\Omega) \times \mathcal{T}^1 \qquad \forall t \geq 0.$$

Concerning the spaces $\mathcal{V}_a^1$ and $\mathcal{V}_a^2$ from above, it is important to note that although the injection $\mathcal{V}_a^1 \hookleftarrow \mathcal{V}_a^2$ is compact, it does not follow that the injection $\mathcal{M}^0 \hookleftarrow \mathcal{M}^1$ is. Indeed, see [27] for a counterexample. Moreover, this means the embedding $\mathcal{H}^1 \hookrightarrow \mathcal{H}^0$ is not compact. Such compactness between the "natural phase spaces" is essential to obtaining further regularity for the global attractors and even for the construction of finite dimensional exponential attractors. To alleviate this issue we follow [20, 27] (also see [11, 18]) and define the so-called *tail function* of $\eta \in \mathcal{M}^0$ by, for all $\tau \geq 0$,

$$\mathbb{T}(\tau; \eta) := \int\limits_{(0, 1/\tau) \cup (\tau, \infty)} g(s) \|\nabla \eta(s)\|^2 ds.$$

With this we set,

$$\mathcal{T}^1 := \left\{ \eta \in \mathcal{M}^1 : \eta_s \in \mathcal{M}^0, \ \eta(0) = 0, \ \sup_{\tau \geq 1} \tau \mathbb{T}(\tau; \eta) < \infty \right\}.$$

The space $\mathcal{T}^1$ is Banach with the norm whose square is defined by

$$\|\eta\|_{\mathcal{T}^1}^2 := \|\eta\|_{\mathcal{M}^1}^2 + \|\eta_s\|_{\mathcal{M}^0}^2 + \sup_{\tau \geq 1} \tau \mathbb{T}(\tau; \eta). \tag{2.16}$$

Importantly, the embedding $\mathcal{T}^1 \hookrightarrow \mathcal{M}^0$ is compact. (We should mention that although the works [11, 18] treat PDE with an *integrated past history* variable, the compactness issue still applies to

models with a relative displacement history variable, such as (1.4) here. In fact, the compactness issue is more delicate in this setting; one must introduce so-called "tail functions," cf. [11, Lemma 3.1] or [18, Proposition 5.4]). Hence, let us now also define the space

$$\mathcal{K}^1 := H^2(\Omega) \times H^1(\Omega) \times \mathcal{T}^1,$$

and the desired *compact* embedding $\mathcal{K}^1 \hookrightarrow \mathcal{H}^0$ holds. Again, each space is equipped with the corresponding graph norm whose square is defined by, for all $U = (u, v, \eta) \in \mathcal{K}^1$,

$$\|U\|_{\mathcal{K}^1}^2 := \|u\|_{H^2(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2 + \|\eta\|_{\mathcal{T}^1}^2.$$

Concerning the IVP (2.13)–(2.14), we will also call upon the following (cf. [11, Lemmas 3.6]).

Lemma 1. *Let $\eta_0 \in D(T_r)$. Assume there is $\rho > 0$ such that, for all $t \geq 0$, $\|\nabla u(t)\| \leq \rho$. Then there is a constant $C > 0$ such that, for all $t \geq 0$,*

$$\sup_{\tau \geq 1} \tau \mathbb{T}(\tau; \eta^t) \leq 2(t + 2) e^{-\delta t} \sup_{\tau \geq 1} \tau \mathbb{T}(\tau; \eta_0) + C\rho^2.$$

We now report some results from [4] who only need to assume (H1)–(H4) hold. The following result is from [4, Theorem 2.1]. The proof follows by relying on classical semigroup theory; namely, the operator $\mathcal{L}$ is the infinitesimal generator of a C^0-semigroup of contractions $e^{\mathcal{L}t}$ in $\mathcal{H}^0$ (cf. [4, Lemma 3.1]) and the local Lipschitz continuity of $\mathcal{F} : \mathcal{H}^0 \to \mathcal{H}^0$.

Theorem 1. *Given $h \in L^2(\Omega)$ and $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}^0$, problem P possesses a unique global mild solution satisfying the regularity*

$$u \in C([0, \infty); H_0^1(\Omega)), \quad u_t \in C([0, \infty); L^2(\Omega)) \quad and \quad \eta^t \in C([0, \infty); \mathcal{M}^0). \qquad (2.17)$$

If $U_0 = (u_0, u_1, \eta_0) \in D(\mathcal{L})$, the solution is regular and satisfies

$$U \in C([0, \infty); D(\mathcal{L})).$$

In addition, if $Z^i(t) = (u^i(t), u_t^i(t), \eta^{i,t})$, $i = 1, 2$, are any two mild solutions to problem P corresponding to the initial data $Z_0^1, Z_0^2 \in \mathcal{H}^0$, respectively, where $\|Z_0^1\|_{\mathcal{H}^0} \leq R$ and $\|Z_0^2\|_{\mathcal{H}^0} \leq R$ for some $R > 0$, then for any $T > 0$ and for all $t \in [0, T]$,

$$\|Z^1(t) - Z^2(t)\|_{\mathcal{H}^0} \leq e^{Q(R)T} \|Z^1(0) - Z^2(0)\|_{\mathcal{H}^0}$$

for some positive nondecreasing function $Q(\cdot)$.

The next result depends on [4, Lemma 3.3]. For this we define the "energy functional" which is used to extend local solutions to global ones, as well as demonstrate the gradient structure of problem P.

$$E(t) := \|u_t(t)\|^2 + \int_\Omega (1 - k_0 a(x))|\nabla u(t)|^2 dx + \|\eta^t\|_{\mathcal{M}^0}^2 + 2 \int_\Omega (F(u(t)) - h(x)u(t)) \, dx. \qquad (2.18)$$

Lemma 2. *The energy $E(t)$ is non-increasing along any solution $U(t) = (u(t), u_t(t), \eta^t)$. In addition, there exists $\delta_0, C_{fh} > 0$, independent of U, such that for all $t \geq 0$,*

$$E(t) \leq \delta_0 \|(u(t), u_t(t), \eta^t)\|_{\mathcal{H}^0}^2 - C_{fh}.$$

The following is [4, Theorem 2.2].

Theorem 2. *Let $h \in L^2(\Omega)$ and $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}^0$. The dynamical system $(\mathcal{H}^0, S(t))$ generated by the mild solutions of Problem P is gradient and possesses a global attractor $\mathcal{A}$ which has finite (fractal) dimension and coincides with the unstable manifold $\mathbb{M}^n(\mathcal{N})$ of stationary solutions of problem P.*

The final two results here will be useful in the next section. Each result follows from the existence of a (bounded) attractor in $\mathcal{H}^0$. The first result provides a uniform bound on the mild solutions of problem P and some extremely important dissipation integrals, and the second provides the existence of an absorbing set in a natural way.

Corollary 1. *For each $R > 0$ and every $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}^0$ such that $\|U_0\|_{\mathcal{H}^0} \leq R$, there exists a positive nondecreasing function $Q(\cdot)$ such that, for all $t \geq 0$,*

$$\|S(t)U_0\|_{\mathcal{H}^0} \leq Q(R). \tag{2.19}$$

In addition, there exists a function $Q(\cdot)$ such that

$$\int_0^\infty \left(\|\sqrt{b(x)}u_t(\tau)\|^2 + \delta\|\eta^\tau\|_{\mathcal{M}^0}^2 \right) d\tau \leq Q(R). \tag{2.20}$$

Consequently, there also holds

$$\int_0^\infty \|u_t(\tau)\|^2 d\tau \leq Q(R). \tag{2.21}$$

P r o o f. The first result is a consequence of the existence of a global/universal attractor.

To show (2.20), let $R > 0$ be given and $U_0 \in \mathcal{H}^0$ be such that $\|U_0\|_{\mathcal{H}^0} \leq R$. Next we formally derive the "energy identity" associated with problem P by multiplying (1.5) by $2u_t$ to then integrate over Ω; this yields (cf. [4, Equation (3.7)]),

$$\frac{d}{dt}E + 2\int_0^\infty g(s) \int_\Omega a(x)\nabla\eta^t(s) \cdot \nabla u_t dx ds + 2\|\sqrt{b(x)}u_t\|^2 = 0.$$

where E is the energy functional (2.18). Observe, thanks to (2.19), (2.4) and (2.9), we readily find $C(R) > 0$ such that, for all $t \geq 0$,

$$|E(t)| \leq C(R). \tag{2.22}$$

Next we note that with $(3.4)_2$ there holds,

$$2\int_0^\infty g(s) \int_\Omega a(x)\nabla\eta^t(s) \cdot \nabla u_t dx ds = \frac{d}{dt}\|\eta^t\|_{\mathcal{M}^0}^2 + \int_0^\infty g(s)\frac{d}{ds}\|\eta^t\|_{\mathcal{V}_a^1}^2 ds,$$

and applying (2.2) yields,

$$\int_0^\infty g(s)\frac{d}{ds}\|\eta^t(s)\|_{\mathcal{V}_a^1}^2 ds = -\int_0^\infty g'(s)\|\eta^t(s)\|_{\mathcal{V}_a^1}^2 ds \geq \delta\int_0^\infty g(s)\|\eta^t(s)\|_{\mathcal{V}_a^1}^2 ds. \tag{2.23}$$

Hence, we have

$$\frac{d}{dt}E + \delta\|\eta^t\|_{\mathcal{M}^0}^2 + 2\|\sqrt{b(x)}u_t\|^2 \leq 0. \tag{2.24}$$

Thus, integrating (2.24) over $(0, t)$ produces (2.20).

Now we show (2.21) easily follows from (2.20). Indeed, using the Mean Value Theorem for Definite Integrals, for each $\tau \geq 0$, there is $\xi_\tau \in \Omega$ so that

$$\|\sqrt{b(x)}u_t\|^2 = \int_\Omega b(x)|u_t(\tau)|^2 dx = b(\xi_\tau)\|u_t(\tau)\|^2.$$

Now consider

$$\int_0^\infty b(\xi_\tau)\|u_t(\tau)\|^2 d\tau = \int_0^\infty \|\sqrt{b(x)}u_t(\tau)\|^2 d\tau,$$

where $b(x) \not\equiv 0$, that is, where $b(x)$ is not identically equal to zero on Ω; thus, motivated by the average value, $b(\xi_\tau) > 0$ for each $\tau \geq 0$. Define $b_* := \inf_{\tau \geq 0} b(\xi_\tau) > 0$. So with (2.20) we find

$$\int_0^\infty \|u_t(\tau)\|^2 d\tau \leq \frac{1}{b_*}Q(R).$$

The thesis (2.21) follows with hypotheses (H5). The proof is complete. $\square$

Corollary 2. *The semigroup of solution operators $S(t)$ admits a bounded absorbing set $\mathcal{B}$ in $\mathcal{H}^0$; that is, for any subset $B \subset \mathcal{H}^0$, there exists $t_B \geq 0$ (depending on B) such that for all $t \geq t_B$, $S(t)B \subset \mathcal{B}$.*

P r o o f. The proof follows directly from the fact that the attractor $\mathcal{A}$ is bounded in $\mathcal{H}^0$; e.g., a ball in $\mathcal{H}^0$ of radius $\|\mathcal{A}\|_{\mathcal{H}^0} + 1$ is an absorbing set in $\mathcal{H}^0$. $\square$

Remark 5. Unfortunately we do not know the rate of convergence of any bounded subset in $\mathcal{H}^0$ to the global attractor $\mathcal{A}$. Moreover, there are several applications in the literature (not containing equations with degeneracies in crucial diffusion or damping terms) in which the rate of convergence of any bonded subset B of $\mathcal{H}^0$ *is* exponential in the sense that there is a constant $\varpi > 0$ such that for any nonempty bounded subset $B \subset \mathcal{H}^0$ and for all $t \geq 0$ there holds,

$$\text{dist}_{\mathcal{H}^0}(S(t)B, \mathcal{B}) \leq Q(R)e^{-\varpi t}.$$

Here, given two subsets U and V of a Banach space X, the *Hausdorff semidistance* between them is

$$\text{dist}_X(U, V) := \sup_{u \in U} \inf_{v \in V} \|u - v\|_X.$$

3. Regularity

The aim of this section, and indeed the aim of this article, is to show the existence of a smooth compact subset of $\mathcal{H}^0$ containing the global attractor $\mathcal{A}$. This is achieved by finding a suitable subset $\mathcal{C}$ of $\mathcal{K}^1 \hookrightarrow \mathcal{H}^0$; hence, $\mathcal{C}$ is compact in $\mathcal{H}^0$. To this end we decompose the semigroup of solution operators by showing it splits into uniformly decaying to zero and uniformly compact parts. With this we obtain asymptotic compactness for the associated semigroup of solution operators. The procedure requires some technical lemmas and a suitable Grönwall type inequality; the presentation follows [14, 16]. The argument developed here will also be relied on to establish the existence of a compact attracting set. As a reminder to the reader, throughout this section we assume the hypotheses (H1r) and (H4r) hold in addition to (H1)–(H4).

The main result in this section is the following.

Theorem 3. *Assume hypotheses (H1)–(H4), (H1r) and (H4r) hold. There exists a closed and bounded subset $\mathcal{C} \subset \mathcal{K}^1$ and a constant $\omega > 0$ such that for every nonempty bounded subset $B \subset \mathcal{H}^0$ and for all $t \geq 0$, there holds*

$$\text{dist}_{\mathcal{H}^0}(S(t)B, \mathcal{C}) \leq Q(\|B\|_{\mathcal{H}^0})e^{-\omega t}.$$

Consequently, the global attractor $\mathcal{A}$ (cf. Theorem 2) is bounded in $\mathcal{K}^1$ and trajectories on $\mathcal{A}$ are regular solutions of the form

$$u \in C([0,\infty); H^2(\Omega)), \quad u_t \in C([0,\infty); H^1(\Omega)) \quad and \quad \eta^t \in C([0,\infty); \mathcal{T}^1). \tag{3.1}$$

The proof of Theorem 3 requires several lemmas.

Step 1. The semigroup of solution operators is decomposed into two operators $S(t) = K(t) + Z(t)$ for all $t \geq 0$.

Step 2. In Lemma 3 we establish the global existence for the associated operators $K(t)$ as well as provide a uniform bound on $K(t)$ in $\mathcal{H}^0$, and, rather importantly, provides various dissipation integrals for various terms. These dissipation results are key to the method for obtaining compactness.

Step 3. Next, Lemma 4 establishes the global existence for the associated operators $Z(t)$. We also show that the operators $Z(t)$ are uniformly decaying to zero in $\mathcal{H}^0$.

Step 4. Upon differentiating the problem corresponding to the operators $K(t)$ we establish a higher-order uniform bound on $\partial_t K(t)$ in $\mathcal{H}^0$ in Lemma 5. This argument is crucial for obtaining the asymptotic compactness for the non-memory terms of the operators $K(t)$.

Step 5. The remaining Lemma 6 and Lemma 7 establish the appropriate bounds on the memory variable to complete the asymptotic compactness of $K(t)$. It is certainly nontrivial to establish asymptotic compactness for solution operators that involve problems with memory terms. (Indeed, recall the embedding $\mathcal{H}^1 \hookrightarrow \mathcal{H}^0$ is not compact.)

Step 6. The proof of Theorem 3 follows. This result ultimately provides a higher-order bound on the global attractor demonstrated in the prequel.

Set
$$\psi(s) := f(s) + \beta s \quad \text{with} \quad \beta \geq \vartheta \quad \text{so that} \quad \psi'(s) \geq 0 \tag{3.2}$$
and set $\Psi(s) := \int_0^s \psi(\sigma)d\sigma$. (We remind the reader of (2.11).) Let $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}^0$. Decompose (1.5)–(1.7) into the functions v, w, ξ and ζ where $v + w = u$ and $\xi + \zeta = \eta$ satisfy, respectively, problem V and problem W which are given by

$$\begin{cases} v_{tt} - \mathrm{div}[(1 - k_0 a(x))\nabla v] - \int_0^\infty g(s)\mathrm{div}[a(x)\nabla \xi^t(s)]ds + b(x)v_t + \psi(u) - \psi(w) = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \xi_t^t = -\xi_s^t + v_t & \text{in } \Omega \times \mathbb{R}^+, \\ v(x,t) = 0, \quad \xi^t(x,s) = 0 & \text{on } \Gamma \times \mathbb{R}^+, \\ v(x,0) = u_0(x), \quad v_t(x,0) = u_1(x), \quad \xi^t(x,0) = 0, \quad \xi^0(x,s) = \eta_0(x,s) & \text{at } \Omega \times \{0\} \end{cases} \tag{3.3}$$

and

$$\begin{cases} w_{tt} - \mathrm{div}[(1 - k_0 a(x))\nabla w] - \int_0^\infty g(s)\mathrm{div}[a(x)\nabla \zeta^t(s)]ds + b(x)w_t + \psi(w) = h(x) + \beta u & \text{in } \Omega \times \mathbb{R}^+, \\ \zeta_t^t = -\zeta_s^t + w_t & \text{in } \Omega \times \mathbb{R}^+, \\ w(x,t) = 0, \quad \zeta^t(x,s) = 0 & \text{on } \Gamma \times \mathbb{R}^+, \\ w(x,0) = 0, \quad w_t(x,0) = 0, \quad \zeta^t(x,0) = 0, \quad \zeta^0(x,s) = 0 & \text{at } \Omega \times \{0\}. \end{cases} \tag{3.4}$$

We now define the operators $K(t)U_0 := (w(t), w_t(t), \zeta^t)$ and $Z(t)U_0 := (v(t), v_t(t), \xi^t)$ using the associated global mild solutions to problem V and problem W (the existence of such solutions follows in a similar manor to the semigroup methods used to establish the well-posedness for problem P; cf. Theorem 1 and the regularity described in (2.17)).

The first of the subsequent lemmas shows that the operators $K(t)$ are bounded bounded on $\mathcal{H}^0$.

The following lemma provides an estimate that will be extremely important later in this section.

Lemma 3. *Assume the hypotheses of Theorem 3 hold. For each $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}^0$ there exists a unique global weak solution*

$$W := (w, w_t, \zeta^t) \in C([0, \infty); \mathcal{H}^0) \tag{3.5}$$

to problem W. Moreover, for each $R > 0$ and for all $U_0 \in \mathcal{H}^0$ with $\|U_0\|_{\mathcal{H}^0} \leq R$, there holds, for all $t \geq 0$,

$$\|K(t)U_0\|_{\mathcal{H}^0} \leq Q(R) \tag{3.6}$$

for some nonnegative increasing function $Q(\cdot)$. There also holds

$$\int_0^\infty \|w_t(\tau)\|^2 d\tau \leq Q(R). \tag{3.7}$$

In addition, for every $\varepsilon > 0$ there exists a function $Q(\cdot)$ such that for every $0 \leq s \leq t$, $R > 0$ and $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}^0$ with $\|U_0\|_{\mathcal{H}^0} \leq R$, there holds

$$\int_s^t \left(\|u_t(\tau)\|^2 + \|\sqrt{b(x)}u_t(\tau)\|^2 + \delta\|\eta^\tau\|_{\mathcal{M}^0}^2 + \|w_t(\tau)\|^2 + \|\sqrt{b(x)}w_t(\tau)\|^2 + \delta\|\zeta^\tau\|_{\mathcal{M}^0}^2 \right) d\tau$$
$$\leq \frac{\varepsilon}{2}(t - s) + \frac{1}{\varepsilon}Q(R). \tag{3.8}$$

Finally, there holds

$$\int_t^{t+1} \left(\|u_t(\tau)\|^2 + \delta\|\eta^\tau\|_{\mathcal{M}^0}^2 + \|\sqrt{b(x)}w_t(\tau)\|^2 + \|w_t(\tau)\|^2 + \delta\|\zeta^\tau\|_{\mathcal{M}^0}^2 \right) d\tau \leq Q(R). \tag{3.9}$$

P r o o f. As we have already stated above, the existence of global mild solutions satisfying (3.5) follows by arguing as in the proof of Theorem 1. The bound (3.6) essentially follows from the existence of a global attractor for problem P (cf. Corollary 1). The dissipation property (3.7) follows by arguing exactly as in the proof of Corollary 1 keeping in mind both $u^{(1)}$ and $u^{(b)}$ make sense, and that we are able to utilize the bound (2.21) for either one.

We are now interested in establishing (3.8). Indeed, multiplying $(3.4)_1$ by $2w_t$ and integrating over Ω, applying $(3.4)_2$ and applying an estimate like (2.23), all with w and ζ in place of u and η, respectively, and E_w denoting the corresponding functional E, produces (in place of (2.24))

$$\frac{d}{dt}E_w + \delta\|\zeta\|_{\mathcal{M}^0}^2 + 2\|\sqrt{b(x)}w_t\|^2 \leq 2\beta(u, w_t). \tag{3.10}$$

Since

$$2\beta(u, w_t) = 2\beta(u_t, w) + 2\beta\frac{d}{dt}(u, w)$$

and by (3.6)

$$2\beta(u_t, w) \leq \beta^2 C(R)\|u_t\| \leq \varepsilon + C_\varepsilon\|u_t\|^2,$$

so the differential inequality (3.10) becomes

$$\frac{d}{dt}\{E_w - 2\beta(u, w)\} + \delta\|\zeta^\tau\|_{\mathcal{M}^0}^2 + 2\|\sqrt{b(x)}w_t\|^2 \leq \varepsilon + C_\varepsilon\|u_t\|^2. \tag{3.11}$$

In light of (2.20) and (2.21), adding $\|u_t\|^2 + \|\sqrt{b(x)}u_t(\tau)\|^2 + \delta\|\eta^\tau\|_{\mathcal{M}^0}^2$ to both sides of (3.11) and integrating the result over (s, t) then applying (2.19), (3.6) and (2.22) for problem W produces the desired estimate (3.8).

To show (3.9), we now add in the bound $\|u_t\|^2 + \delta\|\eta\|^2_{\mathcal{M}^0} + 2\|w_t\|^2 \le C(R)$ into (3.10), and this time estimate the right-hand side with $C(R) + \|w_t\|^2$ to obtain

$$\frac{d}{dt}E_w + \|u_t(\tau)\|^2 + \delta\|\eta^\tau\|^2_{\mathcal{M}^0} + \|\sqrt{b(x)}w_t(\tau)\|^2 + \|w_t(\tau)\|^2 + \delta\|\zeta^\tau\|^2_{\mathcal{M}^0} \le C(R). \qquad (3.12)$$

Integrating (3.12) over $(t, t+1)$ and applying (2.22) for problem W yields (3.9). $\qquad\qquad\square$

Lemma 4. *Assume the hypotheses of Theorem 3 hold. For each $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}^0$ there exists a unique global weak solution*

$$V := (v, v_t, \xi^t) \in C([0, \infty); \mathcal{H}^0) \qquad (3.13)$$

to problem V. Moreover, for each $R > 0$ and for all $U_0 \in \mathcal{H}^0$ with $\|U_0\|_{\mathcal{H}^0} \le R$, there exists $\omega_1 > 0$ such that, for all $t \ge 0$,

$$\|Z(t)U_0\|_{\mathcal{H}^0} \le Q(R)e^{-\omega_1 t} \qquad (3.14)$$

for some positive nondecreasing function $Q(\cdot)$. Thus, the operators $Z(t)$ are uniformly decaying to zero in $\mathcal{H}^0$.

P r o o f. As we have already stated above, the existence of global mild solutions satisfying (3.13) follows by arguing as in the proof of Theorem 1. It suffices to show (3.14).

Let $R > 0$ and $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}^0$ be such that $\|U_0\|_{\mathcal{H}^0} \le R$. Next we rewrite the term $b(x)v_t$ in equation $(3.3)_1$ as $(b(x) + 1)v_t - v_t$. Then multiply the result in $L^2(\Omega)$ by $v_t + \varepsilon v$, where $\varepsilon > 0$ will be chosen below. When we include the basic identity

$$(\psi(u) - \psi(w), v_t) = \frac{d}{dt}\left\{(\psi(u) - \psi(w), v) - \frac{1}{2}(\psi'(u)v, v)\right\} - ((\psi'(u) - \psi'(w))w_t, v) + \frac{1}{2}(\psi''(u)u_t, v^2)$$

to the result and use $(3.3)_2$, we find that there holds, for almost all $t \ge 0$,

$$\begin{aligned}
\frac{d}{dt}\Bigg\{&\|v_t\|^2 + 2\varepsilon(v_t, v) + \int_\Omega (1 - k_0 a(x))|\nabla v|^2 dx + \|\xi^t\|^2_{\mathcal{M}^0} + \varepsilon\|\sqrt{b(x)}v\|^2 \\
&+ 2(\psi(u) - \psi(w), v) - (\psi'(u)v, v)\Bigg\} \\
&- 2\varepsilon\|v_t\|^2 + 2\varepsilon\int_\Omega (1 - k_0 a(x))|\nabla v|^2 dx - \int_0^\infty g'(s)\|\xi^t(s)\|^2_{\mathcal{V}_a^1} ds \\
&+ 2\varepsilon\int_0^\infty g(s)\int_\Omega a(x)\nabla\xi^t(s)\cdot\nabla v\, dx\, ds + 2\|\sqrt{b(x)}v_t\|^2 \\
&- 2(\psi'(u) - \psi'(w))w_t, v) + (\psi''(u)u_t, v^2) + 2\varepsilon(\psi(u) - \psi(w), v) = 0.
\end{aligned} \qquad (3.15)$$

We now consider the functional defined by

$$\begin{aligned}
\mathbb{V}(t) := &\|v_t(t)\|^2 + 2\varepsilon(v_t(t), v(t)) + \int_\Omega (1 - k_0 a(x))|\nabla v(t)|^2 dx + \|\xi^t\|^2_{\mathcal{M}^0} + \varepsilon\|\sqrt{b(x)}v(t)\|^2 \\
&+ 2(\psi(u(t)) - \psi(w(t)), v(t)) - (\psi'(u(t))v(t), v(t))
\end{aligned}$$

We now will show that, given $U(t) = (u(t), u_t(t), \eta^t), W(t) = (w(t), w_t(t), \zeta^t) \in \mathcal{H}^0$ are uniformly bounded with respect to $t \ge 0$ by some $R > 0$, there are constants $C_1, C_2 > 0$, independent of t, in which for all $V(t) = (v(t), v_t(t), \xi^t) \in \mathcal{H}^0$,

$$C_1\|V(t)\|^2_{\mathcal{H}^0} \le \mathbb{V}(t) \le C_2\|V(t)\|^2_{\mathcal{H}^0}. \qquad (3.16)$$

To this end we begin by estimating the following product with (2.1),

$$2\varepsilon|(v_t, v)| \leq \varepsilon\|v_t\|^2 + \varepsilon\|v\|^2 \leq \varepsilon\|v_t\|^2 + \frac{\varepsilon}{\lambda_1}\|\nabla v\|^2, \tag{3.17}$$

and

$$\varepsilon\|\sqrt{b(x)}v\|^2 \leq \varepsilon\|\sqrt{b}\|_\infty^2\|v\|^2 \leq \frac{\varepsilon}{\lambda_1}\|b\|_\infty\|\nabla v\|^2. \tag{3.18}$$

Concerning the terms in the functional $\mathbb{V}$ that involve the nonlinear term ψ, using (3.2), (2.5), (2.6) and the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, and also (2.19), there holds

$$|(\psi'(u)v, v)| \leq C\left(1 + \|\nabla u\|^2\right)\|\nabla v\|\|v\| \leq \varepsilon\|\nabla v\|^2 + C_\varepsilon(R)\|v\|^2, \tag{3.19}$$

where the constant $0 < C_\varepsilon \sim \varepsilon^{-1}$. From assumption (2.11) and (3.2)

$$2(\psi(u) - \psi(w), v) \geq 2(\beta - \vartheta)\|v\|^2. \tag{3.20}$$

Hence, for $\beta = \beta(\varepsilon)$ sufficiently large, the combination of (3.19) and (3.20) produces,

$$2(\psi(u) - \psi(w), v) - (\psi'(u)v, v) \geq 2(\beta - \vartheta)\|v\|^2 - \varepsilon\|\nabla v\|^2 - C_\varepsilon(R)\|v\|^2 \geq -\varepsilon\|\nabla v\|^2. \tag{3.21}$$

With (3.17), (3.18) and (3.21) we attain the lower bound for the functional $\mathbb{V}$,

$$\mathbb{V} \geq \left(\ell_0 - \frac{\varepsilon}{\lambda_1}(2 + \|b\|_\infty) - \varepsilon\right)\|\nabla v\|^2 + (1 - \varepsilon)\|v_t\|^2 + \|\xi^t\|_{\mathcal{M}^0}^2.$$

So for a sufficiently small $\varepsilon > 0$ fixed (which also fixes the choice of β), there is $m_0 > 0$ in which, for all $t \geq 0$, we have that

$$\mathbb{V}(t) \geq m_0\|(v(t), v_t(t), \xi^t)\|_{\mathcal{H}^0}^2.$$

Now by the (local) Lipschitz continuity of f, the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, the uniform bounds on u and w, and the Poincaré inequality (2.1), it is easy to check that with (2.7) there holds

$$2(\psi(u) - \psi(w), v) \leq 2\|\psi(u) - \psi(w)\|\|v\| \leq C(R)\|\nabla v\|^2. \tag{3.22}$$

Also, using (3.2), (2.5), (2.6) and the bound (2.19), there also holds

$$|(\psi'(u)v, v)| \leq C(R)\|\nabla v\|^2. \tag{3.23}$$

Thus, with (3.22), (3.23) and referring to some of the above estimates, the right-hand side of (3.16) also follows.

Moving forward, we now work on (3.15). In light of the estimates

$$2|((\psi'(u) - \psi'(w))w_t, v)| \leq C(1 + \|\nabla u\| + \|\nabla w\|)\|w_t\|\|v\|^2 \leq \frac{1}{2\beta}\|v\|^2 + C(R)\|w_t\|^2\mathbb{V}, \tag{3.24}$$

and

$$|(\psi''(u)u_t, v^2)| \leq C(1 + \|\nabla u\|)\|u_t\|\|v\|^2 \leq \frac{1}{2\beta}\|v\|^2 + C(R)\|u_t\|^2\mathbb{V}, \tag{3.25}$$

(here the constants $C(R) > 0$ also depend on $\beta > 0$) we see that with (3.24), (3.25), as well as (2.4), (2.2) and (3.20), the differential identity (3.15) becomes

$$\frac{d}{dt}\mathbb{V} + \varepsilon\|v_t\|^2 + 2\varepsilon\ell_0\|\nabla v\|^2 + \delta\|\xi^t\|_{\mathcal{M}^0}^2$$

$$+2\varepsilon\int_0^\infty g(s)\int_\Omega a(x)\nabla\xi^t(s)\cdot\nabla v\,dx\,ds + 2\|\sqrt{b(x)}v_t\|^2 + \left(2\varepsilon(\beta - \vartheta) - \frac{1}{\beta}\right)\|v\|^2 \tag{3.26}$$

$$\leq C(R)\left(\|u_t\|^2 + \|w_t\|^2\right)\mathbb{V} + 3\varepsilon\mathbb{V},$$

where we also added $3\varepsilon\|v_t\|^2$ to both sides (observe, $3\varepsilon\|v_t\|^2 \leq 3\varepsilon\mathbb{V}$). We now seek a suitable control on the product

$$\left| 2\varepsilon \int_0^\infty g(s) \int_\Omega a(x)\nabla\xi^t(s) \cdot \nabla v dx ds \right| \leq 2\varepsilon \int_0^\infty g(s) \left| \int_\Omega a(x)\nabla\xi^t(s) \cdot \nabla v dx \right| ds$$

$$= 2\varepsilon \int_0^\infty g(s) \left| (\xi^t(s), v)_{\mathcal{V}_a^1} \right| ds \leq 2\varepsilon \|\xi^t\|_{\mathcal{M}^0} \|\nabla v\| \leq 2\sqrt{\varepsilon}\|\xi^t\|_{\mathcal{M}^0}^2 + \frac{\varepsilon\sqrt{\varepsilon}}{2}\|\nabla v\|^2. \tag{3.27}$$

For sufficiently large $\beta > 0$, we may omit the positive terms $2\|\sqrt{b(x)}v_t\|^2 + (2\varepsilon(\beta - \vartheta) - 1/\beta)\|v\|^2$ from the left-hand side of (3.26) so that it becomes, with (3.27),

$$\frac{d}{dt}\mathbb{V} + \varepsilon\|v_t\|^2 + \varepsilon\left(2\ell_0 - \frac{\sqrt{\varepsilon}}{2}\right)\|\nabla v\|^2 + \left(\delta - 2\sqrt{\varepsilon}\right)\|\xi^t\|_{\mathcal{M}^0}^2 \leq C(R)\left(\|u_t\|^2 + \|w_t\|^2 + 3\varepsilon\right)\mathbb{V}. \tag{3.28}$$

For any $\varepsilon > 0$ sufficiently small so that

$$2\ell_0 - \frac{\sqrt{\varepsilon}}{2} > 0 \quad \text{and} \quad \delta - 2\sqrt{\varepsilon} > 0,$$

we can find a constant $m_1 > 0$, thanks to (3.16), such that (3.28) can be written as the following differential inequality, to hold for almost all $t \geq 0$,

$$\frac{d}{dt}\mathbb{V} + \varepsilon m_1 \mathbb{V} \leq C(R)\left(\|u_t\|^2 + \|w_t\|^2 + 3\varepsilon\right)\mathbb{V}. \tag{3.29}$$

Here we recall Proposition 2 and Lemma 3. Applying these to (3.29) yields, for all $t \geq 0$,

$$\mathbb{V}(t) \leq \mathbb{V}(0)e^{Q(R)}e^{-m_1 t/2}, \tag{3.30}$$

for some positive nondecreasing function $Q(\cdot)$. By virtue of (3.16) and the initial conditions provided in (3.3),

$$\mathbb{V}(0) \leq C_2(R)\|(v(0), v_t(0), \xi^0)\|_{\mathcal{H}^0}^2 \leq C_2(R)\left(\|\nabla u_0\|^2 + \|u_1\|^2 + \|\eta_0\|_{\mathcal{M}^0}^2\right) \leq Q(R).$$

Therefore (3.30) shows that the operators $Z(t)$ are uniformly decaying to zero. The proof is finished.   □

   The remaining lemmas will show that the operators $K(t)$ are asymptotically compact on $\mathcal{H}^0$. In order to establish this, we prove that the operators $K(t)$ are uniformly bounded in $\mathcal{K}^1 \hookrightarrow \mathcal{H}^0$.
   Due to the nature of the proof of the following lemma, we also need to assign the past history for the term w_t. Indeed, from below we need to consider the initial condition

$$\zeta_t^0(x, s) = -\zeta_s^0(x, s) = -w_t(x, 0 - s).$$

However, since $u = v + w$, we can write

$$-u_t(x, 0 - s) = -v_t(x, 0 - s) - w_t(x, 0 - s)$$

and hence assume that

$$v_t(x, 0 - s) = u_t(x, 0 - s) = -\eta_t^0(x, s) \quad \text{and} \quad w_t(x, 0 - s) = 0. \tag{3.31}$$

Lemma 5. *Assume the hypotheses of Theorem 3 hold.   For each $R > 0$ and for all $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}^0$ such that $\|U_0\|_{\mathcal{H}^0} \leq R$, there holds for all $t \geq 0$*

$$\|\partial_t K(t)U_0\|_{\mathcal{H}^0}^2 = \|\nabla w_t(t)\|^2 + \|w_{tt}(t)\|^2 + \|\zeta_t^t\|_{\mathcal{M}^0}^2 \leq Q(R) \tag{3.32}$$

for some positive nondecreasing function $Q(\cdot)$.

P r o o f. For all $x \in \Omega$ and $t, s \in \mathbb{R}^+$, set $H(x,t) := w_t(x,t)$ and $X^t := \zeta_t^t(s)$. Differentiating problem W with respect to t yields the system

$$\begin{cases} H_{tt} - \text{div}[(1 - k_0 a(x))\nabla H] - \int_0^\infty g(s)\text{div}[a(x)\nabla X^t(s)]ds + b(x)H_t + \psi'(w)H = \beta u_t & \text{in } \Omega \times \mathbb{R}^+, \\ X_t^t = -X_s^t + H_t & \text{in } \Omega \times \mathbb{R}^+, \\ H(x,t) = w_t(x,t) = 0, \quad X^t(x,s) = \zeta_t^t(x,s) & \text{on } \Gamma \times \mathbb{R}^+, \\ H(x,0) = w_t(x,0) = 0, \quad H_t(x,0) = w_{tt}(x,0) = -f(0) - u_1 \quad \text{(from (3.4))} & \text{at } \Omega \times \{0\}, \\ X^t(x,0) = w_t(x,t) - w_t(x, t-0) = 0, \quad X^0(x,s) = 0 \quad \text{(see (3.31))} & \text{at } \Omega \times \{0\}. \end{cases} \tag{3.33}$$

Multiply equation $(3.33)_1$ by $H_t + \varepsilon H$ for some $\varepsilon > 0$ to be chosen below. To this result we apply the identities

$$(\psi'(w)H, H_t) = \frac{1}{2}\frac{d}{dt}(\psi'(w)H, H) - \frac{1}{2}(\psi''(w)w_t, H^2),$$

and (here we rely on $(3.33)_2$)

$$\int_0^\infty g(s)\int_\Omega a(x)\nabla X^t(s)\nabla H_t(t)dxds = \frac{1}{2}\frac{d}{dt}\|X^t\|_{\mathcal{M}^0}^2 + \int_0^\infty g(s)\frac{d}{ds}\|X^t(s)\|_{\mathcal{V}_a^1}^2 ds$$

$$= \frac{1}{2}\frac{d}{dt}\|X^t\|_{\mathcal{M}^0}^2 - \int_0^\infty g'(s)\|X^t(s)\|_{\mathcal{V}_a^1}^2 ds$$

so that together we find

$$\frac{d}{dt}\Big\{\|H_t\|^2 + 2\varepsilon(H_t, H) + \int_\Omega (1 - k_0 a(x))|\nabla H|^2 dx + \|X^t\|_{\mathcal{M}^0}^2 + (\psi'(w)H, H)\Big\}$$

$$-2\varepsilon\|H_t\|^2 + 2\varepsilon\|\sqrt{b(x)}H_t\|^2 + 2\varepsilon(b(x)H_t, H) + 2\varepsilon\int_\Omega (1 - k_0 a(x))|\nabla H|^2 dx + 2\varepsilon(\psi'(w)H, H) \tag{3.34}$$

$$-2\int_0^\infty g'(s)\|X^t(s)\|_{\mathcal{V}_a^1}^2 ds + 2\varepsilon\int_0^\infty g(s)\int_\Omega a(x)\nabla X^t(s) \cdot \nabla H(t)dxds$$

$$= (\psi''(w)w_t, H^2) + 2\beta(u_t, H_t) + 2\beta\varepsilon(u_t, H).$$

Next we recall (2.2) and find

$$-2\int_0^\infty g'(s)\|X^t(s)\|_{\mathcal{V}_a^1}^2 ds \geq 2\delta\|X^t\|_{\mathcal{M}^0}^2, \tag{3.35}$$

and

$$2\varepsilon\int_0^\infty g(s)\int_\Omega a(x)\nabla X^t(s) \cdot \nabla H(t)dxds \geq -\delta\|X^t\|_{\mathcal{M}^0}^2 - \frac{\varepsilon^2}{\delta}\|\nabla H\|^2, \tag{3.36}$$

where the last inequality follows from (2.3). For all $\varepsilon > 0$ and $t \geq 0$, define the functional

$$\mathbb{I}(t) := \|H_t(t)\|^2 + 2\varepsilon(H_t(t), H(t)) + \int_\Omega (1 - k_0 a(x))|\nabla H(t)|^2 dx + \|X^t\|_{\mathcal{M}^0}^2 + (\psi'(w)H(t), H(t)). \tag{3.37}$$

Thanks to (2.4) and since $\psi' > 0$, there is a constant $C > 0$, sufficiently small, so that

$$C \left(\|H_t(t)\|^2 + \ell_0 \|\nabla H(t)\|^2 + \|X^t\|^2_{\mathcal{M}^0} \right) \leq \mathbb{I}(t).$$

At this point we can write (3.34)–(3.36) with (3.37) as

$$
\frac{d}{dt}\mathbb{I} - 2\varepsilon\|H_t\|^2 + 2\varepsilon\|\sqrt{b(x)}H_t\|^2 + 2\varepsilon(b(x)H_t, H) + \left(2\varepsilon\ell_0 - \frac{\varepsilon^2}{\delta}\right)\|\nabla H\|^2 \\
+ \delta\|X^t\|^2_{\mathcal{M}^0} + 2\varepsilon(\psi'(w)H, H) \leq 2(\psi''(w)w_t, H^2) + 2\beta(u_t, H_t) + 2\beta\varepsilon(u_t, H). \tag{3.38}
$$

Next, let us rely on the uniform bounds (2.19) and (3.14) to estimate the products on the right-hand side

$$
2|(\psi''(w)w_t, H^2)| \leq 2\|\psi''(w)w_t H^2\|_1 \leq 2\|\psi''(w)w_t\|_{3/2}\|H\|^2_6 \leq 2\|\psi''(w)\|_6\|w_t\|\|H\|^2_6 \\
\leq C(R)\|w_t\|\|\nabla H\|^2 \leq C(R)\|w_t\|\mathbb{I}, \tag{3.39}
$$

$$
2\beta|(u_t, H_t) + \varepsilon(u_t, H)| \leq C(R)\|H_t\| + C(R)\|\nabla H\| \leq C_\varepsilon(R) + \varepsilon\|H_t\|^2 + \varepsilon^2\|\nabla H\|^2, \tag{3.40}
$$

where $C_\varepsilon \sim \varepsilon^{-1} \wedge \varepsilon^{-2}$. Also, we know

$$2\varepsilon(\psi'(w)H, H) \geq 2\varepsilon^2(\beta - \vartheta)\|H\|^2 > 0. \tag{3.41}$$

Thus, combining (3.38)–(3.41) yields

$$
\frac{d}{dt}\mathbb{I} - 3\varepsilon\|H_t\|^2 + 2\varepsilon\|\sqrt{b(x)}H_t\|^2 + \varepsilon\left(2\ell_0 - \varepsilon\left(\frac{1}{\delta} + 1\right)\right)\|\nabla H\|^2 + \delta\|X^t\|^2_{\mathcal{M}^0} \\
\leq C(R)\|w_t\|\mathbb{I} + C_\varepsilon(R). \tag{3.42}
$$

Since $4\varepsilon\|H_t\|^2 \leq 4\varepsilon\mathbb{I}$, adding this to (3.42) makes the differential inequality (we also omit $2\varepsilon\|\sqrt{b(x)}H_t\|^2$)

$$
\frac{d}{dt}\mathbb{I} + \varepsilon\|H_t\|^2 + \varepsilon\left(2\ell_0 - \varepsilon\left(\frac{1}{\delta} + 1\right)\right)\|\nabla H\|^2 + \delta\|X^t\|^2_{\mathcal{M}^0} \leq C(R)(\|w_t\| + \varepsilon)\mathbb{I} + C_\varepsilon(R).
$$

We now find that for any $\varepsilon > 0$ small so that

$$2\ell_0 - \varepsilon\left(\frac{1}{\delta} + 1\right) > 0,$$

then

$$\frac{d}{dt}\mathbb{I} + \varepsilon\mathbb{I} \leq C(R)(\|w_t\| + \varepsilon)\mathbb{I} + C_\varepsilon(R)$$

to which we now apply Proposition 3 and the bounds (3.8) and (3.9) to conclude that, for all $t \geq 0$, there holds

$$\mathbb{I}(t) \leq C(R)\mathbb{I}(0)e^{-\varepsilon t/2} + C_\varepsilon(R).$$

Moreover, with (3.37) and the initial conditions in (3.33) we find that there is a constant $C > 0$ (with $\varepsilon > 0$ now fixed) in which

$$\|H_t(t)\|^2 + \|\nabla H(t)\|^2 + \|X^t\|^2_{\mathcal{M}^0} \leq C(R).$$

This establishes (3.32) and completes the proof. $\qquad\square$

We derive the immediate consequence of (3.4) and (3.32).

Corollary 3. *Under the assumptions of Lemma 5, there holds for all $t \geq 0$,*

$$\|\zeta_s^t\|_{\mathcal{M}^0} \leq Q(R). \tag{3.43}$$

Before we continue, we derive a further estimate for ζ^t.

Lemma 6. *Under the assumptions of Lemma 5, there holds for all $t \geq 0$,*

$$\|\nabla \zeta^t\|_{L_g^2(\mathbb{R}^+;L^2(\Omega))} \leq C_\delta. \tag{3.44}$$

P r o o f. Formally multiplying $(3.4)_2$ in $L_g^2(\mathbb{R}^+;L^2(\Omega))$ by $-\Delta\zeta^t(s)$ and estimating the result yields the differential inequality

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\zeta^t\|_{L_g^2(\mathbb{R}^+;L^2(\Omega))}^2 &= -\int_0^\infty g(s)\frac{d}{ds}\|\nabla\zeta^t(s)\|^2 ds + (\nabla w_t, \nabla\zeta^t)_{L_g^2(\mathbb{R}^+;L^2(\Omega))} \\
&= \int_0^\infty g'(s)\|\nabla\zeta^t(s)\|^2 ds + (\nabla w_t, \nabla\zeta^t)_{L_g^2(\mathbb{R}^+;L^2(\Omega))} \\
&\leq -\delta\int_0^\infty g(s)\|\nabla\zeta^t(s)\|^2 ds + \frac{2}{\delta}\|\nabla w_t\|^2 + \frac{\delta}{2}\|\nabla\zeta^t\|_{L_g^2(\mathbb{R}^+;L^2(\Omega))}^2 \\
&= -\frac{\delta}{2}\|\nabla\zeta^t\|_{L_g^2(\mathbb{R}^+;L^2(\Omega))}^2 + \frac{2}{\delta}\|\nabla w_t\|^2.
\end{aligned}
\tag{3.45}
$$

Hence, applying the bound (3.32) to (3.45), we find the differential inequality which holds for almost all $t \geq 0$

$$\frac{d}{dt}\|\nabla\zeta^t\|_{L_g^2(\mathbb{R}^+;L^2(\Omega))}^2 + \frac{\delta}{2}\|\nabla\zeta^t\|_{L_g^2(\mathbb{R}^+;L^2(\Omega))}^2 \leq C_\delta$$

where $0 < C_\delta \sim \delta^{-1}$. Applying a straight-forward Grönwall inequality and the initial conditions in (3.4) produces the desired bound (3.44). This concludes the proof. $\qquad\square$

Lemma 7. *Under the assumptions of Lemma 5, the following holds for all $t > 0$,*

$$\|K(t)U_0\|_{\mathcal{K}^1} \leq Q(R), \tag{3.46}$$

for some positive nondecreasing function $Q(\cdot)$. Furthermore, the operators $K(t)$ are uniformly compact in $\mathcal{H}^0$.

P r o o f. The proof consists of several parts. In the first part, we derive further bounds for some higher order terms. We begin by rewriting/expanding (3.4) as

$$
\begin{aligned}
w_{tt} + k_0\nabla a(x)\cdot\nabla w + (1 - k_0 a(x))(-\Delta)w& \\
-\int_0^\infty g(s)\nabla a(x)\cdot\nabla\zeta^t(s)ds + \int_0^\infty g(s)a(x)(-\Delta)\zeta^t(s)ds + b(x)w_t + \psi(w) = \beta u.&
\end{aligned}
\tag{3.47}
$$

Next, Using the relative displacement history definition of the memory space term

$$\zeta^t(s) := w(x,t) - w(x,t-s),$$

we rewrite the integral

$$\int_0^\infty g(s)a(x)(-\Delta)\zeta^t(s)ds = k_0 a(x)(-\Delta)w - \int_0^\infty g(s)a(x)(-\Delta)w(t-s)ds. \tag{3.48}$$

Combining (3.47) and (3.48) shows (3.4) takes the useful alternate form

$$w_{tt} - \Delta w - \int_0^\infty g(s)a(x)(-\Delta)w(t-s)ds + b(x)w_t + \psi(w)$$

$$+k_0\nabla a(x)\cdot\nabla w - \int_0^\infty g(s)\nabla a(x)\cdot\nabla\zeta^t(s)ds = \beta u. \tag{3.49}$$

We now report six identities that will be used below:

$$(w_{tt}, (-\Delta)w) = \frac{d}{dt}(\nabla w_t, \nabla w) - \|\nabla w_t\|^2,$$

$$-\int_0^\infty g(s)(a(x)(-\Delta)\underbrace{w(t-s)}_{=w(t)-\zeta^t(s)}, (-\Delta)w_t(t))ds$$

$$= -\int_0^\infty g(s)(a(x)(-\Delta)w(t), (-\Delta)w_t(t))ds + \int_0^\infty g(s)(a(x)(-\Delta)\zeta^t(s), (-\Delta)\underbrace{w_t(t)}_{=\zeta_t^t(s)+\zeta_s^t(s)})ds \tag{3.50}$$

$$= -\frac{k_0}{2}\frac{d}{dt}\|w\|_{\mathcal{V}_a^2}^2 + \frac{1}{2}\frac{d}{dt}\|\zeta^t\|_{\mathcal{M}^1}^2 + \frac{1}{2}\int_0^\infty g(s)\frac{d}{ds}\|\zeta^t(s)\|_{\mathcal{V}_a^2}^2 ds,$$

$$-\int_0^\infty g(s)(a(x)(-\Delta)\underbrace{w(t-s)}_{=w(t)-\zeta^t(s)}, (-\Delta)w(t))ds$$

$$= -k_0\|w\|_{\mathcal{V}_a^2}^2 + \int_0^\infty g(s)(a(x)(-\Delta)\zeta^t(s), (-\Delta)w(t))ds, \tag{3.51}$$

$$(b(x)w_t, (-\Delta)w_t) = \frac{d}{dt}(b(x)w_t, (-\Delta)w) - (b(x)w_{tt}, (-\Delta)w), \tag{3.52}$$

$$k_0(\nabla a(x)\cdot\nabla w, (-\Delta)w_t) = \frac{d}{dt}k_0(\nabla a(x)\cdot\nabla w, (-\Delta)w) - k_0(\nabla a(x)\cdot\nabla w_t, (-\Delta)w), \tag{3.53}$$

and

$$-\int_0^\infty g(s)(\nabla a(x)\cdot\nabla\zeta^t(s), (-\Delta)w_t(t))ds$$

$$= -\frac{d}{dt}\int_0^\infty g(s)(\nabla a(x)\cdot\nabla\zeta^t(s), (-\Delta)w(t))ds + \int_0^\infty g(s)(\nabla a(x)\cdot\nabla\zeta_t^t(s), (-\Delta)w(t))ds. \tag{3.54}$$

Next we multiply (3.49) in $L^2(\Omega)$ by $(-\Delta)w_t + (-\Delta)w$ to obtain, in light of (3.50)–(3.54), the differential identity

$$\frac{d}{dt}\Big\{\|\nabla w_t\|^2 + 2(\nabla w_t, \nabla w) + \|\Delta w\|^2 - k_0\|w\|_{\mathcal{V}_a^2}^2 + \|\zeta^t\|_{\mathcal{M}^1}^2$$

$$+2(b(x)w_t, (-\Delta)w) + 2k_0(\nabla a(x)\cdot\nabla w, (-\Delta)w) - 2\int_0^\infty g(s)(\nabla a(x)\cdot\nabla\zeta^t(s), (-\Delta)w(t))ds\Big\}$$

$$-2\|\nabla w_t\|^2 + 2\|\Delta w\|^2 + \int_0^\infty g(s)\frac{d}{ds}\|\zeta^t(s)\|_{\mathcal{V}_a^2}^2 ds - 2k_0\|w\|_{\mathcal{V}_a^2}^2$$

$$+2\int_0^\infty g(s)(a(x)(-\Delta)\zeta^t(s), (-\Delta)w(t))ds - 2(b(x)w_{tt}, (-\Delta)w) + 2(b(x)w_t, (-\Delta)w) \tag{3.55}$$

$$+2(\psi'(w)\nabla w, \nabla w_t) + 2(\psi(w), (-\Delta)w) - 2k_0(\nabla a(x)\cdot\nabla w_t, (-\Delta)w) + 2k_0(\nabla a(x)\cdot\nabla w, (-\Delta)w)$$

$$+2\int_0^\infty g(s)(\nabla a(x)\cdot\nabla\zeta_t^t(s), (-\Delta)w(t))ds - 2\int_0^\infty g(s)(\nabla a(x)\cdot\nabla\zeta^t(s), (-\Delta)w(t))ds$$

$$= 2\beta(\nabla u, \nabla w_t) + 2\beta(u, (-\Delta)w).$$

We now seek a constant $m_2 > 0$ sufficiently small so that we can write the above differential identity in the following form

$$\frac{d}{dt}\Phi + cm_2\Phi \leq Q(R) \tag{3.56}$$

where

$$\begin{aligned}
\Phi(t) &:= \|\nabla w_t(t)\|^2 + 2(\nabla w_t(t), \nabla w(t)) + \|\Delta w(t)\|^2 - k_0\|w(t)\|_{\mathcal{V}_a^2}^2 + \|\zeta^t\|_{\mathcal{M}^1}^2 \\
&\quad + 2(b(x)w_t(t), (-\Delta)w(t)) + 2k_0(\nabla a(x) \cdot \nabla w(t), (-\Delta)w(t)) \\
&\quad - 2\int_0^\infty g(s)(\nabla a(x) \cdot \nabla \zeta^t(s), (-\Delta)w(t))ds.
\end{aligned} \tag{3.57}$$

The important lower bound holds

$$\Phi \geq C_1(\|\Delta w\|^2 + \|\nabla w_t\|^2 + \|\zeta^t\|_{\mathcal{M}^1}^2) - C_2(R) \tag{3.58}$$

for some constants $C_1, C_2(R) > 0$, and essentially follows from some basic estimates, the bounds (2.19), (3.14), (3.32), (3.44), the Poincaré inequality (2.1) and with the assumptions on the functions a and b. Indeed, we estimate, for all $\varepsilon > 0$,

$$2|(\nabla w_t, \nabla w)| \leq \varepsilon\|\nabla w_t\|^2 + \frac{1}{\varepsilon}\|\nabla w\|^2 \leq \varepsilon\|\nabla w_t\|^2 + C_\varepsilon(R), \tag{3.59}$$

$$-k_0\|w\|_{\mathcal{V}_a^2}^2 = -k_0\int_\Omega a(x)|\Delta w|^2 dx \geq -k_0\|a\|_\infty\|\Delta w\|^2, \tag{3.60}$$

$$2|(b(x)w_t, (-\Delta)w)| \leq \frac{1}{\varepsilon}\|b(x)w_t\| + \varepsilon\|\Delta w\|^2 \leq C_\varepsilon(R) + \varepsilon\|\Delta w\|^2, \tag{3.61}$$

$$2k_0|(\nabla a(x) \cdot \nabla w, (-\Delta)w)| \leq \frac{k_0^2}{\varepsilon}\|\nabla a(x) \cdot \nabla w\|^2 + \varepsilon\|\Delta w\|^2 \leq C_\varepsilon(R) + \varepsilon\|\Delta w\|^2, \tag{3.62}$$

and

$$\begin{aligned}
2\int_0^\infty &g(s)|(\nabla a(x) \cdot \nabla\zeta^t(s), (-\Delta)w(t))|ds \\
&\leq \int_0^\infty g(s)\left(\frac{1}{\varepsilon}\|\nabla a(x) \cdot \nabla\zeta^t(s)\|^2 + \varepsilon\|\Delta w(t)\|^2\right)ds \\
&\leq \frac{1}{\varepsilon}\int_0^\infty g(s)\|\nabla a\|_\infty^2\|\nabla\zeta^t(s)\|^2 ds + \varepsilon\int_0^\infty g(s)\|\Delta w(t)\|^2 ds \\
&\leq \frac{1}{\varepsilon}\|\nabla a\|_\infty^2\|\nabla\zeta^t\|_{L_g^2(\mathbb{R}^+;L^2(\Omega))}^2 + \varepsilon k_0\|\Delta w\|^2 \\
&\leq C_\varepsilon(R) + \varepsilon k_0\|\Delta w\|^2.
\end{aligned} \tag{3.63}$$

Applying (3.59)–(3.63) to (3.57) gives us the lower bound for all $\varepsilon > 0$,

$$\Phi \geq (1-\varepsilon)\|\nabla w_t\|^2 + (\ell_0 - (2+k_0)\varepsilon)\|\Delta w\|^2 + \|\zeta^t\|_{\mathcal{M}^1}^2 - C_\varepsilon(R).$$

For any fixed $0 < \varepsilon < \min\{1, \ell_0/(2+k_0)\}$, we obtain (3.58).

Returning to the aim of (3.56), we first add

$$3\|\nabla w_t\|^2 + 2(\nabla w_t, \nabla w)$$

to both sides of (3.55), and also insert

$$\int_0^\infty g(s)\frac{d}{ds}\|\zeta^t(s)\|_{\mathcal{V}_a^2}^2 ds = -\int_0^\infty g'(s)\|\zeta^t(s)\|_{\mathcal{V}_a^2}^2 ds \geq \delta\int_0^\infty g(s)\|\zeta^t(s)\|_{\mathcal{V}_a^2}^2 ds = \delta\|\zeta^t\|_{\mathcal{M}^1}^2.$$

Putting these together and using the second inequality in (2.2), (3.55) becomes the differential inequality

$$\frac{d}{dt}\Phi + \|\nabla w_t\|^2 + 2(\nabla w_t, \nabla w) + 2\|\Delta w\|^2 - 2k_0\|w\|_{\mathcal{V}_a^2}^2 + \delta\|\zeta^t\|_{\mathcal{M}^1}^2$$

$$+2(b(x)w_t, (-\Delta)w) + 2k_0(\nabla a(x) \cdot \nabla w, (-\Delta)w) - 2\int_0^\infty g(s)(\nabla a(x) \cdot \nabla \zeta^t(s), (-\Delta)w(t))ds$$

$$\leq 3\|\nabla w_t\|^2 + 2(\nabla w_t, \nabla w) + 2(b(x)w_{tt}, (-\Delta)w) + 2k_0(\nabla a(x) \cdot \nabla w_t, (-\Delta)w) \quad (3.64)$$

$$-2\int_0^\infty g(s)(a(x)(-\Delta)\zeta^t(s), (-\Delta)w(t))ds - 2\int_0^\infty g(s)(\nabla a(x) \cdot \nabla \zeta_t^t(s), (-\Delta)w(t))ds$$

$$-2(\psi'(w)\nabla w, \nabla w_t) - 2(\psi(w), (-\Delta)w) + 2\beta(\nabla u, \nabla w_t) + 2\beta(u, (-\Delta)w).$$

(We should mention that the final bound of (3.32) is now realized to control the $\nabla \zeta_t^t$ term appearing on the right-hand side.) Next we employ some basic inequalities, the assumptions on a and b, the assumptions (2.5)–(2.7), the bounds (2.19), (3.6) and (3.32), and finally even the continuous embedding $\mathcal{V}_a^2 \hookrightarrow H_0^1(\Omega)$ of (H1r) to control the right-hand side of (3.64) with the estimates

$$3\|\nabla w_t\|^2 + 2(\nabla w_t, \nabla w) - 2(\psi'(w)\nabla w, \nabla w_t) + 2\beta(\nabla u, \nabla w_t) \leq C(R), \quad (3.65)$$

$$2(b(x)w_{tt}, (-\Delta)w) \leq C(R) + \frac{1}{4}\|\Delta w\|^2, \quad (3.66)$$

$$2k_0(\nabla a(x) \cdot \nabla w_t, (-\Delta)w) \leq C(R) + \frac{1}{4}\|\Delta w\|^2, \quad (3.67)$$

$$-2\int_0^\infty g(s)(a(x)(-\Delta)\zeta^t(s), (-\Delta)w(t))ds = -2\int_0^\infty g(s)(\zeta^t(s), w(t))_{\mathcal{V}_a^2}ds$$

$$\leq 2\int_0^\infty g(s)\|\zeta^t(s)\|_{\mathcal{V}_a^2}\|w(t)\|_{\mathcal{V}_a^2}ds \leq \varepsilon\|\zeta^t\|_{\mathcal{M}^1}^2 + \frac{2}{\varepsilon k_0}k_0\|w\|_{\mathcal{V}_a^2}^2, \quad (3.68)$$

$$-2\int_0^\infty g(s)(\nabla a(x) \cdot \nabla \zeta_t^t(s), (-\Delta)w(t))ds \leq 2\int_0^\infty g(s)\|\nabla a(x) \cdot \nabla \zeta_t^t(s)\|\|\Delta w(t)\|ds$$

$$\leq 2\int_0^\infty g(s)\|\nabla a\|_\infty\|\nabla \zeta_t^t(s)\|\|\Delta w(t)\|ds$$

$$\leq \frac{1}{\varepsilon}\|\nabla a\|_\infty^2 \int_0^\infty g(s)\|\nabla \zeta_t^t(s)\|^2 ds + \varepsilon \int_0^\infty g(s)\|\Delta w(t)\|^2 ds \quad (3.69)$$

$$= \frac{1}{\varepsilon}\|\nabla a\|_\infty^2\|\zeta_t^t\|_{L_g^2(\mathbb{R}^+; H_0^1(\Omega))}^2 + \varepsilon k_0\|\Delta w\|^2$$

$$\leq C_\varepsilon(R)\|\zeta_t^t\|_{\mathcal{M}^0}^2 + \varepsilon k_0\|\Delta w\|^2 \leq C_\varepsilon(R) + \varepsilon k_0\|\Delta w\|^2,$$

$$-2(\psi(w), (-\Delta)w) \leq C(R) + \frac{1}{4}\|\Delta w\|^2, \quad (3.70)$$

and

$$2\beta(u, (-\Delta)w) \leq C(R) + \frac{1}{4}\|\Delta w\|^2. \quad (3.71)$$

Hence, (3.65)–(3.71) show the right-hand side of (3.64) is controlled with, for all $\varepsilon > 0$,

$$C_\varepsilon(R) + (1 + \varepsilon k_0)\|\Delta w\|^2 + \frac{2}{\varepsilon k_0}k_0\|w\|_{\mathcal{V}_a^2}^2 + \varepsilon\|\zeta^t\|_{\mathcal{M}^1}^2.$$

Now fixing $0 < \varepsilon < \min\{1/k_0, \delta\}$ and setting

$$m_2 = m_2(k_0, \delta) := \min\{1 - \varepsilon k_0, \delta - \varepsilon\} > 0 \quad \text{and} \quad c = c(k_0) := 2\left(1 + \frac{1}{\varepsilon k_0}\right)$$

we arrive at the desired estimate (3.56).

So now we integrate the linear differential inequality (3.56) and apply $\Phi(0) = 0$. Thus,

$$\|\Delta w(t)\|^2 + \|\nabla w_t(t)\|^2 + \|\zeta^t\|^2_{\mathcal{M}^1} \leq Q_\delta(R), \qquad (3.72)$$

for some positive nondecreasing function $Q_\delta(\cdot) \sim \delta^{-1}$. By combining (3.72), (3.32) and the Poincaré inequality (2.1), we see that, with the H^2-elliptic regularity estimate (2.12), we have with uniform bounds

$$w(t) \in H^2(\Omega) \quad \text{and} \quad w_t(t) \in H^1(\Omega) \quad \forall \, t > 0.$$

Additionally, collecting the bounds (3.72) and (3.43) establishes that, for all $t \geq 0$,

$$\|\zeta^t\|^2_{\mathcal{M}^1} + \|\zeta^t_s\|^2_{\mathcal{M}^0} \leq Q_\delta(R). \qquad (3.73)$$

Lastly, to show (3.46) holds we need to control the last term of the norm (2.16). With the bound (2.19), we apply the conclusion of Lemma 1 here in the form

$$\sup_{\tau \geq 1} \tau \mathbb{T}(\tau; \zeta^t) \leq 2\,(t+2)\,e^{-\delta t} \sup_{\tau \geq 1} \tau \mathbb{T}(\tau; \zeta_0) + C(R) \leq C(R). \qquad (3.74)$$

where the last inequality follows from the null initial condition given in $(3.4)_4$. Together, the estimates (3.72)–(3.74) show that (3.46) holds. This completes the proof. $\qquad \square$

We now prove the main theorem.

P r o o f. [Proof of Theorem 3.] Define the subset $\mathcal{C}$ of $\mathcal{K}^1$ by

$$\mathcal{C} := \{U = (u, v, \eta) \in \mathcal{K}^1 : \|U\|_{\mathcal{K}^1} \leq Q(R)\},$$

where $Q(R) > 0$ is the function from Lemma 7, and $R > 0$ is such that $\|U_0\|_{\mathcal{H}^0} \leq R$. Let now $U_0 = (u_0, u_1, \eta_0) \in \mathcal{B}$ (the bounded absorbing set of Corollary 2 endowed with the topology of $\mathcal{H}^0$). Then, for all $t \geq 0$ and for all $U_0 \in \mathcal{B}$, $S(t)U_0 = Z(t)U_0 + K(t)U_0$, where $Z(t)$ is uniformly and exponentially decaying to zero by Lemma 4, and, by Lemma 7, $K(t)$ is uniformly bounded in $\mathcal{K}^1$. In particular, there holds

$$\text{dist}_{\mathcal{H}^0}(S(t)\mathcal{B}, \mathcal{C}) \leq Q(R)e^{-\omega t}.$$

The proof is finished. $\qquad \square$

4. Conclusions

We have show that the global attractors associated with a wave equation with degenerate viscoelastic dissipation in the form of degenerate memory possesses more regularity than previously obtained in [4]. This is established under reasonable assumptions by showing the existence of a compact attracting set to which global attractor resides. Moreover, the global attractor consists of regular solutions. The main difficulties encountered here are due to the degeneracy of the dissipation term as well as obtaining compactness for the memory term.

A. Appendix

We include two frequently used Grönwall-type inequalities that are important to this paper. The first can be found in [26, Lemma 5]; the second in [22, Lemma 2.2].

Proposition 2. *Let $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ be an absolutely continuous function satisfying*

$$\frac{d}{dt}\Lambda(t) + 2\eta\Lambda(t) \leq h(t)\Lambda(t) + k,$$

where $\eta > 0$, $k \geq 0$ and $\int_s^t h(\tau)d\tau \leq \eta(t - s) + m$, for all $t \geq s \geq 0$ and some $m \geq 0$. Then, for all $t \geq 0$,

$$\Lambda(t) \leq \Lambda(0)e^m e^{-\eta t} + \frac{ke^m}{\eta}.$$

Proposition 3. *Let $\Phi : [0, \infty) \to [0, \infty)$ be an absolutely continuous function such that, for some $\varepsilon > 0$,*

$$\frac{d}{dt}\Phi(t) + 2\varepsilon\Phi(t) \leq f(t)\Phi(t) + h(t)$$

for almost every $t \in [0, \infty)$, where f and h are functions on $[0, \infty)$ such that

$$\int_s^t |f(\tau)|d\tau \leq \alpha(1 + (t - s)^\lambda), \quad \sup_{t \geq 0} \int_t^{t+1} |h(\tau)|d\tau \leq \beta$$

for some $\alpha, \beta \geq 0$ and $\lambda \in [0, 1)$. Then

$$\Phi(t) \leq \gamma\Phi(0)e^{-\varepsilon t} + K$$

for every $t \in [0, \infty)$, for some $\gamma = \gamma(f, \varepsilon, \lambda) \geq 1$ and $K = K(\varepsilon, \lambda, f, h) \geq 0$.

Acknowledgments

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A NEW ROOT-FINDING ALGORITHM USING EXPONENTIAL SERIES

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Abstract: In this paper, we present a new root-finding algorithm to compute a non-zero real root of the transcendental equations using exponential series. Indeed, the new proposed algorithm is based on the exponential series and in which Secant method is special case. The proposed algorithm produces better approximate root than bisection method, regula-falsi method, Newton-Raphson method and secant method. The implementation of the proposed algorithm in Matlab and Maple also presented. Certain numerical examples are presented to validate the efficiency of the proposed algorithm. This algorithm will help to implement in the commercial package for finding a real root of a given transcendental equation.

Keywords: Algebraic equations, Transcendental equations, Exponential series, Secant method.

Introduction

The root finding algorithms are the most relevant computational problems in science, engineering. The applications of root finding algorithms arise in many practical applications of Biosciences, Physics, Engineering, Chemistry etc. As mentioned in [1], the finding of any unknown appearing implicitly in engineering or scientific formulas, gives rise to root finding problem. A root of a function $f(x)$ is a number 'α' such that $f(\alpha) = 0$. Generally, the roots of transcendental functions cannot be expressed in closed form or cannot be computed exactly. The root-finding algorithms give us approximations to the roots, these approximations are expressed either as small isolating intervals or as floating point numbers. In the literature, there are several root finding algorithms are available, see for example, [1–11]. The basic root-finding methods are Bisection, False position, Newton-Raphson, Secant methods etc. Most of the algorithms use iteration, producing a sequence of numbers that hopefully converge towards the root as a limit. The rates of converge of different algorithms are different. That is, some algorithms are faster converges to the root than others algorithms. The purpose of existing algorithms is to provide higher order convergence with guaranteed root. Many existing algorithms do not guarantee that they will find all the roots; in particular, if such an algorithm does not find any root, that does not mean that no root exists. There are many well known root finding algorithms available to find an approximate root of algebraic or transcendental equations., see for example, [1, 5, 6, 8, 9, 11].

In this work, we propose a new algorithm based on exponential series and secant method becomes a special case of the proposed algorithm. This algorithm provides faster roots in comparison of the previous methods. The new proposed algorithm will be useful for computing a real root of transcendental equations. The rest of the paper is as follows: Section 1 describes the proposed method, their mathematical formulation, calculation steps and flow chart; implementation of the proposed algorithm in Maple is presented in Section 2 with sample computations; and Section 3

discuss some numerical examples to illustrate the algorithm and comparisons are made to show efficiency of the new algorithm.

1. New algorithm using exponential series

The new iterative formula using exponential series is proposed as follows, for any two initial approximations x_0, x_1 of the root,

$$x_{n+1} = x_n \exp\left(\frac{x_{n-1}f(x_n) - x_nf(x_n)}{x_nf(x_n) - x_nf(x_{n-1})}\right), \quad n = 1, 2, \ldots. \tag{1.1}$$

By expanding this iterative formula, one can obtain the standard secant method as in first two terms, and many methods are obtained based on series truncation. Indeed,

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}. \tag{1.2}$$

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} + \frac{1}{2x_n}\left(\frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}\right)^2. \tag{1.3}$$

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} + \frac{1}{2x_n}\left(\frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}\right)^2 - \frac{1}{6x_n^2}\left(\frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}\right)^3. \tag{1.4}$$

This is shown in the following theorem.

Theorem 1. *Suppose $\alpha \neq 0$ is a real exact root of $f(x)$ and θ is a sufficiently small neighbourhood of α. Let $f''(x)$ exist and $f'(x) \neq 0$ in θ. Then the iterative formula given in equation* (1.1) *produces a sequence of iterations $\{x_n : n = 1, 2, 3, \ldots\}$ with order of convergence $p \geq (1 + \sqrt{5})/2$.*

P r o o f. The iterative formula given in equation (1.1) can be expressed in the following form

$$x_{n+1} = x_n \exp\left(\frac{x_{n-1}f(x_n) - x_nf(x_n)}{x_nf(x_n) - x_nf(x_{n-1})}\right).$$

Since

$$\lim_{x_n \to \alpha} \exp\left(\frac{x_{n-1}f(x_n) - x_nf(x_n)}{x_nf(x_n) - x_nf(x_{n-1})}\right) = 1,$$

and hence $x_{n+1} = \alpha$.

Using the standard expansion of e^x as

$$\exp(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots \tag{1.5}$$

and from equations (1.1) and (1.5), we have

$$x_{n+1} = x_n \exp\left(\frac{x_{n-1}f(x_n) - x_nf(x_n)}{x_nf(x_n) - x_nf(x_{n-1})}\right)$$

$$= x_n\left(1 + \left(\frac{x_{n-1}f(x_n) - x_nf(x_n)}{x_nf(x_n) - x_nf(x_{n-1})}\right) + \frac{1}{2}\left(\frac{x_{n-1}f(x_n) - x_nf(x_n)}{x_nf(x_n) - x_nf(x_{n-1})}\right)^2\right.$$

$$\left. + \frac{1}{6}\left(\frac{x_{n-1}f(x_n) - x_nf(x_n)}{x_nf(x_n) - x_nf(x_{n-1})}\right)^3 + \cdots\right)$$

$$= x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} + \frac{1}{2x_n}\left(\frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}\right)^2$$

$$- \frac{1}{6x_n^2}\left(\frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}\right)^3 + o\left(\frac{1}{24x_n^3}\left(\frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}\right)^4\right).$$

Since $f(x_n) \approx 0$, when we neglect higher order terms, then the above equation gives secant method in first two terms. Indeed, we have the following formulae obtained from first two terms, three terms and four terms of the expansion respectively as given in equations (1.2)–(1.4).

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} + \frac{1}{2x_n}\left(\frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}\right)^2.$$

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} + \frac{1}{2x_n}\left(\frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}\right)^2 - \frac{1}{6x_n^2}\left(\frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}\right)^3.$$

In the above equations, we obtained secant method having $(1 + \sqrt{5})/2$ convergence in first two terms. Therefore, the order of convergence of proposed methods (1.1), (1.3) and (1.4) are at least $p \geq (1 + \sqrt{5})/2$. $\square$

1.1. Steps for computing root

I. Select two approximations x_0 and $x_1 \neq 0$.

II. Compute $f(x_0)$ and $f(x_1)$.

III. Compute the next approximate root using formula given in (1.1).

IV. Repeat Step II and III until we get desired approximate root.

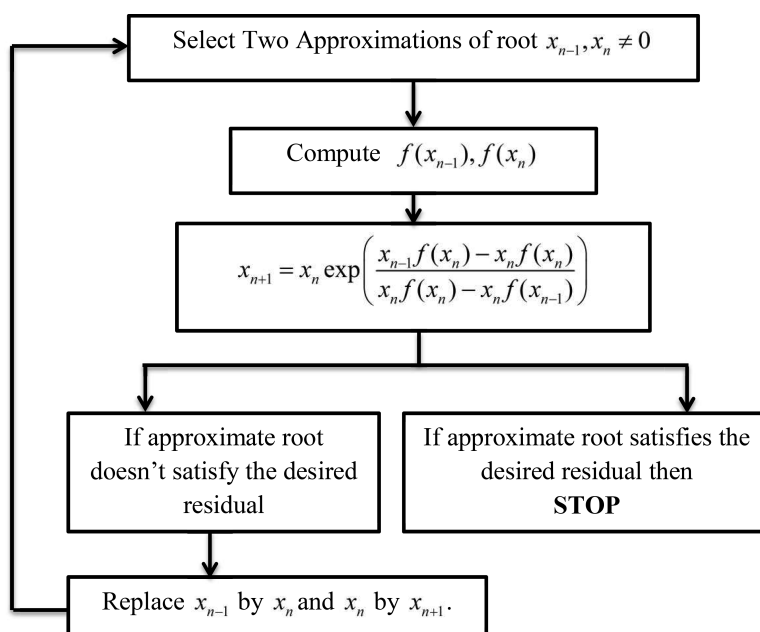Flow chat of the proposed algorithm is presented in Figure 1.

Figure 1. Flow chart for proposed algorithm

2. Implementation of proposed algorithm

2.1. Proposed algorithm in MATLAB

In this section present MATLAB implementation of the proposed algorithm as follows.

```
a=input('Given Function:','s');
f=inline(a);     % Given function is storing in f

x(1)=input('Enter x0:  ');    % Initial approximation x0
x(2)=input('Enter x1:  ');    % Initial approximation x1
n=input('Enter Allowed Error:  ');   % n is allowed error
iteration=0;

for i=3:1000
   x(i) = x(i-1)*exp((x(i-2)*f(x(i-1))-x(i-1)*f(x(i-1)))/
      (x(i-1)*f(x(i-1))-x(i-1)*f(x(i-2)))); % main eq (1.1)
    iteration=iteration+1;
    if abs((x(i)-x(i-1))/x(i))*100<n
          root=x(i)
          iteration=iteration
          break    % breaking if abs error ≥ n
    end
end
```

Sample computations using the implementation of the proposed algorithm are presented in Section 3.

2.2. Proposed algorithm in MAPLE

In this section, we provide the implementation of the proposed method in Maple as follows. To execute the maple code, one should enter the required date at `type` text.

```
eps_step := type;
eps_abs := type;
f(x):= type;
x[0] := type;
x[1] := type;
n:= type;
for i from 2 to n do
   printf("Iteration No:  %g", i-1);
   x[i] := x[i-1]*exp((x[i-2]*f(x[i-1])-x[i-1]*f(x[i-1]))/
        (x[i-1]*f(x[i-1])-x[i-1]*f(x[i-2])));
   if abs(x[i]-x[i-1]) < eps_step and
         abs(f(x[i])) < eps_abs then
      break;
   end if;
end do;
```

Sample computations using the implementation of the proposed algorithm are presented in Section 3.

3. Numerical examples

This section provides some numerical examples to discuss the algorithm presented in Section 1 and comparisons are taken into account to conform that the algorithm is more efficient than other existing methods.

Example 3.1. Consider an equation

$$x^6 - x - 1 = 0. \tag{3.1}$$

This equation has two real roots -0.7780895987 and 1.134724138. The following Table 1 shows the comparison between various existing methods and proposed method at accurate to within $\epsilon = 0.00001$ with initial approximations $x_0 = 1$ and $x_1 = 1.5$.

Table 1. Comparing approximate root using various existing methods

Ite No.	Bisection method	Ite No.	Secant method	Ite No.	Regula-Falsi method	Ite No.	Proposed method
1	1.25	1	1.05055292	1	1.05055292	1	1.111637022
2	1.125	2	1.08362707	2	1.08362707	2	1.121248067
3	1.1875	3	1.14718724	3	1.10430109	3	1.135602993
4	1.15625	4	1.13311087	4	1.11683267	4	1.134695420
5	1.140625	5	1.13467619	5	1.12428166	5	1.134724078
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
16	1.13472748	7	1.13472414	18	1.13471575	6	1.134724138

One can observe from the Table 1 that the proposed algorithm gives approximate root quicker than the other existing methods.

Example 3.2. Consider the following transcendental equations. We compare the number of iterations required to get approximation root. The numerical results are provided in Table 2.

(i) $f(x) = e^x - x - 2$, with initial approximation 1 and 2 with accuracy of 10^{-5}.

(ii) $f(x) = 2x^3 + 11x^2 + 12x - 9$, with initial approximations -5 and -1 with accuracy of 10^{-10}.

(iii) $f(x) = 8 - 4.5(x - \sin x)$, with initial approximations 2 and 3 with accuracy of 10^{-10}.

(iv) $f(x) = xe^{-x} - 0.1$, with initial approximation -0.9 and 0.9 with accuracy of 10^{-10}.

The numerical results given in Table 2 shows that the proposed method is more efficient than other methods.

Example 3.3. Recall the Example 3.1 for the sample computations using Matlab and Maple implementation as described in Section 2.

$$f(x) = x^6 - x - 1$$

Table 2. Comparing No. of iterations by different methods

Fun.	Exact Root	Bisection method	Regula-Falsi method	Secant method	Proposed method
(i)	1.146193221	18	14	7	6
(ii)	-3.00000000	37	56	8	8
(iii)	2.43046574	34	11	7	6
(iv)	0.11183256	36	55	15	11

with initial approximations 1 and 1.5.

Using Matlab implementation, we have the following computations.

```
>> ExpSecant
Given Function:x^6-x-1
Enter x0:  1.0
Enter x1:  1.5
Enter Allowed Error:  0.00001
root=
```

$$1.1347$$

```
iteration=
```

$$6$$

Using Maple implementation, we have the following computations.

```
> eps_step := 1e-5:
> eps_abs := 1e-5:
> f(x):= x^6-x-1:
> x[0] := 1.0:
> x[1] := 1.5:
> n:= 100:
> for i from 2 to n do
>   printf("Iteration No:  %g", i-1);
>   x[i] := x[i-1]*exp((x[i-2]*f(x[i-1])-x[i-1]*f(x[i-1]))/
      (x[i-1]*f(x[i-1])-x[i-1]*f(x[i-2])));
>   if abs(x[i]-x[i-1]) < eps_step and
>   abs(f(x[i])) < eps_abs then
> break;
> end if;
> end do;
```

Iteration No : 1

$$1.111637022$$

Iteration No : 2

$$1.121248067$$

Iteration No : 3

$$1.135602993$$

Iteration No : 4

$$1.134695420$$

Iteration No : 5

$$1.134724078$$

Iteration No : 6

$$1.134724138$$

One can use the implementation of the proposed algorithm to speed up the manual calculations.

4. Conclusion

In this work, we presented a new algorithm to compute an approximate root of a given transcendental function better than previous existing methods as illustrated. The proposed new algorithm was based on exponential series having better convergence than previous existing methods (for example, Bisection, Regula–Falsi, Secant methods etc.). This proposed algorithm is useful for solving the complex real life problems. Implementation of the proposed algorithm in Matlab and Maple is also discussed presented sample computations.

Acknowledgment

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HARMONIC INTERPOLATING WAVELETS IN NEUMANN BOUNDARY VALUE PROBLEM IN A CIRCLE

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Abstract: The Neumann boundary value problem (BVP) in a unit circle is discussed. For the solution of the Neumann BVP, we built a method employing series representation of given 2π-periodic continuous boundary function by interpolating wavelets consisting of trigonometric polynomials. It is convenient to use the method due to the fact that such series is easy to extend to harmonic polynomials inside a circle. Moreover, coefficients of the series have an easy-to-calculate form. The representation by the interpolating wavelets is constructed by using an interpolation projection to subspaces of a multiresolution analysis with basis 2π-periodic scaling functions (more exactly, their binary rational compressions and shifts). That functions were developed by Subbotin and Chernykh on the basis of Meyer-type wavelets. We will use three kinds of such functions, where two out of the three generates systems, which are orthogonal and simultaneous interpolating on uniform grids of the corresponding scale and the last one generates only interpolating on the same uniform grids system. As a result, using the interpolation property of wavelets mentioned above, we obtain the exact representation of the solution for the Neumann BVP by series of that wavelets and numerical bound of the approximation of solution by partial sum of such series.

Keywords: Wavelets, Interpolating wavelets, Harmonic functions, Neumann boundary value problem.

Introduction

Subbotin and Chernykh [1] constructed real 2π-periodic orthogonal wavelets and applied them to represent and analyze solutions of Dirichlet, Neumann, and Poisson boundary value problems for harmonic and biharmonic functions. In [2] the Dirichlet BVP in a unit circle was solved by means of interpolating-orthogonal periodic wavelets from [3]. In the present paper, we propose to use the same wavelets for solving the Neumann BVP in a unit circle. Moreover, our main interest is the exact representation of the solution for the Neumann BVP by series of wavelet bases and behavior of partial sums of such series. For the sake of convenience, we give the reader an adequate background for further study and partially repeat sections with interpolating and interpolating-orthogonal 2π-periodic wavelet construction from [1, 3].

1. Preliminaries

Consideration of autocorrelation functions for orthonormal scaling functions instead of orthonormal scaling functions is commonly used construction technique for interpolating wavelets in $\mathbb{R}$. It is equivalent to replacement of scaling $\varphi(x)$ function by function, which Fourier transform coincides with $|\widehat{\varphi}(\omega)|^2$.

Let ε be a fixed number from $(0, 1/3]$ and let $\widehat{\varphi}_\varepsilon(\omega)$ be a Fourier transform of Meyer-type (see [4, 5]) function:

$$
\begin{cases}
\widehat{\varphi}_\varepsilon(\omega) = 0, & |\omega| > (1+\varepsilon)/2; \\
\widehat{\varphi}_\varepsilon(\omega) = 1, & |\omega| \le (1-\varepsilon)/2; \\
\widehat{\varphi}_\varepsilon^2(\omega) + \widehat{\varphi}_\varepsilon^2(\omega - 1) = 1, & (1-\varepsilon)/2 < \omega < (1+\varepsilon)/2.
\end{cases}
$$

We also require that the function $\widehat{\varphi}_\varepsilon^2(\omega)$ is even and smooth on $\mathbb{R}$ with the symmetry center of its graph on the interval $((1-\varepsilon)/2, (1+\varepsilon)/2)$ at the point $\omega = 1/2$. Define functions $\varphi_s(x)$ $(s = 1, 2)$ as in [1] and function $\varphi_3(x)$ as in [2] such that:

$$\widehat{\varphi}_1(\omega) = \frac{1}{2}\begin{cases} (1 + \widehat{\varphi}_\varepsilon(\omega) - \widehat{\varphi}_\varepsilon(\omega - 1) - \widehat{\varphi}_\varepsilon(\omega + 1)) + i(\text{sign}\,\omega)\sqrt{2\beta(\omega)}, \\ |\omega| < (1+\varepsilon)/2, \\ 0, \\ |\omega| \geq (1+\varepsilon)/2, \end{cases}$$

$$\widehat{\varphi}_2(\omega) = \widehat{\varphi}_\varepsilon^2(\omega) + i(\text{sign}\,\omega)\beta(\omega), \quad \beta(\omega) = \widehat{\varphi}_\varepsilon(\omega)\big(\widehat{\varphi}_\varepsilon(\omega - 1) + \widehat{\varphi}_\varepsilon(\omega + 1)\big),$$

$$\widehat{\varphi}_3(\omega) = \widehat{\varphi}_\varepsilon^2(\omega).$$

Here $\beta(\omega)$ is a smooth even function on $\mathbb{R}$ vanishing together with its derivative at the points $\omega = (-1\pm\varepsilon)/2$ and $\omega = (1\pm\varepsilon)/2$, with the support $\{((-1-\varepsilon)/2, (-1+\varepsilon)/2) \cup ((1-\varepsilon)/2, (1+\varepsilon)/2)\}$ and even on intervals $\pm((1-\varepsilon)/2, (1+\varepsilon)/2)$ with respect to their centers $\omega = \pm 1/2$. Functions $\varphi_s(x)(s = 1, 2, 3)$ generates interpolating in $C(\mathbb{R})$ systems $\{\varphi_s(2^j x - k) : k \in \mathbb{Z}\}$ $(j \in \mathbb{Z})$ on the grids $\{l/2^j : l \in \mathbb{Z}\}$ $(j \in \mathbb{Z})$. For $s = 1, 2$ these systems are also orthogonal in $L^2(\mathbb{R})$. Unless otherwise stipulated, throughout the paper $s = 1, 2, 3$.

The 1-periodization process of the function $\varphi_s(2^j x)$

$$\text{Pe}_1 \varphi_s(2^j x) = \sum_{\mu \in \mathbb{Z}} \varphi_s(2^j(x + \mu)) =: \Phi_s^{j,0}(2\pi x), \quad j \in \mathbb{Z} \tag{1.1}$$

converges uniformly on the interval $[-1/2, 1/2]$ (see [1]). Calculating the coefficients a_ν in the expansion of the function $\Phi_s^{j,0}(2\pi x)$ by the trigonometric system $\{e^{2\pi i\nu x} : \nu \in \mathbb{Z}\}$, we get

$$\Phi_s^{j,0}(2\pi x) = \sum_{\nu \in \mathbb{Z}} a_\nu e^{2\pi i\nu x}, \quad j \in \mathbb{Z}. \tag{1.2}$$

Using (1.1), we find all coefficients a_ν $(\nu \in \mathbb{Z})$

$$a_\nu = \int_0^1 \sum_{\mu \in \mathbb{Z}} \varphi_s(2^j(x + \mu))e^{-2\pi i\nu x}dx = \sum_{\mu \in \mathbb{Z}} \int_0^1 \varphi_s(2^j(x + \mu))e^{-2\pi i\nu x}dx =$$

$$= [\text{substitution: } x + \mu = t] \sum_{\mu \in \mathbb{Z}} \int_\mu^{\mu+1} \varphi_s(2^j t)e^{-2\pi i\nu(t-\mu)}dt =$$

$$= \int_\mathbb{R} \varphi_s(2^j t)e^{-2\pi i\nu t}dt = \widehat{\varphi_s(2^j t)} = 2^{-j}\widehat{\varphi}_s\Big(\frac{\nu}{2^j}\Big).$$

Substituting the coefficients a_ν $(\nu \in \mathbb{Z})$ in (1.2), we obtain:

$$\Phi_s^{j,0}(2\pi x) = 2^{-j} \sum_{\nu \in \Delta_\varepsilon^j \cap \mathbb{Z}} \widehat{\varphi}_s\Big(\frac{\nu}{2^j}\Big)e^{2\pi i\nu x}, \quad j \in \mathbb{Z},$$

where $\Delta_\varepsilon^j = 2^j\big((-1-\varepsilon)/2, (1+\varepsilon)/2\big)$. Replacing the variable x by $x/(2\pi)$, we obtain 2π-periodic wavelet systems

$$\Big\{\Phi_s^{j,k}(x) := \Phi_s^{j,0}\Big(x - \frac{2\pi k}{2^j}\Big) = 2^{-j} \sum_{\nu \in \Delta_\varepsilon^j \cap \mathbb{Z}} \widehat{\varphi}_s\Big(\frac{\nu}{2^j}\Big)e^{i\nu(x - 2\pi k/2^j)} : k \in \mathbb{Z}\Big\}, \quad j \in \mathbb{Z}, \tag{1.3}$$

which are interpolating on the grids $\{x_j^l := 2\pi l/2^j : l = \overline{0, 2^j - 1}\}$ for $s = 1, 2, 3$ and orthogonal in $L^2(\mathbb{R})$ for $s = 1, 2$.

It is easy to see, that for $n \in \mathbb{Z}$

$$\Phi_s^{j,k+2^j n}(x) = 2^{-j} \sum_{\nu \in \Delta_\varepsilon^j \cap \mathbb{Z}} \widehat{\varphi}_s\left(\frac{\nu}{2^j}\right) e^{i\nu(x - 2\pi k/2^j - 2\pi n)} = \Phi_s^{j,k}(x).$$

So the sequence of spaces (1.3) has only 2^j distinct linearly independent terms. Hence, we can assume in the following discussion that $k = \overline{0, 2^j - 1}$.

Define system of spaces $\{V_s^j := \text{span}\{\Phi_s^{j,k}(x) : k = \overline{0, 2^j - 1}\} : j \in \mathbb{Z}\}$. As follows from $\Delta_\varepsilon^j \cap \mathbb{Z} = \{0\}$ for $j \leq 0$ and $\widehat{\varphi}_s(0) = 1$, we see that

$$\Phi_s^{0,0}(x) = \sum_{\nu \in \Delta_\varepsilon^0 \cap \mathbb{Z}} \widehat{\varphi}_s(\nu) e^{i\nu x} = 1$$

and

$$\Phi_s^{j,0}(x) = 2^{-j} \sum_{\nu \in \Delta_\varepsilon^j \cap \mathbb{Z}} \widehat{\varphi}_s\left(\frac{\nu}{2^j}\right) e^{i\nu x} = 2^{-j}, \quad j < 0,$$

i.e., for all integers such that $j \leq 0$ and for all $k \in \mathbb{Z}$ relation $\Phi_s^{j,k}(x) = \Phi_s^{j,0}(x) = \text{const}$ holds and thus we can consider the system of spaces $\{V_s^j\}$ only for $j \in \mathbb{N} \cup \{0\}$. Further, for $j \in \mathbb{N} \cup \{0\}$ define spaces W_s^j as direct complement of V_s^j to V_{s+1}^j with the interpolation system $\{\Psi_s^{j,k}(x) : k = \overline{0, 2^j - 1}\}$ on the grid $\{x_{j+1}^{2l+1} : l = \overline{0, 2^j - 1}\}$, which is interpolating basis of 2π-periodic continuous functions.

Show that the $\Psi_s^{j,k}(x) = \Phi_s^{j+1,2k+1}(x)$ holds for all $j \in \mathbb{N} \cup \{0\}$ and for all $k = \overline{0, 2^j - 1}$. Since $V_s^j \subset V_s^{j+1}$ $(j \in \mathbb{N} \cup \{0\})$, we see that

$$\Phi_s^{j,k}(x) = \sum_{n=0}^{2^{j+1}-1} b_n \Phi_s^{j+1,n}(x), \quad j \in \mathbb{N} \cup \{0\}, \quad k = \overline{0, 2^j - 1}. \tag{1.4}$$

Using interpolating condition of basis $\{\Phi_s^{j+1,k}(x) : k = \overline{0, 2^{j+1} - 1}\}$ on the grid $\{x_{j+1}^l : l = \overline{0, 2^{j+1} - 1}\}$ and assuming $x := 2\pi l/2^{j+1}$ in (1.4), we find the coefficients b_n $(n = \overline{0, 2^{j+1} - 1})$:

$$\Phi_s^{j,k}\left(\frac{2\pi l}{2^{j+1}}\right) = \sum_{n=0}^{2^{j+1}-1} b_n \Phi_s^{j+1,n}\left(\frac{2\pi l}{2^{j+1}}\right) = \sum_{n=0}^{2^{j+1}-1} b_n \delta_{n,l}, \quad l = \overline{0, 2^{j+1} - 1},$$

so

$$b_n = \Phi_s^{j,k}\left(\frac{2\pi n}{2^{j+1}}\right), \quad n = \overline{0, 2^{j+1} - 1}.$$

In view of b_n obtained, the sum on the right side of the expression (1.4) may be written as two sums over even and odd indices

$$\Phi_s^{j,k}(x) = \sum_{n=0}^{2^{j+1}-1} \Phi_s^{j,k}\left(\frac{2\pi n}{2^{j+1}}\right) \Phi_s^{j+1,n}(x) = \sum_{n=0}^{2^j-1} \Phi_s^{j,k}\left(\frac{2\pi n}{2^j}\right) \Phi_s^{j+1,2n}(x) +$$

$$+ \sum_{n=0}^{2^j-1} \Phi_s^{j,k}\left(\frac{2\pi(2n+1)}{2^{j+1}}\right) \Phi_s^{j+1,2n+1}(x) = \underbrace{\Phi_s^{j,k}\left(\frac{2\pi k}{2^j}\right)}_{=1} \Phi_s^{j+1,2k}(x) +$$

$$+ \sum_{n=0}^{2^j-1} \Phi_s^{j,k}\left(\frac{2\pi(2n+1)}{2^{j+1}}\right) \Phi_s^{j+1,2n+1}(x).$$

As a result, we have

$$\Phi_s^{j,k}(x) = \Phi_s^{j+1,2k}(x) + \sum_{n=0}^{2^j-1} \Phi_s^{j,k}\left(\frac{2\pi(2n+1)}{2^{j+1}}\right)\Phi_s^{j+1,2n+1}(x), \quad j \in \mathbb{N} \cup \{0\}, \quad k = \overline{0, 2^j - 1},$$

i.e.,

$$\underbrace{\Phi_s^{j+1,2k}(x)}_{\in V_s^{j+1}} = \underbrace{\Phi_s^{j,k}(x)}_{\in V_s^j} - \sum_{n=0}^{2^j-1} \Phi_s^{j,k}\left(\frac{2\pi(2n+1)}{2^{j+1}}\right)\Phi_s^{j+1,2n+1}(x), \quad j \in \mathbb{N} \cup \{0\}, \quad k = \overline{0, 2^j - 1},$$

and it implies that

$$V_s^{j+1} = \left\{ \sum_{k=0}^{2^{j+1}-1} c_{j+1,k}\Phi_s^{j+1,k}(x) : c_{j+1,k} \in \mathbb{R} \right\} =$$

$$= \left\{ \sum_{k=0}^{2^j-1} c_{j+1,2k}\Phi_s^{j+1,2k}(x) + \sum_{k=0}^{2^j-1} c_{j+1,2k+1}\Phi_s^{j+1,2k+1}(x) \right\} = \left\{ \sum_{k=0}^{2^j-1} c_{j+1,2k}\Phi_s^{j,k}(x) - \right.$$

$$\left. - \sum_{k=0}^{2^j-1} c_{j+1,2k} \sum_{n=0}^{2^j-1} \Phi_s^{j,k}\left(\frac{2\pi(2n+1)}{2^{j+1}}\right)\Phi_s^{j+1,2n+1}(x) + \sum_{k=0}^{2^j-1} c_{j+1,2k+1}\Phi_s^{j+1,2k+1}(x) \right\} =$$

$$= \left\{ \sum_{k=0}^{2^j-1} c_{j+1,2k}\Phi_s^{j,k}(x) - \sum_{n=0}^{2^j-1} \Phi_s^{j+1,2n+1}(x) \sum_{k=0}^{2^j-1} c_{j+1,2k}\Phi_s^{j,k}\left(\frac{2\pi(2n+1)}{2^{j+1}}\right) + \right.$$

$$\left. + \sum_{n=0}^{2^j-1} c_{j+1,2n+1}\Phi_s^{j+1,2n+1}(x) \right\} = \left\{ \sum_{k=0}^{2^j-1} c_{j+1,2k}\Phi_s^{j,k}(x) + \sum_{n=0}^{2^j-1} d_{j,n}\Phi_s^{j+1,2n+1}(x) : \right.$$

$$\left. d_{j,n} = - \sum_{k=0}^{2^j-1} c_{j+1,2k}\Phi_s^{j,k}\left(\frac{2\pi(2n+1)}{2^{j+1}}\right) + c_{j+1,2n+1} \right\} = V_s^j \oplus W_s^j.$$

In view of definitions of spaces V_s^j and W_s^j, for all $j \in \mathbb{N} \cup \{0\}$ and for all $k = \overline{0, 2^j - 1}$ relation

$$\Psi_s^{j,k}(x) = \Phi_s^{j+1,2k+1}(x)$$

holds.

Denote the interpolation projection of a function $f \in C_{2\pi}$ (the space of continuous 2π-periodic functions) onto the V_s^j by

$$S_{s,2^j}(x; f) = \sum_{k=0}^{2^j-1} f\left(\frac{2\pi k}{2^j}\right)\Phi_s^{j,k}(x), \quad j \in \mathbb{N} \cup \{0\}. \tag{1.5}$$

Since $\overline{\bigcup_{j=0}^{\infty} V_s^j} = C_{2\pi}$, for $f \in C_{2\pi}$ we have

$$S_{s,2^j}(x; f) \underset{\mathbb{R}}{\rightrightarrows} f(x), \tag{1.6}$$

$$f(x) = f(0) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1} c_{j,k}\Psi_s^{j,k}(x) = f(0) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1} c_{j,k}\Phi_s^{j+1,2k+1}(x). \tag{1.7}$$

Find all coefficients $c_{j,k}$, $(j \in \mathbb{N} \cup \{0\}$, $k = \overline{0, 2^j - 1})$ from (1.7). Because of $S_{s,2^j}(x; f) \in V_s^j$, $S_{s,2^{j+1}}(x; f) \in V_s^{j+1}$ and definition of spaces W_s^j we have $S_{s,2^{j+1}}(x; f) - S_{s,2^j}(x; f) \in W_s^j$, i.e.,

$$\left(S_{s,2^{j+1}}(x; f) - S_{s,2^j}(x; f)\right)\Big|_{x=x_{j+1}^{2l+1}} = \sum_{k=0}^{2^j-1} c_{j,k}\Phi_s^{j+1,2k+1}(x_{j+1}^{2l+1}) = \sum_{k=0}^{2^j-1} c_{j,k}\delta_{k,l} = c_{j,l},$$

where $j \in \mathbb{N} \cup \{0\}$ and $l = \overline{0, 2^j - 1}$. Using definition (1.5), we rewrite $S_{s,2^{j+1}}(x; f)$ and take $x := x_{j+1}^{2l+1}$

$$S_{s,2^{j+1}}(x; f)\Big|_{x=x_{j+1}^{2l+1}} = \sum_{k=0}^{2^{j+1}-1} f\left(\frac{2\pi k}{2^{j+1}}\right)\Phi_s^{j+1,k}(x_{j+1}^{2l+1}) = \sum_{k=0}^{2^{j+1}-1} f\left(\frac{2\pi k}{2^{j+1}}\right)\delta_{k,2l+1} = f(x_{j+1}^{2l+1}).$$

Consequently,

$$c_{j,l} = S_{s,2^{j+1}}(x_{j+1}^{2l+1}; f) - S_{s,2^j}(x_{j+1}^{2l+1}; f) = f(x_{j+1}^{2l+1}) - S_{s,2^j}(x_{j+1}^{2l+1}; f), \quad j \in \mathbb{N} \cup \{0\}, \quad l = \overline{0, 2^j - 1}.$$

With (1.3), (1.7) and preceding expression the following relation holds for a function $f \in C_{2\pi}$

$$f(x) = f(0) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1} c_{j,k}\Psi_s^{j,k}(x) = f(0) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}\left(f(x_{j+1}^{2k+1}) - S_{s,2^j}(x_{j+1}^{2k+1}; f)\right) \times$$

$$\times \; \Phi_s^{j+1,2k+1}(x) = f(0) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}\left(f(x_{j+1}^{2k+1}) - S_{s,2^j}(x_{j+1}^{2k+1}; f)\right)2^{-(j+1)} \times \qquad (1.8)$$

$$\times \sum_{\nu \in \Delta_\varepsilon^{j+1} \cap \mathbb{Z}} \widehat{\varphi}_s\left(\frac{\nu}{2^{j+1}}\right)e^{i\nu(x - 2\pi(2k+1)/2^{j+1})}.$$

The definition of W_s^j imply $V_s^j = V_s^0 \oplus (\oplus_{l=0}^{j-1} W_s^l)$. Then $S_{s,2^j}(x; f)$ is the partial sum of order 2^j for (1.7) and from (1.6) series (1.7) converges uniformly. Thus for $J \in \mathbb{Z}$

$$S_{s,2^J}(x; f) = f(0) + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\left(f(x_{j+1}^{2k+1}) - S_{s,2^j}(x_{j+1}^{2k+1}; f)\right)\Phi_s^{j+1,2k+1}(x) \qquad (1.9)$$

and as $J \to \infty$

$$S_{s,2^J}(x; f) \underset{\mathbb{R}}{\rightrightarrows} f(x).$$

2. Application to the solution of the Neumann BVP in a circle

Setting of the Neumann BVP in the unit circle K_1 (see, for example, [6]):

$$\begin{cases} \Delta U(r, x) = \dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 U}{\partial x^2} = 0, \quad U \in C^{(1)}(\overline{K_1}) \cap C^{(2)}(K_1), \\[2mm] \dfrac{\partial U}{\partial r}(1, x) = g_1(x) \in C_{2\pi}, \end{cases} \qquad (2.1)$$

where re^{ix} $(0 \le r < 1$, $0 \le x < 2\pi)$ are points of the unit circle K_1 centered at the origin of the polar coordinate system. It has been well known that necessary condition of solvability of the Neumann problem is

$$\int_0^{2\pi} g_1(x)dx = 0, \qquad (2.2)$$

and the problem have a unique solution up to an additive constant.

Define harmonic in the unit circle polynomials $\Phi_s^{j,k}(r,x)$:

$$\Phi_s^{j,k}(r,x) := 2^{-j} \sum_{\nu \in \Delta_\varepsilon^j \cap \mathbb{Z}} \widehat{\varphi}_s\left(\frac{\nu}{2^j}\right) r^{|\nu|} e^{i\nu(x-2\pi k/2^j)}, \quad j \in \mathbb{N} \cup \{0\}, \quad k = \overline{0, 2^j - 1}$$

and consider series

$$U(1,0) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \Big(U(1,\cdot) - S_{s,2^j}(1,\cdot;U(1,\cdot))\Big)(x_{j+1}^{2k+1}) \Phi_s^{j+1,2k+1}(1,x).$$

Since $U(r,x)$ is a harmonic in the unit circle function with continuous boundary value $U(1,x)$, it follows that the above series converges uniformly on the boundary of K_1 by taking into account (1.8) and (1.9) (where for $f(x)$ we take $U(1,x)$). Because of maximum principle for harmonic functions, we obtain the following representation for $U(r,x)$ in form of uniformly convergent in $\overline{K}_1$ series

$$U(r,x) = U(1,0) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \Big(U(1,\cdot) - S_{s,2^j}(1,\cdot;U(1,\cdot))\Big)(x_{j+1}^{2k+1}) \Phi_s^{j+1,2k+1}(r,x) =$$

$$= U(1,0) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \Big(U(1,\cdot) - S_{s,2^j}(1,\cdot;U(1,\cdot))\Big)(x_{j+1}^{2k+1}) 2^{-(j+1)} \times \tag{2.3}$$

$$\times \sum_{\nu \in \Delta_\varepsilon^{j+1} \cap \mathbb{Z}} \widehat{\varphi}_s\left(\frac{\nu}{2^{j+1}}\right) r^{|\nu|} e^{i\nu(x-2\pi(2k+1)/2^{j+1})}.$$

Using (1.8), we have the following representation for function $g_1(x) \in C_{2\pi}$ in form of uniformly convergent in $\overline{K}_1$ series

$$g_1(x) = g_1(0) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \Big(g_1(\cdot) - S_{s,2^j}(\cdot;g_1)\Big)(x_{j+1}^{2k+1}) 2^{-(j+1)} \sum_{\nu \in \Delta_\varepsilon^{j+1} \cap \mathbb{Z}} \widehat{\varphi}_s\left(\frac{\nu}{2^{j+1}}\right) e^{i\nu(x-2\pi(2k+1)/2^{j+1})}.$$

We may extend terms of the series into the interior of the unit circle to harmonic polynomials $c_{j,k}(g_1)\Phi_s^{j,k}(r,x)$ and, consequently, we may extend the series into the interior of the unit circle to harmonic in K_1 and in continuous $\overline{K}_1$ function.

$$g_1(r,x) := g_1(0) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \Big(g_1(\cdot) - S_{s,2^j}(\cdot;g_1)\Big)(x_{j+1}^{2k+1}) 2^{-(j+1)} \times$$

$$\times \sum_{\nu \in \Delta_\varepsilon^{j+1} \cap \mathbb{Z}} \widehat{\varphi}_s\left(\frac{\nu}{2^{j+1}}\right) r^{|\nu|} e^{i\nu(x-2\pi(2k+1)/2^{j+1})}. \tag{2.4}$$

Because of series in (2.3) converges uniformly, we can perform a term-by-term differentiation with respect to r and multiplication by r and as result we get

$$r\frac{\partial U}{\partial r}(r,x) = \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \Big(U(1,\cdot) - S_{s,2^j}(1,\cdot;U(1,\cdot))\Big)(x_{j+1}^{2k+1}) 2^{-(j+1)} \times$$

$$\times \sum_{\nu \in \Delta_\varepsilon^{j+1} \cap \mathbb{Z}} \widehat{\varphi}_s\left(\frac{\nu}{2^{j+1}}\right) |\nu| r^{|\nu|} e^{i\nu(x-2\pi(2k+1)/2^{j+1})}.$$

As is easy to see that this function is harmonic in K_1. In view of setting of the Neumann BVP, we have $\frac{\partial U}{\partial r}(r,x)\big|_{r=1} = g_1(x)$, this implies that for $0 \leq r < 1$ the equality $r\frac{\partial U}{\partial r}(r,x) = g_1(r,x)$ holds as equality of two harmonic functions which are equal at the boundary of K_1. Hence

$$r\frac{\partial U}{\partial r}(r,x) = g_1(0) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}\Big(g_1(\cdot) - S_{s,2^j}(\cdot;g_1)\Big)(x_{j+1}^{2k+1})2^{-(j+1)}\times$$

$$\times \sum_{\nu\in\Delta_\varepsilon^{j+1}\cap\mathbb{Z}} \widehat{\varphi}_s\Big(\frac{\nu}{2^{j+1}}\Big)r^{|\nu|}e^{i\nu(x-2\pi(2k+1)/2^{j+1})}.$$

In consequence of (2.2), we also have

$$\int_0^{2\pi} g_1(r,x)dx = 0. \tag{2.5}$$

Indeed if we expand function $g_1(r,x) = r\frac{\partial U}{\partial r}(r,x)$ in a series by system $\{r^{|n|}e^{inx} : n \in \mathbb{Z}\}$ (for instance, with the use of Poisson kernel), then we get for $0 \leq r < 1$

$$g_1(r,x) = \frac{1}{2\pi}\int_0^{2\pi} g_1(1,t)P_r(x-t)dt = \frac{1}{2\pi}\int_0^{2\pi}\sum_{n\in\mathbb{Z}} g_1(t)r^{|n|}e^{in(x-t)}dt.$$

Interchanging of integration and summation and using (2.2), we arrive at

$$\sum_{n\in\mathbb{Z}\backslash\{0\}}\Big(\frac{1}{2\pi}\int_0^{2\pi} g_1(1,t)e^{-int}dt\Big)r^{|n|}e^{inx},$$

resulting in (2.5).

Thus, using (2.5) and taking into account $\widehat{\varphi}_s(0) = 1$, we obtain

$$g_1(0) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}\Big(g_1(\cdot) - S_{s,2^j}(\cdot;g_1)\Big)(x_{j+1}^{2k+1})2^{-(j+1)} = 0,$$

and numerical series on the left side of the equality converges. Consequently, the following equality holds

$$g_1(r,x) = \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}\Big(g_1(\cdot) - S_{s,2^j}(\cdot;g_1)\Big)(x_{j+1}^{2k+1})2^{-(j+1)}\sum_{\nu\in\Delta_\varepsilon^{j+1}\cap\mathbb{Z}\backslash\{0\}}\widehat{\varphi}_s\Big(\frac{\nu}{2^{j+1}}\Big)r^{|\nu|}e^{i\nu(x-2\pi(2k+1)/2^{j+1})}.$$

Therefore, by setting

$$\Phi_s^{j+1,2k+1,0}(r,x) := \Phi_s^{j+1,2k+1}(r,x) - \frac{1}{2\pi}\int_0^{2\pi}\Phi_s^{j+1,2k+1}(1,x)dx =$$

$$= 2^{-(j+1)}\sum_{\nu\in\Delta_\varepsilon^{j+1}\cap\mathbb{Z}\backslash\{0\}}\widehat{\varphi}_s\Big(\frac{\nu}{2^{j+1}}\Big)r^{|\nu|}e^{i\nu(x-2\pi(2k+1)/2^{j+1})},$$

we obtain

$$\frac{\partial U}{\partial r}(r,x) = \frac{g_1(r,x)}{r} = \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}\Big(g_1(\cdot) - S_{s,2^j}(\cdot;g_1)\Big)(x_{j+1}^{2k+1})\frac{\Phi_s^{j+1,2k+1,0}(r,x)}{r} =$$

$$= \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}\Big(g_1(\cdot) - S_{s,2^j}(\cdot;g_1)\Big)(x_{j+1}^{2k+1})2^{-(j+1)}\sum_{\nu\in\Delta_\varepsilon^{j+1}\cap\mathbb{Z}\setminus\{0\}}\widehat{\varphi}_s\Big(\frac{\nu}{2^{j+1}}\Big)r^{|\nu|-1}e^{i\nu(x-2\pi(2k+1)/2^{j+1})},$$

where the series converges uniformly in $\overline{K}_1$. Setting

$$\Psi_s^{j,k,-1}(r,x) := \int\limits_0^r \frac{\Phi_s^{j+1,2k+1,0}(r',x)}{r'}dr' =$$

$$= 2^{-(j+1)}\sum_{\nu\in\Delta_\varepsilon^{j+1}\cap\mathbb{Z}\setminus\{0\}}\widehat{\varphi}_s\Big(\frac{\nu}{2^{j+1}}\Big)\frac{r^{|\nu|}}{|\nu|}e^{i\nu(x-2\pi(2k+1)/2^{j+1})},\quad j\in\mathbb{N}\cup\{0\},\quad k=\overline{0,2^j-1},$$

and calculating the $U(r,x)$ from the preceding equality, we formulate the following theorem.

Theorem 1. *Under conditions of setting of the Neumann BVP* (2.1) *we obtain for* $s=1,2,3$

$$U(r,x) = U(1,0) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}\Big(g_1(\cdot) - S_{s,2^j}(\cdot;g_1)\Big)(x_{j+1}^{2k+1})\Psi_s^{j,k,-1}(r,x),\quad re^{ix}\in\overline{K}_1. \qquad (2.6)$$

Series in (2.6) *converges uniformly in* $\overline{K}_1$ *and* $U(1,0)$ *is a constant.*

P r o o f follows from preceding equations.   □

Also we obtain the error for approximation of solution $U(r,x)$ of the problem (2.1) by partial sums of series (2.6) denoted by $S_{s,2^J}(r,x;U,\Psi^{-1})$. Denote by $E_{N_{\varepsilon,J}^-}(f)_{C_{2\pi}}$ the best approximation of a function f in $C_{2\pi}$ by trigonometric polynomials of order $N_{\varepsilon,J}^- = \lfloor 2^{J-1}(1-\varepsilon)\rfloor$.

Theorem 2. *Under conditions of setting of the Neumann BVP* (2.1) *for* $s = 1,2,3$ *and* $J\in Z_+ := \{j\in\mathbb{Z} : j\geq 0\}$ *the function* $S_{s,2^J}(r,x;U,\Psi)$ *approximates the solution* $U(r,x)$ *of problem* (2.1) *with accuracy guaranteed by the inequality*

$$\|U(r,x) - S_{s,2^J}(r,x;U,\Psi^{-1})\|_{C(K_1)} \leq \frac{\pi}{\sqrt{3}}\big(1 + \|S_{s,2^J}\|\big)E_{N_{\varepsilon,J}^-}(g_1)_{C_{2\pi}},$$

Estimates for norm of the operator $S_{s,2^J}$ *(interpolation projection onto the subspace* V_j^s*) can be found in Theorem from* [2].

P r o o f. For convenience introduce the following notation:

$$c_{j,k}(g_1) = \big(g_1(\cdot) - S_{s,2^j}(\cdot;g_1)\big)(x_{j+1}^{2k+1}),\quad j\in\mathbb{N}\cup\{0\},\quad k=\overline{0,2^j-1}.$$

Using Euler's formula, we can represent (2.6) in the form

$$U(r,x) = U(0,0) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}c_{j,k}(g_1)2^{-j}\sum_{\nu\in\Delta_\varepsilon^{j+1}\cap\mathbb{N}}\widehat{\varphi}_s\Big(\frac{\nu}{2^{j+1}}\Big)\frac{r^\nu}{\nu}\cos\Big(\nu\big(x-\frac{2\pi(2k+1)}{2^{j+1}}\big)\Big)$$

and partial sum $S_{s,2^J}(r,x;U,\Psi^{-1})$ in the form

$$S_{s,2^J}(r,x;U,\Psi^{-1}) = U(0,0) + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1} c_{j,k}(g_1)2^{-j} \times$$

$$\times \sum_{\nu\in\Delta_\varepsilon^{j+1}\cap\mathbb{N}} \widehat{\varphi}_s\Big(\frac{\nu}{2^{j+1}}\Big)\frac{r^\nu}{\nu}\cos\Big(\nu\big(x-\frac{2\pi(2k+1)}{2^{j+1}}\big)\Big).$$

Note that the following representations hold

$$U(r,x) = \frac{1}{\pi}\int_0^{2\pi}\sum_{\mu=1}^{+\infty}\frac{\cos(\mu(x-\xi))}{\mu}\Big(U(1,0) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1}c_{j,k}(g_1)\Phi_s^{j+1,2k+1}(r,\xi)\Big)d\xi,$$

$$S_{s,2^J}(r,x;U,\Psi^{-1}) = \frac{1}{\pi}\int_0^{2\pi}\sum_{\mu=1}^{+\infty}\frac{\cos(\mu(x-\xi))}{\mu}\Big(U(1,0) + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}c_{j,k}(g_1)\Phi_s^{j+1,2k+1}(r,\xi)\Big)d\xi. \qquad (2.7)$$

It follows from

$$\frac{1}{\pi}\int_0^{2\pi}\sum_{\mu=1}^{+\infty}\frac{\cos(\mu(x-\xi))}{\mu}\Phi_s^{j+1,2k+1}(r,\xi)d\xi =$$

$$= \frac{1}{\pi}\int_0^{2\pi}\sum_{\mu=1}^{N_{\varepsilon,j}^+}\frac{\cos(\mu(x-\xi))}{\mu}2^{-j}\sum_{\nu\in\Delta_\varepsilon^{j+1}\cap\mathbb{Z}^+}\widehat{\varphi}_s\Big(\frac{\nu}{2^{j+1}}\Big)r^\nu\frac{\cos\Big(\nu\big(\xi-\frac{2\pi(2k+1)}{2^{j+1}}\big)\Big)}{\nu}d\xi =$$

$$= 2^{-j}\sum_{\nu\in\Delta_\varepsilon^{j+1}\cap\mathbb{N}}\widehat{\varphi}_s\Big(\frac{\nu}{2^{j+1}}\Big)\frac{r^\nu}{\nu}\cos\Big(\nu\big(x-\frac{2\pi(2k+1)}{2^{j+1}}\big)\Big) = \Psi_s^{j,k,-1}(r,x),$$

where $N_{\varepsilon,j}^+ = \lceil 2^j(1+\varepsilon)\rceil$ and the second equality holds in view of

$$\frac{1}{\pi}\int_0^{2\pi}\cos(\mu(x-\xi))\cos\Big(\nu\big(\xi-\frac{2\pi(2k+1)}{2^{j+1}}\big)\Big)d\xi =$$

$$= \delta_{\mu,\nu}\cos\Big(\nu\big(x-\frac{2\pi(2k+1)}{2^{j+1}}\big)\Big), \quad \nu,\mu\in\Delta_\varepsilon^{j+1}\cap\mathbb{Z}^+.$$

Let $S_{s,2^J}(r,x;g_1)$ be a partial sum of series in (2.4), then

$$|U(r,x) - S_{s,2^J}(r,x;U,\Psi^{-1})| = \Big|\frac{1}{\pi}\int_0^{2\pi}\sum_{\mu=1}^{+\infty}\frac{\cos(\mu(x-\xi))}{\mu}\Big(g_1(r,\xi)) - S_{s,2^J}(r,\xi;g_1)\Big)d\xi\Big| \le$$

$$\le \frac{1}{\pi}\int_0^{2\pi}\Big|\sum_{\mu=1}^{+\infty}\frac{\cos(\mu(x-\xi))}{\mu}\Big|\cdot\Big|g_1(r,x)) - S_{s,2^J}(r,x;g_1)\Big|d\xi \le$$

$$\le \frac{1}{\pi}\Big(\int_0^{2\pi}1d\xi\Big)^{1/2}\Big(\int_0^{2\pi}\Big|\sum_{\mu=1}^{+\infty}\frac{\cos(\mu(x-\xi))}{\mu}\Big|^2 d\xi\Big)^{1/2}\|g_1(x)) - S_{s,2^J}(r,x;g_1)\|_{C(2\pi)} =$$

$$= \frac{1}{\pi}\sqrt{2\pi}\Big(\pi\sum_{\mu=1}^{+\infty}\frac{1}{\mu^2}\Big)^{1/2}\|g_1(x)) - S_{s,2^J}(r,x;g_1)\|_{C(2\pi)} \le \frac{\pi}{\sqrt{3}}\big(1 + \|S_{s,2^J}\|\big)E_{N_{\varepsilon,J}}(g_1)_{C_{2\pi}},$$

where the first equality follows from (2.7), the second equality follows from Parseval's identity, the second inequality follows from Hölder's inequality and the last inequality follows from Theorem in [2]. As the final result we have

$$\|U(r,x) - S_{s,2^J}(r,x;U,\Psi^{-1})\|_{C(K_1)} \leq \frac{\pi}{\sqrt{3}}\big(1 + \|S_{s,2^J}\|\big)E_{N^-_{\varepsilon,J}}(g_1)_{C_{2\pi}}.$$

$\square$

3. Conclusion

Theorem 1 gives the solution (2.6) (up to an additive constant) of the problem (2.1) in form of uniformly convergent in $\overline{K}_1$ series of harmonic interpolating 2π-periodic wavelets. In this case, coefficients of series in (2.6) have an easy-to-calculate form in preference to calculating coefficients (integrals) in case of implementing orthogonal 2π-periodic wavelets. This useful fact simplify the numerical implementation of the suggested method.

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ASYMPTOTIC SOLUTIONS OF A PARABOLIC EQUATION NEAR SINGULAR POINTS OF A AND B TYPES

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Abstract: The Cauchy problem for a quasi-linear parabolic equation with a small parameter multiplying a higher derivative is considered in two cases when the solution of the limit problem has a point of gradient catastrophe. Asymptotic solutions are found by using the Cole–Hopf transform. The integrals determining the asymptotic solutions correspond to the Lagrange singularities of type A and the boundary singularities of type B. The behavior of the asymptotic solutions is described in terms of the weighted Sobolev spaces.

Keywords: Quasi-linear parabolic equation, Cole–Hopf transform, Singular points, Asymptotic solutions, Whitney fold singularity, Il'in's universal solution, Weighted Sobolev spaces.

Introduction

Consider the Cauchy problem for a quasi-linear parabolic equation

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \tag{0.1}$$

$$u(x, t_0) = q(x), \quad x \in \mathbb{R}, \tag{0.2}$$

with a small parameter $\varepsilon > 0$ in such cases that the solution of the limit problem (for $\varepsilon = 0$) has a point of gradient catastrophe. The interest of studying the behavior of solutions near singular points is explained, in particular, by the fact that such singular events themselves occupy very small time, however, in many respects determining all subsequent behavior of the system. The asymptotic behavior of solutions in neighborhoods of singular points is directly connected with constructing an approximation in neighborhoods of shock waves in physical media with a small nonzero viscosity.

Although the types of singular points of solutions are classified in detail [1, ch. 2] and processes of the shock waves formation are studied [2], constructing asymptotic series in the small viscosity parameter ε for an equation of the general form (0.1) is a separate problem in every specific case. First substantial results for several types of singularities, including the Whitney fold singularity A_3, were obtained by A. M. Il'in [3, ch. VI], who constructed a complete asymptotics and derived a universal solution near the fold singularity; see formulas (1.1)–(1.2) in the next section.

In the present paper, the statements about asymptotic solutions of the preliminary notice [4] for Lagrange singularities of types A_{2n+1} and boundary singularities of types B_{2n+1} with any $n \geqslant 2$ are proved.

1. Fold singularity

In Il'in's pioneer paper [5], problem (0.1)–(0.2) was first studied in the case when in the strip $\{t_0 \leqslant t \leqslant T, \ x \in \mathbb{R}\}$ the solution of the limit problem is a function which is smooth everywhere

except for one smooth line of jump discontinuity $x = s(t)$, $t \geqslant t^*$. It is supposed that $\varphi \in C^\infty(\mathbb{R})$, $\varphi''(u) > 0$, $\varphi(0) = \varphi'(0) = 0$, and $t_0 = -1$. For an appropriate choice of the bounded initial function $q \in C^\infty(\mathbb{R})$, the singular point $(s(t^*), t^*)$ coincides with the origin and in its neighborhood the stretched variables $\xi = \varepsilon^{-3/4} x$ and $\tau = \varepsilon^{-1/2} t$ are introduced. An asymptotic expansion of the solution was obtained in the form of the series

$$u(x, t, \varepsilon) = \sum_{k=1}^{\infty} \varepsilon^{k/4} \sum_{j=0}^{k-1} w_{k,j}(\xi, \tau) \ln^j \varepsilon^{1/4}, \quad \varepsilon \to +0. \tag{1.1}$$

Observe properties of the leading term of the expansion $\varepsilon^{1/4} w_{1,0}(\xi, \tau)$. The coefficient $w_{1,0}(\xi, \tau)$ is found using the Cole–Hopf transform

$$w_{1,0}(\xi, \tau) = -\frac{2}{\varphi''(0)\Lambda(\xi, \tau)} \frac{\partial \Lambda(\xi, \tau)}{\partial \xi},$$

where

$$\Lambda(\xi, \tau) = \int\limits_{-\infty}^{+\infty} \exp\left(-\frac{1}{8}(z^4 - 2z^2\tau + 4z\xi)\right) dz. \tag{1.2}$$

The argument of the exponent is a generating family of the Lagrange singularity A_3, see [1, 6].

Theorem 1. *The function $w_{1,0}$ satisfies the asymptotic relations*

$$w_{1,0}(\xi, \tau) = [\varphi''(0)]^{-1} H(\xi, \tau) + \sum_{l=1}^{\infty} h_{1-4l}(\xi, \tau), \quad 3[H(\xi, \tau)]^2 - \tau \to \infty,$$

$(\xi, \tau) \in \Omega_1 = \mathbb{R}^2 \setminus \{|\xi| < \tau^{\gamma_1 - 1/2}, \ \tau > 0, \ 0 < \gamma_1 < 2\}$, *where $H(\xi, \tau)$ is the Whitney fold function,* $H^3 - \tau H + \xi = 0$, $h_{1-4l}(\xi, \tau)$ *are homogeneous functions of power $1 - 4l$, relative to $H(\xi, \tau)$, $|\tau|^{1/2}$ and $\sqrt{3[H(\xi, \tau)]^2 - \tau}$, which are polynomials in $H(\xi, \tau)$, τ and $(3[H(\xi, \tau)]^2 - \tau)^{-1}$,*

$$w_{1,0}(\xi, \tau) = \sqrt{\tau}\left(-\frac{\operatorname{th} z}{\varphi''(0)} + \sum_{k=1}^{\infty} \tau^{-2k} q_k(z)\right), \quad \tau \to +\infty,$$

$(\xi, \tau) \in \Omega_2 = \{|\xi|\tau^{1/2} < \tau^{\gamma_2}, \ \gamma_1 < \gamma_2 < 2\}$, *where $z = \xi\sqrt{\tau}/2$, and the coefficients of the series for $k \geqslant 1$ satisfy the estimates $|q_k(z)| \leqslant M_k(1 + |z|^k)$.*
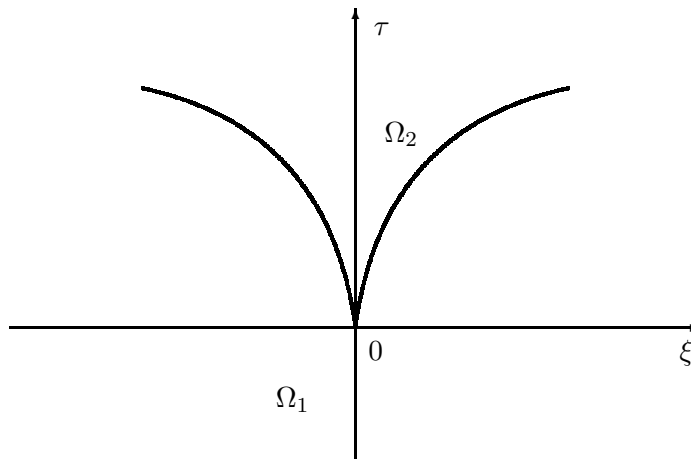

Fig. 1. Domains Ω_1 and Ω_2.

P r o o f of the theorem is based on the calculation of the asymptotics of the integral $\Lambda(\xi, \tau)$ by Laplace's method. In the domain Ω_1, see Figure 1, the essential contribution into the asymptotics

is given by one local maximum and in the domain Ω_2 by two local maxima; see Theorem 4.1 and Lemma 6.1 in [3, ch. VI]. □

Il'in investigated the problem under condition that

$$\varphi'(q(x)) = -x + x^3 + O(x^4), \quad x \to 0.$$

Now consider the following condition

$$\varphi'(q(x)) = -x + x^{2n+1} + O(x^{2n+2}), \quad x \to 0. \tag{1.3}$$

Let us clear out scales of the inner variables, which are introduced using the change

$$x = \eta\varepsilon^\sigma, \quad t = \theta\varepsilon^\mu, \quad u(x,t,\varepsilon) = \varepsilon^\varkappa u^*(\eta,\theta,\varepsilon). \tag{1.4}$$

Since all terms in equation (0.1) should be of the same order, we obtain the relations

$$-\mu = \varkappa - \sigma = 1 - 2\sigma. \tag{1.5}$$

From the characteristic equation $x = y + (t+1)\varphi'(q(y))$ with $y \in \mathbb{R}$ being an arbitrary parameter and condition (1.3) we have $x \approx -yt - y^{2n+1}$, whence by change (1.4) and the relation $u \sim -y$, we obtain $\sigma = \varkappa + \mu = (2n+1)\varkappa$. From these equalities and relations (1.5) we find

$$\sigma = \frac{2n+1}{2n+2}, \quad \mu = \frac{n}{n+1}, \quad \varkappa = \frac{1}{2n+2}. \tag{1.6}$$

Since $u = O(\varepsilon^{1/(2n+2)})$, equation (0.1) becomes the Burgers equation, whose solution can be written in the form of the Cole–Hopf transform. Moreover, the coefficient at s^{2n+2} is determined from the condition of matching the inner asymptotics and the outer expansion

$$u_{\text{out}} \sim \frac{\varepsilon^\varkappa U_0(\eta,\theta)}{\varphi''(0)}, \quad U_0^{2n+1} - \theta U_0 + \eta = 0.$$

To give some description of the behavior of asymptotic solutions, we will use the weighted Sobolev spaces $W_{p,q}^m(\mathbb{R})$ with the norm

$$\|u\|_{W_{p,q}^m(\mathbb{R})} = \sum_{l=0}^m \left(\int_{-\infty}^{+\infty} (1+|x|)^{-q} \left| \frac{\partial^l u(x)}{\partial x^l} \right|^p dx \right)^{1/p} \tag{1.7}$$

and also the proposed in [7] approach to the definition of an asymptotic solution of an evolutionary differential equation.

Theorem 2. *In the domain $\Omega_\varepsilon = \{(x,t) : |x\varepsilon^{-\varkappa}| + |t| < K\varepsilon^\mu, \ K > 0\}$ for any natural $n \geqslant 2$ the function*

$$u_{\text{in}}(x,t,\varepsilon) = -2\varepsilon[\varphi''(0)V(x,t,\varepsilon)]^{-1}\frac{\partial V(x,t,\varepsilon)}{\partial x}, \tag{1.8}$$

where

$$V(x,t,\varepsilon) = \int_{-\infty}^{+\infty} \exp\left(-\frac{2^{2n}s^{2n+2}}{n+1} + \frac{ts^2}{\varepsilon^\mu} - \frac{xs}{\varepsilon^\sigma} \right) ds, \tag{1.9}$$

$$\sigma = \frac{2n+1}{2n+2}, \quad \mu = \frac{n}{n+1}, \quad \varkappa = \frac{1}{2n+2},$$

is an asymptotic solution of equation (0.1) *in the following sense*:

$$\frac{\partial u_{\text{in}}/\partial t + \partial\varphi(u_{\text{in}})/\partial x - \varepsilon\partial^2 u_{\text{in}}/\partial x^2}{\displaystyle\sup_{(x,t)\in\Omega_\varepsilon}\left\{|\partial u_{\text{in}}/\partial t| + |c\partial\varphi(u_{\text{in}})/\partial x| + |\varepsilon\partial^2 u_{\text{in}}/\partial x^2|\right\}} = O\left(\varepsilon^\varkappa\right), \quad \varepsilon\to +0. \tag{1.10}$$

For $(m+1)p < (2n+1)(q-1)$ and any fixed t, $u_{\text{in}}(\cdot,t,\varepsilon)\in W_{p,q}^m(\mathbb{R})$ and $\|u_{\text{in}}(\cdot,t,\varepsilon)\|_{W_{p,q}^m(\mathbb{R})}\sim\varepsilon^{-m}$ as $\varepsilon\to +0$.

P r o o f. Formula (1.9) implies that the function $V(x,t,\varepsilon)$ satisfies the heat equation

$$\frac{\partial V}{\partial t} = \varepsilon\frac{\partial^2 V}{\partial x^2};$$

it follows that function (1.8) is the exact solution of the Burgers equation:

$$\frac{\partial u_{\text{in}}}{\partial t} + \varphi''(0)u_{\text{in}}\frac{\partial u_{\text{in}}}{\partial x} = \varepsilon\frac{\partial^2 u_{\text{in}}}{\partial x^2},$$

since u_{in} has the form of the Cole–Hopf transform (it can also be established by direct substitition). Then using Taylor's formula $\varphi'(u) = \varphi''(0)u + O(u^2)$, $u\to 0$, we easily obtain the following estimate:

$$\frac{\partial u_{\text{in}}}{\partial t} + \frac{\partial\varphi(u_{\text{in}})}{\partial x} - \varepsilon\frac{\partial^2 u_{\text{in}}}{\partial x^2} = \varphi'(u_{\text{in}})\frac{\partial u_{\text{in}}}{\partial x} - \varphi''(0)u_{\text{in}}\frac{\partial u_{\text{in}}}{\partial x} = O\left(u_{\text{in}}^2\frac{\partial u_{\text{in}}}{\partial x}\right). \tag{1.11}$$

It is also elementary checked that

$$u_{\text{in}}^2 = \frac{4\varepsilon^{2-2\sigma}}{[\varphi''(0)]^2}\frac{(V_\eta)^2}{V^2}, \quad \frac{\partial u_{\text{in}}}{\partial x} = -\frac{2\varepsilon^{1-2\sigma}}{\varphi''(0)}\left(\frac{V_{\eta\eta}}{V} - \frac{(V_\eta)^2}{V^2}\right), \quad \frac{\partial u_{\text{in}}}{\partial t} = -\frac{2\varepsilon^{1-\mu-\sigma}}{\varphi''(0)}\left(\frac{V_{\eta\theta}}{V} - \frac{V_\eta V_\theta}{V^2}\right),$$

in terms of the inner variables $\eta = \dfrac{x}{\varepsilon^\sigma}$ and $\theta = \dfrac{t}{\varepsilon^\mu}$, where the functional factors of the powers of ε have a finite order in the domain Ω_ε, since

$$|x\varepsilon^{-\varkappa}| + |t| = |\eta\varepsilon^{\sigma-\varkappa}| + |\theta\varepsilon^\mu| = \varepsilon^\mu(|\eta| + |\theta|).$$

Then estimate (1.11) becomes

$$\frac{\partial u_{\text{in}}}{\partial t} + \frac{\partial\varphi(u_{\text{in}})}{\partial x} - \varepsilon\frac{\partial^2 u_{\text{in}}}{\partial x^2} = O(\varepsilon^{3-4\sigma}), \quad \varepsilon\to +0,$$

and, taking into account that $1-\mu-\sigma = 2-3\sigma$ by (1.6), we find the exact order $\varepsilon^{2-3\sigma}$ of derivatives entering into the denominator in formula (1.10). Thus, we obtain the necessary estimate in the right-hand side of (1.10), since $1-\sigma = \varkappa$ by (1.6).

Further, differentiating (1.8) with respect to x and passing to the inner variable η, we derive the relation

$$\frac{\partial^l u_{\text{in}}}{\partial x^l} = \frac{\varepsilon}{V^{l+1}}\sum_{m_0+\cdots+m_l=l+1}a_{m_0\ldots m_l}\frac{\partial^{m_0}V}{\partial x^{m_0}}\cdots\frac{\partial^{m_l}V}{\partial x^{m_l}} = \frac{\varepsilon^{1-(l+1)\sigma}}{V^{l+1}}\sum_{m_0+\cdots+m_l=l+1}a_{m_0\ldots m_l}\frac{\partial^{m_0}V}{\partial\eta^{m_0}}\cdots\frac{\partial^{m_l}V}{\partial\eta^{m_l}}$$

with some constant coefficients $a_{m_0\ldots m_l}$. Using Laplace's method for the integral $V(x,t,\varepsilon)$, we write the equation

$$4^n(S(\eta,\theta))^{2n+1} - \theta S(\eta,\theta) + \frac{1}{2}\eta = 0,$$

for critical points by equating the derivative of the argument of the exponent in (1.9) to zero and taking into account the change of variables (1.4). For $\theta = \text{const}$, as $\eta \to -\infty$ we obviously obtain a unique critical point $s = S(\eta, \theta) > 0$, while as $\eta \to +\infty$ we obtain a unique critical point $s = S(\eta, \theta) < 0$. Then according to the standard formula [8, ch. II, sect. 2.4], we find the following asymptotics:

$$\frac{\partial^{m_j} V}{\partial \eta^{m_j}} \sim |\eta|^{m_j/(2n+1)}, \quad \eta \to \infty, \quad \frac{\partial^l u_{\text{in}}}{\partial x^l} \sim \varepsilon^{-l} |x|^{(l+1)/(2n+1)}, \quad x \to \infty.$$

Since in the domain Ω_ε the inner variables η, θ are finite and the function V (together with its derivatives) does not depend on ε in the leading order, by the definition (1.7) of the norm in the weighted Sobolev spaces $W^m_{p,q}(\mathbb{R})$, the above relations imply the asymptotic equivalence $\|u_{\text{in}}(\cdot, t, \varepsilon)\|_{W^m_{p,q}(\mathbb{R})} \sim \varepsilon^{-m}$ as $\varepsilon \to +0$ for any p, q, and m satisfying the inequality $(m+1)p < (2n+1)(q-1)$. Theorem 2 is proved. $\qquad\square$

2. Transition of a weak discontinuity into a strong one

In papers [9, 10] the solution of problem (0.1)–(0.2) is studied in the case when the initial function is smooth everywhere except for one point, at which it is continuous and has a jump discontinuity of the first derivative:

$$q(x) = -(x + ax^2) \, \Theta(-x) \, (1 + g_1(x)), \quad t_0 = -1,$$

where $a > 0$, $g_1 \in C^\infty(\mathbb{R})$, $g_1(x) = 0$ in some neighborhood of zero, Θ denotes the Heaviside function. Then $\varphi'(q(x)) = -(x + bx^2) \, \Theta(-x) \, (1 + g_2(x))$, where $b = a - \varphi'''(0)/2 > 0$, and such a weak discontinuity in the limit problem propagates a finite time along a characteristic, and then turns into a shock wave.

In a neighborhood of the singular point $(x = 0, \ t = 0)$ we introduce the stretched inner variables

$$\xi = \varepsilon^{-2/3} x, \quad \tau = \varepsilon^{-1/3} t.$$

The asymptotics of the solution has the form of the series

$$u(x, t, \varepsilon) = \sum_{p=2}^{\infty} \varepsilon^{p/6} \sum_{s=0}^{[p/2]-1} w_{p,s}(\xi, \tau) \ln^s \varepsilon, \quad \varepsilon \to +0.$$

The leading term of the expansion $\varepsilon^{1/3} w_{2,0}(\xi, \tau)$ is found using the Cole–Hopf transform (under the assumption that $\varphi''(0) = 1$)

$$w_{2,0}(\xi, \tau) = -\frac{2}{\Phi(\xi, \tau)} \frac{\partial \Phi(\xi, \tau)}{\partial \xi},$$

$$\Phi(\xi, \tau) = \int_0^{+\infty} \exp\left(-\frac{4b}{3} s^3 + \tau s^2 - \xi s\right) ds. \tag{2.1}$$

The argument of the exponent corresponds to a versal deformation for the boundary singularity B_3, whose general form is $s^3 + \lambda_1 s^2 + \lambda_2 s + \lambda_3$, see [11], however, the factor e^{λ_3} does not play a role in this case due to the form of the Cole–Hopf transform.

To describe the behavior of the functions $w_{p,s}(\xi, \tau)$ as $\xi^2 + \tau^2 \to \infty$, in the plane of independent variables the following overlapping domains are introduced:

$$X^0 = \{(\xi, \tau): \ |\xi| < |\tau|^{1-\gamma}, \ \tau < 0\}, \quad X^s = \{(\xi, \tau): |\xi - 3\tau^2/16b| < \tau^{2-\nu}, \ \tau > 0\}.$$

$$X^- = \{(\xi,\tau) : \xi < 0, \; -|\xi|^{2-\alpha} < \tau\} \cup \{(\xi,\tau) : \tau > 0, \; 16b\xi < 3\tau^2 - \tau^{\alpha-1}\},$$

$$X^+ = \{(\xi,\tau) : \xi > 0, \; -\xi^{2-\alpha} < \tau \leqslant 0\} \cup \{(\xi,\tau) : \tau > 0, \; 16b\xi > 3\tau^2 + \tau^{\alpha-1}\},$$

where $0 < \gamma < 1/2$, parameters α and ν are chosen so that $0 < \nu < \alpha < (1-2\gamma)/(1-\gamma) < 1$. Then the union $X^+ \cup X^0 \cup X^- \cup X^{\mathrm{s}}$ is a neighborhood of infinity in $\mathbb{R}^2$, see Figure 2.

Note that the domain X^{s} contains the parabola $16b\xi - 3\tau^2 = 0$, where the inner local maximum of the integrand in the right-hand side of (2.1) equals the boundary maximum. In each of these four domains the functions $w_{p,s}(\xi,\tau)$ have specific asymptotic behavior; corresponding series for the leading term $w_{2,0}(\xi,\tau)$ are given: in the domain X^0 by [10, Theorem 1], in the domain X^- by [10, Theorem 3], in the domain X^{s} by [10, Theorem 4], in the domain X^+ by [10, Theorem 5].

Now, let us consider the following case, where the initial function is such that

$$\varphi'(q(x)) = -(x + bx^{2n})\,\Theta(-x)\,(1 + g_{2n}(x)), \quad n \geqslant 1.$$

From the characteristic equation $x = y + (t+1)\varphi'(q(y))$ with y being a negative parameter and the relation $u \sim -y$ we have $x \approx -yt - by^{2n}$. Making a new change of variables (1.4), we obtain the equalities $\sigma = \varkappa + \mu = 2n\varkappa$. Taking into account relations (1.5), we find

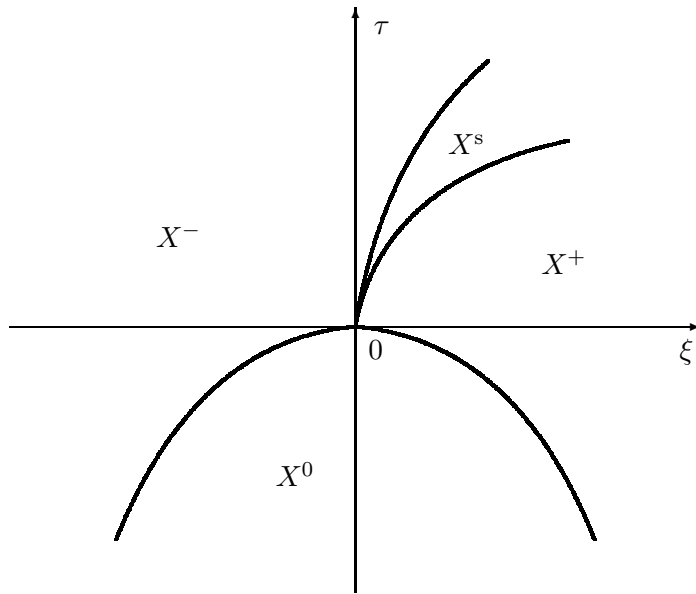$$\sigma = \frac{2n}{2n+1}, \quad \mu = \frac{2n-1}{2n+1}, \quad \varkappa = \frac{1}{2n+1}. \tag{2.2}$$

Fig. 2. Domains X^+, X^0, X^-, and X^{s}.

By analogy with Theorem 2 we obtain the following result.

Theorem 3. *In the domain* $\Omega'_\varepsilon = \{(x,t) : |x\varepsilon^{-\varkappa}| + |t| < K\varepsilon^\mu, \; K > 0\}$ *for any natural* $n \geqslant 2$ *the function*

$$u_{\mathrm{as}}(x,t,\varepsilon) = -2\varepsilon[W(x,t,\varepsilon)]^{-1}\frac{\partial W(x,t,\varepsilon)}{\partial x}, \tag{2.3}$$

$$W(x,t,\varepsilon) = \int\limits_0^{+\infty} \exp\left(-\frac{2^{2n}b}{2n+1}s^{2n+1} + \frac{ts^2}{\varepsilon^\mu} - \frac{xs}{\varepsilon^\sigma}\right)ds, \tag{2.4}$$

where numbers σ, μ, and $\varkappa$ are defined by formulas (2.2), *is an asymptotic solution of equation* (0.1) *in the following sense:*

$$\frac{\partial u_{\text{as}}/\partial t + \partial\varphi(u_{\text{as}})/\partial x - \varepsilon\partial^2 u_{\text{as}}/\partial x^2}{\sup\limits_{(x,t)\in\Omega_\varepsilon'}\left\{|\partial u_{\text{as}}/\partial t| + |\partial\varphi(u_{\text{as}})/\partial x| + \left|\varepsilon\partial^2 u_{\text{as}}/\partial x^2\right|\right\}} = O\left(\varepsilon^\varkappa\right), \quad \varepsilon \to +0. \tag{2.5}$$

For $(m+1)p < 2n(q-1)$ *and any fixed* t, $u_{\text{as}}(\cdot, t, \varepsilon) \in W_{p,q}^m(\mathbb{R})$ *and* $\|u_{\text{as}}(\cdot, t, \varepsilon)\|_{W_{p,q}^m(\mathbb{R})} \sim \varepsilon^{-m}$ *as* $\varepsilon \to +0$.

P r o o f. Formula (2.4) implies that the function $W(x, t, \varepsilon)$ satisfies the heat equation

$$\frac{\partial W}{\partial t} = \varepsilon\frac{\partial^2 W}{\partial x^2};$$

it follows that function (2.3) is the exact solution of the Burgers equation:

$$\frac{\partial u_{\text{as}}}{\partial t} + u_{\text{as}}\frac{\partial u_{\text{as}}}{\partial x} = \varepsilon\frac{\partial^2 u_{\text{as}}}{\partial x^2}.$$

Then, using the relation $\varphi'(u) = u + O(u^2)$, $u \to 0$, we easily obtain the following estimate:

$$\frac{\partial u_{\text{as}}}{\partial t} + \frac{\partial\varphi(u_{\text{as}})}{\partial x} - \varepsilon\frac{\partial^2 u_{\text{as}}}{\partial x^2} = \varphi'(u_{\text{as}})\frac{\partial u_{\text{as}}}{\partial x} - u_{\text{as}}\frac{\partial u_{\text{as}}}{\partial x} = O\left(u_{\text{as}}^2\frac{\partial u_{\text{as}}}{\partial x}\right). \tag{2.6}$$

It is also elementary checked that

$$u_{\text{as}}^2 = 4\varepsilon^{2-2\sigma}\frac{(W_\eta)^2}{W^2}, \quad \frac{\partial u_{\text{as}}}{\partial x} = -2\varepsilon^{1-2\sigma}\left(\frac{W_{\eta\eta}}{W} - \frac{(W_\eta)^2}{W^2}\right), \quad \frac{\partial u_{\text{as}}}{\partial t} = -2\varepsilon^{1-\mu-\sigma}\left(\frac{W_{\eta\theta}}{W} - \frac{W_\eta W_\theta}{W^2}\right)$$

in terms of the inner variable $\eta = x/\varepsilon^\sigma$ and $\theta = t/\varepsilon^\mu$, where the functional factors of the powers of ε have a finite order in the domain Ω_ε', since $|x\varepsilon^{-\varkappa}| + |t| = |\eta\varepsilon^{\sigma-\varkappa}| + |\theta\varepsilon^\mu| = \varepsilon^\mu(|\eta| + |\theta|)$. Then, estimate (2.6) becomes

$$\frac{\partial u_{\text{as}}}{\partial t} + \frac{\partial\varphi(u_{\text{as}})}{\partial x} - \varepsilon\frac{\partial^2 u_{\text{as}}}{\partial x^2} = O(\varepsilon^{3-4\sigma}), \quad \varepsilon \to +0,$$

and, taking into account that $1 - \mu - \sigma = 2 - 3\sigma$ by (2.2), we find the exact order $\varepsilon^{2-3\sigma}$ of derivatives entering into the denominator in formula (2.5). Thus, we obtain the necessary estimate in the right-hand side of (2.5), since $1 - \sigma = \varkappa$ by (2.2).

The estimate of the weighted norm $\|u_{\text{as}}(\cdot, t, \varepsilon)\|_{W_{p,q}^m(\mathbb{R})} \sim \varepsilon^{-m}$ as $\varepsilon \to +0$ is obtained almost exactly in the same way as in the proof of Theorem 2. $\square$

In conclusion, it should be said that by using a priori estimates of solutions of the parabolic equation (0.1) one can prove that the asymptotic solutions defined by formulas (1.8), (1.9) and (2.3), (2.4) are close to some exact solutions; however, such a proof needs uniform formal asymptotic solutions of equation (0.1) in a whole strip $\{t_0 \leqslant t \leqslant T, \ x \in \mathbb{R}\}$ containing a singular point to be constructed.

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A MATHEMATICAL MODEL
OF AN ARTERIAL BIFURCATION

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Abstract: An asymptotic model of an arterial bifurcation is presented. We propose a simple approximate method of calculation of the pressure drop matrix. The entries of this matrix are included in the modified transmission conditions, which were introduced earlier by Kozlov and Nazarov, and which give better approximation of 3D flow by 1D flow near a bifurcation of an artery as compared to the classical Kirchhoff conditions. The present modeling takes into account the heuristic Murrey cubic law.

Keywords: Stokes' flow, Bifurcation of a blood vessel, Modified Kirchhoff conditions, Pressure drop matrix, Murrey's law.

Introduction

The circulatory blood system in the human body contains about a billion bifurcation nodes, which help to distribute blood throughout the body and return to the heart. Most of the nodes belong to the capillary system, the smallest blood vessels, which are peculiar to their functioning laws, cf. [11]. Arterial and venous systems have also a considerable large number of nodes. The arterial nodes, in which the blood flows under significant pressure, are the most vulnerable elements of the circulatory system. Changes in the vessel wall geometry cause stress concentrations, provoking local damage, which in turn could create an obstacle to blood flow.

Our goal is to propose a simple approximate method of calculation *the pressure drop matrix Q* [18]. This is achieved by using classical shape optimization techniques. The pressure drop matrix was introduced in [12, 18] as an integral characteristic of a junction of several pipes with absolutely rigid walls. It appears that the elements of this matrix are included in the modified Kirchhoff transmission conditions, which describe more adequately the total pressure loss at the bifurcation point of the flow passed through the corresponding junction of the pipes, see [3, 9, 10].

In [10], a one-dimensional model of Stokes' flow at a junction of thin vessels with rigid walls for a fixed flow of the fluid at the inlet cross-sections and fixed peripheral pressure at the outlet cross-sections was developed. With the help of the pressure drop matrix Q apart the Neumann conditions (given flux) and the Dirichlet conditions (given pressure) at the external vertices, the one-dimensional Reynolds equations on the edges of the graph are supplied with transmission conditions at the internal vertices containing a small parameter h, where h is the thickness of the vessels, and passing into the classical Kirchhoff transmission conditions as $h \to +0$. It has been established that the pre-limit transmission conditions ensure an exponentially small relative error $O(e^{-\rho/h})$, $\rho > 0$, in the calculation of the solution to the three-dimensional problem, but the limit Kirchhoff conditions only give polynomially small relative error $O(h)$ for the pressure and $O(h^3)$ for the velocity vector.

For obvious reasons (cf. the discussion in [9]), the bifurcation node functioning is predetermined by two of its characteristics, that is, the lumen shape and the elastic wall properties. Moreover,

the influence of each of them causes distortion for the classical Kirchhoff conditions (that are only valid in the first approximation) in a one-dimensional model. The Kirchhoff transmission conditions require the vanishing of the total flux and the pressure matching at the adjacent ends of the vessels. So shrinking/dilatation of walls violates the first Kirchhoff law as a result of provoking the local currents. This aspect of the node functioning remains outside the scope of this paper. At the same time, it is easy to take into account the uniform shrinking/dilatation of the lumen of the elastic node (the change of coordinates $x \mapsto hx$ transforms the pressure drop matrix $Q \mapsto h^{-3}Q$, see the discussion in [9]) and therefore, neglecting elastic side effects, we restrict the study to a connected channel system with rigid walls. Of course, the problem considered finds applications in other matters of fluid mechanics, not only in hemodynamics.

In Section 1, we consider the Stokes system in an unbounded domain with cylindrical outlets to infinity (see, e.g., [1, 2, 4, 14, 21]) and prove the unique solvability of the problem stated in the class of "energy" solutions, see [12] for details. For obtaining the asymptotic behavior of the solution, we exploit special homogeneous solutions to the Stokes problem with non-zero flux and with a linear growth in the pressure at infinity (see [18]). As a consequence, we obtain a definition of the symmetric pressure drop matrix Q, which plays a crucial role in the functioning of the bifurcation node.

In Section 2, we apply asymptotic analysis for the study of the bifurcation of thin vessels with rigid walls. Using the method of matched asymptotic expansions (see, e.g., [6, 22]), the modified Kirchhoff transmission conditions have been derived.

Section 3 refers to some basics for implementation in computer modeling. The introduced matrix Q of pressure drops depends on the shape of a bifurcation region and possibly on its elastic properties, see Assertion 1 in Section 1. To find such dependence for real vessels it is possible only in solving inverse problems based on data obtained by the MRI method. Nevertheless, we propose an elementary procedure of finding these coefficients in the case of a three-dimensional Stokes system (cf. [9] in the case of a two-dimensional Stokes system).

The heuristic Murray law is used to describe the radii of bifurcating vessels (see [16] and [5, Ch. 3.3]). By minimizing the cost function, which is the sum of the friction power loss and the metabolic energy proportional to blood volume, Murray derived an optimal condition for a vascular bifurcation. Murray's law states that the cube of the radius of a parent vessel equals the sum of the cubes of the radii of the daughters, namely,

$$R_0^3 = R_+^3 + R_-^3. \tag{0.1}$$

It means that the radius of at least one of the two vessels $R_\pm$ after bifurcation differs a little from the radius R_0 of the vessel before the bifurcation. This law was verified experimentally [7, 8] but it did not get any theoretical justification. However, a number of optimization problems in branching arteries were solved by using this law (see, e.g., [5, Ch. 3.3]). We emphasize that the third powers of the radii often appears in haemodynamics.

Murray's law was a reason for several attempts (see [5, Ch. 3.3]) to derive a dynamic law of distribution of the blood flux in a bifurcating artery but should not be considered as successful. Such local law cannot exist since the flow leaving the main vessel is always defined by global data. The most frequently used in applied models of arterial flows are the classical Kirchhoff transmission conditions and their interpretations (see [17] on this topic and reference therein, and some reviews in [5, 20]). In any case, such conditions take into account neither the elastic properties of a vessel nor the most of the geometric characteristics of the bifurcation region. The modified Kirchhoff conditions have no these lacks.

In Section 4 it will be explained why the modification of the second Kirchhoff law by means of the pressure drop matrix unexpectedly deeply increases the accuracy of approach for three-dimensional fluid flow in a system of thin channels by the one-dimensional Reynolds–Poiseuille

model. In addition, we present a simple relation for a distributional law of Murrey's type.

1. Statement of the problem

1.1. Domains with cylindrical outlets to infinity and functional spaces

We introduce the domain Ω with three cylindrical outlets to infinity (see Fig. 1). Let Ω be an open unbounded domain with Lipschitz boundary $\partial\Omega$ admitting the representation

$$\Omega = \Omega' \cup \Omega^0 \cup \Omega^+ \cup \Omega^-, \quad \text{where} \quad \Omega^\alpha \cap \Omega^\beta = \varnothing \quad \text{for} \quad \alpha \neq \beta, \quad \alpha, \beta = 0, \pm.$$

Here, $\Omega^\alpha = \{x^\alpha = (y^\alpha, z^\alpha) : y^\alpha \in \omega_\alpha, z^\alpha > L_\alpha\}$ in a certain Cartesian coordinate system $x^\alpha = (y^\alpha, z^\alpha)$ in $\mathbb{R}^3$, where y^α are the variables in the cross-section of the outlet Ω^α, z^α is the variable along the axis of Ω^α, and ω_α is a bounded domain in $\mathbb{R}^2$. The bounded domain Ω' is given by $\Omega' = \{x \in \Omega : z^\alpha < L\}$ for certain L, $L > \max_\alpha L_\alpha$. Henceforth, $x = (x_1, x_2, x_3)$ is a global coordinate system in $\mathbb{R}^3$ related to the whole domain Ω.
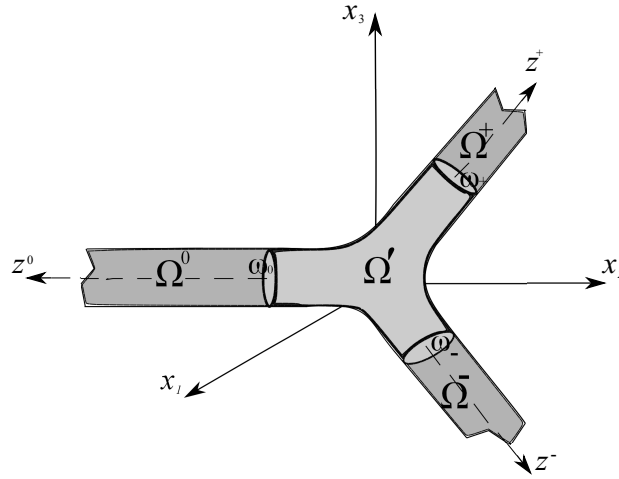

Figure 1. Artery bifurcation (the domain Ω).

We define $L_{2,\beta}(\Omega)$ as the space of measurable functions in Ω with a finite norm

$$||u||_{L_{2,\beta}(\Omega)} = \left(\int_{\Omega'} |u(x)|^2 dx + \sum_{\alpha=0,\pm} \int_{\Omega^\alpha} |z^\alpha|^{2\beta} |u(y^\alpha, z^\alpha)|^2 dy^\alpha dz^\alpha \right)^{1/2}.$$

If $\beta = 0$, we will use the usual notation $L_2(\Omega)$ for this space.

By using the Sobolev space $H^1(\Omega)$ together with $L_{2,1}(\Omega)$, we introduce the space of real-valued vector-functions in Ω

$$\mathcal{H}(\Omega) = \left\{ u = (u_1, u_2, u_3) \in (H^1(\Omega))^3 | \text{ div } u \in L_{2,1}(\Omega) \right\}$$

with the norm

$$||u||^2_{\mathcal{H}(\Omega)} = \int_\Omega (|\nabla u(x)|^2 + |u(x)|^2) dx + \sum_{\alpha=0,\pm} \int_{\Omega^\alpha} |z^\alpha|^2 |\text{div } u(y^\alpha, z^\alpha)|^2 dy^\alpha dz^\alpha.$$

Let also $\mathcal{H}_0(\Omega)$ be the subspace in $\mathcal{H}(\Omega)$ consisting of vector-functions equal to zero on $\partial\Omega$. The dual space of $\mathcal{H}_0(\Omega)$ is denoted by $(\mathcal{H}_0(\Omega))^*$.

1.2. Formulation of the problem

Consider the Dirichlet problem for the stationary Stokes system with nonzero divergence

$$-\nu\Delta u(x) + \nabla p(x) = F(x), \quad -\operatorname{div} u(x) = G(x), \quad x \in \Omega, \tag{1.1}$$

$$u(x) = 0, \quad x \in \partial\Omega. \tag{1.2}$$

Here, $u(x) = (u_1(x), u_2(x), u_3(x))$ is the velocity field, $p(x)$ is the pressure, and $\nu > 0$ is the viscosity of the fluid, which is assumed to be constant.

In order to define a weak solution of problem (1.1), (1.2), we introduce a bilinear form on $\mathcal{H}(\Omega)$:

$$a(u, w) = \sum_{j=1}^{3} \int_{\Omega} \nabla u_j \nabla w_j dx.$$

So, if (u,p) is a classical solution of (1.1), (1.2), then multiplying the first equation in (1.1) by $w \in \mathcal{H}_0(\Omega)$ and integrating over Ω, we obtain

$$\nu a(u, w) - \int_{\Omega} p \operatorname{div} w \, dx = \int_{\Omega} Fw \, dx \quad \text{for any} \quad w \in \mathcal{H}_0(\Omega). \tag{1.3}$$

A *weak solution* of problem (1.1), (1.2) is a pair $(u, p) \in \mathcal{H}_0(\Omega) \times L_{2,-1}(\Omega)$ satisfying the integral identity (1.3) for all $w \in \mathcal{H}_0(\Omega)$ and the equation $-\operatorname{div} u = G$ in Ω, where $F \in (\mathcal{H}_0(\Omega))^*$ and $G \in L_{2,1}(\Omega)$ are given.

To prove the main result of this section, we need the following statement.

Lemma 1. *For arbitrary $g \in L_{2,1}(\Omega)$ subject to*

$$\int_{\Omega} g(x)dx = 0, \tag{1.4}$$

there exists a vector-function $u \in \mathcal{H}_0(\Omega)$ such that $-\operatorname{div} u = g$ in Ω and

$$||u||_{\mathcal{H}(\Omega)} \leq c||g||_{L_{2,1}(\Omega)}.$$

Here, c is a constant independent of g.

P r o o f. We first consider an auxiliary problem; namely, we look for a solution of the boundary value problem

$$-\operatorname{div} u_\alpha(x^\alpha) = \eta_\alpha(y^\alpha)G_\alpha(z^\alpha), \quad x^\alpha \in \Omega^\alpha, \tag{1.5}$$

$$u_\alpha(x^\alpha) = 0, \quad x^\alpha \in \partial\Omega^\alpha, \tag{1.6}$$

where

$$\eta_\alpha(y^\alpha) \in C_0^\infty(\omega^\alpha), \quad \int_{\omega_\alpha} \eta_\alpha(y^\alpha)dy^\alpha = 1 \quad \text{and} \quad G_\alpha(z^\alpha) = \int_{\omega_\alpha} g(y^\alpha, z^\alpha)dy^\alpha.$$

One can verify directly that the vector-function

$$u_\alpha(x^\alpha) = (0, 0, \eta_\alpha(y^\alpha)w_\alpha(z^\alpha)), \quad w_\alpha(z^\alpha) = -\int_{z^\alpha}^{\infty} G_\alpha(t)dt,$$

solves problem (1.5), (1.6). Moreover, by Hardy's inequality (see [13, Theorem 5.2])

$$||w_\alpha||_{L_2(L_\alpha, \infty)} \leq c \int_{L_\alpha}^{\infty} |z^\alpha|^2 |G_\alpha|^2 dz^\alpha \leq C||g||_{L_{2,1}(L_\alpha, \infty)}. \tag{1.7}$$

Using (1.7), we obtain
$$\|u_\alpha\|_{\mathcal{H}(\Omega^\alpha)} \leq c_\alpha \|g\|_{L_{2,1}(\Omega^\alpha)}.$$

We are looking for a solution to $-\operatorname{div} u = g$ in the form

$$u = \sum_{\alpha=0,\pm} \chi_\alpha(z^\alpha) u_\alpha(y^\alpha, z^\alpha) + U(x) \tag{1.8}$$

for the equation $-\operatorname{div} u = g$ in Ω. Here, the cut-off functions χ_α are defined by

$$\chi_\alpha(z^\alpha) = 1 \quad \text{for} \quad z^\alpha > 2L_\alpha, \quad \chi_\alpha(z^\alpha) = 0 \quad \text{for} \quad z^\alpha < L_\alpha.$$

The function U in (1.8) satisfies the equation

$$-\operatorname{div} U = g + \operatorname{div}\Big(\sum_{\alpha=0,\pm} \chi_\alpha(z^\alpha) u_\alpha(y^\alpha, z^\alpha) \Big) \equiv \hat{g} \in L_{2,1}(\Omega)$$

in Ω. It is easy to see that

$$\int_{\omega^\alpha} \hat{g} \, dy^\alpha = 0 \quad \text{for} \quad z^\alpha > L_\alpha. \tag{1.9}$$

Let us introduce a local covering of Ω. Let

$$\Omega_j^\alpha = \{(y^\alpha, z^\alpha) : y^\alpha \in \omega_\alpha, \, L_\alpha + j - 1 < z^\alpha < L_\alpha + j + 3/2\}, \quad j = 1, \ldots$$

Then

$$\Omega = \Omega' + \sum_\alpha \sum_{j=1}^\infty \Omega_j^\alpha.$$

Let us consider the partition of unity corresponding to this covering:

$$1 = \phi'(x) + \sum_\alpha \sum_{j=1}^\infty \phi_j^\alpha(z^\alpha), \quad \text{where} \quad \phi_j^\alpha \in C_0^\infty(L_\alpha + j - 1, L_\alpha + j + 3/2)$$

and ϕ' is a smooth function supported in $\overline{\Omega'}$. We can write

$$\hat{g} = \hat{g}' + \sum_\alpha \sum_{j=1}^\infty \hat{g}_j^\alpha, \quad \text{where} \quad \hat{g}_j^\alpha = \phi_j^\alpha \hat{g}, \quad \hat{g}' = \phi'(x) \hat{g}.$$

By (1.9) and (1.4), we have

$$\int_{\Omega_j^\alpha} \hat{g}_j^\alpha \, dx^\alpha = 0, \quad \text{and,} \quad \int_{\Omega'} \hat{g}' \, dx = 0.$$

So, we obtain the problem in each bounded domain Ω_j^α:

$$-\operatorname{div} U_j^\alpha = \hat{g}_j^\alpha \quad \text{in} \quad \Omega_j^\alpha, \tag{1.10}$$

$$U_j^\alpha = 0 \quad \text{on} \quad \partial\Omega_j^\alpha, \tag{1.11}$$

and a similar problem for U' in Ω'. These problems have solutions $U_j^\alpha \in \mathcal{H}_0(\Omega_j^\alpha)$ (see [21]) which satisfy

$$\|U_j^\alpha\|_{\mathcal{H}(\Omega_j^\alpha)} \leq C \|\hat{g}_j^\alpha\|_{L_{2,1}(\Omega_j^\alpha)}.$$

Thus, the vector-function

$$U = U' + \sum_{\alpha} \sum_{j=1}^{\infty} U_j^{\alpha} \in \mathcal{H}_0(\Omega)$$

solves problem (1.10), (1.11) and satisfies

$$\|U\|_{\mathcal{H}(\Omega)} \le C_1\Big(\|U'\|_{\mathcal{H}(\Omega')} + \sum_{\alpha} \sum_{j=1}^{\infty} \|U_j^{\alpha}\|_{\mathcal{H}(\Omega_j^{\alpha})}\Big) \le C_2\|g\|_{L_{2,1}(\Omega)}.$$

The proof is complete.   □

Now we prove the existence and uniqueness of weak solutions to the boundary value problem (1.1)–(1.2), cf. [12].

Theorem 1. *Suppose that $F \in (\mathcal{H}_0(\Omega))^*$ and $G \in L_{2,1}(\Omega)$ is such that*

$$\int_{\Omega} G(x)dx = 0. \tag{1.12}$$

Then there exists a uniquely (up to an additive constant in the pressure p) determined weak solution $(u, p) \in \mathcal{H}_0(\Omega) \times L_{2,-1}(\Omega)$ of problem (1.1), (1.2). Furthermore,

$$\|u\|_{\mathcal{H}(\Omega)} + \|p\|_{L_{2,-1}(\Omega)} \le c\left(\|F\|_{(\mathcal{H}_0(\Omega))^*} + \|G\|_{L_{2,1}(\Omega)}\right).$$

Here, c is a constant independent of F and G.

P r o o f. Let $V(x) = u(x) - w(x)$, $x \in \Omega$, where $u(x)$ is a solution to the problem (1.1), (1.2) and the conditions of Lemma 1 for $w(x)$ are satisfied, namely,

$$-\mathrm{div}\, w(x) = g(x), \quad x \in \Omega, \quad w(x) = 0, \quad x \in \partial\Omega.$$

Then, we arrive at the Stokes problem

$$-\nu\Delta V(x) + \nabla p(x) = \hat{F}, \quad -\mathrm{div}\, V(x) = 0, \quad x \in \Omega, \tag{1.13}$$

$$V(x) = 0, \quad x \in \partial\Omega, \tag{1.14}$$

where $(\mathcal{H}_0(\Omega))^* \ni \hat{F}(x) = F(x) + \nu\Delta w(x)$. The weak formulation of problem (1.13), (1.14) is as follows: find $V \in \mathcal{H}_0^{div}(\Omega)$ such that

$$\nu a(V, W) = \int_{\Omega} \hat{F}\, dx \quad \text{for any} \quad W \in \mathcal{H}_0^{div}(\Omega). \tag{1.15}$$

Here, $\mathcal{H}_0^{div}(\Omega) = \left\{W \in H_0^1(\Omega): \ \mathrm{div}\, W = 0 \text{ in } \Omega\right\}$. By the Riesz theorem, there exists a uniquely determined vector-function $V \in \mathcal{H}_0^{div}(\Omega)$ such that (1.15) is satisfied and

$$\|V\|_{\mathcal{H}(\Omega)} \le c\|\hat{F}\|_{(\mathcal{H}_0(\Omega))^*}.$$

Here, we introduce

$$\mathcal{L}_{2,1}(\Omega) = \left\{g(x) \in L_{2,1}(\Omega): \ \int_{\Omega} g(x)dx = 0\right\}.$$

By Lemma 1, for any $q \in \mathcal{L}_{2,1}(\Omega)$, there exists a vector-function $u_q \in \mathcal{H}_0(\Omega)$ such that $-\operatorname{div} u_q = q$ in Ω and

$$||u_q||_{\mathcal{H}(\Omega)} \leq c||q||_{L_{2,1}(\Omega)}.$$

We consider the functional

$$G(q) = \int_\Omega \hat{F} u_q \, dx - \nu a(V, u_q) \tag{1.16}$$

on $L_{2,1}(\Omega)$. In virtue of

$$|G(q)| \leq c\left(||\hat{F}||_{(\mathcal{H}_0(\Omega))^*} + ||V||_{\mathcal{H}(\Omega)}\right)||u_q||_{\mathcal{H}(\Omega)} \leq c||\hat{F}||_{(\mathcal{H}_0(\Omega))^*}||q||_{L_{2,1}(\Omega)},$$

the linear functional $G(q)$ is continuous on $L_{2,1}(\Omega)$ and, by the Riesz theorem, there exist a unique element p of the dual space $L_{2,-1}(\Omega)$ such that

$$G(q) = \int_\Omega pq \, dx \quad \text{for all } q \in \mathcal{L}_{2,1}(\Omega).$$

Then, by (1.16), we get

$$\nu a(V, u_q) + \int_\Omega pq \, dx = \int_\Omega \hat{F} u_q \, dx,$$

or

$$\nu a(V, u_q) - \int_\Omega p \operatorname{div} u_q \, dx = \int_\Omega \hat{F} u_q \, dx \quad \text{for all } u_q \in \mathcal{H}_0(\Omega).$$

We have the estimate for

$$||p||_{L_{2,-1}(\Omega)} = \sup_{\mathcal{L}_{2,1}(\Omega):\; ||q||_{L_{2,1}(\Omega)}=1} \left| \int_\Omega pq \, dx \right|$$

$$= \sup_{\mathcal{L}_{2,1}(\Omega):\; ||q||_{L_{2,1}(\Omega)}=1} \left| \int_\Omega \hat{F} u_q \, dx - \nu a(V, u_q) \right| \leq c||\hat{F}||_{(\mathcal{H}_0(\Omega))^*}.$$

Finally, then $(u, p) = (V + w, p)$ is the required weak solution, and the estimate

$$||u||_{\mathcal{H}(\Omega)} + ||p||_{L_{2,-1}(\Omega)} \leq c\left(||F||_{(\mathcal{H}_0(\Omega))^*} + ||G||_{L_{2,1}(\Omega)}\right)$$

is fullfilled. $\qquad \square$

Remark 1. Consider a non-homogeneous Dirichlet problem for Stokes system, i.e., equations (1.1) are supplied with the boundary condition

$$u(x) = H, \quad x \in \partial\Omega, \tag{1.17}$$

where $H \in \mathcal{H}(\Omega)$, and, instead of (1.4), we require

$$\int_\Omega G(x)dx + \int_{\partial\Omega} H(x) \cdot n \, d\Gamma = 0,$$

where n is the unit outward normal to $\partial\Omega$. Substituting $u(x) = v(x) + H(x)$ into (1.1), (1.17), we obtain

$$-\nu\Delta v(x) + \nabla p(x) = f(x), \quad -\operatorname{div} v(x) = g(x), \quad x \in \Omega,$$

$$v(x) = 0, \quad x \in \partial\Omega,$$

where $f(x) = F(x) + \nu\Delta H(x) \in (\mathcal{H}_0(\Omega))^*$ and $g(x) = G(x) + \operatorname{div} H(x) \in L_{2,1}(\Omega)$ verifies (1.4). Now, the application of the previous theorem gives the existence of a pair $(v, p) \in \mathcal{H}_0(\Omega) \times L_{2,-1}(\Omega)$ solving problem (1.1), (1.17) and satisfying the estimate

$$||v||_{\mathcal{H}(\Omega)} + ||p||_{L_{2,-1}(\Omega)} \leq c\left(||f||_{(\mathcal{H}_0(\Omega))^*} + ||g||_{L_{2,1}(\Omega)} + ||H||_{\mathcal{H}(\Omega)}\right).$$

Moreover, p is defined up to an additive constant.

1.3. Asymptotics of the variational solution

Let the right-hand sides in (1.1), (1.2) satisfy

$$\int_{\Omega'} |F(x)|^2 dx + \sum_\alpha \int_{\Omega^\alpha} |F(x^\alpha)|^2 e^{2az^\alpha} dx^\alpha < \infty \tag{1.18}$$

and

$$\int_{\Omega'} |G(x)|^2 dx + \sum_\alpha \int_{\Omega^\alpha} |G(x^\alpha)|^2 e^{2az^\alpha} dx^\alpha < \infty, \tag{1.19}$$

where a is a positive number. Let also G be subject to (1.12). Then, according to Theorem 1, problem (1.1), (1.2) has a solution $(u, p) \in \mathcal{H}_0(\Omega) \times L_{2,-1}(\Omega)$. We can conclude that this solution satisfies the following asymptotic representation at infinity:

$$(u, p) = \sum_{\alpha = 0, \pm} \chi_\alpha c_\alpha(0, 1) + (\tilde{v}, \tilde{p}), \tag{1.20}$$

where $\chi_\alpha = \chi_\alpha(z^\alpha)$ are smooth functions equal to 1 for $z^\alpha > L_\alpha + 1$ and 0 for $z^\alpha < L_\alpha$, $(\tilde{v}, \tilde{p})$ are exponentially decaying terms as $z^\alpha \to \infty$, and c_α are real constants. Since this solution is defined up to an additive constant, we can (and will) assume $c_0 = 0$. Then the solution is unique, see Remark 1.

The remaining part of this section is devoted to finding formulas for evaluation of the constants c_+ and c_-. For this purpose, we need solutions of homogeneous problem (1.1), (1.2), which have a linear growth at infinity. More precisely, we introduce two linear independent solutions $(V^\pm, P^\pm)$ which have the following asymptotic representations (see [18]):

$$(V^\pm, P^\pm) = -\chi_0(\mathcal{V}^0, \mathcal{P}^0) + \chi_\pm(\mathcal{V}^\pm, \mathcal{P}^\pm) + (v^\pm, p^\pm), \tag{1.21}$$

where $(\mathcal{V}^\alpha, \mathcal{P}^\alpha)$ is the Poiseuille flow in the cylinder Ω^α, i.e., $\mathcal{V}_{y_i^\alpha}^\alpha(x) = 0$, $i = 1, 2$, $\mathcal{P}^\alpha(x) = -A_\alpha^{-1} z^\alpha$, and $\mathcal{V}_{z^\alpha}^\alpha = \mathcal{V}_{z^\alpha}^\alpha(y^\alpha)$ solves the following Dirichlet problem in ω_α:

$$\Delta \mathcal{V}_{z^\alpha}^\alpha = -A_\alpha^{-1} \quad \text{in } \omega_\alpha, \quad \mathcal{V}_{z^\alpha}^\alpha = 0 \ \text{ on } \partial\omega_\alpha. \tag{1.22}$$

The normalizing constant c_α is choosing to satisfy

$$\int_{\omega_\alpha} \mathcal{V}_{z^\alpha}^\alpha(y^\alpha) dy^\alpha = 1. \tag{1.23}$$

In the most important case of the circular cylinder, i.e., $\omega_\alpha = \{y^\alpha : |y^\alpha| < r_\alpha\}$, we have

$$\mathcal{V}_{z^\alpha}^\alpha(x) = \frac{2(r_\alpha^2 - |y^\alpha|^2)}{\pi r_\alpha^4}, \quad \mathcal{P}^\alpha(x) = \frac{-8\nu}{\pi r_\alpha^4} z^\alpha. \tag{1.24}$$

The remainder term $(v^\pm, p^\pm)$ in (1.21) satisfies the problem

$$-\nu\Delta v^\pm + \nabla p^\pm = f^\pm, \quad -\text{div}\, v^\pm = g^\pm \text{ in } \Omega,$$

$$v^\pm = 0 \quad \text{on } \partial\Omega,$$

where the right-hand sides

$$f^\pm := \nu\Delta(\chi_0 \mathcal{V}^0 - \chi_\pm \mathcal{V}^\pm) - \nabla(\chi_0 \mathcal{P}^0 - \chi_\pm \mathcal{P}^\pm) \tag{1.25}$$

and

$$g^\pm := \text{div}(\chi_0 \mathcal{V}^0 - \chi_\pm \mathcal{V}^\pm) \tag{1.26}$$

have compact supports. To verify condition (1.2) in Theorem 1 for $g^{\pm}$, we apply the Gauss theorem for the domain $\Omega_R = \{x \in \Omega : z^a < R\}$, where R is a sufficiently large number, and obtain

$$\int_{\Omega} g^{\pm} dx = \lim_{R \to \infty} \int_{\Omega_R} \mathrm{div}(\chi_0 \mathcal{V}^0 - \chi_{\pm} \mathcal{V}^{\pm}) dx = \int_{\omega_R^0} \mathcal{V}_{z^0}^0 (y^0) d\Sigma - \int_{\omega_R^{\pm}} \mathcal{V}_{z^{\pm}}^{\pm} (y^{\pm}) d\Sigma = 0.$$

Here we used the normalization condition (1.23). Therefore, $(v^{\pm}, p^{\pm})$ admits the asymptotic representation (1.20), where $c_0 = 0$ and $(\tilde{v}, \tilde{p})$ exponentially tends to zero as $z^{\alpha} \to \infty$.

Now we can present formulas for calculation of coefficients in (1.20)

Theorem 2. *Let the functions F and G satisfy* (1.18)–(1.19) *and let the asymptotic formula* (1.20) *be valid with $c_0 = 0$. Then*

$$c_{\pm} = \int_{\Omega} \left(FV^{\pm} + GP^{\pm} \right) dx. \tag{1.27}$$

P r o o f. Let Ω_R be the same domain as before. Multiplying equations (1.1), (1.2) by $(V^{\pm}, P^{\pm})$, integrating over Ω_R and using Green's formula, we obtain

$$\int_{\Omega_R} \left((-\nu \Delta v + \nabla p) V^{\pm} - \mathrm{div} v P^{\pm} \right) dx$$

$$= \sum_{\alpha} \int_{\omega_{\alpha}} \left(-\nu \left(V^{\pm} \partial_{z^{\alpha}} v_{z^{\alpha}} - v \partial_{z^{\alpha}} V_{z^{\alpha}}^{\pm} \right) + \left(p V_{z^{\alpha}}^{\pm} - P^{\pm} v_{z^{\alpha}} \right) \right) \Big|_{z^{\alpha} = R} dy^{\alpha}.$$

Taking here limit and using asymptotic formulas for $(V^{\pm}, P^{\pm})$ and (1.20) we arrive at (1.27). □

Applying formula (1.27) to the solution $(v^{\pm}, p^{\pm})$ of problem (1.1), (1.2) with the right hand sides given by (1.25), (1.26), we obtain the representations

$$(v^{\pm}, p^{\pm}) = \chi_{\pm} Q_{\pm\pm}(0, 1) + \chi_{\mp} Q_{\pm\mp}(0, 1) + (\tilde{v}_{\pm}, \tilde{p}_{\pm}), \tag{1.28}$$

where the coefficients are evaluated according to the formula

$$Q_{\gamma\tau} = \int_{\Omega} (f^{\gamma} V^{\tau} + g^{\gamma} P^{\tau}) dx, \quad \gamma, \tau = \pm. \tag{1.29}$$

From (1.21) and (1.28), we get the following representations:

$$(V^{\pm}, P^{\pm}) = -\chi_0 (\mathcal{V}^0, \mathcal{P}^0) + \chi_{\pm} (\mathcal{V}^{\pm}, \mathcal{P}^{\pm})$$
$$+ \chi_{\pm} Q_{\pm\pm}(0, 1) + \chi_{\mp} Q_{\pm\mp}(0, 1) + (\tilde{v}_{\pm}, \tilde{p}_{\pm}) \tag{1.30}$$

with the remainders $(\tilde{v}_{\pm}, \tilde{p}_{\pm})$ exponentially decaying at infinity. Note that a straightforward calculation gives the equality $Q_{\gamma\tau} = Q_{\tau\gamma}$. The coefficients $Q_{\gamma\tau}$ in expansion (1.30) of the pressure at infinity in $\Omega_{\pm}$ form the symmetric (2×2)-matrix Q called *the pressure drop matrix*. Another approach to introducing the matrix Q was presented in [18].

Assertion 1. *Since the replacement $p \mapsto \mathbf{p} = \nu^{-1} p$ eliminates the viscosity ν in the Stokes system* (1.1), *the matrix Q admits the representation $Q = \nu \mathbf{Q}$, where $\mathbf{Q}$ depends only on the geometry of the set Ω' (the bifurcation node).*

Remark 2. Solution (1.30) describes the flow of the fluid (blood) with unit flow through the cross-section ω_0 in the direction of the bifurcation node and its outflow through the cross-section $\omega_\pm$, that is, the flow through the other cross-section $\omega_\mp$ is zero. The constant component in the representation of the pressure at infinity in the supplying blood vessel Ω_0 is also made to be zero, but $P^\pm$ contains the linear component $A_0^{-1}z^0$, which together with $-A_\pm^{-1}z^\pm$ ensures a pressure drop between the infinitely distant cross-sections in the blood vessels Ω_0 and $\Omega_\pm$. The summands with the coefficients $Q_{\gamma\tau}$ in (1.30) also make their contribution, however, it is finite and, therefore, the (2×2)-matrix $Q = (Q_{\gamma\tau})_{\gamma,\tau=\pm}$ composed of them is called the pressure drop matrix. We note that the absence of the summand $\chi_0 Q_{0\pm}$ in (1.30) fixes the pressure $P_\pm$ and, therefore, the matrix Q itself, while earlier the arbitrariness in the solution to problem (1.1), (1.2) was eliminated by the condition $c_0 = 0$. Due to conditions (1.18), (1.19), we apply the fact that some external actions F and G in the model are negligible on cylindrical outputs at infinity, since we study in principal the problem with the bifurcation of an artery.

2. Asymptotic analysis of the bifurcation of thin channels with rigid walls

Let the bifurcation be characterized by a common small parameter $h > 0$. More precisely, suppose that

$$\Omega_h = \Omega' \cup \Omega_h^0 \cup \Omega_h^+ \cup \Omega_h^-,$$

$$\Omega_h^\alpha = \{x : \ |y^\alpha| < hr_\alpha, \ z^\alpha < l_\alpha\}, \quad \alpha = 0, \pm,$$

$r_\alpha > 0$ and $l_\alpha > 0$ are certain fixed radii and lengths, respectively, the radii r_0 and $r_\pm$ are comparable, and $h << \max\{1, r_0^{-1}r_\pm\}$.

In the domain Ω_h, consider a stationary Stokes flow. The velocity vector u^h and the pressure p^h satisfy the Stokes system

$$-\nu\Delta_x u^h(x) + \nabla_x p^h(x) = 0, \quad -\text{div } u^h(x) = 0, \quad x \in \Omega_h, \tag{2.1}$$

with the no-slip conditions

$$u^h(x) = 0, \quad x \in \Sigma_h = \partial\Omega_h \cap \partial\Omega. \tag{2.2}$$

on the lateral boundary.

Nonhomogeneous boundary conditions at the ends $\Gamma_h^\alpha = \{x \in \partial\Omega_h : z^\alpha = l_\alpha\}$, describing the inflow and outflow of the fluid, will not be used, but we can take them the same as in Section 4. Asymptotic ansatzes for a flow in a pipe Ω_h^α (in what follows, it will be called the Reynolds–Poiseuille ansatz) have the form

$$p^h(x) = p_\alpha(z^\alpha) + \ldots, \tag{2.3}$$

$$u_{y_i^\alpha}^h(x) = 0 + \cdots, \ i = 1, 2, \quad u_{z^\alpha}^h(x) = \frac{h^2}{4\nu}(r_\alpha^2 - h^{-2}|y_\alpha|^2)\partial_{z^\alpha}p_\alpha(z^\alpha) + \cdots \tag{2.4}$$

Here, $u_{y_i^\alpha}^h$ and $u_{z^\alpha}^h$ are the velocities along the axes y_i^α and z^α. The flux through the cross-section in the positive direction z^α is evaluated as

$$\int\limits_{|y^\alpha|<hr^\alpha} u_{z^\alpha}^h(x)|_{z^\alpha=z}dy^\alpha = h^4\frac{\pi r_\alpha^4}{8\nu} =: h^4 A_\alpha. \tag{2.5}$$

The unknown functions p_α, $\alpha = 0, \pm$, satisfy the classical one-dimensional Reynolds equations

$$-A_\alpha\partial_{z^\alpha}^2 p_\alpha(z^\alpha) = 0, \quad z^\alpha \in (0, l_\alpha). \tag{2.6}$$

Boundary conditions at the endpoints $z^\alpha = l_\alpha$ are not needed, but to adjust the solutions $p_\alpha(z^\alpha)$ at the common point $\mathcal{O}$ with coordinates $z^\alpha = 0$, we apply the method of matched asymptotic expansions (see, e.g., [6, 22]).

Theorem 3. *The solutions p_α, $\alpha = 0, \pm$, of the Reynolds Eqs. (2.6) satisfy the modified Kirchhoff transmission conditions at the bifurcation point $z^\alpha = 0$*

$$p_0(0) = p_\pm(0) + h \sum_{\tau=\pm} Q_{\pm\tau} A_\tau \partial_{z^\tau} p_\tau(0), \tag{2.7}$$

$$\sum_{\alpha=0,\pm} A_\alpha \partial_{z^\alpha} p_\alpha(0) = 0. \tag{2.8}$$

P r o o f. We stretch the coordinates with respect to the center of the bifurcation node:

$$x \longmapsto \xi = h^{-1}(x - \mathcal{O}). \tag{2.9}$$

Taking the formal limit at $h = 0$, we transform Ω_h into the unbounded domain Ω_1 with three outlets to infinity having the shape of the semicylinders $\Omega^{1\alpha} = \{\xi : |\eta^\alpha| < r_\alpha, \zeta^\alpha > 0\}$, where $\alpha = 0, \pm$, and $\xi^\alpha = (\eta^\alpha, \zeta^\alpha)$ are local coordinates obtained by stretching the coordinates (y^α, z^α) (cf. (2.9)).

As usual, for the application of the method of matched asymptotic expansions, it is required to find all solutions to the Dirichlet problem for the Stokes system in Ω_1, which may grow at infinity no faster than linear functions. One of such solutions is obvious, that is, the constant pressure

$$P^0(\xi) = 1, \quad V^0(\xi) = 0.$$

There are two more solutions (1.30) generated by the unit flux of fluid in Ω^{10} from infinity, which is compensated by an equal flux in $\Omega^{1\pm}$ at infinity as well. Now we will use the procedure of matching inner and outer expansions. As an *inner expansion*, we take the linear combinations

$$h \sum_{\pm} a_\pm^h V^\pm(\xi), \tag{2.10}$$

$$a_0^h P^0(\xi) + \sum_{\pm} a_\pm^h P^\pm(\xi). \tag{2.11}$$

The additional factor h is used in the velocity expansion (2.10) according to the general algorithms of construction of boundary layers (see, e.g., [15, Ch. 4]); it is needed, because the vector u^h in (2.1) is differentiated twice, but the scalar p^h only once.

By using the Taylor formula with respect to z^α, we rewrite the outer expansions (2.3) and (2.4) in the form

$$p^h(x) = p_\alpha(0) + z^\alpha \partial_{z^\alpha} p_\alpha(0) + \ldots = p_\alpha(0) + h\zeta^\alpha \partial_{z^\alpha} p_\alpha(0) + \ldots, \tag{2.12}$$

$$u^h(x) = \frac{h^2}{4\nu}(r_\alpha^2 - \frac{|y_\alpha|^2}{h^2})\partial_{z^\alpha} p_\alpha(0) + \ldots = \frac{h^2}{4\nu}(r_\alpha^2 - |\eta^\alpha|^2)\partial_{z^\alpha} p_\alpha(0) + \ldots \tag{2.13}$$

Comparing the coefficients of constant and linear functions in representations (2.12) and (2.11), we take into account (1.21)–(1.24) and (1.30) and arrive at the relations

$$p_0(0) = a_0^h, \quad p_\pm(0) = a_0^h + Q_{\pm+}a_+^h + Q_{\pm-}a_-^h,$$

$$hA_0\partial_{z^\alpha} p_0(0) = a_+^h + a_+^h, \quad hA_\pm\partial_{z^\alpha} p_\pm(0) = -a_\pm^h,$$

where the quantities A_α from (2.5) and (2.6) and (1.21)–(1.24) are used. These relations guarantee the matching of the leading terms in the asymptotics (2.4), (2.10), and (2.13) and imply the relations

$$a_0^h = p_0(0), \quad a_\pm^h = -hA_\pm\partial_{z^\alpha} p_\pm(0)$$

and (2.7), (2.8).   □

If we logically take the limit as $h \to 0$ in relations (2.7) and (2.8), then the latter terms in (2.7) vanish and we obtain the classical Kirchhoff transmission conditions

$$p_\pm(0) = p_0(0),$$
$$\sum_{\alpha=0,\pm} A_\alpha \partial_{z^\alpha} p_\alpha(0) = 0,$$

which mean the continuity of the pressure and the vanishing of the total flux from the bifurcation point (cf. (2.5) and (1.23)).

Let us explain why it is preferable to keep the last two terms in (2.7) despite they are small, see the discussions in [9, 10]. The solutions p^α to the differential equations (2.6) are linear functions

$$p^\alpha(z^\alpha) = c_\alpha^0(h) + z^\alpha c_\alpha^1(h), \quad \alpha = 0, \pm. \tag{2.14}$$

The coefficients c_α^n, $n = 1, 2$, can be found from the Kirchhoff conditions (2.7), (2.8), and boundary conditions at $z^\alpha = l_\alpha$, see Section 4. The quantities $c_\alpha^n = c_\alpha^n(h)$, $n = 1, 2$, are rational functions of the small parameter h, since they are determined from a system of linear algebraic equations with invertible matrix, polynomially (linear) dependent on h. The functions

$$p^h(x) = p^\alpha(z^\alpha) = c_\alpha^0(h) + z^\alpha c_\alpha^1(h),$$

$$u_{y_i^\alpha}^h(x) = 0, \quad i = 1, 2, \quad u_{z^\alpha}^h(x) = \frac{h^2}{4\nu}(r_\alpha^2 - h^{-2}|y_\alpha|^2)c_\alpha^1(h)$$

constructed according to (2.12), (2.13), and (2.14) form the Poiseuille flow in the cylindric parts of the vessels Ω_h^α and, hence, they exactly satisfy the Stokes system (2.1) and the no-slip condition (2.2) restricted to the corresponding parts of the literal surface.

Near the bifurcation region of Ω_h, there appears a boundary layer (we go over from the method of matched asymptotic expansions to the method of compound asymptotic expansions; cf. [15, Ch. 2]). By definition, relations (2.7) and (2.8) become conditions of an exponential decay for the boundary layer. As a result, we derive that the constructed approximate solution leaves in the problem (2.1), (2.2) exponentially small discrepancies as $h \to 0$. Thus, using the more complicated coupling conditions (2.7), (2.8), we obtain estimates of the asymptotic remainders with majorants $Ce^{-\rho/h}$, $\rho > 0$, whereas the application of the Kirchhoff conditions delivers remainders of order $O(h)$ in the asymptotic representation of pressure and of order $O(h^3)$ for the velocity.

3. Asymptotics of the pressure drop matrix

3.1. Elementary procedure for finding the pressure drop matrix. Murrey's law

Consider an infinite three-dimensional symmetric channel Υ with a bifurcation, which is depicted in Fig. 2, where its geometric parameters are shown. The walls are assumed to be rigid and the angle θ is small. The vertical dot-and-dash divide Υ into four parts: three semi-infinite cylinders $\Omega^\pm$ and Ω^0 and a middle part $\Omega^\bullet$, which is located between the points $z = 0$ and $z = -L_\bullet$ in the figure. The length and width of the middle section $\Omega^\bullet$ are evaluated as follows:

$$L_\bullet = (\sin\theta)^{-1}(2R - R_0\cos\theta) \approx (\sin\theta)^{-1}(2R - R_0), \quad R = R_\pm,$$
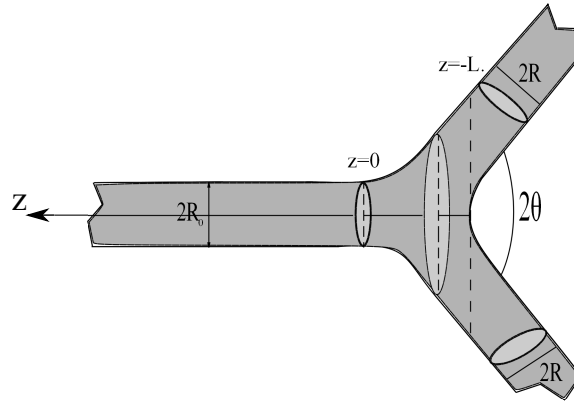$$2a_\bullet(z) = 2(R_0 - z\sin\theta). \tag{3.1}$$

Figure 2. Symmetric bifurcation of a three-dimensional channel.

We assume that the cross-section of the middle part $\Omega^\bullet$ is an ellipse whose semi-axes are connected by the relation

$$b_\bullet(z) = (1 + \beta z)a_\bullet(z), \quad z \in [-L, 0], \quad \text{with } \beta = L_\bullet^{-1}(2 + \sqrt{2})^{-1}.$$

Let us find approximate formulas for solutions (1.30) of problem (1.1), (1.2) in the three-dimensional junction Υ. Its symmetry with respect to the horizontal axis allows us to consider only one of the solutions, e.g., $(V, P) = (V^+, P^+)$.

Theorem 4. *Let the radii R_α, $\alpha = 0, \pm$, of the supplying Ω^0 and accepting $\Omega^\pm$ channels obey Murrey's law (0.1). Then the pressure drop matrix Q in (1.30) is positive definite.*

P r o o f. Owing to the large parameter $(\sin\theta)^{-1}$, we can use a three-dimensional Reynolds–Poiseuille ansatz (2.3), (2.4) ignoring the boundary layer effect, which brings an error of order $O(1)$, small with respect to $O((\sin\theta)^{-1})$. Thus, we seek the one-dimensional distribution of the pressure in the form

$$\begin{aligned}
P(x) &\approx P^0(z) = A_0^{-1}z & \text{for} \quad z \in (0, +\infty), \\
P(x) &\approx P^+(z) = A^{-1}z + Q_{++} & \text{for} \quad z \in (-\infty, -L_\bullet), \\
P(x) &\approx P^-(z) = Q_{+-} & \text{for} \quad z \in (-\infty, -L_\bullet), \\
P(x) &\approx P^\bullet(z) & \text{for} \quad z \in (-L_\bullet, 0).
\end{aligned} \tag{3.2}$$

In the three-dimensional case, the coefficients, cf. (2.5) and (1.21)–(1.24), are found by the formulas

$$A_\bullet(z) = \frac{\pi}{4\nu} \frac{a_\bullet^4(z)(1 + \beta z)^3}{(1 + (1 + \beta z)^2)}, \quad A_0 = \frac{\pi R_0^4}{8\nu}, \quad A = \frac{\pi R^4}{8\nu},$$

(see, e.g., [19] for calculation of an explicit formula in case of ellipse of the coefficient $A_\bullet$, which is proportional to *the torsion rigidity of a section*), and the Reynolds equation on $(-L_\bullet, 0)$ takes the form

$$-\partial_z(A_\bullet(z)\partial_z P^\bullet(z)) = 0. \tag{3.3}$$

Taking into account (3.2), solving the Reynolds Eq. (3.3) for unknown function $P^\bullet$ and satisfying the Kirchhoff transmission conditions at $z = 0$ and $z = -L_\bullet$, we find the quantities $Q_{\gamma\tau}$, $\gamma, \tau = \pm$, in (1.30):

$$\begin{aligned}
Q_{+-} &\approx P^\bullet(-L_\bullet) \approx -\frac{4\nu}{\pi\sin\theta}\phi_{+-}(R, R_0), \\
Q_{++} &\approx P^\bullet(-L_\bullet) + A^{-1}L_\bullet \approx -\frac{4\nu}{\pi\sin\theta}\phi_{++}(R, R_0).
\end{aligned} \tag{3.4}$$

Here, the functions $\phi_{+\pm}$ have explicit but cumbersome expressions and do not depend on the small parameter $\sin\theta$.

Using Sylvester's criterion for the pressure drop matrix Q, we state the assertion of Theorem 4.□
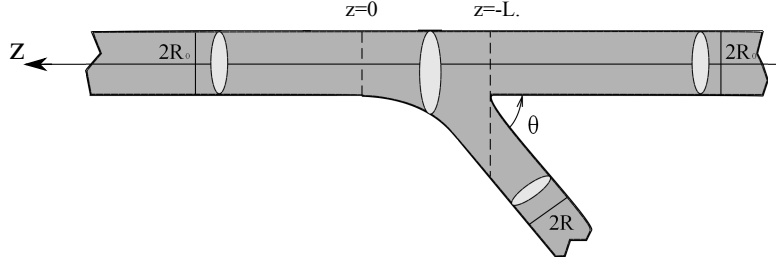

Figure 3. Asymmetric bifurcation of a three-dimensional channel.

A three-dimensional channel Υ with branching is depicted in Fig. 3, where all geometric parameters can be found. We accept the notation and assumptions from the symmetric bifurcation. Formulas (3.1) are replaced by

$$L_\bullet \approx (\sin\theta)^{-1}2R, \quad 2R_\bullet(z) = 2R_0 - z\sin\theta. \tag{3.5}$$

Here we assume that the cross-section of the middle part $\Omega^\bullet$ is a disc of radius $R_\bullet(z)$, $z \in [-L_\bullet, 0]$. Since there is no symmetry now, we must construct both solutions $(V_\pm, P_\pm)$.

Proposition 1. *The same Theorem 4 is valid for the asymmetric bifurcation (see Fig. 3).*

P r o o f. We start with $(V, P) = (V^+, P^+)$, the flux of which goes to the left channel Ω^+ of width $2R_0$ and, similarly to (3.2), we obtain

$$
\begin{aligned}
P(x) &\approx P^0(z) = A_0^{-1}z && \text{for} \quad z \in (0, +\infty), \\
P(x) &\approx P^+(z) = A_0^{-1}z + Q_{++} && \text{for} \quad z \in (-\infty, -L_\bullet), \\
P(x) &\approx P^-(z) = Q_{+-} && \text{for} \quad z \in (-\infty, -L_\bullet), \\
P(x) &\approx P^\bullet(z) && \text{for} \quad z \in (-L_\bullet, 0).
\end{aligned} \tag{3.6}
$$

We have the relations

$$A_\bullet(-L_\bullet) = \frac{\pi R_\bullet^4(z)}{8\nu}, \quad A_0 = \frac{\pi R_0^4}{8\nu}, \quad A = \frac{\pi R^4}{8\nu}.$$

Equation (3.3) is valid, and the Kirchhoff transmission conditions at $z = 0$ and $z = -L_\bullet$ show that

$$Q_{+-} \approx \frac{8\nu}{3\pi\sin\theta}\left(\frac{1}{(R_0+R)^3} - \frac{1}{R_0^3}\right), \quad Q_{++} \approx \frac{8\nu}{3\pi\sin\theta}\left(\frac{1}{(R_0+R)^3} - \frac{1}{R_0^3} + \frac{6R}{R_0^4}\right). \tag{3.7}$$

For the solution $(V, P) = (V^-, P^-)$, the flux of which goes to the lower (see Fig. 3) channel of width $2R$, the two middle relations in (3.6) must be changed according to

$$
\begin{aligned}
P(x) &\approx P^-(z) = A^{-1}z + Q_{--} && \text{for} \quad z \in (-\infty, -L_\bullet), \\
P(x) &\approx P^+(z) = Q_{-+} && \text{for} \quad z \in (-\infty, -L_\bullet).
\end{aligned}
$$

In this case, we obtain

$$Q_{-+} \approx \frac{8\nu}{3\pi\sin\theta}\left(\frac{1}{(R_0+R)^3} - \frac{1}{R_0^3}\right), \quad Q_{--} \approx \frac{8\nu}{3\pi\sin\theta}\left(\frac{1}{(R_0+R)^3} - \frac{1}{R_0^3} + \frac{6}{R^3}\right). \tag{3.8}$$

Using Sylvester's criterion, we state that the pressure drop matrix Q is positive definite. $\quad\square$

Remark 1. A possibility to obtain approximate formulas (3.4) and (3.7), (3.8) for the pressure drop matrix Q is provided by the fact that, for small θ, the junction elongates and has length $O((\sin\theta)^{-1})$ (see expressions (3.1) and (3.5) for $L_\bullet$). At the same time, the limit passage $\theta \to +0$ fails, i.e., one can use these results only for "not very small" angles θ.

Interesting to note that the entries (3.4) and (3.7), (3.8) of the matrix of pressure drops are inversely proportional to the cube of the radii in accordance with Murrey's distributional law (0.1).

4. Pressure drop matrix and modified Kirchhoff transmission conditions

Let us truncate cylindrical outlets in Ω and assume

$$\Omega_h = \Omega' \cup \Omega_h^0 \cup \Omega_h^+ \cup \Omega_h^-,$$
$$\Omega_h^\alpha = \left\{ x : \ |y^\alpha| < r_\alpha, \ z^\alpha < h^{-1} l_\alpha \right\}, \quad \alpha = 0, \pm, \tag{4.1}$$

where $h > 0$ is a small dimensionless parameter, and $r_\alpha > 0$ and $l_\alpha > 0$ are certain fixed radii and lengths, respectively. In the domain Ω_h, we define the homogeneous ($F = 0$, $G = 0$) Stokes equations (1.1) and, on its lateral surface $\Sigma_h = \partial\Omega_h \cap \partial\Omega$, we impose the homogeneous ($H = 0$) no-slip conditions (1.2) (hereinafter, we refer to these relations implying that they are restricted to these sets). On the truncated surfaces $\Gamma_h^\alpha = \left\{ x : \ |y^\alpha| < r_\alpha, \ z^\alpha = h^{-1} l_\alpha \right\}$, assign the following conditions:

$$u_{y_i^0}^h(x) = 0, \quad i = 1, 2, \quad u_{z^0}^h(x) = -\Psi_0^h(y^0), \quad x \in \Gamma_h^0, \tag{4.2}$$
$$u_{y_i^\tau}^h(x) = 0, \quad i = 1, 2, \quad -\nu\partial_{z^\tau} u_{z^\tau}^h(x) + p^h(x) = p^\infty, \quad x \in \Gamma_h^\tau, \quad \tau = \pm. \tag{4.3}$$

Here, Ψ_0^h is the Prandl function, that is, the solution of the Dirichlet problem for the Poisson equation (1.22). In other words, at the inlet cross-section of the vessel Ω_h^0, the incoming unit flux of fluid is assigned and, on the allocated ends of the outlet cross-sections of the vessels $\Omega_h^\pm$, peripheral pressure p_∞ is set. At the same time, the compression of coordinates by h^{-1} times transforms the problem stated to the usual problem of the blood flow through the bifurcation node of thin vessels, which walls, as already explained, it is assumed to be rigid (see Section 2, cf. [9]). In the new coordinates, the vessels become the smaller radii hr_α and the fixed lengths l_α. We emphasize that the problem (1.1), (1.2), (4.1), (4.2), (4.3) is still included in the symmetric Green formula in Ω_h. Its interpretation in the framework of the weighted spaces technique with detached asymptotics is given in [18]. The conventional Reynolds–Poiseuille ansatzes (see Section 2)

$$u^h(x) = -(8\nu)^{-1}\pi\Psi_\alpha^h(y^\alpha)e_{z^\alpha}\partial_{z^\alpha}p_\alpha(z^\alpha) + \dots, \quad p^h(x) = p_\alpha(z^\alpha) + \dots \tag{4.4}$$

(where e_{z^α} is the unit vector of the axis z^α directed away from the node) on the vessels Ω_h^α, after the substitution into (1.1), (1.2), (4.2), (4.3), generate the differential equations and boundary conditions

$$-A_\alpha\partial_{z^\alpha}^2 p_\alpha = 0, \text{ for } z^\alpha \in (0, h^{-1}l_\alpha), \quad A_\alpha = (8\nu)^{-1}\pi r_\alpha^4,$$
$$-A_0\partial_{z^0}p_0 = 1 \quad \text{at } z^0 = h^{-1}l_0, \quad p_\pm = p^\infty \quad \text{at } z^\pm = h^{-1}l_\pm$$

for unknown functions p_α, $\alpha = 0, \pm$. The latter functions linearly depend on the longitudinal variable z_α, and, thus, it is possible to remove the dots replacing lower-order asymptotic terms in the right-hand sides of Eqs. (4.4).

As an approximate solution of the problem stated in Ω_h, we take the sums

$$\hat{u}^h = ha^h_+ V^+ + ha^h_- V^-, \quad \hat{p}^h = a^h_+ P^+ + a^h_- P^- + a^h_0, \tag{4.5}$$

where $(V^\pm, P^\pm)$ are introduced special solutions (1.21) and the latter term refers to the constant pressure. Using the method of matched asymptotic expansions in the interpretation of ansatzes (4.4) as the outer expansions and the linear combination (4.5) as the inner expansions (see Section 2, cf. [9]), we satisfy the boundary conditions (4.2), (4.3) up to exponentially small terms as $h \to +0$ and get the following relations:

$$1 = a^h_+ + a^h_-, \tag{4.6}$$

$$p^\infty = a^h_0 - h^{-1}\mathcal{L}_\tau a^h_\tau + \sum_{\alpha=\pm} Q_{\tau\alpha} a^h_\alpha, \quad \tau = \pm, \tag{4.7}$$

where $\mathcal{L}_\alpha = \dfrac{8\nu}{\pi r^4_\alpha} l_\alpha,\ \alpha = 0, \pm$, and henceforth $\mathcal{L} = \mathrm{diag}\{\mathcal{L}_+, \mathcal{L}_-\}$.

Let $\mathbf{e} = (1, 1)$ and $\mathbf{a}^h = (a^h_+, a^h_-)$ be columns. In virtue of (4.7), we deduce

$$(p^\infty - a^h_0)\mathbf{e} = (Q - h^{-1}\mathcal{L})\mathbf{a}^h,$$

hence,

$$\mathbf{a}^h = (h^{-1}\mathcal{L} - Q)^{-1}\mathbf{e}(a^h_0 - p^\infty),$$

and thus equality (4.6) rewritten in the form $\mathbf{e} \cdot \mathbf{a}^h = 1$ leads to the relations

$$1 = T_h(a^h_0 - p^\infty),$$

$$T_h = \mathbf{e} \cdot (h^{-1}\mathcal{L} - Q)^{-1}\mathbf{e} = h^{-1}\mathbf{e} \cdot \mathcal{L}^{-1}(I - hQ\mathcal{L}^{-1})^{-1}\mathbf{e} = h^{-1}\mathbf{e} \cdot (L^{-1} + hL^{-1}QL^{-1} + O(h^2))\mathbf{e}.$$

We finally find that

$$a^h_0 = p^\infty + T^{-1}_h,$$

$$T_h = h^{-1}(t_0 + ht_1 + O(h^2)), \quad t_0 = \mathbf{e} \cdot \mathcal{L}^{-1}\mathbf{e} > 0, \quad t_1 = \mathbf{e} \cdot \mathcal{L}^{-1}Q\mathcal{L}^{-1}\mathbf{e}, \tag{4.8}$$

so, we get

$$a^h_0 = p^\infty + ht^{-1}_0(1 - ht^{-1}_0 t_1 + O(h^2)). \tag{4.9}$$

Thus, the pressure at the "input" Γ^0_h up to the smaller terms is equal to

$$h^{-1}\mathcal{L}_0 + p^\infty + ht^{-1}_0 - h^2 t^{-2}_0 t_1 + O(h^3). \tag{4.10}$$

The first term of (4.10) is the pressure drop, which provides the unit flux delivery to the artery bifurcation, the second term is also positive, it is necessary to supply the fluxes to the points $z^\pm = h^{-1}l_\pm$, and the third term, the sign of which depends on the pressure drop matrix Q, corresponds to just the shape of the node.

Proposition 2. *Let $a^h_\pm = 1/2 \pm b^h/2$ in accordance with (4.6), and let $b^h \in [-1, 1]$ be the factor of flux distribution. Then the partial Murrey cubic law*

$$b^0 = -\frac{1/r^3_+ - 1/r^3_-}{1/r^3_+ + 1/r^3_-} \tag{4.11}$$

is satisfied provided that $l_+/r_+ = l_-/r_-$.

P r o o f. From (4.7), we deduce

$$0 = -h^{-1}L_+(1 + b^h) + h^{-1}L_-(1 - b^h) + Q_{++}(1 + b^h)$$

$$+Q^{+-}(1 - b^h) - Q_{--}(1 - b^h) - Q_{-+}(1 + b^h),$$

or

$$-h^{-1}b^h(L_+ + L_-) + b^h(Q_{++} + Q_{--} - 2Q_{+-}) = h^{-1}(L_+ - L_-) - (Q_{++} - Q_{--}),$$

Hence,

$$b^h = -\frac{L_+ - L_- - h(Q_{++} - Q_{--})}{(L_+ + L_-) - h(Q_{++} + Q_{--} - 2Q_{+-})}.$$

If we put $l_+/r_+ = l_-/r_-$, then we arrive at (4.11) as $h \to +0$ or, similarly, the flux ratio is

$$\frac{1 + b^h}{1 - b^h} \approx \frac{1 - \dfrac{L_+ - L_-}{L_+ + L_-}}{1 + \dfrac{L_+ - L_-}{L_+ + L_-}} = \frac{L_-}{L_+} = \frac{l_-}{l_+}\frac{r_+^4}{r_-^4} = \frac{r_+^3}{r_-^3}.$$

□

5. Conclusion

For the arterial tree, under the assumption that the walls of the blood vessels are rigid, for every bifurcation node, a (2×2)-pressure drop matrix Q appears, and its influence on the Kirchhoff transmission conditions is taken into account. The modified Kirchhoff transmission conditions via the matrix Q depend on the geometry of the bifurcation region. We do not know the exact value of the matrix Q for any concrete bifurcation of an artery. In Section 3, we have produced the calculation scheme of an approximate determination of this matrix. With the help of Murrey's cubic law, we indicate general properties (sign definiteness) of the characteristics introduced for the certain nodes in which some information about the matrix is nevertheless accessible.

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